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The Higher Riemann-Hilbert Correspondence and Multiholomorphic Mappings

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Abstract
This thesis consists in two chapters. In the first part we describe an A_∞-quasi-equivalence of dg-categories between Block's $\Perf$, corresponding to the de Rham dga $\As$ of a compact manifold M and the dg-category of infinity-local systems on M. We understand this as a generalization of the Riemann-Hilbert correspondence to $\Z$-graded connections (superconnections in some circles). In one formulation an infinity-local system is an $(\infty,1)$-functor between the $(\infty,1)$-categories ${\pi}_{\infty}M$ and a repackaging of the dg-category of cochain complexes by virtue of the simplicial nerve and Dold-Kan. This theory makes crucial use of Igusa's notion of higher holonomy transport for $\Z$-graded connections which is a derivative of Chen's main idea of generalized holonomy. In the appendix we describe some alternate perspectives on these ideas and some technical observations.

The second chapter is concerned with the development of the theory of \emph{multiholomorphic maps}. This is a generalization in a particular direction of the theory of pseudoholomorphic curves. We first present the geometric framework of compatible n-triads, from which follows naturally the definition of a multiholomorphic mapping. We develop some of the essential analytic and differential-geometric facts about these maps in general, and then focus on a special case of the theory which pertains to the calibrated geometry of G_2-manifolds. This work builds toward the realization of invariants generated by the topology of the moduli spaces of multiholomorphic maps. Because this study is relatively fundamental, there will be many instances where questions/conjectures are put forward, or directions of further research are described.

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This thesis is dedicated to my parents, who let me play with fire.

ABSTRACT

The Higher Riemann-Hilbert Correspondence and Multiholomorphic Mappings

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Jonathan Block, Advisor

This thesis consists in two chapters. In the first part we describe an A_∞-quasi-equivalence of dg-categories between Block's $\mathcal{P}_{\mathsf{A}}$, corresponding to the de Rham dga A of a compact manifold M and the dg-category of infinity-local systems on M. We understand this as a generalization of the Riemann-Hilbert correspondence to $\mathbb{Z}$-graded connections (superconnections in some circles). In one formulation an infinity-local system is an $(\infty, 1)$-functor between the $(\infty, 1)$-categories $\pi_\infty M$ and a repackaging of the dg-category of cochain complexes by virtue of the simplicial nerve and Dold-Kan. This theory makes crucial use of Igusa's notion of higher holonomy transport for $\mathbb{Z}$-graded connections which is a derivative of Chen's main idea of generalized holonomy. In the appendix we describe some alternate perspectives on these ideas and some technical observations.

The second chapter is concerned with the development of the theory of *multiholomorphic maps*. This is a generalization in a particular direction of the theory of pseudoholomorphic curves. We first present the geometric framework of compatible n-triads, from which follows naturally the definition of a multiholomorphic mapping. We develop some of the essential analytic and differential-geometric facts about these maps in general, and then focus on a special case of the theory which pertains to the calibrated geometry of G_2-manifolds. This work builds toward the realization of invariants generated by the topology of the moduli spaces of multiholomorphic maps. Because this study is relatively fundamental, there will be many instances where questions/conjectures are put forward, or directions of further research are described.

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Chapter 1

The Higher Riemann-Hilbert Correspondence

1.1 Introduction and Summary

Given a compact manifold M, the classical Riemann-Hilbert correspondence gives an equivalence of categories between $Rep(\pi_1(M))$ and the category $Flat(M)$ of vector bundles with flat connection on M. While beautiful, this correspondence has the primary drawback that it concerns the truncated object π_1, which in most cases contains only a small part of the data which comprises the homotopy type of M. From the perspective of (smooth) homotopy theory the manifold M can be replaced by its infinity-groupoid $Sing_\bullet^\infty M := \pi_\infty M$ of smooth simplices. Considering the correct notion of a representation of this object will allow us to produce an untruncated Riemann-Hilbert theory. More specifically, we define an infinity-local system to be a map of simplicial sets which to each simplex of $\pi_\infty M$ assigns a homotopy coherence in the category of chain complexes over $\mathbb{R}, \mathcal{C} := Ch(\mathbb{R})$. Our main theorem is an A_∞-quasi-equivalence

$$\mathcal{RH} : \mathcal{P}_{\mathsf{A}} \to \mathsf{Loc}^{\mathcal{C}}(\pi_\infty M) \tag{1.1.1}$$

Where, $\mathcal{P}_{\mathsf{A}}$ is the dg-category of graded bundles on M with flat $\mathbb{Z}$-graded connection and $\mathsf{Loc}^{\mathcal{C}}(\pi_\infty M)$ is the dg-category of infinity-local systems on M.

In the classical Riemann-Hilbert equivalence the map

$$Flat(M) \to Rep(\pi_1(M)) \tag{1.1.2}$$

is developed by calculating the holonomy of a flat connection. The holonomy descends to a representation of $\pi_1(M)$ as a result of the flatness. The other direction

$$Rep(\pi_1(M)) \to Flat(M) \tag{1.1.3}$$

is achieved by the associated bundle construction.

In the first case our correspondence proceeds analogously by a calculation of the holonomy of a flat $\mathbb{Z}$-graded connection. The technology of iterated integrals suggests a precise and rather natural notion of such holonomy. Given a vector bundle V over M with connection, the usual parallel transport can be understood as a form of degree 0 on the path space PM taking values in the bundle $\mathrm{Hom}(ev_1^*V, ev_0^*V)$. The higher holonomy is then a string of forms of total degree 0 on the path space of M taking values in the same bundle. Such a form can be integrated over cycles in PM , and the flatness of the connection implies that such a pairing induces a representation of π_∞ as desired. This is the functor

$$\mathcal{RH} : \mathcal{P}_{\mathsf{A}} \to \mathsf{Loc}^{\mathcal{C}}(\pi_\infty M). \tag{1.1.4}$$

It would be an interesting problem in its own right to define an inverse functor which makes use of a kind of associated bundle construction. However we chose instead to prove quasi-essential surjectivity of the above functor. Given an infinity-local system (F, f) one can form a complex of sheaves over X by considering $\mathsf{Loc}^{\mathcal{C}}(\pi_\infty U)(\mathbb{R}, F)$. This complex is quasi-isomorphic to the sheaf obtained by extending by the sheaf of C^∞ functions and then tensoring with the de Rham sheaf. Making use of a theorem of Illusie we construct from this data a perfect complex of $\mathcal{A}^0$-modules quasi-isomorphic to the zero-component of the connection in $\mathcal{RH}(F)$. Finally we follow an argument of [Bl1] to complete this to an element of $\mathcal{P}_{\mathsf{A}}$ which is quasi isomorphic to $\mathcal{RH}(F)$.

We reserve the appendix to work out some of the more conceptual aspects of the theory as it intersects with our understanding of homotopical/derived algebraic geometry (in the parlance of Lurie, Simpson, Toen-Vezzosi, et. al.). One straightforward extension of this theory is to take representations in *any* linear ∞-category. In fact, considering representations of $\pi_\infty M$ in the category of A_∞-algebra leads to a fruitful generalization of recent work of Emma Smith-Zbarsky [S-Z] who has considered the action of a group G on families of A_∞ algebras over a $K(G,1)$.

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1.2 Infinity-Local Systems

1.2.1 ∞-categories

We first give a short commentary on some of the terminology of higher category theory that appears in this paper. All of the language and notation can be accessed in more complete form in [Lu2]. *Initially*, by ∞-category one means an abstract higher categorical structure which possesses a set of objects and a set of k-morphisms for each natural number k. The morphisms in the k-th level can be understood as morphisms between $(k-1)$-morphisms and so on. Then by (∞, n)-category one denotes such a structure for which all morphisms of level greater than n are in fact invertible up to higher level morphisms. One typically needs to deal with concrete mathematical objects which instantiate these kinds of structures. Hence there are several extant *models* for higher categories in the literature. We pick one for good so that there is no confusion as to what exactly we are dealing with when we talk about ∞-categories.

By ∞-category we will *henceforth* mean a weak Kan complex (also called a quasicategory). This is a particularly concrete model for $(\infty, 1)$-categories. All higher categorical structures we will see here have the invertibility condition above level 1. Hopefully this will not cause confusion as this naming convention is relatively standard in the literature. Suppose that Δ^k is the standard combinatorial k-simplex. By the q^{th} k-horn Λ^k_q we mean the simplicial set which consists of Δ^k with its k-face and the codimension-1 face $\partial_q \Delta^k$ removed. If $q \neq 0, k$ we call Λ^k_q an inner horn.

Definition 1.2.1. *A weak Kan complex is a simplicial set K which satisfies the property that any sSet map T: $\Lambda^k_q \to S$ from an inner horn to S, can be extended to a map from the entire simplex $\hat{T} : \Delta^k \to S$.*

This *weak Kan extension property* elegantly marries homotopy theory to higher category theory. In the case of a two-simplex, the weak Kan extension criterion guarantees the existence of a kind of weak composition between the two faces that make up the inner horn. See [Lu2] for elaboration on this philosophy.

In the same vein,

Definition 1.2.2. *A Kan complex is a simplicial set K which satisfies the Kan extension property of the previous definition for all horn inclusions, both inner and outer.*

A Kan complex models the theory of $(\infty, 0)$-categories because the outer horn extension conditions imply invertibility for the 1-morphisms. Hence, these are also referred to as ∞-*groupoids*. Later on we will refer to the ∞-groupoid of a space M.

1.2.2 Infinity Local Systems

Now we develop an higher version of a local system. These objects will be almost the same as the A_∞-functors of [Ig02], but tailored to suit our equivalence result. We want to emphasize the analogy with classical local systems. It will also be clear that there is yet another perspective in which these could be regarded as akin to twisting cochains. Let $\mathcal{C}$ be a pre-triangulated dg-category over k a characteristic 0 field (which we are implicitly regarding at $\mathbb{R}$ in this paper), and K a simplicial set which is a ∞-category. Fix a map $F : K_0 \to Ob\,\mathcal{C}$. Then define:

$$\mathcal{C}^k_F(K) := \bigoplus_{i+j=k, i\geq 0} \mathcal{C}^{i,j}_F, \tag{1.2.1}$$

with,

$$\mathcal{C}^{i,j}_F := \{\text{maps } f : K_i \to \mathcal{C}^j \mid\ f(\sigma) \in \mathcal{C}^j(F(\sigma_{(i)}), F(\sigma_{(0)}))\} \tag{1.2.2}$$

There are some obvious gradings to keep track of. For $f \in \mathcal{C}^{p,q}_F$ define

$$T(f) := (-1)^{p+q} f =: (-1)^{|f|} f, \quad K(f) := (-1)^q f, \quad J(f) := (-1)^p f \tag{1.2.3}$$

With respect to the simplicial degree in $\mathcal{C}^k_F$, we write

$$f = f^1 + f^2 + \ldots, \quad f^i \in \mathcal{C}^{i,\bullet}_F. \tag{1.2.4}$$

We define some operations on these maps:

$$(df^i)(\sigma_i) := d(f^i(\sigma_i)) \tag{1.2.5}$$

$$(\delta f^i)(\sigma_{i+1}) := \sum_{l=1}^{i} (-1)^l f^i(\partial_l(\sigma)) \tag{1.2.6}$$

$$\hat{\delta} = \delta \circ T(\bullet) \tag{1.2.7}$$

and for $g^p \in \mathcal{C}_F^{p,q}$,

$$(f^i \cup g^p)(\sigma \in K_{i+p}) := (-1)^{i(p+q)} f^i(\sigma_{(0\dots i)}) g^p(\sigma_{(i\dots p+i)}). \tag{1.2.8}$$

Extend by linearity to sums in $\oplus \mathcal{C}^s(K)$ so that the cup product is defined as the sum of the cups across all internal pairs of faces:

$$(f \cup g)(\sigma_k) := \sum_{t=1}^{k-1} (-1)^{t|g^{k-t}|} f^t(\sigma_{(0\dots t)}) g^{k-t}(\sigma_{(t\dots k)}) \tag{1.2.9}$$

We could suggestively write $f \cup g := \mu \circ (f \otimes g) \circ \Delta$ were Δ is the usual comultiplication which splits a simplex into a sum over all possible splittings into two faces:

$$\Delta(\sigma_k) = \sum_{p+q=k, p,q \geq 1} \sigma_p \otimes \sigma_q \tag{1.2.10}$$

and μ is the composition $\mathcal{C}^\bullet \otimes \mathcal{C}^\bullet \to \mathcal{C}^\bullet$. However, strictly speaking there is no Δ operator on a general simplicial set because one doesn't have a linear structure. The sign above appears because an i-simplex passes an element of total degree $p+q$ –consistent with the Koszul conventions.

Definition 1.2.3. *A pair (F, f) with $f \in \mathcal{C}_F^1(K)$ such that $0 = \hat{\delta} f + df + f \cup f$ is called an infinity-local system. The set of infinity-local systems valued in $\mathcal{C}$ is denoted $\mathsf{Loc}_\infty^{k,\mathcal{C}}(K)$*

We will often denote an infinity-local system (F, f) by just F if no confusion will arise.

In [I], Igusa defines a notion of an A_∞*-functor* associated to a flat, $\mathbb{Z}$-graded connection on a manifold. This is essentially the same thing as an infinity-local system on M, with only minor differences.

Example 1.2.4. *If F denotes an ordinary local system, then it naturally defines an infinity-local system.*

Proof. Exercise. □

Often it will be the case that the differential in $\mathcal{C}$ will be given by commutation with some family of degree-1 elements $d_x \in \mathcal{C}^1(x, x)$ i.e.,

$$df(\sigma_k) = d_{F(\sigma_{(0)})} \circ f_k(\sigma_{(0\ldots k)}) - (-1)^{|f_k|} f_k(\sigma_{(0\ldots k)}) \circ d_{F(\sigma_{(k)})} \tag{1.2.11}$$

A more conceptual description is the following

Definition 1.2.5. *(alternate) An infinity-local system on K valued in $\mathcal{C}$ is an element of $s\mathcal{S}et(K, \mathcal{C}_\infty)$, i.e. the set of $s\mathcal{S}et$-maps from K to $\mathcal{C}_\infty$.*

Here $\mathcal{C}_\infty$ is the simplicial set of homotopy coherent simplices in $\mathcal{C}$(appendix).

Definition 1.2.6. *We denote by $\mathsf{Loc}^{\mathcal{C}}_\infty(K)$, the category of infinity-local systems. The objects are infinity-local systems on the simplicial set K valued in the dg-category $\mathcal{C}$. We define a complex of morphisms between two infinity-local systems F ,G:*

$$\mathsf{Loc}^{k,\mathcal{C}}_\infty(K)(F, G) := \bigoplus_{i+j=k} \{\phi : K_i \to \mathcal{C}^j | \phi(\sigma) \in \mathcal{C}^j(F(\sigma_{(i)}), G(\sigma_{(0)}))\} \tag{1.2.12}$$

with a differential D:

$$D\phi := \hat{\delta}\phi + d\phi + G \cup \phi - (-1)^{|\phi|}\phi \cup F. \tag{1.2.13}$$

In the above, $\phi = \phi^0 + \phi^1 + \ldots$ is of total degree $|\phi| = p$, and

$$(\hat{\delta}\phi)(\sigma_k) := \delta \circ T = \sum_{j=1}^{k-1} (-1)^{j+|\phi|} \phi^{k-1}(\partial_j(\sigma_k)) \tag{1.2.14}$$

$$d\phi(\sigma_k) := d_{G(\sigma_{(0)})} \circ \phi(\sigma_{(0\ldots k)}) - \phi(\sigma_{(0\ldots k)}) \circ d_{F(\sigma_{(k)})} \tag{1.2.15}$$

Following the alternate definition above one might be tempted to define

Definition 1.2.7. *We define the space (actually a stable ∞-category) of infinity- local systems to be the usual simplicial mapping space*

$$\mathcal{F}un_{s\mathcal{S}et}(A, B)_k := s\mathcal{S}et(\Delta^k \times A, B) \tag{1.2.16}$$

Since this ∞-category is k-linear, it can be massaged into a dg-category coming from an application of a version of the Dold-Kan correspondence (specifically, applying $I \circ \tilde{M} \circ \mathfrak{C}[]$). However, this is certainly not the same as the dg-category $\mathsf{Loc}^{\mathcal{C}}_{\infty}(K)$, so as of this version we will not write down a definition in homotopical notation which directly enhances the definition of the objects.

Proposition 1.2.8. *$\mathsf{Loc}^{\mathcal{C}}_{\infty}(K)$ is a dg-category.*

Proof. $D^2 = 0$ follows from the following observations:

$$\hat{\delta}[F, G] = [\hat{\delta}F, G] + (-1)^{|F|}[F, \hat{\delta}G] \tag{1.2.17}$$

where $[,]$ is the graded commutator $[A, B] = A \cup B + (-1)^{|A||B|} B \cup A$,

$$[F, [G, H]] = [[F, G], H] + (-1)^{|F||G|}[G, [F, H]], \tag{1.2.18}$$

and the fact F and G are local systems:

$$\hat{\delta}F + dF + F \cup F = \hat{\delta}G + dG + G \cup G = 0 \tag{1.2.19}$$

□

We can define a shift functor in $\mathsf{Loc}^{\mathcal{C}}_{\infty}(K)$. Given $F \in \mathsf{Loc}^{\mathcal{C}}_{\infty}(K)$, define $F[q]$ via, $F[q](x \in K_0) := F(x)[q]$ and $F[q](\sigma_k) := (-1)^{q(k-1)} F(\sigma_k)$. For a morphism ϕ: $\phi[q](\sigma_k) = (-1)^{qk}\phi$.

And a cone construction: Given a morphism $\phi \in \mathsf{Loc}^{\mathcal{C}}_{\infty}(K)(F, G)$ of total degree q, define the map $C(\phi) : K_0 \to Ob\mathcal{C}$ by the assignment $x \mapsto F[1-q](x) \oplus G(x)$. And define the element $c(\phi)$ of $\mathcal{C}^1_{C(\phi)}$ via,

$$c(\phi) = \begin{pmatrix} f[1-q] & 0 \\ \phi[1-q] & g \end{pmatrix} \tag{1.2.20}$$

Unless ϕ is closed, this cone will not be an element of $\mathsf{Loc}_\infty^{\mathcal{C}}(K)$, but this useful construction will appear in our calculations later.

Evidently the Maurer-Cartan equation is preserved under this shift, and the resulting category is a pre-triangulated dg-category[Bondal-Kapranov].

Definition 1.2.9. *A degree* 0 *closed morphism* ϕ *between two infinity-local systems* F, G *over* K *is a homotopy equivalence if it induces an isomorphism in* $\mathrm{Ho}\,\mathsf{Loc}_\infty^{\mathcal{C}}(K)$.

We want to give a simple criterion for ϕ to define such a homotopy equivalence. On the complex $\mathsf{Loc}_\infty^{\bullet;\mathcal{C}}(K)(F,G)$ define a decreasing filtration by

$$F^k\mathsf{Loc}_\infty^{\bullet;\mathcal{C}}(K)(F,G) = \{\phi \in \mathsf{Loc}_\infty^{\bullet;\mathcal{C}}(K)(F,G))|\ \phi^i = 0 \text{ for } i < k\}$$

Proposition 1.2.10. *There is a spectral sequence*

$$E_0^{pq} \Rightarrow H^{p+q}(\mathsf{Loc}_\infty^{\bullet;\mathcal{C}}(K)(F,G) \tag{1.2.21}$$

where

$$E_0^{pq} = gr\ (\mathsf{Loc}_\infty^{\bullet;\mathcal{C}}(K)(F,G)) = \{\phi : K_p \to \mathcal{C}^q | \phi(\sigma) \in \mathcal{C}^q(F(\sigma_{(i)}), G(\sigma_{(0)}))\}$$

with differential

$$d_0(\phi^p) = d_G \circ \phi^p - (-1)^{p+q}\phi^p \circ d_F$$

Corollary 1.2.11. *For two infinity-local systems* F *and* G*, the* E_1*-term of the spectral sequence is a local system in the ordinary sense.*

Proposition 1.2.12. *A closed morphism* $\phi \in \mathsf{Loc}_\infty^{0,\mathcal{C}}(K)(F,G)$ *is a homotopy equivalence if and only if* $\phi^0 : (F_x, d_F) \to (G_x, d_G)$ *is a quasi-isomorphism of complexes for all* $x \in K_0$.

Proof. The proof follows as in the proof of Proposition 2.5.2 in [Bl1]. $\square$

1.3 Iterated Integrals and Holonomy of $\mathbb{Z}$-graded Connections

Now let $\mathsf{A} = (\mathcal{A}^\bullet(M), d)$ be the de Rham differential graded algebra (DGA) of a compact, C^∞-manifold M.

Definition 1.3.1. *$\pi_\infty M$ —the* ∞-groupoid *of M— is $Sing_\bullet^\infty M$, the simplicial set over $k = \mathbb{R}$ of C^∞-simplices.*

By $\mathcal{C}$ we denote the dg-category of (cohomological) complexes over $\mathbb{R}$. Our goal is to derive an A_∞-quasi-equivalence between $\mathcal{P}_{\mathsf{A}}$ and $\mathsf{Loc}_\infty^{\mathcal{C}}(\pi_\infty M)$. The former is a dg-category of *cohesive modules* [Bl1]; to wit, an object of $\mathcal{P}_{\mathsf{A}}$ is a pair $(E^\bullet, \mathbb{E})$ where $E^\bullet$ is a $\mathbb{Z}$-graded (bounded), finitely-generated, projective, right $\mathcal{A}^0$-module and $\mathbb{E}$ is a $\mathbb{Z}$-connection with the flatness condition $\mathbb{E} \circ \mathbb{E} = 0$. This category should not be confused with the category of dg-modules over the one-object dg-category $\mathcal{A}$. $\mathcal{P}_{\mathsf{A}}$ is a much fined invariant.

The second dg-category, $\mathsf{Loc}_\infty^{\mathcal{C}}(K)$ should be thought of as representations of $\pi_\infty M$. It's objects are coherent associations between simplices in M and simplices in the category of cochain complexes. Specifically, to any dg-category $\mathcal{C}$ over k we can associate the construction $\mathcal{C}_\infty$ which is a simplicial set whose simplices are *homotopy coherent* simplices in $\mathcal{C}$. Then an infty-local system is simplicial set map between the ∞-groupoid of M and this $\mathcal{C}_\infty$. The dg-categorical structure comes in an obvious way from the dg-category $\mathcal{C}$.

One reason for bringing out the homotopical language is for the sake of describing the Riemann-Hilbert correspondence as a kind of representability theorem. Since an infinity-local system is an element of $s\mathcal{S}et(\pi_\infty M, \mathcal{C}_\infty)$, and since $\pi_\infty M$ is a simplicial proxy for M itself, one could conceivably then describe this higher Riemann-Hilbert correspondence as a enriched representability of the functor which sends a space to its (linear) ∞-category (alternately its dg-category) of "homotopy-locally constant" sheaves of vector spaces (in our setup $M \mapsto (\mathcal{P}_{\mathsf{A}})_\infty$). In fact Toen and Vezzosi present something similar in [TV1] and [To1]. Their primary concern is a Segal Tannakian theory, and they show that for a CW complex X, $\pi_\infty X$ can be recovered from the category of simplicial local systems. Our Riemann-Hilbert correspondence concerns smooth manifolds (and uses iterated integrals) and so we have not developed a theorem which applies to CW complexes.

By the Serre-Swan correspondence, an object of $\mathcal{P}_{\mathsf{A}}$ corresponds to the smooth sections of a $\mathbb{Z}$-graded vector bundle $V^\bullet$ over M with the given flat $\mathbb{Z}$-connection. In [I] Kiyoshi Igusa presents from scratch a notion of higher parallel transport for a $\mathbb{Z}$-connection. This is a tweaked example of Chen's higher transport outlined in [Ch] which makes crucial use of his theory of iterated integrals. We slightly reformulate and extend this idea to produce a functor from $\mathcal{P}_{\mathsf{A}}$ to $\mathsf{Loc}^{\mathcal{C}}_\infty(K)$ which is an A_∞-quasi-equivalence. To start we present a version of iterated integrals valued in a graded endomorphism bundle.

1.3.1 Sign Conventions

Let V be a graded vector bundle on M, then $End(V)$ is a graded algebra bundle on M. The symbols T, J, K will be used to denote an alternating sign with respect to the total degree, form degree, and bundle-grading degree of a form valued in a graded bundle. For instance if $\omega \in V^q \otimes_{\mathcal{A}^0} \mathcal{A}^p$, then $T\omega = (-1)^{p+q}\omega, K\omega = (-1)^q\omega$, and $J\omega = (-1)^p\omega$. The similar convention carries over for forms valued in the $End(V)$ which has an obvious grading. Also, an element of $f \in End^k(V)$ can be broken into a sum $f = \sum_i f_i$ where $f_i : V^i \to V^{i+k}$. If we want to pick out this index, we write $I(f_i) := i$.

Given a form $A = (f \otimes \eta) \in End^k(V) \otimes_{\mathcal{A}^0} \mathcal{A}^p$ it is understood as a homomorphism

$$V^\bullet \otimes_{\mathcal{A}^0} \mathcal{A}^\bullet \to V^\bullet \otimes_{\mathcal{A}^0} \mathcal{A}^\bullet \tag{1.3.1}$$

via

$$(f \otimes \eta)(v \otimes \alpha) \mapsto (-1)^{|v||\eta|}(f(v) \otimes \eta \wedge \alpha) \tag{1.3.2}$$

Hence composition in $End^{\bullet}V \otimes_{\mathcal{A}^0} \mathcal{A}^{\bullet}$ is calculated,

$$(f \otimes \eta) \circ (g \otimes \rho) := (-1)^{|\eta||g|}(f \circ g \otimes \eta \wedge \rho) \tag{1.3.3}$$

There are obvious left and right actions of $\mathcal{A}^{\bullet}$

$$L_{\beta}(v \otimes \alpha) := (-1)^{|v||\beta|} v \otimes \beta \wedge \alpha \tag{1.3.4}$$

$$R_{\beta}(v \otimes \alpha) := v \otimes \alpha \wedge \beta \tag{1.3.5}$$

We will use the same notation for left and right contraction since context should separate them:

$$L : \Gamma(M, TM^{\otimes k}) \to Hom(End^{\bullet}V \otimes_{\mathcal{A}^0} \mathcal{A}^{\bullet}, End^{\bullet}V \otimes_{\mathcal{A}^0} \mathcal{A}^{\bullet - k}) \tag{1.3.6}$$

via

$$\xi = \xi_1 \otimes \ldots \otimes \xi_k \mapsto (f \otimes \alpha \mapsto f \otimes \alpha(\xi, \bullet)) \tag{1.3.7}$$

$$R : \Gamma(M, TM^{\otimes k}) \to Hom(End^{\bullet}V \otimes_{\mathcal{A}^0} \mathcal{A}^{\bullet}, End^{\bullet}V \otimes_{\mathcal{A}^0} \mathcal{A}^{\bullet - k}) \tag{1.3.8}$$

via

$$\xi = \xi_1 \otimes \ldots \otimes \xi_k \mapsto (f \otimes \alpha \mapsto f \otimes \alpha(\bullet, \xi)) \tag{1.3.9}$$

With these definitions we have the relations

$$L_{\xi}(A \circ B) = (L_{\xi}A) \circ KB = JA \circ (L_{\xi}B) \tag{1.3.10}$$

$$(A \circ B)R_{\xi} = A \circ (BR_{\xi}) = (AR_{\xi}) \circ TB \tag{1.3.11}$$

which shows that right contraction is much more natural.

With local coordinates in a trivializing patch, we have a differential d which acts on sections of V,

$$dv = (\frac{\partial}{\partial x^i} \otimes dx^i)(v) = (-1)^{|v|} \frac{\partial v}{\partial x^i} \otimes dx^i \tag{1.3.12}$$

and hence extends to V-valued forms via:

$$d(v \otimes \alpha) = (dv) \cdot \alpha + ((-1)^{|v|} v \otimes d\alpha). \tag{1.3.13}$$

With this definition, d satisfies the desideratum:

$$[d, L_\beta] := d \circ L_\beta - (-1)^{|\beta|} L_\beta \circ d = L_{d\beta}. \tag{1.3.14}$$

This d also defines a differential on sections of the endomorphism bundle via

$$df = (\frac{\partial}{\partial x^i} \otimes dx^i)(f) = (-1)^{|f|}(\frac{\partial f}{\partial x^i} \otimes dx^i), \tag{1.3.15}$$

which then defines a differential which acts on an endomorphism-valued form $A = \phi \otimes \eta$ via

$$d(A) := d \circ A - T(A) \circ d = d\phi \cdot \eta + (-1)^{|\phi|+|\eta|} \phi \otimes d\eta. \tag{1.3.16}$$

So d satisfies the graded Leibniz rule

$$[d, A] := d \circ A - TA \circ d = dA, \tag{1.3.17}$$

and it follows

$$d(A \circ B) = dA \circ B + TA \circ dB. \tag{1.3.18}$$

We also have a $\mathbb{Z}$-graded Stokes' Theorem:

$$\int_{M^n} (-1)^{n|f|} d(f \otimes \alpha) = \int_{\partial M} (-1)^{(n-1)|f|)} f \otimes \alpha \tag{1.3.19}$$

For this reason, if we have a projection $\pi : \Delta^k \times X \to X$, we define the push-forward

$$\pi_* : f \otimes \alpha \mapsto \int_{\Delta^k} (-1)^{k|f|)} f \otimes \alpha \tag{1.3.20}$$

so that with respect to this alternating definition of an integral, we get the usual Stokes' formula.

A $\mathbb{Z}$-graded connection in $\mathcal{P}_{\mathsf{A}}$ shifts by the formula $\mathbb{E}[q] = (-1)^q \mathbb{E}$. So, supposing that $\mathbb{E} = d - A_0 - A_1 \ldots - A_n$ in a local coordinate patch, we get

$$d[q] = (-1)^q d, \quad A_k[q] = (-1)^q A_k \tag{1.3.21}$$

1.3.2 Path Space Calculus

In [Ch] and earlier works, Chen defined a notion of a differentiable space –the archetypal differentiable space being PM for some smooth manifold M. This is a space whose topological structure is defined in terms of an atlas of plots –maps of convex neighborhoods of the origin in $\mathbb{R}^n$ into the space which cohere with composition by smooth maps– and the relevant analytic and topological constructs are defined in terms of how they pull back onto the plots. In particular one can construct a reasonable definition of vector bundles over a differentiable space as well as differential forms. One can likewise define an exterior differential, and subsequently a so-called Chen de Rham complex [Ha]. We will try to make transparent use of these constructions, but we defer the reader to the existing discussions of these matters in [Ch],[Ha],[I],[BH].

The primary reason that path-space calculus is relevant to our discussion is that the holonomy of a $\mathbb{Z}$-graded connection on V can be defined as a sequence of smooth forms on PM with values in the bundle $Hom(p_1^*V, p_0^*V)$. The usual parallel transport will be the 0-form part of the holonomy form. The higher terms will constitute the so-called *higher holonomy forms.*

1.3.3 Iterated Integrals

We opt to describe iterated integrals in a different but equivalent form than the recursive definition of Chen.

Let us parametrize the k-simplex by k-tuples $t = (1 \geq t_1 \geq t_2 \geq \ldots \geq t_k \geq 0)$. Then we define the obvious evaluation and projection maps:

$$ev_k : PM \times \Delta^k \to M^k : (\gamma, (t_1, \ldots, t_k)) \mapsto (\gamma(t_1), \gamma(t_2), \ldots, \gamma(t_k)) \tag{1.3.22}$$

$$\pi : PM \times \Delta^k \to PM \tag{1.3.23}$$

Let V be a graded bundle on PM, and in a trivializing patch we identify $End(V)$ as a graded matrix bundle $E := Mat^\bullet(V)$. Define ι to be the embedding

$$\iota : (\Gamma(E \otimes \Lambda^\bullet T^*X))^{\otimes_{\mathbb{R}} k} \to \Gamma(E^{\boxtimes k} \otimes \Lambda^\bullet T^* X^{\boxtimes k})$$

$$(a_1 \otimes_{\mathbb{R}} \ldots \otimes_{\mathbb{R}} a_k) \mapsto (a_1 \boxtimes \ldots \boxtimes a_k). \tag{1.3.24}$$

Given the space of forms $ev_k^*(\Gamma E^{\boxtimes k} \otimes \Lambda^\bullet T^*(X)^{\boxtimes k})$ we can use the multiplication in the fibers of E and $\Lambda^\bullet T^*(X)$ to define

$$\mu : ev_k^*\Gamma(E^{\boxtimes k} \otimes (\Lambda^\bullet T^* X)^{\boxtimes k}) \to \Gamma(p_0^* E \otimes \Lambda^\bullet T^*(PM \times \Delta^k)). \tag{1.3.25}$$

Definition 1.3.2. *The iterated integral map is the composition*

$$\int a_1 a_2 \ldots a_k := (-1)^{\spadesuit} \pi_*(\mu(ev_k^*(\iota(a_1 \otimes_{\mathbb{R}} \ldots \otimes_{\mathbb{R}} a_k)))) \tag{1.3.26}$$

with

$$\spadesuit = \sum_{1 \leq i < k} (T(a_i) - 1)(k - i). \tag{1.3.27}$$

Since E is graded, the elements $\{a_i\}$ are bi-graded as usual, with $T(\bullet)$ denoting the total degree.

1.3.4 $\mathbb{Z}$-graded Connection Holonomy

Suppose V has a $\mathbb{Z}$-connection $\mathbb{E}$. Locally $\mathbb{E}$ is of the form $d - [A^0 + A^1 + \ldots + A^m]$. (With the above conventions, $(-1)^k d$ is locally the trivial connection on E^k) Let $\omega = A^0 + A^1 + \ldots + A^m$. This is a form of total degree 1, i.e. in $\oplus End^{1-i}(V) \otimes_{\mathcal{A}^0} \mathcal{A}^i$. To any such form we can associate its holonomy/parallel transport

$$\Psi := I + \int \omega + \int \omega\omega + \int \omega\omega\omega + \ldots \tag{1.3.28}$$

which breaks further into its components with respect to the "form-grading". For instance,

$$\Psi_k = \int A^{k+1} + \sum_{i+j=k+2} \int A^i A^j + \sum_{i_1+i_2+i_3=k+3} \int A^{i_1} A^{i_2} A^{i_3} + \ldots \tag{1.3.29}$$

Chen calculated the differential of a holonomy form (without the graded changes we have worked into our definition)

$$d\Psi = -\int \kappa + (-\int \kappa\omega + \int J\omega\kappa) + \dots \\ + \sum_{i+j=r-1} (-1)^{i+1} \int (J\omega)^i \kappa \omega^j + \dots + -p_0^*\omega \wedge \Psi + J\Psi \wedge p_1^*\omega. \quad (1.3.30)$$

where $\kappa = d\omega - J\omega \wedge \omega$ defines the curvature of ω, and the integral itself is defined with different sign conventions.

An analog of the above calculation can be proved with two basic lemmas which are modifications of Chen. Let $\partial\pi$ is the composition

$$\pi \circ (in \times id): \quad \partial\Delta^k \times X \xrightarrow{in \times id} \Delta^k \times X \xrightarrow{\pi} X. \quad (1.3.31)$$

and,

$$\pi_*(f(t,x)_I \otimes dVol_{\Delta^k} \wedge dx^I) := (\int_{\Delta^k} (-1)^{k|f|} f(t,x)_I dVol_{\Delta^k}) \otimes dx^I \quad (1.3.32)$$

the alternating integration along the fiber as in the previous section. Then,

Lemma 1.3.3.

$$\pi_* \circ d - (-1)^k d \circ \pi_* = (\partial\pi)_* \circ (in \times id)^* \quad (1.3.33)$$

And for any $A \in End^\bullet(V^\bullet) \otimes \mathcal{A}(PM)$, $B \in End^\bullet(V^\bullet) \otimes \mathcal{A}(\Delta^k \times PM)$,

$$\begin{aligned} \pi_*(\pi^*(A) \circ B) &= (-1)^{kT(A)} A \circ \pi_* B \\ \pi_*(B \circ \pi^* A) &= (\pi_* B) \circ A \end{aligned} \quad (1.3.34)$$

Proof. Exercise □

Proposition 1.3.4.

$$d\int \omega_1 \dots \omega_r =$$
$$= \textstyle\sum_{i=1}^{r}(-1)^i \int T\omega_1 \dots d\omega_i\omega_{i+1}\dots\omega_r + \sum_{i=1}^{r-1}(-1)^i \int T\omega_1 \dots (T\omega_i \circ \omega_{i+1})\omega_{i+2}\dots\omega_r +$$
$$+p_1^*\omega_1 \circ \textstyle\int \omega_2 \dots \omega_r - T(\int \omega_1 \dots \omega_{r-1}) \circ p_0^*\omega_r \tag{1.3.35}$$

Proof. Using the previous lemma, and the definition of iterated integrals, we have

$$d\int \omega_1 \dots \omega_r = (-1)^{\spadesuit} d \circ \pi_* \circ \mu \circ ev_r^* \circ \iota(\omega_1 \otimes \dots \otimes \omega_r) =$$
$$= (-1)^{\spadesuit + r}\pi_* \circ d \circ \mu \circ ev_r^* \circ \iota(\omega_1 \otimes \dots \otimes \omega_r) - (-1)^{\spadesuit + r}(\partial\pi)_* \circ (in \times id)^* \circ \mu \circ ev_r^* \circ \iota(\omega_1 \otimes \dots \otimes \omega_r) \tag{1.3.36}$$

Note that the faces of Δ^k are the sets $\{t_1 = 1\}$, $\{t_k = 0\}$, and $\{t_i = t_{i+1}\}$ and that d commutes with μ in the (Koszul) graded sense. Hence this formula expands/reduces to

$$(-1)^{\spadesuit + r}\pi_* \circ \mu \circ ev_r^* \circ \iota(\sum_i T\omega_1 \otimes \dots \otimes d\omega_i \otimes \dots \otimes \omega_r)$$
$$- (-1)^{\spadesuit + r}(\partial\pi)_* \circ (in \times id)^* \circ \mu \circ ev_r^* \circ \iota(\omega_1 \otimes \dots \otimes \omega_r) \tag{1.3.37}$$

$$= \sum_i^r (-1)^i \int T\omega_1 \dots d\omega_i \dots \omega_r$$
$$- (-1)^{\spadesuit + r}[\int_{t_1 = 1} \mu(p_1^*\omega_1 \otimes ev_{r-1}^* \circ \iota(\omega_2 \otimes \dots \otimes \omega_r))$$
$$+ \int_{t_k = 0} \mu(ev_{r-1}^* \circ \iota(\omega_1 \otimes \dots \otimes \omega_{r-1}) \otimes p_0^*\omega_r)$$
$$+ \sum_i \int_{t_i = t_{i+1}} \mu(ev_{r-1}^* \circ \iota(\omega_1 \otimes \dots \otimes \omega_i \circ \omega_{i+1} \otimes \dots \otimes \omega_r))] \tag{1.3.38}$$

$$= \sum_{i}^{r}(-1)^i \int T\omega_1 \dots d\omega_i \dots \omega_r$$

$$- (-1)^{\spadesuit + r}[(-1)^{T(\omega_1)(r-1)} p_1^*\omega_1 \circ \int_{t_1=1} \mu(ev_{r-1}^* \circ \iota(\omega_2 \otimes \dots \otimes \omega_r))$$

$$+ \int_{t_k=0} \mu(ev_{r-1}^* \circ \iota(\omega_1 \otimes \dots \otimes \omega_{r-1})) \circ p_0^*\omega_r$$

$$+ \sum_i \int_{t_i=t_{i+1}} \mu(ev_{r-1}^* \circ \iota(\omega_1 \otimes \dots \otimes \omega_i \circ \omega_{i+1} \otimes \dots \otimes \omega_r))] \quad (1.3.39)$$

$$= \sum_{i}^{r}(-1)^i \int T\omega_1 \dots d\omega_i \dots \omega_r + p_1^*\omega_1 \circ \int \omega_2 \dots \omega_r$$

$$-(-1)^{\sum_{i=1}^{r-1}(T(\omega_i)-1)} \int \omega_1 \dots \omega_{r-1} \circ p_0^*\omega_r - \sum_i (-1)^{\sum_{j=1}^{i}(T(\omega_j)-1)} \int \omega_1 \dots (\omega_i \circ \omega_{i+1}) \dots \omega_r$$

$$(1.3.40)$$

$$= \sum_{i}^{r}(-1)^i \int T\omega_1 \dots d\omega_i \dots \omega_r + p_1^*\omega_1 \circ \int \omega_2 \dots \omega_r$$

$$- T(\int \omega_1 \dots \omega_{r-1}) \circ p_0^*\omega_r + \sum_{i=1}^{r} \int T\omega_1 \dots (T\omega_i \circ \omega_{i+1}) \dots \omega_r \quad (1.3.41)$$

On the subset of the path space with fixed endpoints, $PM(x_0, x_1)$ the pullbacks p_i^* kill all but 0-forms hence we get

$$d \int \omega_1 \dots \omega_r =$$

$$\sum_{i}^{r}(-1)^i \int T\omega_1 \dots d\omega_i \dots \omega_r + p_1^*\omega_1^0 \circ \int \omega_2 \dots \omega_r$$

$$- T(\int \omega_1 \dots \omega_{r-1}) \circ p_0^*\omega_r^0 + \sum_{i=1}^{r}(-1)^i \int T\omega_1 \dots (T\omega_i \circ \omega_{i+1}) \dots \omega_r \quad (1.3.42)$$

□

If ω has total degree 1, then

$$d \int (\omega)^r = \sum_{i+j+1=r} \int (\omega)^i d\omega (\omega)^j + \sum_{i+j+2=r} \int (\omega)^i (\omega \circ \omega)(\omega)^j +$$

$$+ p_1^*\omega^0 \circ \int (\omega)^{r-1} - \int (\omega)^{r-1} \circ p_0^*\omega^0 \quad (1.3.43)$$

So for ω in $\Gamma End^{1-k}(V) \otimes_{\mathcal{A}^0} \mathcal{A}^k$, with Ψ the holonomy of the local connection $d - \omega$,

$$d\Psi =$$

$$[\int \varkappa + (\int \varkappa\omega + \int \omega\varkappa) + \ldots + \sum_{i+j=r-1} \int (\omega)^i \varkappa \omega^j + \ldots] + -p_1^*\omega \circ \Psi + \Psi \circ p_0^*\omega. \tag{1.3.44}$$

where here $\varkappa := (d - \omega) \circ (d - \omega) = -d\omega - T\omega \circ \omega = -d\omega + \omega \circ \omega$ is the curvature.

Note that if $\varkappa = 0$ then we have

$$d\Psi = -p_1^*\omega \circ \Psi + \Psi \circ p_0^*\omega \tag{1.3.45}$$

and on $PM(x_0, x_1)$ this reduces further to

$$d\Psi = -p_0^* A^0 \circ \Psi + \Psi \circ p_1^* A^0 \tag{1.3.46}$$

The condition $\varkappa = 0$ locally amounts to the series of equations

$$\begin{gathered} A^0 \circ A^0 = 0 \\ A^0 \circ A^1 + A^1 \circ A^0 = (dA^0) \\ \cdots \\ \sum_{i=0}^{q+1} A^i \circ A^{q-i+1} = (dA^q) \\ \cdots \end{gathered} \tag{1.3.47}$$

which is identical to the flatness condition $\mathbb{E} \circ \mathbb{E} = 0$ in $\mathcal{P}_{\mathsf{A}}$. With the help of the Stokes' formula the equation (1.3.46) is equivalent to the integral form

$$-A^0_{x_1} \circ \int_{I^q} h^*\Psi_q + (-1)^q (\int_{I^q} h^*\Psi_q) \circ A^0_{x_0} = \int_{\partial I^q} h^*\Psi_{q-1} \tag{1.3.48}$$

for any q-family of paths $h : I^q \to P(M, x_0, x_1)$ inside a trivializing patch.

1.3.5 Holonomy With Respect to the Pre-triangulated Structure

Holonomy with Respect to the Shift

Let $(E^\bullet, \mathbb{E})$ be an element of $\mathcal{P}_{\mathsf{A}}$, $d - A$ a local coordinate description, and Ψ its associated holonomy form. It is not hard to see that holonomy form commutes with the shift functor. Considering a particular term in the holonomy with form degree k,

$$\begin{aligned}\int_{I^k}(\int (A[q])^j)v[q] = (-1)^{(|v|+q)(j+k)}\int_{I^k}(\int (A[q]^j(v[q]))) = \\ = (-1)^{|v|(j+k)+qk}\int_{I^k}(\int A^j(v)) = (-1)^{qk}\int_{I^k}(\int A^j)(v)[q] \quad (1.3.49)\end{aligned}$$

where the first sign shows up because the form-degree of the integrand is reduced by j in the integral. Later we will see I^k as a cube in PM induced in a particular way from a simplex σ_{k+1} in M. Likewise, supposing ϕ is a degree p morphism between E,F locally represented by the matrix-valued forms A, B respectively, we see

$$\begin{aligned}&\int_{I^k}(\int B[q]^i\phi[p+q-1]A[p+q-1])v[p+q-1] = \\ &= (-1)^{(|v|+p+q-1)(i+j+1+k)}\textstyle\int_{I^k}(\int (B[q]^i\phi[p+q-1]A[p+q-1]^j(v[p+q-1]))) = \\ &= (-1)^{(|v|+p-1)(i+j+1+k)+q(k+1)}\textstyle\int_{I^k}(\int B^i\phi[p-1]A^i[p-1]v[p-1]) = \\ &= (-1)^{q(k+1)}\textstyle\int_{I^k}(\int B^i\phi[p-1]A^i[p-1])v[p-1]\end{aligned} \quad (1.3.50)$$

Holonomy of a Cone

Suppose we have a morphism in $\mathcal{P}_{\mathsf{A}}$, i.e. an element ϕ of total degree q of

$$\mathrm{Hom}^q_{\mathcal{P}_{\mathsf{A}}}(E_1, E_2) = \{\phi : E_1 \otimes_{\mathcal{A}^0} \mathcal{A}^\bullet \to E_2 \otimes_{\mathcal{A}^0} \mathcal{A}^\bullet | \phi(ea) = (-1)^{q|a|}\phi(e)a\} \quad (1.3.51)$$

The differential is defined

$$d\phi := E_2 \circ \phi - (-1)^{|\phi|}\phi \circ E_1 \quad (1.3.52)$$

We can construct the cone complex associated to ϕ, $C(\phi)$.

$$C(\phi)^k = (E_1^{k+1-q} \oplus E_2^k) \tag{1.3.53}$$

with the differential (this has total degree 1):

$$D^\phi = \begin{pmatrix} (-1)^{1-q}\mathbb{E}_1 & 0 \\ \phi & \mathbb{E}_2 \end{pmatrix} \tag{1.3.54}$$

Note that D^ϕ is flat iff ϕ is a closed morphism.

In a trivializing coordinate patch write $D^\phi = d - \omega$ and denote the corresponding $\mathbb{Z}$-connection holonomy by Ψ^ϕ. Then applying our Chen-formula for $d\Psi$ on $PM(x_0, x_1)$, we calculate,

$$d\Psi^\phi = \left[\int \varkappa + (\int \varkappa\omega + \int \omega\varkappa) + \ldots + \sum_{i+j=r-1} \int \omega^i \varkappa \omega^j + \ldots\right] - p_1^*\omega^0 \circ \Psi^\phi + \Psi^\phi \circ p_0^*\omega^0 \tag{1.3.55}$$

And since D_{11}^ϕ, and D_{22}^ϕ are flat, it is evident that

$$\varkappa = -d\omega + \omega \circ \omega = \left(\begin{smallmatrix} 0 & 0 \\ d\phi & 0 \end{smallmatrix}\right) \tag{1.3.56}$$

Then the 21-component is:

$$d\Psi_{21}^\phi = \left[\int d\phi + \ldots + \sum_{i+j+2=r} \int (B)^i d\phi((-1)^{1-q}A)^j + \ldots\right] - -p_1^* B^0 \circ \Psi_{21}^\phi + (-1)^{1-q}\Psi_{21}^\phi \circ p_0^* A^0 \tag{1.3.57}$$

Alternately, if we first take $d\phi$ in $\mathcal{P}_{\mathsf{A}}$ and take the holonomy of its cone $(C(d\phi), D^{d\phi})$. We already showed that since $D^{d\phi}$ is flat, $d\Psi^{d\phi} = -p_1^*\omega^{d\phi,0} \circ \Psi^{d\phi} + \Psi^{d\phi} \circ p_0^*\omega^{d\phi,0}$. And we have,

$$\Psi_{21}^{d\phi} = \int d\phi + \ldots \sum_{i+j+1=r} \int (B)^i d\phi((-1)^q A)^j + \ldots =$$

$$= \int d\phi + \ldots \sum_{i+j+1=r} (-1)^j \int (B)^i d\phi((-1)^{1-q} A)^j + \ldots \quad (1.3.58)$$

But according to the signs in our definition of iterated integrals, the sign changes by $j+1$ if we switch $d\phi$ from degree 1 to degree 2. Thus, the above formula becomes,

$$\Psi_{21}^{d\phi} = -\int d\phi - \ldots - \sum_{i+j+1=r} \int (B)^i d\phi((-1)^{1-q} A)^j + \ldots \quad (1.3.59)$$

And consequently,

$$d\Psi_{21}^{\phi} = -\Psi_{21}^{d\phi} - p_1^* B^0 \circ \Psi_{21}^{\phi} + (-1)^{|\phi|-1} \Psi_{21}^{\phi} \circ p_0^* A^0 \quad (1.3.60)$$

Going further, we consider generalized cones associated to any string of morphisms

$$\phi_n \otimes \ldots \otimes \phi_1 \in \mathcal{P}_{\mathsf{A}}(E_{n-1}, E_n) \otimes \ldots \otimes \mathcal{P}_{\mathsf{A}}(E_0, E_1), \quad (1.3.61)$$

Let p_i denote the total degree of ϕ_i, and define $D^{\phi_n \otimes \ldots \otimes \phi_1}$ to be the total degree 1 endomorphism of

$$E_0^\bullet[n - \sum_1^n p_i] \oplus E_1^\bullet[n - 1 - \sum_2^n p_i] \oplus \ldots \oplus E_n^\bullet \quad (1.3.62)$$

by

$$D^{\phi_n \otimes \ldots \otimes \phi_1} = \begin{pmatrix} \mathbb{E}_0[n-\sum_1^n p_i] & 0 & 0 & \ldots & \ldots & 0 & 0 \\ \phi_1[n-\sum_1^n p_i] & \mathbb{E}_1[n-1-\sum_2^n p_i] & 0 & \ldots & \ldots & 0 & 0 \\ 0 & \phi_2[n-1-\sum_2^n p_i] & \mathbb{E}_2[n-2-\sum_3^n p_i] & \ldots & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \ldots & \phi_{n-1}[2-p_{n-1}-p_n] & \mathbb{E}_{n-1}[1-p_n] & 0 \\ 0 & 0 & \ldots & \ldots & 0 & \phi_n[1-p_n] & \mathbb{E}_n \end{pmatrix} \quad (1.3.63)$$

We call $C(\phi_n \otimes \ldots \otimes \phi_1) = (E_0^\bullet[n - \sum_1^n p_i] \oplus E_1^\bullet[n-1-\sum_2^n p_i] \oplus \ldots \oplus E_n^\bullet, D^{\phi_n \otimes \ldots \otimes \phi_1})$ the *generalized homological cone* associated to this n-tuple of morphisms.

In a local trivialization: $D^{\phi_n \otimes \ldots \otimes \phi_1} = d - \omega$,

$$\omega = \begin{pmatrix} A_0[n-\sum_1^n p_i] & 0 & 0 & \ldots & \ldots & 0 & 0 \\ \phi_1[n-\sum_1^n p_i] & A_1[n-1-\sum_2^n p_i] & 0 & \ldots & \ldots & 0 & 0 \\ 0 & \phi_2[n-1-\sum_2^n p_i] & A_2[n-2-\sum_3^n p_i] & \ldots & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \ldots & \phi_{n-1}[2-p_{n-1}-p_n] & A_{n-1}[1-p_n] & 0 \\ 0 & 0 & \ldots & \ldots & 0 & \phi_n[1-p_n] & A_n \end{pmatrix} \quad (1.3.64)$$

So that the curvature is

$$\varkappa =$$

$$\begin{pmatrix} 0 & 0 & \ldots & 0 & 0 & 0 \\ (-1)^{n-1-\sum_2^n p_i} d\phi_1[n-\sum_1^n p_i] & 0 & \ldots & 0 & 0 & 0 \\ \phi_2 \circ \phi_1[n-\sum_1^n p_i] & \ldots & \ldots & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & (-1)^{1-p_n} d\phi_{n-1}[2-p_{n-1}-p_n] & 0 & 0 \\ 0 & 0 & \ldots & \phi_n \circ \phi_{n-1}[2-p_{n-1}-p_n] & d\phi_n[1-p_n] & 0 \end{pmatrix} \quad (1.3.65)$$

We call the holonomy of this connection $\Psi^{\phi_n \otimes \ldots \otimes \phi_1}$. Using the modified Chen formula above, we consider the $n+1, 1$-component of the differential of this holonomy form:

$$d\Psi^{\phi_n \otimes \ldots \otimes \phi_1}_{n+1,1} = [\sum_{i,j} \int \omega^i \varkappa \omega^j]_{n+1,1} - p_1^* \omega^0_{n+1,1} \circ \Psi^{\phi_n \otimes \ldots \otimes \phi_1}_{n+1,1} + \Psi^{\phi_n \otimes \ldots \otimes \phi_1}_{n+1,1} \circ p_0^* \omega^0_{n+1,1} \quad (1.3.66)$$

Considering the terms in the first part of the sum, (the shifting is suppressed for clarity)

$$\sum_{i,j} [\int \omega^i \varkappa \omega^j]_{n+1,1} =$$

$$\sum_{k=1}^{n-1} \sum_{(i_0,\ldots,\hat{i_k},\ldots,i_n)} \int A_n^{i_n} \phi_n A_{n-1}^{i_{n-1}} \phi_{n-1} \ldots \phi_{k+2} A_{k+1}^{i_{k+1}} (\phi_{k+1} \circ \phi_k) A_{k-1}^{i_{k-1}} \phi_{k-1} \ldots \phi_1 A_0^{i_0} +$$

$$+ \sum_{k=1}^{n} \sum_{(i_0,\ldots,i_n)} \int A_n^{i_n} \phi_n A_{n-1}^{i_{n-1}} \phi_{n-1} \ldots \phi_{k+1} A_k^{i_k} d\phi_k A_{k-1}^{i_{k-1}} \phi_{k-1} \ldots \phi_1 A_0^{i_0} \quad (1.3.67)$$

Now, we can recognize inside this series the terms of

$$\Psi^{\phi_n \otimes \ldots \otimes d\phi_k \otimes \ldots \otimes \phi_1}_{n+1,1} \text{ and } \Psi^{\phi_n \otimes \ldots \otimes \phi_{k+1} \circ \phi_k \otimes \ldots \otimes \phi_1}_{n+1,1}.$$

These are the holonomies of the cone $D^{\phi_n \otimes \ldots \otimes d\phi_k \otimes \ldots \otimes \phi_1} = d - \omega$ with

$$\omega =$$

$$\begin{pmatrix} A_0[n-\sum_1^n p_i+1] & 0 & \ldots & \ldots & 0 & 0 & 0 \\ \phi_1[n-\sum_1^n p_i+1] & \ldots & \ldots & \ldots & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & d\phi_k[n-k+1-\sum_k^n p_i+1] & A_k[n-k+1-\sum_{k+1}^n p_i] & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \ldots & \ldots & A_{n-1}[1-p_n] & 0 \\ 0 & 0 & \ldots & \ldots & 0 & \phi_n[1-p_n] & A_n \end{pmatrix} \quad (1.3.68)$$

As before, shifting the terms to the right of $d\phi_k$ has the effect of changing the sign on all of the A's to the left of $d\phi_k$ and leaving the sign on the $\phi's$ unchanged. But in doing so we change the degree of $d\phi_k$ by one, thus introducing a sign change for every term to the right of $d\phi_k$ and one more from the alternation of the integral. So the total change is $(-1)^{n-k+1}$. But the sign in the definition of the iterated integral again accounts for this $n-k$. All together then,

$$\begin{aligned} d\Psi_{n+1,1}^{\phi_n\otimes\ldots\otimes\phi_1} = & -\textstyle\sum_{k=1}^{n-1}(-1)^{n-k-1-|\phi_n\otimes\ldots\otimes\phi_{k+2}|}\Psi_{n+1,1}^{\phi_n\otimes\ldots\otimes\phi_{k+1}\circ\phi_k\otimes\ldots\otimes\phi_1} \\ & -\textstyle\sum_{k=1}^{n}(-1)^{n-k-|\phi_n\otimes\ldots\otimes\phi_{k+1}|}\Psi_{n+1,1}^{\phi_n\otimes\ldots\otimes d\phi_k\otimes\ldots\otimes\phi_1} \\ & -p_1^*A_n^0\circ\Psi_{n+1,1}^{\phi_n\otimes\ldots\otimes\phi_1}+(-1)^{|\phi_n\otimes\ldots\otimes\phi_n|}\Psi_{n+1,1}^{\phi_n\otimes\ldots\otimes\phi_1}\circ p_0^*A_0^0 \end{aligned} \tag{1.3.69}$$

1.3.6 Cubes to Simplices

Now we want to integrate over simplices rather than cubes, which will involve realizing any simplex as a family of paths with fixed endpoints. This construction is described technically in [Ch],[I] and elsewhere and only outlined here. P is the path space functor.

Given a geometric k-simplex, $\sigma : \Delta^k \to M$, we want to realize this as a factor of a $(k-1)$-family of paths into M. That is, we produce a map $\theta : I^k \to \Delta^k$ which then can be viewed as a family of paths $\theta_{(k-1)} : I^{k-1} \to P\Delta^k$. This map is factored into two parts: $I^k \xrightarrow{\lambda} I^k \xrightarrow{\pi_k} \Delta^k$. Here π_k is an order-preserving retraction. λ is given by the map $\lambda_w : I \to I^k$ parametrized by $w \in I^{k-1}$. The result, is an I^{k-1}-family of paths in I^k (we call this $\lambda_{(k-1)} : I^{k-1} \to PI^k$) each starting at $(w_1, w_2, \ldots, w_{k-1}, 1)$ and ending at $(0, 0, \ldots, 0)$. When post-composed with π_k we get a $(k-1)$-family of paths in Δ^k which start at σ_k and end at σ_0. Define $\theta_{(k-1)} : I^{k-1} \to P\Delta^k$ by $P\pi_k \circ \lambda_{(k-1)}$

We restate the characteristic properties of such a factorization c/o [I]:

- If $x \leq X'$ in the sense that $x_i \leq x_i'$ for all i, then $\pi_k(x) \leq \pi_k(x')$.

  Furthermore, $\pi_k(x) \geq x$.

- π_k sends $\partial_i^+ I^k = \{x \in I^k | x_i = 1\}$ to the back $k-i$ face of Δ^k spanned by and $\{v_i, \ldots, v_k\}$ and given by the equation $y \geq v_i$.

- π_k sends $\partial_i^- I^k = \{x \in I^k | x_i = 0\}$ onto $\partial_i \Delta^k = \{y \in \Delta^k | y_i = y_{i+1}\}$.

- The adjoint of $\theta_{(k)}$ is a piecewise-linear epimorphism $I^k \twoheadrightarrow \Delta^k$
- For each $w \in I^{k-1}$, θ_w is a path from $\theta_w(0) = v_k$ to $\theta_w(1) = v_0$.
- θ_w passes through the vertex v_i iff $w_i = 1$.
- $\theta_{(k)}$ takes each of the 2^{k-1} vertices of I^{k-1} to the shortest path from v_k to v_0 passing through the corresponding subset $\{v_1 \ldots v_{k-1}\}$.

1.4 An A_∞-quasi-equivalence

In this section we establish our Riemann-Hilbert correspondence for infinity-local systems. Recall that $\mathcal{C}$ is the category of cochain complexes over $\mathbb{R}$.

Theorem 1.4.1. *There is an A_∞-functor*

$$\mathcal{RH} : \mathcal{P}_{\mathsf{A}} \to \mathsf{Loc}^{\mathcal{C}}_{\infty}(\pi_\infty M)$$

which is a quasi-equivalence.

1.4.1 The functor $\mathcal{RH} : \mathcal{P}_{\mathsf{A}} \to \mathsf{Loc}^{\mathcal{C}}_{\infty}(\pi_\infty M)$

On objects the functor $\mathcal{RH}_0 : Ob(\mathcal{P}_{\mathsf{A}}) \to Ob(\mathsf{Loc}^{\mathcal{C}}_{\infty}(\pi_\infty M))$ is described as follows. Given an element $(E^\bullet, \mathbb{E}) \in \mathcal{P}_{\mathsf{A}}$ take the corresponding graded bundle V over$M$with a $\mathbb{Z}$-graded connection $\mathbb{E}$. Define an infinity-local system by the following assignment:

$$\mathcal{RH}_0((E^\bullet, \mathbb{E}))(x) = (V_x, \mathbb{E}^0_x) \tag{1.4.1}$$

$$\mathcal{RH}_0((E^\bullet, \mathbb{E}))(\sigma_k) := \int_{I^{k-1}} (-1)^{(k-1)(K\Psi)} \theta^*_{(k-1)} (P\sigma)^* \Psi \tag{1.4.2}$$

i.e. assign to each k-simplex the integral of the higher holonomy integrated over that simplex (understood as a $(k-1)$-family of paths), which is a degree $(1-k)$ homomorphism from the fiber over the endpoint to the fiber over the starting point of the simplex. To a 0-simplex this yields a degree 1 map in the fiber over that point which we shall see will be a differential as a result of the flatness of the $\mathbb{Z}$-graded connection. To a 1-simplex (a path) we get the usual parallel transport of the underlying graded connection. Flatness will imply that this is a cochain map with respect to the differentials on the fibers over the endpoints of the path.

So far we only have a simplicial set map $\pi_\infty M \to \mathcal{C}_\infty$. Call it F. Since we are integrating a flat $\mathbb{Z}$-graded connection,

$$d\Psi = -p_0^* A^0 \circ \Psi + \Psi \circ p_1^* A^0. \tag{1.4.3}$$

So via Stokes' Theorem, F satisfies the local system condition:

$$\begin{aligned} \mathbb{E}^0 \circ F_k(\sigma) - (-1)^k F_k(\sigma) \circ \mathbb{E}^0 = \sum_{i=1}^{k-1} (-1)^i F_{k-1}(\sigma_0, \ldots, \hat{\sigma_i}, \ldots, \sigma_k) + \\ - \sum_{i=1}^{k-1} (-1)^i F_i(\sigma_0, \ldots, \sigma_i) \circ F_{k-i}(\sigma_i, \ldots, \sigma_k) \end{aligned} \tag{1.4.4}$$

which is the Maurer-Cartan/Twisting Cochain equation

$$dF + \hat{\delta} F + F \cup F = 0 \tag{1.4.5}$$

Proving this amounts to the task of figuring out what $\int_{\partial I^{q-1}} h^* \Psi$ is in the case that h is the map constructed above which factors through σ. That is we must relate ∂I^{q-1} to $\partial \Delta^{q-1}$. Igusa works this out elegantly in [I] and obtains (If we write $I(\sigma_k) := \int (-1)^{(k-1)K\Psi} \theta^* (P[\sigma_k])^* (\Psi))$

$$\int (-1)^{(k-1)K(d\Psi)} \theta^* (P[\sigma_k])^* (d\Psi) = -\hat{\delta} I - I \cup I \tag{1.4.6}$$

On tensor-tuples of morphisms we can describe a map $\mathcal{RH}_n$ as follows (we use the subscript to denote the valence):

$$\mathcal{RH}_n : \mathcal{P}_{\mathsf{A}}(E_{n-1}, E_n) \otimes \ldots \otimes \mathcal{P}_{\mathsf{A}}((E_0, E_1)) \to \mathsf{Loc}^{\mathcal{C}}_\infty(K)(X)(\mathcal{RH}_0(E_0), \mathcal{RH}_0(E_n))[1-n] \tag{1.4.7}$$

Given a tuple $\phi_n \otimes \ldots \otimes \phi_1$, assign to it the generalized homological cone of the previous section, and its associated holonomy form $\Psi^{\phi_n \otimes \ldots \otimes \phi_1}$. Then define

$$\mathcal{RH}_n(\phi_n \otimes \ldots \otimes \phi_1)(\sigma_k) := \mathcal{RH}_0(C(\phi_n \otimes \ldots \otimes \phi_1)(\sigma_k))_{n+1,1} \tag{1.4.8}$$

Note that applying $\mathcal{RH}_0$ to the cone $C(\phi_n \otimes \ldots \otimes \phi_1)$ does not necessarily yield an infinity local system. $\mathcal{RH}_0$ is perfectly well-defined as a holonomy map on any $\mathbb{Z}$-connection regardless of flatness. Flatness implies that the image is an infinity-local system.

Theorem 1.4.2. *The maps $\{\mathcal{RH}_i\}$ define an A_∞-functor:*

$$\mathcal{RH} : \mathcal{P}_{\mathsf{A}} \to \mathsf{Loc}^{\mathcal{C}}_\infty(\pi_\infty). \tag{1.4.9}$$

Proof. Given a tuple of morphisms $\phi := \phi_n \otimes \ldots \otimes \phi_1 \in \mathcal{P}_{\mathsf{A}}(E_{n-1}, E_n) \otimes \ldots \otimes \mathcal{P}_{\mathsf{A}}(E_0, E_1)$, denote the holonomy transport associated to the generalized homological cone $C(\phi)$ by $\Psi^{\phi_n \otimes \ldots \otimes \phi_1}$. Locally write $D^\phi = d - \omega$.

We already calculated that (on $PM(x_0, x_1)$),

$$\begin{aligned}
&- d\Psi^{\phi_n \otimes \ldots \otimes \phi_1}_{n+1,1} - p_0^*\omega^0_{n+1,1} \circ \Psi^{\phi_n \otimes \ldots \otimes \phi_1}_{n+1,1} + \Psi^{\phi_n \otimes \ldots \otimes \phi_1}_{n+1,1} \circ p_1^*\omega^0_{n+1,1} = \\
&= \sum_{k=1}^{n-1} (-1)^{n-k-1-|\phi_n \otimes \ldots \otimes \phi_{k+2}|} \Psi^{\phi_n \otimes \ldots \otimes \phi_{k+1} \circ \phi_k \otimes \ldots \otimes \phi_1}_{n+1,1} \\
&\qquad\qquad + \sum_{k=1}^{n} (-1)^{n-k-|\phi_n \otimes \ldots \otimes \phi_{k+1}|} \Psi^{\phi_n \otimes \ldots \otimes d\phi_k \otimes \ldots \otimes \phi_1}_{n+1,1}
\end{aligned} \tag{1.4.10}$$

Thus, applying $\int (-1)^{K(\Psi)J(\bullet)} \theta^*(P[\bullet])^*(\Psi)$ to both sides yields,

$$\begin{aligned}
&\Big[\mathcal{RH}_0(C(\phi)) \cup \mathcal{RH}_0(C(\phi)) + \hat{\delta}\mathcal{RH}_0(C(\phi)) + d\mathcal{RH}_0(C(\phi)) \Big]_{n+1,1} = \\
&= \sum_{k=1}^{n} (-1)^{n-k-|\phi_n \otimes \ldots \otimes \phi_{k+1}|} \mathcal{RH}_{n-1}(\phi_n \otimes \ldots \otimes d\phi_k \otimes \ldots \phi_1) \\
&\qquad + \sum_{k=1}^{n-1} (-1)^{n-k-1-|\phi_n \otimes \ldots \otimes \phi_{k+2}|} \mathcal{RH}_{n-1}(\phi_n \otimes \ldots \otimes (\phi_{k+1} \circ \phi_k) \otimes \ldots \otimes \phi_1)
\end{aligned} \tag{1.4.11}$$

Observe

$$\Big[\mathcal{RH}_0(C(\phi)) \cup \mathcal{RH}_0(C(\phi))\Big]_{n+1,1} =$$

$$= \sum_{i+j=n} \mathcal{RH}_0(C(\phi_n \otimes \ldots \otimes \phi_{i+1}))_{j+1,1} \cup \mathcal{RH}_0(C(\phi_i \otimes \ldots \otimes \phi_1))_{i+1,1}[(j - \sum_{i+1}^{n} p_k)]$$

$$+ \mathcal{RH}_0(E_n) \cup \mathcal{RH}_0(C(\phi))_{n+1,1} + \mathcal{RH}_0(C(\phi))_{n+1,1} \cup \mathcal{RH}_0(E_0[n - |\phi|])$$

$$= \sum_{i+j=n} (-1)^{(j - \sum_{i+1}^{n} p_k)} \mathcal{RH}_j(\phi_n \otimes \ldots \otimes \phi_{i+1}) \cup \mathcal{RH}_i(\phi_i \otimes \ldots \otimes \phi_1)$$

$$+ \mathcal{RH}_0(E_n) \cup \mathcal{RH}_0(C(\phi))_{n+1,1} + (-1)^{n-|\phi|} \mathcal{RH}_0(C(\phi))_{n+1,1} \cup \mathcal{RH}_0(E_0) \quad (1.4.12)$$

By definition,

$$D_{\mathsf{Loc}^{\mathcal{C}}_{\infty}(K)(X)}(\mathcal{RH}_n(\phi))$$

$$= \mathcal{RH}_0(E_n) \cup \mathcal{RH}_n(\phi) + (-1)^{n-|\phi|} \mathcal{RH}_n(\phi) \cup \mathcal{RH}_0(E_0) + \hat{\delta} \mathcal{RH}_n(\phi) + d\mathcal{RH}_n(\phi), \quad (1.4.13)$$

Hence we get,

$$\sum_{i+j=n} (-1)^{j - |\phi_n \otimes \ldots \otimes \phi_{i+1}|} \mathcal{RH}_j(\phi_n \otimes \ldots \otimes \phi_{i+1}) \cup \mathcal{RH}_i(\phi_i \otimes \ldots \otimes \phi_1)$$

$$+ D(\mathcal{RH}_n(\phi_n \otimes \ldots \otimes \phi_1))$$

$$= \sum_{k=1}^{n} (-1)^{n-k-|\phi_n \otimes \ldots \otimes \phi_{k+1}|} \mathcal{RH}_n(\phi_n \otimes \ldots \otimes d\phi_k \otimes \ldots \otimes \phi_1)$$

$$+ \sum_{k=1}^{n-1} (-1)^{n-k-1-|\phi_n \otimes \ldots \otimes \phi_{k+2}|} \mathcal{RH}_{n-1}(\phi_n \otimes \ldots \otimes \phi_{k+1} \circ \phi_k \otimes \ldots \otimes \phi_1). \quad (1.4.14)$$

These are the A_∞-relations for an A_∞-functor between two dg-categories understood as A_∞-categories. □

Proposition 1.4.3. *The functor $\mathcal{RH}$ is A_∞-quasi-fully faithful.*

Proof. Consider two objects $E_i = (E_i^\bullet, \mathbb{E}_i) \in \mathcal{P}_{\mathsf{A}}$, $i = 1, 2$. The chain map

$$\mathcal{RH}_1 : \mathcal{P}_{\mathsf{A}}(E_1, E_2) \to \mathsf{Loc}^{\mathcal{C}}_{\infty}(\pi_\infty M)(\mathcal{RH}_0(E_1), \mathcal{RH}_0(E_2))$$

induces a map on spectral sequences (1.2.21) and [Bl1], Theorem 2.5.1. At the E_1-level on the $\mathcal{P}_{\mathsf{A}}$ side, we have that $H^*((E_i,\mathbb{E}_i^0))$ are both vector bundles with flat connection, while according to 1.2.11, we have $H^*((\mathcal{RH}(E_i),\mathbb{E}_i^0))$ are local systems on M. At the E_2-term the map is

$$H^*(M; Hom(H^*(E_1,\mathbb{E}_1^0),H^*(E_2,\mathbb{E}_2^0))) \to H^*(M; H^*((\mathcal{RH}(E_1),\mathbb{E}_1^0)), H^*((\mathcal{RH}(E_2),\mathbb{E}_2^0)))$$

which is an isomorphism by the ordinary De Rham theorem for local systems. □

1.4.2 $\mathcal{RH}$ is A_∞-essentially surjective

(*This section is due to Jonathan Block*)

We must prove that for any $(F,f)\in \mathsf{Loc}_\infty^{\mathcal{C}}(\pi_\infty M)$, that there is an object $E=(E^\bullet,\mathbb{E})\in \mathcal{P}_{\mathsf{A}}$ such that $\mathcal{RH}_0(E)$ is quasi-isomorphic to (F,f). We first define a complex of sheaves on M. Let $\underline{\mathbb{R}}$ denote the constant local system, and thus an infinity-local system. We also view $\underline{\mathbb{R}}$ as a sheaf of rings with which $(M,\underline{\mathbb{R}})$ becomes a ringed space. For an open subset $U\subset M$, let $(C_F(U),D) = (\mathsf{Loc}_\infty^{\mathcal{C}}(\pi_\infty U)(\underline{\mathbb{R}}|_U, F|_U), D)$. Let $(\underline{C}_F, D)$ denote the associated complex of sheaves. Then $\underline{C}_F$ is soft; see the proof of Theorem 3.15, [W]. By corollary 1.2.11, $\underline{C}_F$ is a perfect complex of sheaves over $\underline{\mathbb{R}}$. Let $\underline{\mathcal{A}}_M$ denote the sheaf of C^∞ functions and $(\underline{\mathcal{A}}^\bullet, d)$ denote the dg sheaf of C^∞ forms on M. Set $\underline{C}_F^\infty = \underline{C}_F\otimes_{\underline{\mathbb{R}}}\underline{\mathcal{A}}_M$. By the flatness of $\underline{\mathcal{A}}_M$ over $\underline{\mathbb{R}}$, $\underline{C}_F^\infty$ is perfect as a sheaf of $\underline{\mathcal{A}}_M$-modules. Now the map

$$(\underline{C}_F^\bullet, D)\to (\underline{C}_F^\infty\otimes_{\underline{\mathcal{A}}_M}\underline{\mathcal{A}}_M^\bullet, D\otimes 1 + 1\otimes d)$$

is a quasi-isomorphism of sheaves of $\underline{\mathbb{R}}$-modules by the flatness of $\underline{\mathcal{A}}_M$ over $\underline{\mathbb{R}}$.

We need the following

Proposition 1.4.4. *Suppose $(X,\underline{\mathcal{S}}_X)$ is a ringed space, where X is compact and $\underline{\mathcal{S}}_X$ is a soft sheaf of rings. Then*

1. *The global sections functor*

$$\Gamma : Mod\text{-}\underline{\mathcal{S}}_X \to Mod\text{-}\underline{\mathcal{S}}_X(X)$$

is exact and establishes an equivalence of categories between the category of sheaves of right $\underline{\mathcal{S}}_X$*-modules and the category of right modules over the global sections* $\underline{\mathcal{S}}_X(X)$.

2. *If* $\underline{M} \in Mod\text{-}\underline{\mathcal{S}}_X$ *locally has finite resolutions by finitely generated free* $\underline{\mathcal{S}}_X$*-modules, then* $\Gamma(X; \underline{M})$ *has a finite resolution by finitely generated projectives.*

3. *The derived category of perfect complexes of sheaves* $D_{perf}(Mod\text{-}\underline{\mathcal{S}}_X)$ *is equivalent the derived category of perfect complexes of modules* $D_{perf}(Mod\text{-}\underline{\mathcal{S}}_X(X))$.

Proof. See Proposition 2.3.2, Exposé II, SGA6, [SGA6]. □

Theorem 1.4.5. *The functor*

$$\mathcal{RH} : \mathcal{P}_{\mathsf{A}} \to \mathsf{Loc}^{\mathcal{C}}_{\infty}(\pi_\infty M)$$

is A_∞*-essentially surjective.*

Proof. By the Proposition , there is a (strictly) perfect complex $(E^\bullet, \mathbb{E}^0)$ of $\mathcal{A}$-modules and quasi-isomorphism $e^0 : (E^\bullet, \mathbb{E}^0) \to (X^\bullet, \mathbb{X}^0) := (\Gamma(M, \underline{C}^\infty_F), D)$. Following the argument of Theorem 3.2.7 of [Bl1], which in turn is based on arguments from [OTT], we construct the higher components $\mathbb{E}^i$ of a $\mathbb{Z}$-graded connection along with the higher components of a morphism e^i at the same time.

We have a $\mathbb{Z}$-graded connection on $X^\bullet$ by

$$\mathbb{X} := D \otimes 1 + 1 \otimes d : X^\bullet \to X^\bullet \otimes_{\mathcal{A}} \mathcal{A}^\bullet$$

Then we have an induced connection

$$\mathbb{H} : H^k(X^\bullet, \mathbb{X}^0) \to H^k(X^\bullet, \mathbb{X}^0) \otimes_{\mathcal{A}} \mathcal{A}^1$$

for each k. We use the quasi-morphism e^0 to transport this connection to a connection, also denoted by $\mathbb{H}$ on $H^k(E^\bullet;\mathbb{E}^0)$

$$
\begin{array}{ccc}
H^k(E^\bullet;\mathbb{E}^0) & \overset{\mathbb{H}}{\to} & H^k(E^\bullet,\mathbb{E}^0)\otimes_{\mathcal{A}}\mathcal{A}^1 \\
\downarrow e^0 & & \downarrow e^0\otimes 1 \\
H^k(X^\bullet,\mathbb{X}^0) & \overset{\mathbb{H}}{\longrightarrow} & H^k(X^\bullet,\mathbb{X}^0)\otimes_{\mathcal{A}}\mathcal{A}^1
\end{array}
\tag{1.4.15}
$$

The right vertical arrow above $e^0\otimes 1$ is a quasi-isomorphism because $\mathcal{A}^\bullet$ is flat over $\mathcal{A}$. The first step is handled by the following lemma.

Lemma 1.4.6. *Given a bounded complex of f.g. projective $\mathcal{A}$ modules $(E^\bullet,\mathbb{E}^0)$ with connections $\mathbb{H}: H^k(E^\bullet;\mathbb{E}^0)\to H^k(E^\bullet,\mathbb{E}^0)\otimes_{\mathcal{A}}\mathcal{A}^1$, for each k, there exist connections*

$$\widetilde{\mathbb{H}}: E^k \to E^k\otimes_{\mathcal{A}}\mathcal{A}^1$$

lifting $\mathbb{H}$. That is,

$$\widetilde{\mathbb{H}}\mathbb{E}^0 = (\mathbb{E}^0\otimes 1)\widetilde{\mathbb{H}}$$

and the connection induced on the cohomology is $\mathbb{H}$.

Proof. (of lemma) Since $E^\bullet$ is a bounded complex of $\mathcal{A}$-modules it lives in some bounded range of degrees $k\in[N,M]$. Pick an arbitrary connection on E^M, ∇. Consider the diagram with exact rows

$$
\begin{array}{cccc}
E^M & \overset{j}{\to} & H^M(E^\bullet,\mathbb{E}^0) & \to 0 \\
\nabla\downarrow & \overset{\theta}{\searrow} & \mathbb{H}\downarrow & \\
E^M\otimes_{\mathcal{A}}\mathcal{A}^1 & \overset{j\otimes 1}{\to} & H^M(E^\bullet,\mathbb{E}^0)\otimes_{\mathcal{A}}\mathcal{A}^1 & \to 0
\end{array}
\tag{1.4.16}
$$

In the diagram, $\theta=\mathbb{H}\circ j-(j\otimes 1)\circ\nabla$ is easily checked to be $\mathcal{A}$-linear and $j\otimes 1$ is surjective by the right exactness of tensor product. By the projectivity of E^M, θ lifts to

$$\widetilde{\theta}: E^M\to E^M\otimes_{\mathcal{A}}\mathcal{A}^1$$

so that $(j \otimes 1)\widetilde{\theta} - \theta$. Set $\widetilde{\mathbb{H}} = \nabla + \widetilde{\theta}$. With $\widetilde{\mathbb{H}}$ in place of ∇, the diagram above commutes.

Now choose on E^{M-1} any connection ∇_{M-1}. But ∇_{M-1} does not necessarily satisfy $\mathbb{E}^0\nabla_{M-1} = \widetilde{\mathbb{H}}\mathbb{E}^0 = 0$. So we correct it as follows. Set $\mu = \widetilde{\mathbb{H}}\mathbb{E}^0 - (\mathbb{E}^0 \otimes 1)\nabla_{M-1}$. Then μ is $\mathcal{A}$-linear. Furthermore, $\operatorname{Im}\mu \subset \operatorname{Im}\mathbb{E}^0 \otimes 1$; this is because $\widetilde{\mathbb{H}}\mathbb{E} \in \operatorname{Im}\mathbb{E} \otimes 1$ since $\widetilde{\mathbb{H}}$ lifts $\mathbb{H}$. So by projectivity it lifts to $\widetilde{\theta} : E^{M-1} \to E^{M-1} \otimes_{\mathcal{A}} \mathcal{A}^1$ such that $(\mathbb{E}^0 \otimes 1) \circ \widetilde{\theta} = \theta$. Set $\widetilde{\mathbb{H}} : E^{M-1} \to E^{M-1} \otimes_{\mathcal{A}} \mathcal{A}^1$ to be $\nabla_{M-1} + \widetilde{\theta}$. Then $\mathbb{E}^0\widetilde{\mathbb{H}} = \widetilde{\mathbb{H}}\mathbb{E}^0$ in the right most square below.

$$\begin{array}{ccccccccc}
E^N & \xrightarrow{\mathbb{E}^0} & E^{N+1} & \xrightarrow{\mathbb{E}^0} \cdots \xrightarrow{\mathbb{E}^0} & E^{M-1} & \xrightarrow{\mathbb{E}^0} & E^M \\
 & & & & \nabla_{M-1}\downarrow & \overset{\mu}{\searrow} & \widetilde{\mathbb{H}}\downarrow \\
E^N \otimes_{\mathcal{A}} \mathcal{A}^1 & \xrightarrow{\mathbb{E}^0\otimes 1} & E^{N+1} \otimes_{\mathcal{A}} \mathcal{A}^1 & \xrightarrow{\mathbb{E}^0\otimes 1} \cdots \xrightarrow{\mathbb{E}^0\otimes 1} & E^{M-1} \otimes_{\mathcal{A}} \mathcal{A}^1 & \xrightarrow{\mathbb{E}^0\otimes 1} & E^M \otimes_{\mathcal{A}} \mathcal{A}^1
\end{array} \tag{1.4.17}$$

Now we continue backwards to construct all $\widetilde{\mathbb{H}} : E^\bullet \to E^\bullet \otimes_{\mathcal{A}} \mathcal{A}^1$ satisfying $(\mathbb{E}^0 \otimes 1)\widetilde{\mathbb{H}} = \widetilde{\mathbb{H}}\mathbb{E}^0 = 0$. This completes the proof of the lemma. ☐

(Proof of the theorem, continued.) Set $\widetilde{\mathbb{E}}^1 = (-1)^k\widetilde{\mathbb{H}}$ on E^k. Then

$$\mathbb{E}^0\widetilde{\mathbb{E}}^1 + \widetilde{\mathbb{E}}^1\mathbb{E}^0 = 0$$

but it is not necessarily true that $e^0\widetilde{\mathbb{E}}^1 - \mathbb{X}^1 e^0 = 0$. We correct this as follows. Consider $\psi = e^0\widetilde{\mathbb{E}}^1 - \mathbb{X}^1 e^0 : E^\bullet \to X^\bullet \otimes_{\mathcal{A}} \mathcal{A}^1$. Check that ψ is $\mathcal{A}$-linear and a map of complexes.

$$\begin{array}{ccc}
 & & (E^\bullet \otimes_{\mathcal{A}} \mathcal{A}^1, \mathbb{E}^0 \otimes 1) \\
 & \overset{\widetilde{\psi}}{\nearrow} & \downarrow e^0 \otimes 1 \\
E^\bullet & \xrightarrow{\psi} & (X^\bullet \otimes_{\mathcal{A}} \mathcal{A}^1, \mathbb{X}^0 \otimes 1)
\end{array} \tag{1.4.18}$$

In the above diagram, $e^0 \otimes 1$ is a quasi-isomorphism e^0 is a homotopy equivalence. So by Lemma 1.2.5 of [OTT] there is a lift $\widetilde{\psi}$ of ψ and a homotopy $e^1 : E^\bullet \to X^{\bullet-1} \otimes_{\mathcal{A}} \mathcal{A}^1$ between $(e^0 \otimes 1)\widetilde{\psi}$ and ψ,

$$\psi - (e^0 \otimes 1)\widetilde{\psi} = (e^1\mathbb{E}^0 + \mathbb{X}^0 e^1)$$

So let $\mathbb{E}^1 = \widetilde{\mathbb{E}}^1 - \widetilde{\psi}$. Then

$$\mathbb{E}^0\mathbb{E}^1 + \mathbb{E}^1\mathbb{E}^0 = 0 \text{ and } e^0\mathbb{E}^1 - \mathbb{X}^1 e^0 = e^1\mathbb{E}^0 + \mathbb{X}^0 e^1. \tag{1.4.19}$$

So we have constructed the first two components $\mathbb{E}^0$ and $\mathbb{E}^1$ of the $\mathbb{Z}$-graded connection and the first components e^0 and e^1 of the quasi-isomorphism $E^\bullet \otimes_{\mathcal{A}} \mathcal{A}^\bullet \to X^\bullet \otimes_{\mathcal{A}} \mathcal{A}^\bullet$.

To construct the rest, consider the mapping cone $L^\bullet$ of e^0. Thus

$$L^\bullet = E[1]^\bullet \oplus X^\bullet$$

Let $\mathbb{L}^0$ be defined as the matrix

$$\mathbb{L}^0 = \begin{pmatrix} \mathbb{E}^0[1] & 0 \\ e^0[1] & \mathbb{X}^0 \end{pmatrix} \tag{1.4.20}$$

Define $\mathbb{L}^1$ as the matrix

$$\mathbb{L}^1 = \begin{pmatrix} \mathbb{E}^1[1] & 0 \\ e^1[1] & \mathbb{X}^1 \end{pmatrix} \tag{1.4.21}$$

Now $\mathbb{L}^0\mathbb{L}^0 = 0$ and $[\mathbb{L}^0, \mathbb{L}^1] = 0$ express the identities (1.4.19). Let

$$D = \mathbb{L}^1\mathbb{L}^1 + \begin{pmatrix} 0 & 0 \\ \mathbb{X}^2 e^0 & [\mathbb{X}^0, \mathbb{X}^2] \end{pmatrix}. \tag{1.4.22}$$

Then, as is easily checked, D is $\mathcal{A}$-linear and

1. $[\mathbb{L}^0, D] = 0$ and

2. $D|_{0\oplus X^\bullet} = 0$.

Since $(L^\bullet, \mathbb{L}^0)$ is the mapping cone of a quasi-isomorphism, it is acyclic and since $\mathcal{A}^\bullet$ is flat over $\mathcal{A}$, $(L^\bullet \otimes_{\mathcal{A}} \mathcal{A}^2, \mathbb{L}^0 \otimes 1)$ is acyclic too. Since $E^\bullet$ is projective, we have that

$$\mathrm{Hom}^\bullet_{\mathcal{A}}((E^\bullet, \mathbb{E}^0), (L^\bullet \otimes_{\mathcal{A}} \mathcal{A}^2, \mathbb{L}^0))$$

is acyclic. Moreover

$$\mathrm{Hom}^\bullet_{\mathcal{A}}((E^\bullet, \mathbb{E}^0), (L^\bullet \otimes_{\mathcal{A}} \mathcal{A}^2, \mathbb{L}^0)) \subset \mathrm{Hom}^\bullet_{\mathcal{A}}(L^\bullet, (L^\bullet \otimes_{\mathcal{A}} \mathcal{A}^2, [\mathbb{L}^0, \cdot]))$$

is a subcomplex. Now we have $D \in \mathrm{Hom}^\bullet_{\mathcal{A}}(E^\bullet, L^\bullet \otimes_{\mathcal{A}} \mathcal{A}^2)$ is a cycle and so there is $\widetilde{\mathbb{L}}^2 \in \mathrm{Hom}^\bullet_{\mathcal{A}}(E^\bullet, L^\bullet \otimes_{\mathcal{A}} \mathcal{A}^2)$ such that $-D = [\mathbb{L}^0, \widetilde{\mathbb{L}}^2]$. Define $\mathbb{L}^2$ on $L^\bullet$ by

$$\mathbb{L}^2 = \widetilde{\mathbb{L}}^2 + \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{X}^2 \end{pmatrix} \tag{1.4.23}$$

Then

$$\begin{aligned} [\mathbb{L}^0, \mathbb{L}^2] =& [\mathbb{L}^0, \widetilde{\mathbb{L}}^2 + \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{X}^2 \end{pmatrix}] \\ =& - D + [\mathbb{L}^0, \widetilde{\mathbb{L}}^2 + \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{X}^2 \end{pmatrix}] \\ =& - \mathbb{L}^1\mathbb{L}^1 \end{aligned} \tag{1.4.24}$$

So

$$\mathbb{L}^0\mathbb{L}^2 + \mathbb{L}^1\mathbb{L}^1 + \mathbb{L}^2\mathbb{L}^0 = 0.$$

We continue by setting

$$D = \mathbb{L}^1\mathbb{L}^2 + \mathbb{L}^2\mathbb{L}^1 + \begin{pmatrix} 0 & 0 \\ \mathbb{X}^3 e^0 & [\mathbb{X}^0, \mathbb{X}^3] \end{pmatrix} \tag{1.4.25}$$

Then $D : L^\bullet \to L^\bullet \otimes_{\mathcal{A}} \mathcal{A}^3$ is $\mathcal{A}$-linear, $D|_{0 \oplus X^\bullet} = 0$ and

$$[\mathbb{L}^0, D] = 0.$$

Hence, by the same reasoning as above, there is $\widetilde{\mathbb{L}}^3 \in \mathrm{Hom}^\bullet_{\mathcal{A}}(E^\bullet, L^\bullet \otimes_{\mathcal{A}} \mathcal{A}^3)$ such that $-D = [\mathbb{L}^0, \widetilde{\mathbb{L}}^3]$. Define

$$\mathbb{L}^3 = \widetilde{\mathbb{L}}^3 + \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{X}^3 \end{pmatrix} \tag{1.4.26}$$

Then one can compute that $\sum_{i=0}^3 \mathbb{L}^i\mathbb{L}^{3-i} = 0$.

Now suppose we have defined $\mathbb{L}^0, \dots, \mathbb{L}^n$ satisfying for $k = 0, 1, \dots, n$

$$\sum_{i=0}^k \mathbb{L}^i\mathbb{L}^{k-i} = 0 \qquad \text{for } k \neq 2$$

and

$$\sum_{i=0}^{2} \mathbb{L}^i \mathbb{L}^{2-i} = 0 \qquad \text{for } k = 2$$

Then define

$$D = \sum_{i=1}^{n} \mathbb{L}^i \mathbb{L}^{n+1-i} + \begin{pmatrix} 0 & 0 \\ \mathbb{X}^{n+1} e^0 & [\mathbb{X}^0, \mathbb{X}^{n+1}] \end{pmatrix} \tag{1.4.27}$$

$D|_{0 \oplus X^\bullet} = 0$ and we may continue the inductive construction of $\mathbb{L}$ to finally arrive at a $\mathbb{Z}$-graded connection satisfying $\mathbb{L}\mathbb{L} = 0$. The components of $\mathbb{L}$ construct both the $\mathbb{Z}$-graded connection on $E^\bullet$ as well as the morphism from $(E^\bullet, \mathbb{E})$ to $(X^\bullet, \mathbb{X})$.

It follows from Proposition 1.2.12 that $\mathcal{RH}((E^\bullet, \mathbb{E})) \xrightarrow{e} (F, f)$ is a quasi-isomorphism.

□

1.5 Extensions

1.5.1 Infinity-local systems taking values in codifferential-coalgebras and A_∞-local systems

For the sake of interest, we present a notion of an A_∞-local system which is an infinity-local system valued in an A_∞-category. In fact an infinity-local system can take values in any k-linear ∞-category as is evident from the linear simplicial nerve construction and its adjoint. We will use almost entirely the same notation as before. Secondly we point out an application of our framework which is concerned with infinity-local systems which take values in the dg-category of codifferential-coalgebras and relate it to work of Smith-Zbarsky [S-Z].

Let $\mathcal{C}$ be an A_∞-category, with multiplications denoted μ_i, and K a simplicial set which is an ∞-category.

As before, an object F consists of a choice of a map $F : K_0 \to Ob\mathcal{C}$ along with an element f of total degree 1 from the set

$$f \in \mathcal{C}^1_F(K) := \underset{i+j=1, i\geq 0}{\oplus} \mathcal{C}^{i,j}_F, \tag{1.5.1}$$

with,

$$\mathcal{C}^{i,j}_F := \{k\text{-linear maps } f : K_i \to \mathcal{C}^j | F(\sigma) \in \mathcal{C}^j(F(\sigma_{(i)}), F(\sigma_{(0)}))\} \tag{1.5.2}$$

and which satisfies a generalized Maurer-Cartan equation.

Morphisms are also as before:

$$\mathsf{Loc}_{A_\infty}(K)^q(F, G) = \{\text{k-linear maps } K_i \to \mathcal{C}^j(F(i), G(0)\} \tag{1.5.3}$$

We define a series of multiplications on composable tuples of morphisms. Consider an $n+1$-tuple of objects $(F_n, \dots, F_0)$ and a corresponding tuple of composable morphisms $(\phi_n \otimes \dots \otimes \phi_0)$.

$$m_n : \otimes_{n \geq i \geq 0} \mathsf{Loc}_{A_\infty}(K)^\bullet(F_{i+1}, F_i) \to \mathsf{Loc}_{A_\infty}(K)(F_0, F_n)^\bullet[2-n] \tag{1.5.4}$$

For $n = 1$,

$$m_1 : \phi \mapsto \mu_1 \circ (\phi) - (-1)^{|\phi|} \phi \circ \mu_1 + (-1)^{|\phi|} (\sum_l (-1)^l \phi \circ \partial_l) \tag{1.5.5}$$

and for $n \geq 1$,

$$m_n : (\phi_n \otimes \dots \otimes \phi_0) \mapsto \mu_n \circ (\phi_n \otimes \dots \otimes \phi_0) \circ \Delta^{(n)} \tag{1.5.6}$$

Definition 1.5.1. *A pair (F, f) with $f \in \mathcal{C}^1_F(K)$ such that $0 = \sum_{i=1}^\infty m_i(f^{\otimes i})$ is called an A_∞-local system. The set of A_∞-local systems is denoted $\mathsf{Loc}^{\mathcal{C}}_{A_\infty}(K)$.*

It is important to note that the Maurer-Cartan equation above is not finite, but has a finite number of terms when evaluated on any simplex due to the fact that $\Delta^n(\sigma) = 0$ for $n >> 0$.

If the multiplications defined above satisfy the constraint that for any tuple of composable morphisms, $(\phi_0, \dots, \phi_N)$, the graded vector space $\oplus_{i,j} \mathsf{Loc}_{A_\infty}(K)^\bullet(F_i, F_j)$ equipped with the direct sums of the above multiplications becomes an A_∞-algebra, then $\mathsf{Loc}_{A_\infty}(K)$ is an A_∞-category. In particular the one-point A_∞-category is just an A_∞-algebra.[K-S]

Proposition 1.5.2. $\mathsf{Loc}^{\mathcal{C}}_{A_\infty}(K)$ *is an* A_∞*-category.*

In a recent pre-print [S-Z] Emma Smith-Zbarsky develops a closely related theory. She considers a $\mathbb{Z}$-graded bundle V over a smooth manifold M and then passes to the bundle which is fiber-wise

$$\mathfrak{g}_x^\bullet = \prod_{n=1}^{\infty} \mathrm{Hom}^\bullet(V_x^{\otimes n}, V_x[1-n]) \tag{1.5.7}$$

If one chooses a usual local system ∇, a total degree 1 form α valued in this bundle which satisfies the relevant Maurer-Cartan equation gives the graded bundle V extra structure. In particular the 0-form part of α turns V into a bundle of A_∞-algebras. In her paper Smith-Zbarsky calculates the first couple terms of the holonomy of such an MC element to show that the MC equation implies that the fiber of V over each point is an A_∞-algebra and that parallel transport is an A_∞-morphism. For a smooth, pointed manifold (M, x) which is a $K(G, 1)$, loops at x act on the fiber of V over x by A_∞-morphisms and a homotopy of paths yields an A_∞-homotopy between the corresponding parallel transports morphisms. Thus Smith-Zbarsky calls this type of action of $\pi_1(M, x)$ a homotopy group action of A_∞ algebras. In forthcoming work Smith-Zbarsky will further develop this A_∞ point of view with applications to Lagrangian Floer theory.

This setup can be fruitfully understood in terms of our formalism. Given the graded bundle V, one can pass to the "bar-bundle" which is the fiber-wise application of the bar construction. The bundle $\mathfrak{g}$ is the bundle whose fiber is

$$\mathrm{Hom}_{\mathrm{CoAlg}}(B(V_x), B(V_x)) \tag{1.5.8}$$

where $B(\bullet)$ denotes the bar construction. Given a usual local system ∇, a $\mathfrak{g}$-valued form

α satisfying

$$d_\nabla \alpha + \alpha \circ \alpha = 0 \tag{1.5.9}$$

yields a flat $\mathbb{Z}$-graded connection via

$$d_\nabla + [\alpha, \ \ ] : \mathfrak{g}^\bullet \otimes_{\mathcal{A}^0} A^\bullet \to \mathfrak{g}^\bullet \otimes_{\mathcal{A}^0} A^\bullet \tag{1.5.10}$$

whose holonomy yields an infinity-local system F satisfying

$$dF + \hat{\delta} F + F \cup F = 0. \tag{1.5.11}$$

(A useful guide is [Me]). This holonomy breaks further according to the components in $\mathfrak{g}$ to give the sum-over-trees formulas that one usually sees in such A_∞ applications. In fact, this construction yields an infinity-local system valued in the dg-category of codifferential coalgebras, and should be equivalent to an A_∞-local system valued in the category of A_∞-algebras. However, one should be careful to note that the starting bundle in this case is not of finite rank, so developing this theory technically involves an extension of our results which we expect to go through without serious hitches.

1.5.2 Coefficients in some $\mathcal{P}_B$

As was already mentioned, one can define infinity-local systems valued in any *dg*-category or linear ∞-category. Given such a target category the question arises of what can be said about the correspondence we have proved. I.e., what dg-category sits on the other side? Here is one interesting example.

Suppose Σ is some complex manifold, and $B := (\Omega^{0,\bullet}(\Sigma), \bar{\partial})$ its Dolbeaux complex. Then it is not too hard to see that our Riemann-Hilbert correspondence extends to

$$\mathcal{P}_{B \hat{\otimes} A(X)} \cong \mathsf{Loc}_\infty^{\mathcal{P}_B}(X). \tag{1.5.12}$$

which is a statement which concerns infinity-local systems valued in $\mathcal{P}_B$ which, in this case is a dg-enhancement of the derived category of sheaves with coherent cohomology on Σ, [Bl1].

1.6 Appendix

1.6.1 The Simplicial Nerve

We describe here a construction called the simplicial nerve functor which was originally introduced by Cordier [Co] and appears in Lurie's book on higher Topoi [Lu1]. The simplicial nerve is a functor

$$N : s\mathcal{C}at \to s\mathcal{S}et \tag{1.6.1}$$

which is defined by the adjunction property:

$$s\mathcal{S}et(\Delta^n, N(\mathcal{C})) = s\mathcal{C}at(\mathfrak{C}[\Delta^n], \mathcal{C}) \tag{1.6.2}$$

where $s\mathcal{C}at$ is the category of simplicial categories all of whose hom-spaces are Kan complexes, and where $\mathfrak{C}[\bullet]$ is a kind of categorification functor which in a sense constructs the free simplicial category generated by a simplicial set. Note that any category of enriched categories such as $s\mathcal{C}at$ is understood to have homsets consisting of *enriched* functors. The k-simplices of $N(\mathcal{C})$ can be understood as homotopy coherent simplices in the category $\mathcal{C}$. This will become apparent as we further describe this functor. This exposition is taken almost verbatim from [Lu1] but we include it for completeness.

Let k be a field of characteristic zero.

Definition 1.6.1. *Let $[n]$ denote the linearly ordered, finite set of elements $\{0, 1, 2, \ldots, n\}$, as well as the corresponding category, and let Δ^n denote the simplicial set which is the* *combinatorial n-simplex.* $\mathfrak{C}[\Delta^n]$ *is the element of sCat given by the following assignments:*

$$Ob\mathfrak{C}[\Delta^n] = Ob[n] \tag{1.6.3}$$

for $i, j \in [n]$,

$$\mathfrak{C}[\Delta^n](i,j) = \begin{cases} \emptyset & \text{for } j > i \\ \\ N(P_{i,j}) & \text{for } i \leq j \end{cases} \tag{1.6.4}$$

Where $N[\bullet]$ *the usual nerve functor, and* $P_{i,j}$ *the partially ordered set of subsets* $I \subset \{i \leq \ldots \leq j\}$ *such that both* $i \in I$*, and* $j \in I$*. For the ordered tuple,* $(i_0 \leq \ldots \leq i_l)$*, the compositions in* $\mathfrak{C}[\Delta^n]$

$$\mathfrak{C}[\Delta^n](i_0, i_1) \times \ldots \times \mathfrak{C}[\Delta^n](i_{l-1}, i_l) \to \mathfrak{C}[\Delta^n](i_0, i_l) \tag{1.6.5}$$

are induced by the poset maps given by the union:

$$P_{i_0,i_1} \times \ldots \times P_{i_{l-1},i_l} \to P_{i_0,i_l} \tag{1.6.6}$$

$$I_0 \times \ldots \times I_l \mapsto I_0 \cup \ldots \cup I_l \tag{1.6.7}$$

And $\mathfrak{C}$ is functorial:

Definition 1.6.2. *For* $f : [n] \to [m]$ *a monotone map of linearly ordered finite sets, we get a morphism*

$$\mathfrak{C}[\Delta^n](f) : \mathfrak{C}[\Delta^n] \to \mathfrak{C}[\Delta^m] \tag{1.6.8}$$

$$v \in Ob\mathfrak{C}[\Delta^n] \mapsto f(v) \in Ob\mathfrak{C}[\Delta^m] \tag{1.6.9}$$

For $i \leq j$ *in* $[n]$*, the map*

$$\mathfrak{C}[\Delta^n](i,j) \to \mathfrak{C}[\Delta^m](f(i), f(j)) \tag{1.6.10}$$

is induced by

$$f : P_{i,j} \to P_{f(i),f(j)}, \; I \mapsto f(I) \tag{1.6.11}$$

by applying $N[\bullet]$.

$\mathfrak{C}$ can be extended uniquely to a functor which preserves small colimits, and so it is easily verified that $\mathfrak{C}[\bullet]$ is a functor

$$\mathfrak{C} : s\mathcal{S}et \to s\mathcal{C}at \tag{1.6.12}$$

We use the notation

Definition 1.6.3. *$(s\mathcal{V}ect, \times, *)$ is the monoidal category of simplicial vector spaces over k, and $s\mathcal{C}at(k)$ the category of small categories enriched on $s\mathcal{V}ect$.*

Definition 1.6.4. *$dg\mathcal{C}at$ consists of small categories enriched over the monoidal category $(Ch(k), \otimes, 0)$ of chain complexes over k.*

1.6.2 k-linear ∞-categories

Naively, we would regard a k-linear (∞, n)-category to be an (∞, n)-category whose j-morphism sets are k-vectorspaces for any $j \geq 0$. From this perspective we are replacing the j-morphism sets with j-morphism vectorspaces, while still maintaining a *set* of objects. One would require that composition of j-morphisms ought to be linear for all j.

From the homotopy theory perspective on higher category theory, particular elements of $s\mathcal{C}at$ and $s\mathcal{S}et$ are models for $(\infty, 1)$-categories. It becomes an interesting question as to what objects could model linear ∞-categories within each of these paradigms. One might start with the models of $s\mathcal{S}et$ and $s\mathcal{C}at$ and attempt to enrich or enhance some of these structures in such way that relevant sets are replaced with k-vectorspaces. $s\mathcal{V}ect$ and $s\mathcal{C}at(k)$ immediately come to mind. Both of these models seem less-than-ideal because they incorporate the condition

The domain (target) maps sending a $(j+1)$-morphism to its domain (target) j-morphism must be k-linear. (1.6.13) which is not intuitive to the author. However, as will be seen later, there is a precedent that points toward choosing $s\mathcal{C}at(k)$ as a model. With this in mind, we can use the simplicial nerve to extend this to the quasi-categorical paradigm.

Definition 1.6.5. *We say an ∞-category K is k-linear if K is in the image of N restricted to $s\mathcal{C}at(k)$ inside $s\mathcal{C}at$.*

This matter of giving a proper description of this quasicategorical definition of a k-linear ∞-category has turned out to be a large undertaking, and is mostly tangential to the primary results of this paper. So this task has been pushed off to another paper, [Sm], and we here will give a sketch of structures which this definition implies. Suppose K is a linear ∞-category. For any ordered pair of points (x_0, x_1), $K_1(x_0, x_1)$ —the set of 1-simplices extending from x_0 to x_1— is a k-vectorspace. We can regard these vectorspaces all at once as fibers of the evident map

$$\begin{array}{ccc} & \xrightarrow{p_1} & \\ K_1 & \times & K_0 \times K_0 \\ & \xrightarrow{p_2} & \end{array} \tag{1.6.14}$$

At the next level, given any 3-tuple of points (x_0, x_1, x_2), the set of 2-simplices which span these points $K_2(x_0, x_1, x_2)$, is not a vectorspace. But there are two k-actions on its elements and two partially-defined ways to sum them. In general we can explain this as follows. Let $(x_0, \ldots, x_n)$ be any $(n+1)$-tuple of points. Then,

$$K_1(x_0, x_1) \times K_1(x_1, x_2) \times \ldots \times K_1(x_{n-1}, x_n) \tag{1.6.15}$$

admits an obvious multilinear structure. So all at once we have k-actions

$$\mu_i : k \times K_1 \times_{K_0} K_1 \times_{K_0} \ldots \times_{K_0} K_1 \to K_1 \times_{K_0} \ldots \times_{K_0} K_1 \tag{1.6.16}$$

multiplying by k in the i-th component. And partially-defined addition operations are defined component-wise

$$+_i : (f_1, \ldots, f_i, \ldots, f_n) + (f_1, \ldots, f_i', \ldots, f_n) = (f_1, \ldots, f_i + f_i', \ldots, f_n) \tag{1.6.17}$$

(which is linear with respect to μ_i) whenever all but one of the components are fixed. There is an obvious fibration of sets

$$\Phi_n : K_n \to K_1 \times_{K_0} \ldots \times_{K_0} K_1, \sigma^n \mapsto (\sigma(01), \sigma(12), \ldots, \sigma(n-1n)) \tag{1.6.18}$$

The weak Kan extension property specifies that these maps are always surjective. If K is linear then K_n will have k-actions

$$\tilde{\mu_i} : k \times K_n \to K_n, \tag{1.6.19}$$

which *cover* the linear components of $K_1 \times_{K_0} \ldots \times_{K_0} K_1$ in the sense that

$$\mu_i \circ (Id \times \Phi_n) = \Phi_n \circ \tilde{\mu_i} \tag{1.6.20}$$

Likewise, given tuples $(f_1, \ldots, f_i, \ldots, f_n)$ and $(f_1, \ldots, f_i', \ldots, f_n)$ differing in only one component, there is an abelian pairing in the fibers of Φ

$$\tilde{+}_i : \Phi_n^{-1}((f_1, \ldots, f_i, \ldots, f_n)) \times \Phi_n^{-1}((f_1, \ldots, f_i', \ldots, f_n)) \to \Phi_n^{-1}((f_1, \ldots, f_i + f_i', \ldots, f_n) \tag{1.6.21}$$

which is linear with respect to $\tilde{\mu_i}$. Furthermore, the face and degeneracy maps will be multi-linear in a specific sense. Let d^i be the face maps and s^i the degeneracies. Define

$$D^i : K_1 \times_{K_0} \ldots \text{ } n \text{ factors } \ldots \times_{K_0} K_1 \to [K_1 \times_{K_0} \ldots \times_{K_0} K_1] \times [K_1 \times_{K_0} \ldots \times_{K_0} K_1] \tag{1.6.22}$$

$$(f_1, \ldots, f_n) \mapsto (f_1 \ldots \hat{f_i} \ldots f_n) \tag{1.6.23}$$

where the target is not necessarily composable between the $(i-1)$-st and i-th components. The fibers of $D^i \circ \Phi_n$ are just the n-simplices which complete the horn defined by

$(g_1, \ldots, g_{i-1}, f, g_i, \ldots g_{n-1})$ where the g's are fixed and only f can vary. If we restrict d^j to this set (which is a vectorspace with multiplication $\tilde{\mu}_i$, and sum $+_i$)

$$d^j := \{ \begin{array}{l} (D^i \circ \Phi_n)^{-1}(g_1, \ldots, g_{n-1}) \to (D^i \circ \Phi_{n-1})^{-1}(g_1, \ldots, g_{i-1}, g_{i+1}, \ldots, g_{n-1}), i < j \\ (D^i \circ \Phi_n)^{-1}(g_1, \ldots, g_{n-1}) \to (D^{i-1} \circ \Phi_{n-1})^{-1}(g_1, \ldots, g_{i-1}, g_{i+1}, \ldots, g_{n-1}), i \geq j \end{array} \tag{1.6.24}$$

is linear. Likewise for the degeneracies which go in the other direction.

Definition 1.6.6. *$k - s\mathcal{S}et$ denotes the category of k-linear ∞-categories. Given elements K, L, and ϕ an element of $s\mathcal{S}et(K, L)$, with $\phi_n : K_n \to L_n$,*

$$\begin{aligned} k - s\mathcal{S}et(K, L) = \quad & \{\phi \in s\mathcal{S}et(K, L) | \\ & \phi_n : (D^i \circ \Phi_n)^{-1}(g_1, \ldots, g_{n-1}) \to (D^i \circ \Phi_n)^{-1}(\phi_1(g_1), \ldots, \phi_1(g_{n-1})) \\ & \textit{is linear for any choice of } i, n, (g_1, \ldots, g_{n-1})\} \end{aligned} \tag{1.6.25}$$

There is also a free construction $k() : s\mathcal{S}et \to k - s\mathcal{S}et$ which is left adjoint to the forgetful functor. So that

$$k - s\mathcal{S}et(k(\Delta^n), K) = s\mathcal{S}et(\Delta^n, K) \tag{1.6.26}$$

allows one to pick out the n-simplices of K.

Some expected propositions from [Sm]

Speculation 1.6.7. *$\mathcal{C}$ is a category enriched in simplicial vectorspaces if and only if $N\mathcal{C}$ (the simplicial nerve of $\mathcal{C}$) has the linear structure described above.*

Proof. Straightforward □

and,

Speculation 1.6.8. *For any simplicial set K,*

$$\mathfrak{C}[k(K)] = k\mathfrak{C}[K] \tag{1.6.27}$$

Where the latter is the category enriched in simplicial vectorspaces freely generated by $\mathfrak{C}[K]$.

Proof. [Sm]. □

This linear quasicategorical picture agrees with another precedent in the literature of describing connective dg-categories (and their A_∞-cousins) as models for linear ∞-categories. The model equivalence of Tabuada [Ta], and Dugger-Shipley [DSh] (described below) lends weight to this precedent. In this paper we will not speculate about what should *really* model linear ∞-categories. Instead we will start with the following assumption: *Categories enriched in simplicial vectorspaces model linear ∞-categories.* (Note that these are a proper subset of $s\mathcal{C}at$ because all simplicial vectorspaces are Kan complexes [Ma]).

One of the main goals of [Sm] is to check

Speculation 1.6.9. *The adjoint equivalence* $(\mathfrak{C}[], \mathcal{N})$ *restricts to an adjoint equivalence between* $s\mathcal{C}at(k)$ *and* $k - s\mathcal{S}et$.

Proof. □

And furthermore, with respect to appropriate model structures, these are Quillen.

1.6.3 Relevant Model Structures

By throwing out non-invertible 1-morphisms one can form the $(\infty, 1)$-category of $(\infty, 1)$-categories. There are explicit model structures on each of these categories which present these structures. These are the model structure on $s\mathcal{C}at$ defined by Bergner [Be], and the model structure of Joyal on $s\mathcal{S}et$. These model categories are denoted $s\mathcal{C}at_B$ and $s\mathcal{S}et_J$ respectively. Likewise there is a model structures on $s\mathcal{C}at(k)$ and $k - s\mathcal{S}et$ which are related to these two.

Proposition 1.6.10. *One can induce a model structure on $k - s\mathcal{S}et$ by virtue of the free-forgetful adjunction between $k - s\mathcal{S}et$ and $s\mathcal{S}et_J$. We call this $k - s\mathcal{S}et_J$.*

Proof. □

And in [Ta] Tabuada presents a model structure on $s\mathcal{C}at(k)$ induced from Bergner's model structure on $s\mathcal{C}at$ by virtue of the lifted free-forgetful adjunction $(\tilde{k()}, \tilde{S}) : \tilde{k()} : s\mathcal{C}at \to s\mathcal{C}at(k) : \tilde{S}$. We will call this $s\mathcal{C}at(k)_{BT}$.

The fibrations are the enriched functors $F : A \to B$ which are level-wise fibrations of underlying simplicial sets

$$F_{x,y} : A(x, y) \twoheadrightarrow B(x, y) \tag{1.6.28}$$

which also satisfy the lifting property:

For any x in A, and any isomorphism $v : \pi_0(F)(x) \to y'$ in $\pi_0(B)$, there exists an isomorphism $u : x \to y$ in A such that $\pi_0(F)u = v$.

The cofibrations are generated by the LLP with respect to trivial fibrations.

This model structure could well have been obtained by the general construction described in ([Lu1], A.3) (for $\mathcal{S} = s\mathcal{V}ect$) of a model category structure on $\mathcal{S} - \mathcal{C}at$ whenever $\mathcal{S}$ is an *excellent monoidal model category* ([Lu1],A). This definition subsumes the examples of Bergner and Tabuada for $\mathcal{S} = s\mathcal{S}et, Ch(k)_{\leq 0}$.

Definition 1.6.11. *[Lu1]A. A symmetric monoidal model category $\mathcal{S}$ is called excellent* *if*

i) $\mathcal{S}$ *is combinatorial.*

ii) *The monomorphism in* $\mathcal{S}$ *are cofibrations, and the cofibrations are closed under products.*

iii) *The weak equivalences of* $\mathcal{S}$ *are stable under filtered limits.*

iv) $\mathcal{S}$ *satisfies the invertibility hypothesis*([*Lu1*], *A*.3.2.12)

$$(1.6.29)$$

In this procedure the weak equivalences are the *Dwyer-Kan equivalences*, i.e. a S-functor $F : A \to B$ such that $F_{x,y} : A(x,y) \to B(x,y)$ is a weak equivalence in S for all x, y, and which induces an equivalence of categories $\pi_0(F) : \pi_0(A) \to \pi_0(B)$, where $\pi_0(A)$, resp. $\pi_0(B)$ is the homotopy category of A resp. B whose morphisms are weak equivalence classes of morphisms in A, resp. B.

The fibrations are the functors $F : A \to B$ which are level-wise fibrations of underlying S-objects

$$F_{x,y} : A(x,y) \twoheadrightarrow B(x,y) \tag{1.6.30}$$

which also satisfy the lifting property:

For any x *in* A*, and any isomorphism* $v : \pi_0(F)(x) \to y'$ *in* $\pi_0(B)$*, there exists an isomorphism* $u : x \to y$ *in* A *such that* $\pi_0(F)u = v$*.*

The cofibrations are generated by the LLP with respect to trivial fibrations.

We can regard $s\mathcal{V}ect$ as a model category with the model structure coming from the Kan structure on $s\mathcal{S}et$. Then,

Proposition 1.6.12. $\mathcal{V}ect_k$ *is excellent.*

Proof. Follows from the results of [DK]. $\square$

Lurie [Lu1], and later Dugger-Spivak [DSp],show that the functors N and $\mathfrak{C}$ constitute a Quillen equivalence

$$N : s\mathcal{C}at_B \leftrightarrows s\mathcal{S}et_J : \mathfrak{C} \tag{1.6.31}$$

And we expect, [Sm], a Quillen equivalence

$$N : s\mathcal{C}at(k)_{BT} \leftrightarrow (k - s\mathcal{S}et)_J : \mathfrak{C} \tag{1.6.32}$$

Hence, following ([Hi] 17.4), this adjunction refines to induce weak equivalences of the mapping spaces (although we have not specified any particular constructions of these mapping spaces)

$$\mathcal{M}aps_{s\mathcal{S}et_J}(\mathfrak{C}[K], S) \approx \mathcal{M}aps_{s\mathcal{C}at_B}(K, N[S]) \tag{1.6.33}$$

$$\mathcal{M}aps_{s\mathcal{V}ect_k}(\mathfrak{C}[K], S) \approx \mathcal{M}aps_{s\mathcal{C}at(k)}(K, N[S]) \tag{1.6.34}$$

Another model for a k-linear ∞-category is a connective dg-category, i.e. an element of $dg\mathcal{C}at$ whose hom-complexes are connective. We will specify this full subcategory of $dg\mathcal{C}at$ by $dg\mathcal{C}at_{\leq 0}$. The "intelligent truncation" functor induces a relation between these two categories.

Definition 1.6.13. *The truncation functor*

$$T : Ch \to Ch_{\leq 0} \tag{1.6.35}$$

$$TV^\bullet = \begin{cases} 0 & \text{for } \bullet > 0 \\ Z(V^\bullet) & \text{for } \bullet = 0 \\ V^\bullet & \text{for } \bullet > 0 \end{cases} \tag{1.6.36}$$

induces a lifted truncation functor (abusing notation)

$$T : dg\mathcal{C}at \to dg\mathcal{C}at_{\leq 0} \tag{1.6.37}$$

$ObT(\mathcal{C}) = Ob\mathcal{C}$, and for $X, Y \in Ob\mathcal{C}$, $T\mathcal{C}(X,Y) = T(\mathcal{C}(X,Y))$ with the appropriate restriction of the differentials. Multiplication is given by applying the truncation functor to the composition maps:

$$\begin{aligned} T\mathcal{C}(X,Y) \otimes T\mathcal{C}(Y,Z) &= T(\mathcal{C}(X,Y)) \otimes T(\mathcal{C}(Y,Z)) \\ &\to T(\mathcal{C}(X,Y) \otimes \mathcal{C}(Y,Z)) \xrightarrow{T(\circ_{\mathcal{C}})} T(\mathcal{C}(X,Z)) = T\mathcal{C}(X,Z) \end{aligned} \tag{1.6.38}$$

The adjoint of T is the inclusion $I : dg\mathcal{C}at_{\leq 0} \to dg\mathcal{C}at$.

And as before, Tabuada has presented a model structure on $dg\mathcal{C}at_{\leq 0}$ which presents the $(\infty,1)$-category of k-linear $(\infty,1)$-categories.

Proposition 1.6.14. *([Ta] 4.18) With the Tabuada model structure, the pair (I,T) is a Quillen adjunction.*

The Lifted Dold-Kan Equivalence of Schwede-Shipley and Tabuada

In the hopes of demonstrating a Quillen equivalence between this model category and $s\mathcal{C}at(k)$, one might hope to naively lift the Dold-Kan equivalence to the level of an equivalence between enriched categories. This does not work, as we will see in what follows (which is largely lifted from [SS] with some commentary).

Definition 1.6.15. *We use the notation*

$$M : s\mathcal{V}ect \leftrightarrows Ch_{\leq 0} : \Gamma \tag{1.6.39}$$

to denote the Dold-Kan equivalence. Throughout we use η and ϵ as the unit and counit of these adjunctions.

The Dold-Kan equivalence is (lax) monoidal with respect to these monoidal structures, and so can be applied on the hom-objects so that every object in $s\mathcal{C}at(k)$ can be regarded as an object of $dg\mathcal{C}at_{\leq 0}$. It can be verified that this map induces a functor $\tilde{M}$ between these categories. This verification makes critical usage of all of the properties of the monoidal functor M (and dually Γ) including the naturality of the monoidal maps: (In the first case the shuffle map)

$$m : M(A) \otimes M(B) \to M(A \otimes B). \tag{1.6.40}$$

and

$$\gamma : \Gamma(C) \otimes \Gamma(D) \to \Gamma(C \otimes D) \tag{1.6.41}$$

Definition 1.6.16. *Lift of M (likewise for Γ)*

$$\tilde{M} : s\mathcal{C}at(k) \leftrightarrows dg\mathcal{C}at_{\leq 0} \tag{1.6.42}$$

for any $\mathcal{A} \in s\mathcal{C}at(k)$,

$$\mathcal{O}b\tilde{M}(\mathcal{A}) = \mathcal{O}b\mathcal{A}, \quad \tilde{M}\mathcal{A}(X,Y) = M(\mathcal{A}(X,Y)) \tag{1.6.43}$$

with composition in $\tilde{M}(\mathcal{A})$ defined by

$$(\circ_{\mathcal{A}})\circ m : M(\mathcal{A})(Y,Z) \otimes M(\mathcal{A})(X,Y) \to M(\mathcal{A}(Y,Z) \otimes \mathcal{A}(X,Y)) \to M(\mathcal{A}(X,Z)) \tag{1.6.44}$$

The unit is specified by

$$1_{\tilde{M}\mathcal{A}} \to M(1_{\mathcal{A}}) \to M(\mathcal{A}(X,X)) \tag{1.6.45}$$

where, the first map comes from the unitary condition on M, the second is M applied to the identity-assigning morphism in $\mathcal{A}$. (the same definitions hold for Γ)

We might ask if this lift is an adjunction or even a natural equivalence of categories. Isomorphisms in $s\mathcal{C}at(k)$ are enriched *isomorphisms* of categories. One could try to define an adjoint equivalence in an obvious way. Consider the unit: we need a natural equivalence $\mu : I_{s\mathcal{C}at(k)} \to \tilde{\Gamma}\tilde{M}$. For any $\mathcal{A} \in s\mathcal{C}at(k)$ μ assigns the isomorphism in $s\mathcal{C}at(k)$:

$$\mu_{\mathcal{A}} : \mathcal{A} \to \tilde{\Gamma}\tilde{M}\mathcal{A} \tag{1.6.46}$$

$$\mathcal{O}b(\mu_{\mathcal{A}}\mathcal{A}) = \mathcal{O}b\mathcal{A}, \quad \mu_{\mathcal{A}}(\mathcal{A}(X,Y)) = \eta_{\mathcal{A}(X,Y)}(\mathcal{A}(X,Y)) \tag{1.6.47}$$

where η is the unit of the Dold-Kan adjunction. And since μ should be a ($s\mathcal{V}ect$-enriched) functor, one would require (amongst other conditions) the following diagram to commute

$$\begin{array}{ccc}
\mathcal{A}(Y,Z)\otimes\mathcal{A}(X,Y) & \overset{\mu_{\mathcal{A}}\otimes\mu_{\mathcal{A}}}{\to} & \tilde{\Gamma}\tilde{M}\mathcal{A}(Y,Z)\otimes\tilde{\Gamma}\tilde{M}\mathcal{A}(X,Y) \\
\downarrow^{I\otimes I} & & \downarrow^{m} \\
\mathcal{A}(Y,Z)\otimes\mathcal{A}(X,Y) & \overset{\mu_{\mathcal{A}}}{\to} & \tilde{\Gamma}\tilde{M}(\mathcal{A}(Y,Z)\otimes\mathcal{A}(X,Y)) \\
\downarrow^{\circ_{\mathcal{A}}} & & \downarrow^{\tilde{\Gamma}\tilde{M}(\circ_{\mathcal{A}})} \\
\mathcal{A}(X,Z) & \overset{\mu_{\tilde{\Gamma}\tilde{M}\mathcal{A}}}{\to} & \tilde{\Gamma}\tilde{M}\mathcal{A}(X,Z)
\end{array}$$

The right column is by definition the composition in $\tilde{\Gamma}\tilde{M}\mathcal{A}$, and the lower square commutes because η is natural. But the upper square does not commute in general. In particular, it is well known that the Dold-Kan functors are not a *monoidal* adjunction. That is, the unit and counits are *not* monoidal natural transformations [SS]. For any A, B in $s\mathcal{V}ect$, the square

$$\begin{array}{ccc}
A\otimes B & \overset{\eta_A\otimes\eta_B}{\to} & \Gamma MA\otimes\Gamma MB \\
\downarrow^{I\otimes I} & & \downarrow^{m} \\
A\otimes B & \overset{\eta_{A\otimes B}}{\to} & \Gamma M(A\otimes B)
\end{array}$$

does not strictly commute. And so our putative counit/unit are not even defined. So much for that. This is the topic of extensive work on the monoidal Dold-Kan correspondence and enrichments in monoidal model categories. [D], [DSh], [SS], [Ta].

In [Ta], building on the work of Shipley, and Shipley-Schwede, Tabuada realizes the goal by constructing a Quillen equivalence

$$L : dg\mathcal{C}at_{\leq 0,T} \rightleftarrows s\mathcal{C}at(k)_{BT} : \tilde{M} \tag{1.6.48}$$

Where the left adjoint of $\tilde{M}$ is constructed abstractly from principles of model category theory, rather than by lifting Γ. We can now define the k-linear simplicial nerve $N(k)$.

1.6.4 The Linear Simplicial Nerve and $\mathcal{C}_\infty$

Denote by S the forgetful functor

$$S : s\mathcal{C}at(k) \to s\mathcal{C}at \tag{1.6.49}$$

Proposition 1.6.17. *([Ta], 6.2) S is adjoint to the free k-vectorspace functor $k()$, and this pair $(k(), S)$ yields a Quillen adjunction between $s\mathcal{C}at(k)_{BT}$ and $s\mathcal{C}at_B$.*

Definition 1.6.18. *By composition with T, L, and S, we define the functor $N(k)$:*

$$N(k) := N \circ S \circ L \circ T : dg\mathcal{C}at \to k - s\mathcal{S}et \subset s\mathcal{S}et \tag{1.6.50}$$

Based on the previous discussion we have

Proposition 1.6.19. *The adjunction*

$$N(k) : dg\mathcal{C}at_T \leftrightarrow s\mathcal{S}et_J : I \circ \tilde{M} \circ k() \circ \mathfrak{C} \tag{1.6.51}$$

is Quillen.

Proof. □

Which then yields a natural bijection:

$$s\mathcal{S}et(\Delta^n, N(k)[\mathcal{C}]) = dg\mathcal{C}at(I \circ \tilde{M} \circ k \circ \mathfrak{C}[\Delta^n], \mathcal{C}) \tag{1.6.52}$$

We now describe $\mathcal{C}_\infty$

Definition 1.6.20. *Given a dg-category $\mathcal{C}$, we define $\mathcal{C}_\infty := N \circ S \circ \tilde{\Gamma} \circ T[\mathcal{C}]$*

This construction is almost identical to the linear simplicial nerve except that we apply the lift of Γ instead of its replacement L. We can give a very explicit description of this construction, but it will not satisfy the adjunction that $N(k)$ does.

Given a dg-category $\mathcal{C}$ in $dg\mathcal{C}at_{\leq 0}$, we produce a simplicial set $\mathcal{C}_\infty$ which is an ∞-category. It is not too hard to observe that this describes explicitly the definition above.

Denote by $Y_i([n])$ the set of length-i (ordered) subsets of $[n]$. We denote an element of $Y_j([n])$ as an ordered tuple $(i_0 < i_1 < \ldots < i_j)$ and make use of this notation below.

$$\mathcal{C}_{\infty\, 0} = \{F = (F_0, f)|$$

$$f : Y_1([0]) \to \mathrm{Ob}\mathcal{C}$$

$$\mathcal{C}_{\infty\, 1} = \{F = (F_0 + F_1, f)|$$

$$f : Y_1([1]) \to \mathrm{Ob}\mathcal{C}$$

$$F_1 : Y_2([1]) \to \mathcal{C}^0(f(i_1), f(i_0))\}$$

$$\ldots$$

$$\mathcal{C}_{\infty\, l} = \{F = (\sum_{i=0}^{l} F_i, f)|$$

$$f : Y_1([1]) \to \mathrm{Ob}\mathcal{C}$$

$$\ldots$$

$$F_j : Y_{j+1}([l]) \to \mathcal{C}^{1-j}(f(i_j), f(i_0)),$$

$$\ldots$$

$$F_l : Y_{l+1}([l]) \to \mathcal{C}^{1-l}(f(i_l), f(i_0))\}$$

such that $dF + \hat{\delta}F + F \cup F = 0$, where

$$dF(i_1 < \ldots < i_j) := d(F(i_1 < \ldots < i_j)) \tag{1.6.53}$$

$$\hat{\delta}F_j(i_0 < \ldots < i_{j+1}) := -\sum_{q=1}^{k-1} (-1)^q F_j(i_0 < \ldots < \hat{i_q} < \ldots < i_{j+1}) \tag{1.6.54}$$

and

$$(F \cup F)_j(i_0 < \ldots < i_j) = \sum_{q=1}^{j-1} (-1)^q F_q(i_0 < \ldots < i_q) \circ F_{j-q}(i_q < \ldots < i_j) \tag{1.6.55}$$

The face maps are defined as follows:

$$\partial_q F_j(i_0 < \ldots < i_{j+1}) = F_j(i_0 < \ldots < \hat{i_q} < \ldots < i_{j+1}) \tag{1.6.56}$$

and degeneracies,

$$s_q F_j(i_0 < \ldots < i_{j-1}) = F_j(i_0 < \ldots < i_q < i_q < \ldots < i_{j-1}). \tag{1.6.57}$$

$\mathcal{C}_\infty$ is a simplicial set, and we need to show that it satisfies the weak Kan extension property.

Proposition 1.6.21. *$N(k)[\mathcal{C}]$ is an ∞-category.*

Proof. Note that $L \circ T \circ \mathcal{C}$ is an element of $s\mathcal{C}at$ by virtue of the fact that the hom-spaces are elements of $s\mathcal{V}ect$, and hence are Kan complexes ([Ma],17.1). Then, by ([Lu1],1.1.5.10), $N \circ L \circ T[\mathcal{C}]$ is a weak Kan complex, an ∞-category. $\square$

Proposition 1.6.22. *$\mathcal{C}_\infty$ is an ∞-category.*

Proof. This can be seen by direct calculation: Let $0 < q < k$, and S be a simplicial set map $S : \Lambda^k_q \to \mathcal{C}_\infty$. We show that this extends to $S : \Delta^k \to \mathcal{C}_\infty$. We first define S on the missing $k-1$-face:

$$\begin{gathered} S_{k-1}(01\ldots\hat{q}\ldots k) := \\ (-1)^{q+1}\sum_{j=1, j\neq q}^{k-1}(-1)^j S_{k-1}(0\ldots\hat{j}\ldots k) + \\ (-1)^q\sum_{j=1}^{k-1}(-1)^j S_j(0\ldots j)S_{k-j}(j\ldots k) \end{gathered} \tag{1.6.58}$$

We then define the new k-face:

$$S_k(01\ldots k) := 0(\text{any closed morphism will work}) \tag{1.6.59}$$

Then by this assignment,

$$(\hat{\delta}S)_k(0\dots k) + (S\cup S)_k(01\dots k) =$$

$$= -(-1)^q S_{k-1}(0\dots\hat{q}\dots k) + \sum_{j=1, j\neq q}^{k-1} (-1)^j S(0\dots\hat{j}\dots k) + \tag{1.6.60}$$

$$+(S\cup S)_k(01\dots k) = 0.$$

$$\text{hence, } dS_k + (\hat{\delta}S)_k + (S\cup S)_k = 0.$$

It remains to show that the new $k-1$-face satisfies the corresponding equation:

$$dS_{k-1}(0\dots\hat{q}\dots k) = -(\hat{\delta}S)_{k-1}(0\dots\hat{q}\dots k) - (S\cup S)_{k-1}(0\dots\hat{q}\dots k) \tag{1.6.61}$$

Expanding the LHS we have:

$$(-1)^{q+1}\sum_{j=1, j\neq q}(-1)^j d(S_{k-1}(0\dots\hat{j}\dots k)) +$$
$$+ (-1)^q\sum_{j=1}^{k-1}(-1)^j d(S_j(0\dots j))\circ S_{k-j}(j\dots k) +$$
$$+ (-1)^q\sum_{j=1}^{k-1}(-1)^j(-1)^{1-j}S_j(0\dots j)\circ d(S_{k-j}(j\dots k)) \tag{1.6.62}$$

Expanding just the first of the three terms above we get:

$$(-1)^{q+1}\sum_{j=1, j\neq q}^{k-1}(-1)^{j+1}(\hat{\delta}S)_{k-1}(0\dots\hat{j}\dots k) +$$
$$(-1)^{q+1}\sum_{j=1, j\neq q}^{k-1}(-1)^{j+1}(S\cup S)_{k-1}(0\dots\hat{j}\dots k) \tag{1.6.63}$$

And the first of the two sums above yields:

$$(-1)^{q+1}\sum_{j=1}^{q-1}(-1)^{j+1}[\sum_{t=1}^{j-1}(-1)^t S_{k-1}(0\dots\hat{t}\dots\hat{j}\dots k) +$$
$$+ \sum_{t=j+1}^{k-1}(-1)^{t+1}S_{k-2}(0\dots\hat{j}\dots\hat{t}\dots k)] +$$
$$+ (-1)^{q+1}\sum_{j=q+1}^{k-1}(-1)^j[\sum_{t=1}^{j-1}(-1)^t S_{k-1}(0\dots\hat{t}\dots\hat{j}\dots k) +$$
$$+ \sum_{t=j+1}^{k-1}(-1)^{t+1}S_{k-1}(0\dots\hat{j}\dots\hat{t}\dots k)] \tag{1.6.64}$$

After some inspection it can be seen that all of the terms here appear in canceling pairs except for when t=q. So this reduces to

$$\sum_{j=1}^{q-1}(-1)^j S_{k-2}(0\dots\hat{j}\dots\hat{q}\dots k) + \sum_{j=q+1}^{k-1}(-1)^{j+1}S_{k-2}(0\dots\hat{q}\dots\hat{j}\dots k), \tag{1.6.65}$$

which is exactly equal to $(-\hat{\delta}S)_{k-2}(0\dots\hat{q}\dots k)$ on the RHS.

We now expand the second sum of (1.6.63):

$$\begin{aligned}(-1)^{q+1}\sum_{j=1,j\neq q}^{k-1}(-1)^{j+1}[\sum_{t=1}^{j-1}(-1)^t S_t(0\dots t)\circ S_{k-t-1}(t\dots\hat{j}\dots k)+ \\ +\sum_{t=j+1}^{k-1}(-1)^{t-1}S_{t-1}(0\dots\hat{j}\dots t)\circ S_{k-t}(t\dots k)]\end{aligned} \tag{1.6.66}$$

And likewise we expand the second two sums of (1.6.62):

$$\begin{aligned}(-1)^q\sum_{j=1}^{k-1}(-1)^j(-(\hat{\delta}S)_j(0\dots j) - (S\cup S)_{k-j}(j\dots k))\circ S_{k-j}(j\dots k)+ \\ +(-1)^q\sum_{j=1}^{k-1}S_j(0\dots j)\circ(-(\hat{\delta}S)_{k-j}(j\dots k) - (S\cup S)_{k-j}(j\dots k))\end{aligned} \tag{1.6.67}$$

Expanding the above sums without the triple composition terms, we get

$$\begin{aligned}(-1)^q\sum_{j=1}^{k-1}\sum_{t=1}^{j-1}(-1)^{t+j}S_{j-1}(0\dots\hat{t}\dots j)\circ S_{k-j}(j\dots k)+ \\ (-1)^q\sum_{j=1}^{k-1}\sum_{t=j+1}^{k-1}(-1)^{t+j}S_j(0\dots j)\circ S_{k-j-1}(j\dots\hat{t}\dots k)\end{aligned} \tag{1.6.68}$$

It can be seen that these pair with the terms in (1.6.66) to cancel all but the terms which give $-(S\cup S)_{k-1}(0\dots\hat{q}\dots k)$ on the RHS. It just remains to analyze the triple composition terms of (1.6.67):

$$\begin{aligned}(-1)^q\sum_{j=1}^{k-1}(-1)^j(S\cup S)_j(01\dots j)\circ S_{k-j}(j\dots k) = \\ = (-1)^q\sum_{j=1}^{k-1}\sum_{t=1}^{j-1}(-1)^{j+t+1}S_t(0\dots j)\circ S_{j-t}(t\dots j)\circ S_{k-j}(j\dots k)\end{aligned} \tag{1.6.69}$$

$$(-1)^q \sum_{j=1}^{k-1} (-1)^{j+1}(-1)^{1-j}(-1) S_j(0 \dots j) \circ (S \cup S)_{k-j}(j \dots k) =$$

$$= (-1)^q \sum_{j=1}^{k-1} \sum_{t=j+1}^{k-1} (-1)^{t+j} S_j(0 \dots j) \circ S_{t-j}(j \dots t) \circ S_{k-t}(t \dots k). \quad (1.6.70)$$

The sum of these above terms can be seen to vanish since each term appears twice with opposite signs which proves the assertion. □

1.6.5 Stability

Proposition 1.6.23. *$\mathcal{C}_\infty$, $N(k)[\mathcal{C}]$, are stable ∞-categories.*

Proof. [Lu2] 13.4. □

Chapter 2

The Theory of Multiholomorphic Maps

2.1 n-triads and the Multi-Cauchy-Riemann Equations

2.1.1 n-triads

A ubiquitous feature in the theory of pseudoholomorphic curves is a choice of a compatible trio of geometric structures on the target symplectic manifold. This takes the form (ω, g, J) where ω is a symplectic form, g, a Riemannian metric, and J an almost complex structure. These elements are compatible as a triad in the sense that any two determine the other via obvious formulae. Alternately, there is a notion of compatibility between any pair which is essentially the stipulation that the pair in question fits into a compatible triad in which the third item is determined by the appropriate formula. These notions originated in Gromov's notions of "taming" [G], and also fit into Kähler geometry, as this is a kind of almost-Kähler condition. The theory of pseudoholomorphic curves studies a PDE (the non-linear Cauchy-Riemann equation) whose solutions consist of maps between two almost complex manifolds which intertwine the complex structures. The real geometric/topological interest of the theory emerges when the complex structures are enhanced to compatible triads.

We start by introducing a more general triadic structure that generalizes Gromov's situation. Such a triad includes three elements: a *nondegenerate* $(n+1)$-form, an n-fold *split product*, and a Riemannian metric along with some relevant compatibility stipulations. The split product is *not* a stand-alone generalization of an almost complex structure: It does not reduce to an almost complex structure when $n = 1$. However, when fit into a 1-triad it does, and hence, it does in every case that matters to us.

Definition 2.1.1. *(Split Product) An n-fold* split product *on a smooth manifold M is a pair (J, K) of bundle homomorphisms (over the identity)*

$$K : TM \to \Lambda^n TM, \quad J : \Lambda^n TM \to TM \tag{2.1.1}$$

such that

$$J \circ K = (-1)^n \lambda I_{TM} \tag{2.1.2}$$

for some positive function $\lambda(m) > 0$. This is the same as saying the exact sequence associated to J,

$$0 \to ker(J) \to \Lambda^n TM \to TM \to 0 \tag{2.1.3}$$

is split by $\frac{(-1)^n}{\lambda} K$.

In the course of this study, we came across the notion of *vector cross product* which were classified in [BG], and are discussed in the context of gauge theory by Lueng, Lueng-Lee, [L1], [LL].

Definition 2.1.2. *(Vector Cross Product) A bundle homomorphism J on a Riemannian manifold (M, g)*

$$J : \Lambda^n TM \to TM \tag{2.1.4}$$

is called an vector cross product *if it generates with g an $(n+1)$-form ω,*

$$g(A_0, J(A_1, \ldots, A_n)) = \omega(A_0, \ldots, A_n), \quad \omega \in \Omega^{n+1}TM \tag{2.1.5}$$

and has comass $= 1$ in the sense

$$\|J(A_1, \ldots, A_n)\|_g^2 = \|A_1 \wedge \ldots \wedge A_n\|_g^2. \tag{2.1.6}$$

Note that this is a condition on simple vectors, and it is not equivalent to the condition that J be an isometry.

There is a close connection between vector cross products and calibrated geometry because a vector cross product on a Riemannian manifold yields a calibrating form via $\omega = g(\bullet, J\bullet)$.

Definition 2.1.3. *(Compatible Triad)*

*An n-*compatible triad *on M is a triple $(\omega, g, (J, K))$ consisting of a non-degenerate $(n+1)$-form, a Riemannian metric, and an n-split product such that*

$$J = g^{-1} \circ \omega, \text{ where } g : TM \cong T^*M, \ \omega : \Lambda^n TM \to T^*M, \text{ i.e.,} \tag{2.1.7}$$

$$\omega(A, B, C) = g(A, J(B, C)).$$

$$K = (-1)^n \Lambda^n g^{-1} \circ \omega, \text{ where } g : TM \cong T^*M, \ \omega : TM \to \Lambda^n T^*M, \text{ i.e.,} \tag{2.1.8}$$

$$\Lambda^n g(\zeta, KA) = (-1)^n \omega(A, \zeta).$$

And J is furthermore required to be vector cross product. The conditions (2.1.7), (2.1.8) imply that J, K are adjoint to each other up to $(-1)^n$ with respect to the metrics g, and $\Lambda^n g$. Along with $J \circ K = (-1)^n \lambda I$ these conditions imply

$$g(A_0, A_1) = \lambda^{-1} \omega(K(A_0), A_1) \tag{2.1.9}$$

Definition 2.1.4. *(n-plectic triad) We say a compatible triad is* n-plectic *if* ω *is closed and non-degenerate.* ω *is non-degenerate if for all non-zero tangent vectors* V*, at any tangent space, the contraction map*

$$\iota_V : T_xM \to \Lambda^n T_x^*M \tag{2.1.10}$$

is injective. We refer to a manifold with such a form as n-plectic*, and a triad containing such a form an* n-plectic triad*.*

The definition of n-plectic or multi-(sym)plectic appears in work of Gotay, Isenberg, Marsden, Montgomery, [GIMM], and some recent papers by Baez, Hoffnung, Rogers [BHR]. (motivated by the canonical n-plectic form on an n-form bundle). Symplectic manifolds are 1-plectic.

And because a triadic manifold is Riemannian it makes sense to consider the covariant derivatives of any of these tensors. Hence,

Definition 2.1.5. *(Parallel Triad) Let* ∇ *be the Riemannian connection associated to* g*. Then, we say* ω*, and* (J, K) *are* parallel *if*

$$\nabla\omega = 0, \quad \textit{and hence } \nabla J = 0, \nabla K = 0. \tag{2.1.11}$$

In the examples we are about to consider, the triads all have the property that J is a vector cross product, and all the relevant tensors are parallel.

Example 2.1.6. *(Conformal Triad) Suppose* M *is an oriented,* $(n+1)$*-dimensional smooth, Riemannian manifold, and consider the case when the n-plectic form is a volume form. Then the metric* g*, and volume form* $dVol_M$ *(which is n-plectic) yields a canonical n-split product* (J, K) *which fits into a triad. Namely, let* J *and* K *be particular multiples of the Hodge dual operator on* TM*. Explicitly,*

$$J = (-1)^n \star, \quad K = (-1)^n \star \tag{2.1.12}$$

This implies

$$K \circ J = (-1)^n I, \quad J \circ K = (-1)^n I \tag{2.1.13}$$

and consequently that the Hodge star operation which acts on forms on M *is given (in degrees* $1, n$*) in terms of precomposition with* $(-1)^n J, (-1)^n K$*.*

This example will become crucial because such a triadic manifold will be the domain in a mapping theory.

Example 2.1.7. *(Associatively calibrated* G_2*-manifolds)(See [HL])*
Consider $Im\mathbb{O}$*, and fix an identification with* $\mathbb{R}^7$*.*

$$a + bI + cJ + dK + e1' + fI' + gJ' + hK' = (a, b, c, d, e, f, g, h) \tag{2.1.14}$$

This is equipped with the usual inner product $g(,)$*, the octionic product* $\cdot$*, the notion of real/imaginary parts, and conjugation:*

$$Re(a + bI + cJ + dK + eI' + fJ' + gK') = a \tag{2.1.15}$$

$$Im(a + bI + cJ + dK + eI' + fJ' + gK') = bI + cJ + dK + eI' + fJ' + gK' \tag{2.1.16}$$

$$\bar{A} := ReA - ImA \tag{2.1.17}$$

$$g(A, B) := Re(A \cdot \bar{B}) \tag{2.1.18}$$

Furthermore, the octonionic product restricts to a product on $\mathbb{R}^7$ *after projection:*

$$J(A, B) := Im(A \cdot B) \tag{2.1.19}$$

From these structures we can define an alternating three-form on $\mathbb{R}^7$ *by*

$$\omega(A, B, C) = g(A, J(B, C)) \tag{2.1.20}$$

In coordinates, we have the standard volume form and metric, and

$$\omega_0 = dx^{123} - dx^1(dy^{23} + dy^{10}) - dx^2(dy^{31} + dy^{20}) - dx^3(dy^1 2 + dy^{30}). \tag{2.1.21}$$

This furnishes a well-known description of the exceptional Lie group G_2:

$$G_2 := \{\sigma \in GL(Im\mathbb{O}) | \sigma^* \omega_0 = \omega_0\} \tag{2.1.22}$$

In [Br1], Bryant proved that G_2 can also be described as the group which preserves the metric and vector cross product. For this reason, a Riemannian 7-manifold with G_2-holonomy is equipped with a parallel 3-form defined via parallel transport, and a split vector cross product given by

$$g(A, J(B, C)) = \omega(A, B, C), \quad g(A, B) = \frac{1}{3}\omega(K(A), B) \tag{2.1.23}$$

So any G_2-manifold has a canonical 2-triad.

Definition 2.1.8. *Such a triad on a G_2-manifold will be called the* associative triad. *In this case $\nabla\omega = 0$. Such a parallel structure we will call a* torsion-free G_2-structure *on M, and is equivalent to the condition that ω is harmonic. If ω is closed but not necessarily co-closed, the structure will be called a* closed G_2-structure, *and we reserve the terminology* G_2-structure *to denote the existence on M of a principal G_2-subbundle of the frame bundle on M. The latter condition implies the existence of a triad which is not necessarily closed. See later sections for a more full discussion of these facts.*

Example 2.1.9. *(Cossociatively Calibrated G_2-manifolds)(See [HL])*

We define the coassociative 4-form on a G_2-manifold via

$$\omega(A_0, \ldots, A_3) := g(A_0, Im[A_1, A_2, A_3]) \tag{2.1.24}$$

where $[,,,]$ is the associator

$$J(A, B, C) := Im[A, B, C] \tag{2.1.25}$$

fits into a canonical 3-plectic-triad, the coassociative triad.

Example 2.1.10. *(Cayley Manifolds)(See [HL])*

On a $Spin_7$-manifold (tangent space identified with $\mathbb{O}$) we can define the Cayley Calibration

$$\Phi(x \wedge y \wedge z \wedge w) := g(x, y \times z \times w) = g(x, J(y, z, w)), \quad J(A, B, C) := A \times B \times C \tag{2.1.26}$$

where the operation above is the triple product in $\mathbb{O}$.

$$x \times y \times z := \frac{1}{2}(x(\bar{y}z) - z(\bar{y}x)) \tag{2.1.27}$$

2.1.2 The Multi-Cauchy-Riemann Equation

Suppose a smooth manifold M is equipped with an n-plectic triad. We denote this as $(M, \omega, g, (J, K))$ as before. We also choose a closed, compact $(n+1)$-manifold X, with an oriented Riemannian structure (hence a conformal n-triad).

We use the notation $\Lambda^n du$ to denote the application of the n-th exterior power functor pointwise to du. Thus, $\Lambda^n du$ is an element of $\Omega^n(X, \Lambda^n u^* TM)$. Given the usual definition of the Λ functor,

$$\begin{aligned} &\Lambda^n : Hom_{\mathbb{R}}(V, W) \to Hom_{\mathbb{R}}(\Lambda^n V, \Lambda^n W) : \\ &\phi \mapsto (v_1 \wedge \ldots \wedge v_n \mapsto n \sum_{\sigma \in S_n} \frac{1}{n!} Sgn(\sigma) \phi(v_{\sigma(1)}) \wedge \ldots \wedge \phi(v_{\sigma(n)})) = n\phi(v_1) \wedge \ldots \wedge \phi(v_n) \end{aligned} \tag{2.1.28}$$

we introduce a dynamical equation which mimics the Cauchy-Riemann equation:

Definition 2.1.11. *(Multi-Cauchy-Riemann equation)*

$$\eth u := \frac{1}{(n+1)^{(n-1)/2}} |du|^{n-1} du - \frac{1}{n} J \circ \Lambda^n du \circ k = 0 \tag{2.1.29}$$

A solution to the above equation will also satisfy the alternative equation

$$\eth u := \|du\|^{n-1} du - \frac{1}{n} J \circ \Lambda^n du \circ k = 0 \tag{2.1.30}$$

where the norm $\|du\|$ is the operator norm on du understood as a linear map pointwise. The norm of du as a vector pointwise in the tensor product space $T_x^*X \otimes u^*T_{u(x)}M$ we denote by $|du|$. This is the *Hilbert-Schmidt norm Warning:* we will often leave context to decide which statements are meant pointwise.

We will sometimes make use of the fact that the latter norm comes from the metric $g \otimes g_X^*$, so that

$$|du| = (g \otimes g_X^*(du, du))^{\frac{1}{2}}. \tag{2.1.31}$$

It should be noted that if u were mapping $(\mathbb{R}^n, g_0, dVol, \star)$ to itself then

$$|du|^2 = tr(du \circ du^t). \tag{2.1.32}$$

It is immediately clear from the right-hand formula that this is an orthogonal invariant of du.

If u is a solution $\eth u = 0$, then these two norms are proportional by a factor of $\frac{1}{\sqrt{(n+1)}}$ on du, but are not the same in general. This issue will arise when we consider the variational perspective on these equations.

Proposition 2.1.12. *The MCR equation is conformally equivariant. Suppose*

$$\phi : X^{n+1} \to X^{n+1}$$

is a conformal diffeomorphism with conformal factor μ^2, $\mu^2 g = \phi^* g$. Then

$$\eth(u \circ \phi) = \mu^{(n-1)}(\eth u) \circ d\phi. \tag{2.1.33}$$

As a result the solution space is conformally invariant.

Proof.

$$\eth(u\circ\phi)=\frac{1}{(n+1)^{(n-1)/2}}|du\circ d\phi|^{n-1}du\circ d\phi-\frac{1}{n}J\circ\Lambda^n(du\circ d\phi)\circ k=$$

$$=\frac{1}{(n+1)^{(n-1)/2}}\mu^{(n-1)}|du|^{n-1}du\circ d\phi-\frac{1}{n}J\circ\Lambda^n du\circ\Lambda^n d\phi\circ k=$$

$$=\frac{1}{(n+1)^{(n-1)/2}}\mu^{(n-1)}|du|^{n-1}du\circ d\phi-\frac{\mu^{(n-1)}}{n}J\circ\Lambda^n du\circ k\circ d\phi \quad (2.1.34)$$

The last equality follows from $\Lambda^n d\phi\circ k=\mu^{(n-1)}k\circ d\phi$, which is just the statement about how the Hodge star transforms under a conformal automorphism of the domain. □

In the above calculation we used the observation that multiholomorphic automorphisms $\phi: X\to X$ are merely conformal automorphisms. This demonstrates an important coherence of the framework of n-triads: when we consider multiholomorphic maps $X\to M$, we can say that the automorphisms of X (defined within the theory only in terms of X) preserve the solutions to the MCR equation between X and M.

So, if X, M are two n-triadic $(n+1)$-dimensional manifolds, then the theory of multiholomorphic maps between them includes the theory of conformal maps between X and M, but a general solution can be singular. In fact the solutions in this case are a particular geometric generalization of the well-studied field of 1-quasiregular mappings. This issue is discussed in a proceeding section. We found in [IM] an excellent presentation of the of modern issues in this direction and a lot of inspiration for this more global differential-geometric approach. In what remains in this section we describe the MCR equations in the presence of isothermal coordinates.

Suppose X is conformally flat. We have local conformal coordinates on given by the chart $\phi_\alpha: U\subset X\to\mathbb{R}^{n+1}$, with $\psi_\alpha:=\phi_\alpha^{-1}$. Then, a map u satisfies the MCR equation iff in local conformal coordinates, $u_\alpha:=u\circ\psi_\alpha$ satisfies the MCR system:

$$\frac{1}{(n+1)^{(n-1)/2}}\Big(\sum_{j=0}^{n}|\partial_j u_\alpha|_g^2\Big)^{\frac{n-1}{2}}\partial_i u_\alpha-(-1)^{n(n-i+1)}J(\partial_{i+1}u_\alpha\wedge\ldots\wedge\partial_{i-1}u_\alpha)=0 \quad (2.1.35)$$

Taking the norm of both sides in TM, and defining the real numbers $|\partial_i u_\alpha| =: A_i$,

$$\frac{1}{(n+1)^{(n-1)/2}}(\sum_{j=0}^{n}|A_i^2)^{\frac{n-1}{2}}A_i = |J(\partial_{i+1}u_\alpha \wedge \ldots \wedge \partial_{i-1}u_\alpha)| \tag{2.1.36}$$

Since J is a vector cross product,

$$C(|\partial_0 u|^2 - |\partial_1 u|^2) = g(\partial_0 u, J(\partial_1 \wedge \ldots \wedge \partial_n u)) + g(\partial_1 u, J(\partial_0 \wedge \ldots \wedge \partial_n u)) = 0 \tag{2.1.37}$$

where the coefficients are subsumed into C. So $|\partial_i u| = |\partial_j u|$ for all i,j. Similarly, by the same kind of argument all of these partials $\partial_i u$ must be mutually orthogonal because of the fact that $\omega() = g(, J)$ is an alternating form. Hence, we can simplify slightly in these conformal coordinates,

Proposition 2.1.13. *In local conformal coordinates $\{x_0, \ldots, x_n\}$, the MCR equation on $u : U \subset \mathbb{R}^n \to M$ is satisfied iff, $\{\partial_i u\}$ is a positively-oriented orthogonal frame in $T_{u(x)}M$ for all x, and $|\partial_i u| = |\partial_j u|$ for all i, j. This means we could write the equations*

$$|\partial_i u|^{\frac{n-1}{2}}\partial_i u = (-1)^{n(n-i+1)}J(\partial_{i+1}u \wedge \ldots \wedge \partial_{i-1}u) \tag{2.1.38}$$

Proof. □

Of course if M is Euclidean this becomes even more simple, as is discussed in the section on quasiregular maps and the Beltrami system. And even in the general situation in which there are obstructions to the existence of isothermal coordinates we can always pick local normal coordinates so that the above equations hold at the center point.

2.1.3 Endomorphisms and Quasi-regular mappings

The theory of multiholomorphic maps subsumes the important and well-studied theory of quasi-regular mappings. Following the nomenclature of [IM], given Ω an open domain in $\mathbb{R}^n$, and $f : \Omega \to \mathbb{R}^n$ a map, $|J(f)|$ the Jacobian determinant,

Definition 2.1.14. *The Cauchy-Riemann system in* $\mathbb{R}^n$ *is the system of equations*

$$Df^t \circ Df = |J(f)|^{\frac{2}{n}} \cdot I \tag{2.1.39}$$

This system is a first-order, fully non-linear system, which has the obvious extension to the Riemannian setup,

Definition 2.1.15. *Let,* $(M, g_X, dVol_X, \star), (N, g, dVol_M, \star)$ *be* $(n+1)$*-dimensional, oriented, Riemannian manifolds with conformal triads, and u a map between them. The Cauchy-Riemann system is the equation*

$$u^*g = [(u^*dVol_M(dVol_X^*)]^{\frac{2}{n+1}} \cdot g_X \tag{2.1.40}$$

where $dVol_X^*$ *is the oriented, unit,* $n+1$*-plane field on* X *dual to* $dVol_X$*. Which in local coordinates is*

$$g_{ij}\frac{\partial u^i}{\partial x^k}\frac{\partial u^j}{\partial x^l} = [\sqrt{\frac{|detg|}{|detg_X|}}\epsilon_{i_0 \ldots i_n}\frac{\partial u^{i_0}}{\partial x^0}\cdot \ldots \cdot \frac{\partial u^{i_n}}{\partial x^n}]^{\frac{2}{n+1}} g_{Xkl} \tag{2.1.41}$$

Following [IM] we call $Dstr$ *the distortion tensor*

$$Dstr(x,u) = \{ \begin{array}{lll} [u^*dVol_M(dVol_X^*)^{\frac{2}{n+1}}g_X]^{-1} \circ u^*g & if & u^*dVol_M(dVol_X^*) \neq 0 \\ I & if & u^*dVol_M(dVol_X^*) = 0 \end{array} \tag{2.1.42}$$

where composition is understood as composition of $(1,1)$*-tensors on* TX*. We are suppressing the fact that these u's will ultimately be chosen to reside in* $W^{1,n+1}_{loc}$*, so that such a disjunctive definition makes sense.*

The motivating theorem in this field is the classical Liouville theorem for C^3 functions ([IM],2.3.1)

Theorem 2.1.16. *(Liouville) Every solution* $f \in C^3(\Omega, \mathbb{R}^n), n \geq 3$*, to the CR-system (2.1.39), where* $|J(f)|$ *does not change sign in* Ω *is of the form*

$$f(x) = b + \frac{\alpha A(x-a)}{|x-a|^{\epsilon}} \tag{2.1.43}$$

In this formula the arbitrary parameters are $a \in \mathbb{R}^n - \Omega, b \in \mathbb{R}^n, \alpha \in \mathbb{R}, A \in O(n)$, *and* $\epsilon = 0$ *or* 2.

This theorem is a strong rigidity result which shows that the $n = 2$ case (in which the space of solutions is infinite dimensional) is exceptional.

This Liouville Theorem was extended to the case of functions of Sobolev class $W^{1,n}_{loc}$ by the labors of Ghering, Reshetnyak, and by different methods, Bojarski-Iwaniec, and Iwaniec-Martin (see [IM] pg. 85 for these references).

From a geometric perspective, solutions of (2.1.39) are maps of directionally symmetric distortion in the sense that its directional derivatives satisfy

$$\max_\alpha |\partial_\alpha f(x)| = \min_\alpha |\partial_\alpha f(x)| \tag{2.1.44}$$

which implies that these solutions are special cases of maps with bounded distortion. In particular they are 1-quasi-regular mappings.

Definition 2.1.17. *A mapping* $f : \Omega \subset \mathbb{R}^n \to \Omega' \subset \mathbb{R}^n$ *is* K*-quasi-regular if it has bounded, finite distortion,*

$$max_\alpha |\partial_\alpha f(x)| \leq K min_\alpha |\partial_\alpha f(x)| \tag{2.1.45}$$

It is easy enough to see that a solution to the MCR equation is 1-quasi-regular. Consider the following pointwise calculations. Let ζ, η are unit tangent vectors at some point of X, and complete these to oriented orthonormal frames $\zeta_0 = \zeta, \ldots, \zeta_n$, $\eta_0 = \eta, \ldots, \eta_n$, so that $\star\zeta = \zeta_1 \wedge \ldots \wedge \zeta_n$. Then,

$$|du|^{n-1} g(du(\zeta), du(\zeta)) = \frac{1}{n} g(du(\zeta), \star\Lambda^n du(\star\zeta)) = u^* dVol_M(\zeta_0 \wedge \ldots \wedge \zeta_n) =$$
$$= u^* dVol_M(\eta_0 \wedge \ldots \wedge \eta_n) = |du|^{n-1} g(du(\eta), du(\eta)) \tag{2.1.46}$$

which says that the distortion is the same in arbitrary directions η, ζ.

Proposition 2.1.18. *Given $(M, h, dVol_M, \star)$, $(N, g, dVol_N)$, two $(n+1)$-dimensional Riemannian manifolds with conformal triads, a map u is a solution of the multi-Cauchy Riemann equations iff it is a solution to the Cauchy-Riemann system.*

Proof. First of all, recall the compatibility conditions on the conformal traids. In particular,

$$dVol_X(\zeta \wedge \star\eta) = g(\zeta, \eta) \tag{2.1.47}$$

Given u an MCR map,

$$\begin{aligned} g(du(\zeta), du(\eta)) = dVol_M(du(\zeta) \wedge \star du(\eta)) = \frac{1}{n|du|^{n-1}} dVol_M(du(\zeta) \wedge \Lambda^n du(\star\eta)) = \\ = \frac{1}{|du|^{n-1}} u^* dVol_M(\zeta \wedge \star\eta) = \frac{1}{|du|^{n-1}} u^* dVol_M(dVol_X^*) g_X(\zeta, \eta) \end{aligned} \tag{2.1.48}$$

Now, again by applying the norm to the MCR equation it is easy to show

$$|du|^{n+1} = u^* dVol_M(dVol_X^*) \tag{2.1.49}$$

so that we get the Cauchy-Riemann system

$$u^* g = [u^* dVol_M(dVol_X^*)]^{\frac{2}{n+1}} g_X \tag{2.1.50}$$

The other implication follows by tracing this argument backwards.

□

Example 2.1.19. *We consider the properties of multiholomorphic endomorphisms of the 3-sphere S^3 with its usual metric and conformal triad. Since this manifold is locally conformally flat, it can be covered by conformal coordinate patches. Therefore any point of S^3 is contained in a small neighborhood which lies inside a conformal patch, and is small enough that almost all of it lands inside a single conformal patch. Therefore, by the Liouville Theorem, this map restricted to this open set is in fact a Mobius transformation,* *and subsequently has no critical points. Then in particular, a smooth example is a covering $u : S^3 \to S^3$. We can easily demonstrate some of these endomorphisms using the identification $S^3 = Units(\mathbb{H})$. Namely, define,*

$$f_k : S^3 \to S^3,\ p \mapsto p^k \tag{2.1.51}$$

This f_k can relatively easily be seen to satisfy the MCR equations and is a $k-$fold cover. These will be in different connected components of the moduli space of such maps as the energy identity shows.

The above example applies to endomorphisms of any locally conformally flat manifold. We now mention a situation which will be instructive in the G_2-geometric case which is studied later.

Example 2.1.20. *Consider smooth $u : \mathbb{R}^3 \to \mathbb{R}^7$ satisfying the MCR equations in which $\mathbb{R}^7$ is equipped with the Euclidean G_2-triad $(g_0, (J, K), \phi_0)$ coming from $Im\mathbb{O}$*

$$J(X, Y) := Im(X \cdot Y), \quad \phi_0(X, Y, Z) := g_0(X, J(Y, Z)). \tag{2.1.52}$$

Then, the associated MCR-system is not linked to a Beltrami-type equation, but still we can mimic it's form by applying du^t to both sides. This produces

$$[\sum_q [\sum_p \frac{\partial u^p}{\partial x^q}]^2]^{\frac{1}{2}} \frac{\partial u^j}{\partial x^i} \frac{\partial u^j}{\partial u^l} = \phi_{jab} \frac{\partial u^j}{\partial x^i} \frac{\partial u^a}{\partial x^{l+1}} \frac{\partial u^b}{\partial x^{l+2}} \epsilon_{i,l+1,l+2} \tag{2.1.53}$$

The previous example applies to any situation in which the domain and target in the theory are locally conformally flat. This just so happens in the case $u : S^3 \to S^7$, which is of significant interest in the recent preprint of Lotay [Lo], which is entirely concerned with associative submanifolds of S^7 with its standard G_2-structure (coming from the spin structure on $\mathbb{R}^8$, and incorporating the round metric [Lo]). These kinds of examples are properly situated in the G_2-geometry sections.

The work of Reshetnyak [Re] established the following strong rigidity result for solutions of the Cauchy-Riemann system in $\mathbb{R}^n$,

Theorem 2.1.21. *(Reshetnyak) Given $u \in W^{1,n}(\Omega, \mathbb{R}^n)$ satisfying the Euclidean CR-system, then u is discrete and open. By this we mean that the preimage of a point in the target is a discrete set, and that u maps open sets to open sets.*

These kind of rigidity theorems are relevant to general multholomorphic mappings because the moduli of such maps $u : X \to M$ where X is equipped with the conformal n-triad and M with any n-triad will locally be foliated by endomorphisms of the domain X, that is, solutions of the afore-mentioned CR system on maps $\phi : X \to X$. If X is locally conformally flat then the results on 1-quasi-regular maps in $\mathbb{R}^{n+1}$ apply directly to this situation. We collect some of these results and then attempt to extend then to the case of non-vanishing Cotten/Weyl tensor on X.

When $n \geq 2$, there is strong rigidity in the presence of higher regularity.

Theorem 2.1.22. *([IM] 16.9.1) If $n \geq 2$, every nonconstant orientation-preserving mapping $f \in C^\infty(\Omega \subset \mathbb{R}^{n+1}, \mathbb{R}^{n+1})$ solution to the Eulidean CR-system (more generlly of bounded distortion) is a local homeomorphism.*

Hence it is clear that for a smooth mapping the preimage of a point is discrete. But we are interested also in the critical locus of such a map. The following proposition will be useful

Proposition 2.1.23. *If $n \geq 2$, and $u : \Omega \subset \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is a solution to the Euclidean CR-system, then there is a global bound on the order of zeros of u depending only on n.*

By the order of a zero of a smooth function u we simply mean the infimum of the degrees of derivatives of the function which are non-zero at the point in question. This notion extends to situations of lesser regularity (such as $L^1_{loc}(\Omega \subset \mathbb{R}^{n+1}, \mathbb{R}^{n+1})$) by defining a zero to be a point x_0 where the limit vanishes

$$\lim_{r \to 0} \frac{1}{r^{n+1}} \int_{B(x_0,r)} |f(x)| dx \tag{2.1.54}$$

and then the order of this zero is

$$sup\{\alpha : \lim_{r \to 0} \frac{1}{r^{n+1+\alpha}} \int_{B(x_0,r)} |f(x)| dx = 0\} \tag{2.1.55}$$

Proof. The method of proof is hinted in [IM]. By the previous theorem, u is a local homeomorphism, and hence locally admits an inverse which will also satisfy the CR-system (weakly). There are Holder estimates for such mappings, which immediately imply this result. □

Hence, given such a bound on the zeros of a smooth map,

Theorem 2.1.24. *A non-constant smooth multiholomorphic map $u : X \to X$ where X is LCF has finite critical locus.*

Proof. This is a local matter, hence we can appeal to the previous theorem after taking local isothermal coordinates. Given that the order of any zeros is bounded, it is a relatively simple matter to see that the critical locus cannot have accumulation points –because the higher derivatives must be continuous at the accumulation point. Then compactness implies finiteness of this discrete locus. □

This theorem specifies, for instance, that all smooth multiholomorphic endomorphisms of S^3 are covers, and ultimately, We would like to establish the following

Speculation 2.1.25. *A non-constant smooth multiholomorphic map $u : X \to M$ where X and M are LCF has finite critical locus.*

and eventually,

Speculation 2.1.26. *A non-constant smooth multiholomorphic map $u : X \to M$ has finite critical locus.*

2.1.4 The Energy Identity

The MCR equation is closely intertwined with a variational problem related to the $(n+1)$-energy:

Definition 2.1.27. *($(n+1)$-Energy) Given a compact smooth manifold X and any smooth manifold M both with a compatible triads $(X, g_X, dVol_X, (j,k))$, $(M, g_M, dM, (J,K))$, the $(n+1)$-energy of a map $u : X \to M$ is defined*

$$E_{n+1}(u) := \frac{1}{(n+1)^{\frac{n+1}{2}}} \int_X |du|^{n+1} dVol_X \tag{2.1.56}$$

In the case where X is not compact, we retain the same definition with the proviso that the energy could be infinite. We will refer to the integrand as the energy density.

Theorem 2.1.28. *(The $(n+1)$-Energy Identity) The energy of a multiholomorphic map whose domain is a compact, n-triadic, $(n+1)$-manifold is a topological invariant. Specifically, let u be a smooth map*

$$u : (X, g_X, dX, (j,k)) \to (M, g, \omega, (J,K)) \tag{2.1.57}$$

between a compact n-triadic $(n+1)$-manifold X and an n-triadic manifold M which satisfies $\eth u = 0$. Then

$$E_{n+1}(u) = \int_X u^* \omega \tag{2.1.58}$$

Proof. Consider

$$\frac{1}{(n+1)^{\frac{n+1}{2}}} \int_X |du|^{n+1} dVol_X = \frac{1}{(n+1)^{\frac{n+1}{2}}} \int_X |du|^{n-1} (g \otimes g_X^*)(du, du) dVol_X =$$
$$= \int_X \frac{1}{(n+1)} (g \otimes g_X^*)(du, \frac{1}{n} J \circ \Lambda^n du \circ k) dVol_X \tag{2.1.59}$$

Since $k = (-1)^n \star$, $g_X^*(\alpha, \beta \circ k) dVol_X = \alpha \wedge \beta$ for any α, β, 1-forms on X. So that

$$g \otimes g_X^*(du, \frac{1}{n} J \circ \Lambda^n du \circ k) dVol_X = g(du, \frac{1}{n} J \circ \Lambda^n du) = (n+1) \cdot u^* \omega \tag{2.1.60}$$

Hence,

$$\frac{1}{(n+1)^{\frac{n+1}{2}}} \int_X |du|^{n+1} dVol_X = \int_X u^* \omega \tag{2.1.61}$$

□

This estimate shows that if one fixes the topological class of a family of multiholomorphic maps then these have a fixed, finite energy. It also corresponds to a uniform $W^{1,n+1}$-bound on such a class. However, it is not immediately obvious that this is a variational identity. After all, in the theory of pseudoholomorphic curves we have the more robust equation for *any* smooth u

$$\frac{1}{2} \int_\Sigma |du|^2 dVol_\Sigma = \frac{1}{2} \int_X |\bar{\partial} u|^2 + \int_X u^* \omega \tag{2.1.62}$$

where both quantities on the right are positive. This means that a pseudoholomorphic curve *minimizes* energy. And hence can be realized as a critical point of the energy functional. This perspective yields, (by virtue of the Euler-Lagrange equations associated to this energy functional) the important fact that a pseudoholomorphic curves are harmonic. In the next section we investigate the corresponding issue with respect to MCR maps. In what follows in this section we consider an alternate notion of energy we call the *mixed energy* which leads to a generalization of the energy identity above and immediately expresses a minimizing property.

Consider the pointwise calculation of the norm of $\frac{\eth u}{|du|^{\frac{n-1}{2}}}$ (which follows from all of the same kind of calculations as above). Note that this term should not be problematic at critical points of u as the Hadamard inequality shows (see the section on variational issues).

$$(n+1)^{\frac{n-1}{2}}|\frac{\eth u}{|du|^{\frac{n-1}{2}}}|^2 = \frac{1}{(n+1)^{\frac{n-1}{2}}}|du|^{n+1} + \frac{(n+1)^{\frac{n-1}{2}}}{n^2|du|^{n-1}}|J\circ\Lambda^n du\circ k|^2 - 2(n+1)u^*\omega \tag{2.1.63}$$

Rewriting and integrating,

$$\int_X [\frac{1}{2(n+1)^{\frac{n+1}{2}}}|du|^{n+1} + \frac{(n+1)^{\frac{n-3}{2}}}{2n^2|du|^{n-1}}|J\circ\Lambda^n du\circ k|^2]dVol_X = \frac{(n+1)^{\frac{n-3}{2}}}{2}\int_X |\frac{\eth u}{|du|^{\frac{n-1}{2}}}|^2 dVol_X + \int_X u^*\omega \tag{2.1.64}$$

In the conformal setup in which J and k are both isometries,

$$\int_X [\frac{1}{2(n+1)^{\frac{n+1}{2}}}|du|^{n+1} + \frac{(n+1)^{\frac{n-3}{2}}}{2n^2|du|^{n-1}}|\Lambda^n du|^2]dVol_X = \frac{(n+1)^{\frac{n-3}{2}}}{2}\int_X |\frac{\eth u}{|du|^{\frac{n-1}{2}}}|^2 dVol_X + \int_X u^*\omega \tag{2.1.65}$$

So, in the conformal case we have on the LHS a kind of energy with mixed linear and multilinear capacity which is minimized by an MCR map (topologically). For the sake of clarity,

Definition 2.1.29. *(Mixed $(n{+}1)$-energy) We define the mixed $(n{+}1)$-energy of a smooth map*

$$E_{mix}(u) := \int_X [\frac{1}{2(n+1)^{\frac{n+1}{2}}}|du|^{n+1} + \frac{(n+1)^{\frac{n-3}{2}}}{2n^2|du|^{n-1}}|J\circ\Lambda^n du\circ k|^2]dVol_X \tag{2.1.66}$$

And the identity

$$\int_X [\frac{1}{2(n+1)^{\frac{n+1}{2}}}|du|^{n+1} + \frac{(n+1)^{\frac{n-3}{2}}}{2n^2|du|^{n-1}}|J\circ\Lambda^n du\circ k|^2]dVol_X = \frac{(n+1)^{\frac{n-3}{2}}}{2}\int_X \frac{|\eth u|^2}{|du|^{n-1}} dVol_X + \int_X u^*\omega \tag{2.1.67}$$

will be called the mixed energy identity*.*

It bears noting what the second term in the mixed energy denotes. It can easily be seen that

$$|J \circ \Lambda^n du \circ k|^2 = |proj_{imK}(\Lambda^n du)|^2 \tag{2.1.68}$$

But, per the definitions, restricted to $imK \subset \Lambda^n TM$, $J, \frac{(-1)^n}{\lambda} K$ are an isometric pair. Thus if $\Lambda^n du$ always lands in imK,

$$|J \circ \Lambda^n du \circ k|^2 = |\Lambda^n du|^2 \tag{2.1.69}$$

so this condition on u is enough to yield the simplified definitions that we would have were we in the doubly-conformal situation mentioned above.

2.1.5 Analytic Setup

In this section we develop the primary analytic framework in which we will be working. Let $\mathcal{B}$ be the space $\mathcal{C}^\infty(X, M)$ –which is a Frechét manifold. And given some $u \in \mathcal{B}, u^*TM$ is a metric bundle over X. Furthermore, we get an L^2-metric on $\Omega^\bullet(X, u^*TM)$ via

$$\phi, \eta \in \Omega^\bullet(X, u^*TM), \quad <\phi, \eta>_{L^2} := \int_X <\phi, \eta> = \int_X g(\phi \wedge \star(\eta)) = \int_X g_X^* \otimes g(\phi, \eta) dVol_X \tag{2.1.70}$$

or any L^p-norm

$$|\phi|^p = \int_X <\phi, \phi>^{\frac{p}{2}} dVol_X. \tag{2.1.71}$$

Define $\mathcal{E} \to \mathcal{B}$ to be the bundle over $\mathcal{B}$ such that

$$\mathcal{E}_u = \Omega^1_{L^{n+1}}(X, u^*TM) := L^{n+1}(X, T^*X \otimes u^*TM) \tag{2.1.72}$$

We can regard $\eth$ as a section of this bundle

$$u \mapsto (u, \eth u). \tag{2.1.73}$$

and consider the vertical differential $D_u\eth$ at some solution u. (over a zero no connection is required to define the vertical differential)

$$D_u\eth : T_u\mathcal{B} = \Omega^0(X, u^*TM) \to T_{(u,0)}\mathcal{E} = T_u\mathcal{B} \oplus \Omega^1(X, u^*TM) \to \Omega^1(X, u^*TM) \quad (2.1.74)$$

Defining a vertical differential in general requires a connection on the bundle $\mathcal{E}$. There is a natural way in which a connection on M induces a connection on $\mathcal{E}$. (Following [McDS] sec. 3.1,) Suppose ∇^M is the Levi-Civita connection on TM. A section α_λ of $\mathcal{E}$ along a curve $\gamma : \mathbb{R} \to \mathcal{B} : \lambda \mapsto u_\lambda$ is parallel with respect to the induced connection on $\mathcal{E}$ if the vector field $\lambda \mapsto \alpha_\lambda(z;\zeta) \in T_{u_\lambda(z)}M$ along the curve $\lambda \mapsto u_\lambda(z)$ is parallel for every $\zeta \in T_zX$.

For $\xi \in \Omega^0(X, u^*TM)$, we can define

$$\Phi_u(\xi) : \Omega^0(X, u^*TM) \to \Omega^0(X, exp_u(\xi)^*TM) \quad (2.1.75)$$

given by parallel transport along the ∇^M-geodesics $s \mapsto exp_{u(z)}(s\xi(z))$. Here $exp_u(\xi)$ is defined by the natural prescription:

$$exp_u(\xi)(z) = exp_{u(z)}(\xi(z)) \quad (2.1.76)$$

We then define

$$\mathcal{F}_u : \Omega^0(X, u^*TM) \to \Omega^1(X, u^*TM),\ \mathcal{F}_u(\xi) := \Phi_u^{-1}(\xi)\eth(exp_u(\xi)) \quad (2.1.77)$$

This is the vertical part of the section $u \mapsto (u, \eth u)$ in the trivialization given by ∇^M on TM. In this framework the vertical differential of this section is given by

$$D_u\eth\xi := d\mathcal{F}_u(0)\xi \quad (2.1.78)$$

Proposition 2.1.30. *Suppose $u : X \to M$, where X is a compact, $(n+1)$-manifold with the conformal triad, and M any compact n-triadic manifold. Then we can describe $(D_u\eth)$*

as follows,

$$
\begin{aligned}
&(D_u\eth)\xi = \eta \mapsto \\
&\mapsto \frac{n-1}{(n+1)^{(n-1)/2}}|du|^{(n-3)} < u^*\nabla^M\xi, du > du(\eta) + \frac{1}{(n+1)^{(n-1)/2}}|du|^{n-1}u^*\nabla^M_\eta \xi - \\
&\qquad - \frac{(-1)^n}{n-1}J_u(\Lambda^{n-1}du \wedge u^*\nabla^M\xi \circ k)(\eta) + u^*\nabla^M_\xi J(\Lambda^n du \circ k)(\eta) \qquad (2.1.79)
\end{aligned}
$$

Proof. Consider the path

$$\mathbb{R} \to C^\infty(X, M) : \lambda \mapsto u_\lambda := exp_u(\lambda\xi). \qquad (2.1.80)$$

Then by the definition of $\mathcal{F}_u$, (using ∇ to denote $u^*\nabla^M$)

$$
\begin{aligned}
D_u\eth\xi =& \frac{d}{d\lambda}\mathcal{F}_u(\lambda\xi)|_{\lambda=0} && (2.1.81)\\
=& \nabla_\lambda\eth(u_\lambda)|_{\lambda=0} && (2.1.82)\\
=& [\frac{1}{(n+1)^{(n-1)/2}}\nabla_\lambda(|du_\lambda|^{n-1}du_\lambda) - \frac{(-1)^n}{n}\nabla_\lambda(J\Lambda^n du_\lambda)\circ k]|_{\lambda=0} && (2.1.83)\\
=& [\frac{n-1}{(n+1)^{(n-1)/2}}|du_\lambda|^{(n-3)}(< u^*\nabla_\lambda du_\lambda, du_\lambda > + && (2.1.84)\\
+& \frac{1}{(n+1)^{(n-1)/2}}|du_\lambda|^{n-1}\nabla_\lambda du_\lambda - (-1)^n\nabla_\lambda J_u(\Lambda^n du_\lambda)\circ k]|_{\lambda=0} && (2.1.85)\\
=& \frac{1}{(n+1)^{(n-1)/2}}|du|^{n-1}u^*\nabla\xi + \frac{n-1}{(n+1)^{(n-1)/2}}|du_\lambda|^{(n-3)} < u^*\nabla\xi, du > du- \\
& && (2.1.86)\\
-& \frac{(-1)^n}{n-1}J_u(\Lambda^{n-1}du \wedge u^*\nabla\xi\circ k) - u^*\nabla_\xi J(\Lambda^n du\circ k) && (2.1.87)
\end{aligned}
$$

The fact that $\nabla_\lambda du_\lambda|_{\lambda=0} = \nabla\xi$ follows because ∇ is torsion-free. We have also made use of the fact that ∇ is compatible with the metric on M: $\nabla g(A,B) = g(\nabla A, B) + g(A, \nabla B)$. And that the Hilbert-Schmidt metric $g_X^* \otimes g$ along this path will only depend on λ in the second factor. Also the following observation was put to use:

$$\nabla_\lambda(\frac{1}{n}\Lambda^n du_\lambda \circ k) = \frac{1}{n-1}(\Lambda^{n-1}du_\lambda \wedge \nabla_\lambda du_\lambda)\circ k \qquad (2.1.88)$$

□

2.1.6 Variational Aspects

We saw that the MCR maps satisfy the energy identity

$$\frac{1}{(n+1)^{\frac{n+1}{2}}}\int_X |du|^{n+1} dVol_X = \int_X u^*\omega \tag{2.1.89}$$

Now, we consider mappings $u : X \to M$ as usual which are in the Sobolev space $W^{1,n+1}_{loc}$, and investigate the Euler-Lagrange equations for the minima of the $(n+1)$-energy functional

$$E(u) = \frac{1}{(n+1)^{\frac{n+1}{2}}}\int_X |du|^{n+1} dVol_X \tag{2.1.90}$$

We note that strictly-speaking there is no coordinate-independent definition of $W^{1,n+1}(X, M)$ in general. This changes when the regularity is strictly greater than the dimension of the domain, or if M is affine.

Proposition 2.1.31. *E is lower semicontinuous, i.e.,*

$$E(u) \leq \liminf_{k\to\infty} E(u_k) \text{ whenever } u_k \to u \text{ weakly in } W^{1,n+1}_{loc}(X, M). \tag{2.1.91}$$

and coercive in the sense that for each u, and any $A \in L^{n+1}(X, Hom(TX, u^*TM))$, the functional

$$E(A) := \int_X g \otimes g_X^*(A, A)^{\frac{n+1}{2}} dVol_X \tag{2.1.92}$$

has

$$E(A) \to \infty, \text{ as} |A|_{n+1} \to \infty. \tag{2.1.93}$$

(In this case this is trivial since $E(A)$ is just a power of the L^{n+1}-norm).

The proof is similar to ([IM] 15.32)

Proof. Since $du_k \to du$ in the L^{n+1} norm,

$$\begin{aligned} E[u] \quad &= \lim_{k\to\infty} \int_X g \otimes g_X^*(du,du)^{\frac{n-1}{2}} g \otimes g_X^*(du,du_k) dVol_X \\ &\leq \liminf_{k\to\infty} \int_X [g\otimes g_X^*(du,du)^{\frac{n-1}{2}}]^{\frac{n-1}{n+1}} [g\otimes g_X^*(du_k,du_k)^{\frac{n+1}{2}}]^{\frac{1}{n+1}} dVol_X \\ &\leq \Big[\int_X |du|^{n+1} dVol_X \Big]^{\frac{n-1}{n+1}} \cdot \liminf_{k\to\infty} \Big[g\otimes g_X^*(du_k,du_k)^{\frac{n+1}{2}} \Big]^{\frac{1}{n+1}} \end{aligned} \tag{2.1.94}$$

where we have used first the Cauchy-Schwarz inequality and then the Holder inequality. The result follows by dividing both sides by the leading term of the product on the RHS and taking the nth power. $\square$

As a first step we can consider the Euler-Lagrange equations in the simplest scenario of interest, that of the Euclidean, conformal case $u : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$. In this case it is easy to calculate that a solution of class $W^{1,n+1}_{loc}$ to the MCRE is in fact a weak solution to the euclidean $n+1$-Laplace equation,

$$\Delta_{n+1} u := d^*(|du|^{n-1} du) = 0. \tag{2.1.95}$$

Going further we consider the Euclidean case of interest in the G_2-situation, that of $u : \mathbb{R}^3 \to \mathbb{R}^7$. Let Ω be an open domain in $\mathbb{R}^3$,

Proposition 2.1.32. *If $u \in C^\infty(\Omega, \mathbb{R}^7)(u \in W^{1,n+1}_{loc}(\Omega,\mathbb{R}^7))$ satisfies $\eth u = 0$ (for the flat conformal structure on $\mathbb{R}^3$ and the flat G_2structure on $\mathbb{R}^7$), then u is a (weak) solution to the $n+1$-Laplace equation $\Delta_{n+1} u = 0$.*

Proof. We will denote the Hilbert-Schmidt norm, $(g_0 \otimes g_0^*)$, by $<,>$ for simplicity. Also even though we can write explicitly

$$J(\Lambda^2 du) = J^p_{ab} \partial_i u^a \partial_j u^b dx^i \wedge dx^j \tag{2.1.96}$$

it will be simpler just to use the abstract notation of J and to note that J is parallel. For any smooth 1-form field $\phi \in \Omega^1(\Omega, u^*T\mathbb{R}^7)$

$$0 = \int_\Omega < \nabla\phi, \eth u > dx = \int_X < \phi, \nabla^*\eth u > dx \tag{2.1.97}$$

That is to say that u weakly satisfies the second-order equation

$$(\nabla^* \circ \eth u) = 0. \tag{2.1.98}$$

In the local Euclidean coordinates, $\nabla_k = \partial_k \cdot dx^k$ and $\nabla^* = \star \circ \nabla \circ \star$. Then,

$$\star\nabla_k \star ([\sum_j |\partial_j u|^2]^{\frac{1}{2}} \partial_i u dx^i - J(\partial_{i+1}u \ldots \partial_{i-1}u)dx^i) = 0 \tag{2.1.99}$$

the first part is the 3-Laplacian of u:

$$\Delta_3 u := \nabla^*(|du|du) \tag{2.1.100}$$

And since $\nabla J = 0$ and ∇ is torsion free in $\mathbb{R}^7$, we have

$$\nabla^*(J\Lambda^2 du \circ k) = \star J\nabla\Lambda^2 du = 0 \tag{2.1.101}$$

This follows from, $\nabla_k\partial_i u = \nabla_i\partial_k u$ combined with the calculation,

$$\sum_i \nabla_i J(\partial_{i+1}u \wedge \ldots \wedge \partial_{i-1}u) \cdot dx^i \wedge dx^{i+1} \wedge \ldots \wedge dx^{i-1} =$$

$$= \sum_i \sum_{j \neq i} J(\partial_{i+1}u \wedge \ldots \wedge \nabla_i\partial_j u \wedge \ldots \wedge \partial_{i-1}u)dx^i \wedge \ldots \wedge dx^{i-1} = 0. \tag{2.1.102}$$

These are distributional equations if u is properly in $W^{1,3}_{loc}$. □

In the most general situation we can do similarly after giving some sense to what the p-Laplacian should be globally. Let u denote a smooth map $u : (X, g_X, (j,k), dVol_X) \to (M, g, (J,K), \omega)$. Given a connection ∇^M on TM, we get $u^*\nabla^M$ on u^*TM, which extends uniquely to an exterior covariant derivative on forms in u^*TM:

$$u^*\nabla^M : \Omega^k(X, u^*TM) \to \Omega^{k+1}(X, u^*TM), \quad (u^*\nabla^M)^2 = F_{u^*\nabla^M} \wedge \bullet \tag{2.1.103}$$

satisfying the Leibnitz rule:

$$u^*\nabla^M(\alpha \otimes v) = d\alpha \otimes v + \alpha \cdot u^*\nabla^M v \tag{2.1.104}$$

But this is not what we need to consider in this situation. Because there are metrics on $\Lambda^k T^*X$ and u^*TM, there is a metric on the tensor product, $\Lambda^k g_X^* \otimes g$, which we will denote by $<,>$. Let ∇^X be the Levi-Civita connection on X and ∇^M the Levi-Civita connection on M. The tensor product connection given by $\nabla := (\nabla^{X*} := g_X \nabla^X g_X^{-1}) \otimes I + I \otimes u^*\nabla^M$ is compatible with $<,>$ in the sense,

$$d < A, B >=< (\nabla^{X*} \otimes I + I \otimes u^*\nabla^M)A, B > + < A, (\nabla^{X*} \otimes I + I \otimes u^*\nabla^M)B > \tag{2.1.105}$$

and so that

$$d(< du, du >)^{\frac{p}{2}} = p < du, du >^{\frac{p-2}{2}} < \nabla du, du >= p|du|^{p-2} < \nabla du, du > \tag{2.1.106}$$

Definition 2.1.33. *Given the setup in the previous calculations, we can define the p-Laplacian,*

$$\Delta_p u := \nabla^*(|du|^{p-2} du) \tag{2.1.107}$$

where ∇^* is the formal adjoint to the connection $\nabla : \Omega^0(u^*TM) \to \Omega^1(u^*TM)$ with respect to the Hilbert-Schmidt metric $<,>:= g_X^* \otimes g$.

Proposition 2.1.34. *On $\mathcal{B} := C^\infty(X, M)$ the $(n+1)$-harmonic maps extremize the functional $E : \mathcal{B} \to \mathbb{R}$.*

Proof. Letting $\gamma : \mathbb{R} \to \mathcal{B}$ be a path $\lambda \mapsto u_\lambda$ such that $\gamma(0) = u$, and $\dot{\gamma}(0) = \zeta$, we consider $dE(0)\zeta$:

$$dE(0)\zeta = \frac{d}{d\lambda}[E(u_\lambda)]|_0 = \int_X \frac{d}{d\lambda}[< du_\lambda, du_\lambda >^{\frac{n+1}{2}} dVol_X]|_0 dVol_X = \tag{2.1.108}$$

$$=(n+1) \int_X |du|^{n-1} < \nabla_\lambda du_\lambda, du_\lambda >|_0 dVol_X \tag{2.1.109}$$

$$=(n+1) \int_X < \nabla\zeta, |du|^{n-1} du > dVol_X \tag{2.1.110}$$

This will be zero for all ζ iff $\Delta_{n+1}u = 0$. $\square$

Theorem 2.1.35. *Suppose as usual $u : X \to M$ is a smooth MCR-map with all the usual notation. Then, u is a solution to an $(n+1)$-Poisson equation, $\Delta_{n+1}u = \eta$. Where*

$$\eta := \frac{1}{n}\nabla(J) \circ \Lambda^n du \circ k \tag{2.1.111}$$

Proof. We use the same notation as the previous calculation. For any smooth 0-form valued in u^*TM we have

$$0 = \int_\Omega < \nabla\phi, \eth u > dVol_X = \int_X < \phi, \nabla^*\eth u > dVol_X \tag{2.1.112}$$

So u weakly satisfies the second-order equation

$$(\nabla^* \circ \eth u) = \frac{1}{(n+1)^{\frac{n-1}{2}}}\nabla^*(|du|^{n-1}du) - \frac{1}{n}\nabla^*(J \circ \Lambda^n du \circ k) = 0. \tag{2.1.113}$$

The first part is proportional to the $(n+1)$-Laplacian of u

$$\Delta_{n+1}u := \nabla^*(|du|^{n-1}du) \tag{2.1.114}$$

Since $\nabla^* = (-1)^n \star \nabla \star = (-1)^n(\circ k)\nabla(\circ j)$,

$$\nabla^*(J \circ \Lambda^n du \circ k) = \nabla(J \circ \Lambda^n du) \circ k \tag{2.1.115}$$

Then,

$$\nabla(J) \circ \Lambda^n du \circ k + J \circ \nabla(\Lambda^n du) \circ k = \nabla(J) \circ \Lambda^n du \circ k \tag{2.1.116}$$

The second term amounts to 0. In local normal coordinates on X centered at some point z, we have because ∇^M is torsion-free, and the metric is diagonal at z, and the Christoffel symbols of g_X vanish there,

$$\nabla_i^M \partial_j u = \nabla_j^M \partial_i u, \text{ and } \nabla du = (I \otimes \nabla^M)(du) \tag{2.1.117}$$

and so,

$$\sum_i \nabla_i^M J(\partial_{i+1}u \wedge \ldots \wedge \partial_{i-1}u) \cdot dx^i \wedge dx^{i+1} \wedge \ldots \wedge dx^{i-1} =$$

$$= \sum_i \sum_{j \neq i} J(\partial_{i+1}u \wedge \ldots \wedge \nabla_i^M \partial_j u \wedge \ldots \wedge \partial_{i-1}u) dx^i \wedge \ldots \wedge dx^{i-1} = 0. \quad (2.1.118)$$

This is true at the origin point for the normal coordinates on X. However, since this is a tensorial equation, the identity is true globally. □

We see the expected fact:

Proposition 2.1.36. *On $\mathcal{B} := C^\infty(X, M)$ the $(n+1)$-harmonic maps extremize the functional $E : \mathcal{B} \to \mathbb{R}$.*

Proof. Letting $\gamma : \mathbb{R} \to \mathcal{B}$ be a path $\lambda \mapsto u_\lambda$ such that $\gamma(0) = u$, and $\dot{\gamma}(0) = \zeta$, we consider $dE(0)\zeta$:

$$dE(0)\zeta = \frac{d}{d\lambda}[E(u_\lambda)]|_0 = \int_X \frac{d}{d\lambda}[< du_\lambda, du_\lambda >^{\frac{n+1}{2}} dVol_X]|_0 dVol_X = \quad (2.1.119)$$

$$= (n+1) \int_X |du|^{n-1} < \nabla_\lambda du_\lambda, du_\lambda >|_0 dVol_X \quad (2.1.120)$$

$$= (n+1) \int_X < \nabla\zeta, |du|^{n-1} du > dVol_X \quad (2.1.121)$$

This will be zero for all ζ iff $\Delta_{n+1} u = 0$. □

But we can also consider the mixed energy:

$$E_{mixed}(u) = \frac{1}{2(n+1)^{\frac{n-1}{2}}} \int_X |du|^{n+1} dVol_X + \frac{(n+1)^{\frac{n-2}{2}}}{2n^2} \int_X \frac{|J \circ \Lambda^n du \circ k|^2}{|du|^{n-1}} dVol_X \quad (2.1.122)$$

Letting $\gamma : \mathbb{R} \to \mathcal{B}$ be a path $\lambda \mapsto u_\lambda$ such that $\gamma(0) = u$, and $\dot{\gamma}(0) = \zeta$, we consider

$dE_{mixes}(0)\zeta$:

$$dE_{mixed}(0)\zeta = \frac{(n+1)}{2}\int_X < \nabla\zeta, |du|^{n-1}du > dVol_X \tag{2.1.123}$$

$$+\frac{(n+1)^{\frac{n-2}{2}}}{n}\int_X \frac{< J\circ\nabla\zeta\wedge\Lambda^{n-1}du\circ k, J\circ\Lambda^n du\circ k >}{|du|^{n-1}} dVol_X \tag{2.1.124}$$

$$-\frac{(n-1)(n+1)^{\frac{n-2}{2}}}{2n^2}\int_X \frac{|J\circ\Lambda^n du\circ k|^2}{|du|^{n+1}} < \nabla\zeta, du > dVol_X \tag{2.1.125}$$

This condition which arises on u by setting this this equal to zero for all ζ is opaque to the author. It is clear that the added terms can not be folded into the first on account of any general identities. In fact the MCR-equation is necessary to make such a move. We suspect that

Speculation 2.1.37. *In some cases there exist $(n+1)$-harmonic maps which form continuous moduli, one of whose points is a MCR-map, but which are not minimal for the mixed Energy functional.*

In any case, if $\eth u = 0$ we have in hand a $n+1$harmonic map and it becomes an interesting question of what regularity we can cull from this fact.

Regularity for the Homogeneous p-Laplace equation

In $\mathbb{R}^{n+1}$ it has been shown in the work of Uhlenbeck [U] and Evans [E], and related work in [T], that *a priori* a solution to the $(n+1)$-Laplace equation on real functions (with constant coefficients) on $\mathbb{R}^{n+1}$ must have $C^{1,\alpha}$-regularity. In principle this is the best one can do as a result of the degeneracy of Δ_p. However it is the hope that with *a priori* control on the critical locus of du this situation improves significantly. Furthermore when the target manifold M is not conformally flat, the variational equations become more general, having variable metric coefficients and inhomogeneous parts.

Speculation 2.1.38. *We can prove smooth interior regularity, and smooth regularity up to the boundary for the inhomogeneous p-Laplace equation with non-constant metric* *coefficients for maps* $u \in W^{1,p>n+1}(X, M)$ *satisfying* $\eth u = 0$. *This generalizes the work of Uhlenbeck and Evans in so far as we treat non-constant coefficients. The fact that their work could be extended to the non-constant case was proposed by Evans and Uhlenbeck in their respective papers, but not carried out. In our case we have the further datum that* $\eth u = 0$. *This condition will be especially useful because it might give a priori topological information about the "dimension" of the critical locus of* u. *(For instance the propositions in this section) Since the critical locus of* u *is the degenerate locus of the* $n+1$*-Laplacian, this data could be leveraged into stronger estimates than those applying in the general case.*

2.2 A Mapping Theory

2.2.1 Main Goal

In this section we introduce the main idea of this theory. The goal is to develop a mapping theory, or field theory (gratuitously) associated to the framework of triadic structures and multiholomorphic maps. The starting data includes a compact, n-triadic manifold $(M, \omega, g, (J, K))$, as well as any compact, $(n+1)$-dimensional Riemannian manifold X equipped with the associated conformal triad. X is regarded as the domain and M the target, and the main object of study will be configurations of multiholomorphic maps from X to M.

$$\{\text{(n+1)-manifolds with conformal triad}\} \xrightarrow{\eth u=0} \{\text{ manifolds with n-triad}\} \tag{2.2.1}$$

It should not come as a surprise at this point that we will really only need the data of conformal class of X, so we expect that any relevant invariants will be conformal invariants. One then considers the moduli spaces of multiholomorphic maps from X

into M, a space which is split into components (by virtue of the energy identity) by the possible topological classes of such maps. We might also consider boundary conditions for non-closed X which lie on "branes," that is, maximal isotropic submanifolds for the multiholomorphic form ω. For example, in the G_2-case the branes are the coassociative submanifolds. There is an obvious choice for the branes for each of the distinct families of n-triad structures. There is preexisting work along these lines in [LL],[L1],[L2], (and other work of Leung), which in turn rests on the work of McLean [McL] with regard to the deformation theory of calibrated submanifolds and the associated branes. The main novelty of this paper's framework is a PDE intertwining compatible structures on domain and target. Prior work seems to be concerned mainly with arbitrary parametrizations of associative submanifolds, a class of maps which is much larger and does not benefit from any regularity results enjoyed by solutions of the MCR equation.

The paradigmatic examples for the kinds of theories we have in mind here are Hamiltonian/Lagrangian Floer Theory and its cousin Quantum cohomology.

2.2.2 Foundational Matters

Much needs to be established in order to get this theory off the ground.

Integrability and Existence

In general, the MCR equations are overdetermined as can be seen from the calculations of the previous section. In general this means that some integrability conditions will need to obtain in order to guarantee that a non-empty moduli space exists. These conditions will be investigated in some particular examples rather than trying to give an over-all theory. After all, we know that in the Kähler case the equations are underdetermined, and in some others obviously overdetermined.

Bubbling

Conformal rescaling a la Gromov results in the same bubbling effects as in symplectic Floer theories. A particularly impressive theorem would be a version of Gromov Compactness

Speculation 2.2.1. *Given a sequence of multiholomorphic maps u_ν such that $sup_\nu E(u_\nu)$ is finite then there exists a subsequence which converges moduli bubbling from a finite number of isolated points.*

This theorem has not yet been pursued because some more basic considerations need to be considered first: namely issues of existence and integrability.

Compactness

Eventually the goal here would be to levereage the Gromov-Compactness speculations above in order to give a complete characterization of the boundary of the moduli space of multiholomorphic maps. With this description in hand one can specify its compactification. Alternately one could identify criteria that imply the moduli space is already compact, not requireing a compactification in the first place. Both of these are methodologies which come into play in the pseudoholomorphic region.

2.2.3 Case: G_2-manifolds with the associative triad

As was pointed out in earlier sections, a G_2-manifold possesses the parallel associative triad. This integrability condition simplifies the analytic difficulties of the theory analogously to the simplification in the theory of pseudoholomorphic curves between Kähler and almost-Kähler manifolds. First we describe more fully the subject of G_2-manifolds and collect some relevant facts. An excellent source is [J1].

G_2-manifolds

Definition 2.2.2. *A G_2-structure on an oriented 7-manifold M is a principal, G_2-subbundle of the oriented frame bundle of M.*

One can regard the frame bundle over some point $x \in M$ as consisting of isomorphisms $\phi : T_xM \to \mathbb{R}^7$. Following [DS] and [J1], we describe an equivalent notion of a G_2-structure more explicitly. Consider V a 7-dimensional real vectorspace with orientation O. There is an open $GL_+(V)$-orbit $P_3 \subset \Lambda^3 V^*$ each element of which has stabilizer $G_2 \subset SL(V)$ and likewise for $P_4 \subset \Lambda^4 V^*$. Now consider an oriented 7-manifold M. At each point $p \in M$ we have open subsets $P_{3,p} \subset \Lambda^3 T_p^* M, P_{4,p} \subset \Lambda^4 T_p^* M$ as before. Then,

Definition 2.2.3. *A G_2*-structure *on an oriented 7-manifold M is a choice of a 3-form ω which lies in $P_{3,p}$ for each p.*

Note that such a structure defines a G_2-structure in the first sense by considering the subbundle of the positive frame bundle consisting of isomorphisms $\alpha : T_xM \to \mathbb{R}^7$ for which $\alpha^*\phi_0 = \phi$. Conversely, one uniquely defines a 3-form, Riemannian structure, and Hodge stars since one can define a metric on a G_2-manifold by pulling back the Euclidean metric, ϕ_0 and the Hodge stars, and noting that these are G_2-invariant/equivariant.

If ω is a 3-form which yields a G_2-structure, then one can consider the Riemannian connection ∇ associated to the metric g determined by ω. We define,

Definition 2.2.4. *If ω is a G_2-structure on M, then the* torsion *of this G_2-structure is defined to be $\nabla\omega$.*

Proposition 2.2.5. *([J1]11.1.3)*

Let ω determine a G_2-structure on an oriented 7-manifold M, and $g$be the associated *metric. Then the following are equivalent:*

$$i)\ \nabla\omega = 0 \tag{2.2.2}$$

$$ii)\ Hol(g) \subset G_2,\ \textit{and } \omega \textit{ is the induced } 3\textit{-form} \tag{2.2.3}$$

$$iii)\ d\omega = d^*\omega = 0 \tag{2.2.4}$$

$$iv)\ \textit{There exists a parallel spinor field on } M. \tag{2.2.5}$$

The last statement is ambiguous unless$M$inherits a particular spin structure from the G_2-structure. This is in fact the case: G_2 embeds in SO_7, which induces an embedding of their universal covers. Because G_2 is a simply-connected, this is a map $\tilde{\iota} : G_2 \to Spin_7$, which is an injective Lie group homomorphism lifting the covering of SO_7. The G_2-structure Q (understood as a principal subbundle of the positive frame bundle) induces an SO_7 structure on M via $P = SO_7 \cdot Q$. The spin structure on M is then given by $\tilde{P} = Q \times_{G_2} Spin_7$ making use of $\tilde{\iota}$ for the G_2-action on $Spin_7$.

There are also strong constraints on the topology of closed G_2-manifolds. As modules over G_2, we have splittings

$$i)\qquad \Lambda^1 T^*M = \Lambda^1_7 \qquad\qquad ii)\qquad \Lambda^2 T^*M = \Lambda^2_7 \oplus \Lambda^2_{14} \tag{2.2.6}$$

$$iii)\qquad \Lambda^3 T^*M = \Lambda^3_1 \oplus \Lambda^3_7 \oplus \Lambda^3_{27} \qquad\qquad iv)\qquad \Lambda^4 T^*M = \Lambda^4_1 \oplus \Lambda^4_7 \oplus \Lambda^4_{27} \tag{2.2.7}$$

$$v)\qquad \Lambda^5 T^*M = \Lambda^5_7 \oplus \Lambda^5_{14} \qquad\qquad vi)\qquad \Lambda^6 T^*M = \Lambda^6_7 \tag{2.2.8}$$

where the lower indices mark off the dimensions of these submodules. The Hodge star gives an isometry between Λ^k_l and Λ^{7-k}_l. $\Lambda^3_1 =< \omega >$, $\Lambda^4_1 =< \star\omega >$, and Λ^k_7 for all k are canonically isomorphic. It is a well-known fact that these decompositions are respected by the Hodge Laplacian, yielding a refinement of the de Rham cohomology, $\oplus_l H^k_l(M, \mathbb{R}) = H^k(M, \mathbb{R})$. We have the theorem

Theorem 2.2.6. *Let (M, ω, g) be a compact G_2-manifold. Then,*

$$i) \qquad H^1(M, \mathbb{R}) = H^6(M, \mathbb{R}) = 0 \tag{2.2.9}$$

$$ii) \qquad H^3_1(M, \mathbb{R}) =< [\omega] >, \quad H^4_1(M, \mathbb{R}) =< [\star\omega] > \tag{2.2.10}$$

$$iii) \qquad H^k_l(M, \mathbb{R}) \cong H^{n-k}_l(M, \mathbb{R}), \tag{2.2.11}$$

$$iv) \qquad \textit{and, if } Hol(g) = G_2, \textit{ then } H^k_7(M, \mathbb{R}) = 0 \textit{ for all } k \tag{2.2.12}$$

Ultimately this means that for a full-holonomy G_2-manifold there are only two Betti numbers to be determined (b^2,b^3). One can find tables of possible Betti numbers in [J1].

And finally one might consider the characteristic classes relevant to a G_2-manifold. The next theorem is largely an application of Chern-Weil theory combined with the previous considerations.

Theorem 2.2.7. *([J1] 11.2.7) Suppose M is a compact 7-manifold admitting metrics with full holonomy G_2. Then M is orientable, spin, has finite fundamental group, and has a non-trivial first Pontrjagin class.*

Calibrated geometry and G_2-manifolds

G_2-geometry is closely related to the topic of calibrated geometry by virtue of the fact that if a manifold M has a closed G_2-structure, i.e. $d\omega = 0$, then ω is a calibration on M. If M is a G_2-manifold (has holonomy inside G_2) then the theorem above implies not only that ω is closed, but that $\star\omega$ is as well. It follows then that $\star\omega$ is a calibration. The relevant *calibrated submanifolds* are the integral submanifolds for the distributions consisting of the unit-multivectors dual to these forms. In what follows we describe this explicitly.

We have already noted that a G_2-manifold is one whose tangent spaces are identified continuously, and orientedly, and generically with $Im\mathbb{O}$. Given the complicated subgroup/subalgebra structure of $\mathbb{O}$, it is not surprising that there will be interesting sub-geometries. In particular, we note that the unusual presentation of $\mathbb{O}$ is by virtue of the Cayley-Dickson process:

$$\mathbb{O} = \mathbb{H} \oplus \mathbb{H} =< 1, i, j, k > \oplus < l, il = I, jl = J, kl = K >, \quad (a, b)(c, d) = (ac - \bar{d}b, da + b\bar{c}) \tag{2.2.13}$$

This product restricts nicely to $Im\mathbb{O}$ where we have defined in the introduction, $J(x, y) := Im(x \cdot y)$. But it is not associative, hence we have the associator:

$$[x, y, z] := (x \cdot y) \cdot z - x \cdot (y \cdot z) \tag{2.2.14}$$

Which is defined on Im$\mathbb{O}$ and takes values in Im$\mathbb{O}$. On Im$\mathbb{O}$ it can be observed that

$$J(x \cdot y) = \frac{1}{2}[x, y] \tag{2.2.15}$$

Where the right-hand side is the usual commutator. However, the lack of associativity of $\mathbb{O}$ means that this commutator doesn't satisfy the Jacobi identity. In fact,

$$J(x, J(yz)) + J(z, J(xy)) + J(y, J(zx)) = -\frac{3}{2}[x, y, z] \tag{2.2.16}$$

There are an important class of 3-planes in $Im\mathbb{O}$ which are the 3-planes isomorphic to $Im\mathbb{H}$ inside $\mathbb{O}$. These are the *associative* 3-planes, the paradigm case being $i \wedge j \wedge k$. Likewise, a 4-plane which is complementary to an associative 3-plane is called *coassociative*, with $l \wedge I \wedge J \wedge K$ an exemplar. Hence according to the above formulas, the associator vanishes on associative 3-planes, and on a G_2-manifold the associator form is proportional to the *Jacobiator* of J (so that J becomes a Lie bracket on associative 3-planes). Let

$$G := \{\zeta = x \wedge y \wedge z \text{ an oriented 3-plane in Im}\mathbb{O} \;|[x, y, z] = 0\} \tag{2.2.17}$$

Likewise, over a G_2-manifold M, we could define the *associative Grassmannian* $G(M)$ which is a fiber bundle over M with fiber over $m \in M$:

$$G_x(M) = \{\zeta = x \wedge y \wedge z \text{ an oriented 3-plane in } T_mM \;|[x, y, z]_m = 0\} \tag{2.2.18}$$

We could do similarly and construct the *coassociative Grassmanian.* Hence we define an *(co)associative submanifold* to be a 3(4)-dimensional submanifold X of M, such that its tangent 3(4)-planes are in the (co)associative Grassmannian. These calibrated submanifolds are highly constrained (for instance they are volume-minimizing), and yet are relatively abundant (for example any analytic surface in a G_2-manifold locally extends uniquely to an associative submanifold) [HL]. For the sake of later usage, we now describe the topology of the associative Grassmanian and the corresponding associative Stiefel manifold. It is proven in [HL] that

$$G \equiv G_2/((Sp_1 \times Sp_1)/\mathbb{Z}_2) \tag{2.2.19}$$

Where $Sp_1 \times Sp_1/\mathbb{Z}_2$ acts on Im$\mathbb{O}$ =Im$\mathbb{H} \oplus \mathbb{H}$ via

$$(a, b) \mapsto (q_1 a\bar{q_1}, q_2 b\bar{q_1}) \tag{2.2.20}$$

where q_1, q_2 are unit quaternions. Define the *associative Stiefel manifold*

$$V := \{(e_1, e_2, e_3) \text{ an orthonormal 3-frame in Im}\mathbb{O} \mid [x, y, z] = 0\} \tag{2.2.21}$$

And likewise the *associative frame bundle*(or *associative Stiefel bundle*) $V(M)$ over M with fiber:

$$V_m(M) := \{(e_1, e_2, e_3) \text{ an orthonormal 3-frame in } T_mM \mid [e_1, e_2, e_3]_m = 0\} \tag{2.2.22}$$

Obviously V is a principal SO_3-bundle over G, and $G(M)$ is a quotient of $V(M)$ by the pointwise SO_3-action.

Proposition 2.2.8.

$$V \cong V_{7,2} \cong G_2/SU_2 \tag{2.2.23}$$

Proof. We follow the identification outlined in [HL]. Namely, note that $V_{7,2}$, the Stiefel manifold of oriented, orthonormal 2-planes in $\mathbb{R}^7$ is mapped onto V by

$$(e_1, e_2) \mapsto (e_1, e_2, e_1 \cdot e_2) = (e_1, e_2, J(e_1, e_2)) \tag{2.2.24}$$

And it is a simple matter to check that this is an isomorphism (this uses the fact the J is a vector cross product). Harvey and Lawson show furthermore that $G_2 \subset SO_7$ acts transitively on $V_{7,2}$. The stabilizer of (i,j) can be recognized as SU_2 in the following manner. Note that if $\sigma \in \mathrm{Stab}_{G_2}((i,j))$, then $\sigma = Id$ restricted to the first factor of $\mathrm{Im}\mathbb{O} = \mathrm{Im}\mathbb{H} \oplus \mathbb{H}$. So we write

$$Stab_{G_2}((i,j)) = Aut_{\mathbb{O}}(\text{span} < e, I, J, K >) \tag{2.2.25}$$

We have identified $\mathbb{O}$ with $\mathbb{H} \oplus \mathbb{H}$ according to the Cayley-Dickson process, so the product restricted to the second component is $(0,b)(0,d) = (-\bar{d}b, 0)$. Since $\bar{d}b = g_{\mathbb{H}}(b,d) + Im(\bar{d}b)$, and $\phi(\bar{d})\phi(b) = \bar{d}b$, we can take real parts and equate

$$g_{\mathbb{H}}(b,d) = g_{\mathbb{H}}(\phi(b), \phi(d)) \tag{2.2.26}$$

Hence, $\phi \in Sp_1 \cong SU_2$. $\square$

Constructions of G_2-manifolds

We mention that there are well-known methods of constructing G_2-manifolds with full G_2-holonomy, first famously due to Joyce [J3], and another due to Kovalev [Ko]. We leave this exploration to the reader.

Multiholomorphic images are associative

Coming back to the main thread, suppose a closed, oriented, Riemannian 3-manifold X maps by u into a closed G_2-manifold M. If $\eth u = 0$ (and u is an embedding for the time being) then the image is an associative submanifold of M. We consider the MCR equation in local normal coordinates $\{s, p, q\}$, at the origin point:

$$|\partial_s u|\partial_s u = J\partial_p u \wedge \partial_q u \tag{2.2.27}$$

$$|\partial_p u|\partial_p u = J\partial_q u \wedge \partial_s u \tag{2.2.28}$$

$$|\partial_q u|\partial_q u = J\partial_s u \wedge \partial_p u \tag{2.2.29}$$

And then since J is alternating,

$$|\partial_s u|J(\partial_s u, J(\partial_p u \wedge \partial_q u)) = 0 \tag{2.2.30}$$

$$|\partial_p u|J(\partial_p u, J(\partial_q u \wedge \partial_s u)) = 0 \tag{2.2.31}$$

$$|\partial_q u|J(\partial_q u, J(\partial_s u \wedge \partial_p u)) = 0 \tag{2.2.32}$$

Hence, $\partial_s u \wedge \partial_p u \wedge \partial_q u$ is an associative 3-plane. This is sufficient to write

$$u^*[\bullet, \bullet, \bullet] = 0 \in \Omega^3(X, u^*TM) \tag{2.2.33}$$

If u is an embedding then $im(u)$ is an associative submanifold of M.

2.2.4 Integrability and Existence

As one can readily see, in general the MCR equations are an overdetermined system. This is the primary issue in the $\mathbb{R}^n \to \mathbb{R}^n$ case of quasi-regular maps (hence the Liouville theorem), and it becomes relevant in other cases as well.

It is exceedingly clear that solutions exist in abundance in the G_2-situation. There is a very rich world of associative submanifolds of a G_2-manifold and several researchers have studied these intensely. To name one example, Lotay's recent work [Lo] begins a more systematic study of the associative submanifolds of S^7. And there is literature pertaining to the existence of G_2-manifolds with conic singularities as well as corresponding associatives with singularities. Given an associative submanifold, one can obtain at least as many multholomorphic maps as there are endomorphisms of this domain. This is the issue we turn to in the next subsection.

Endomorphisms

As we saw earlier, in the case where X is a manifold with a conformal triad, multiholomorphic self-maps are 1-quasiregular. In the case in which X is a 3-manifold the local description of such maps (a la Liouville) is not adequate to understand their behavior because there is an obstruction in dimension 3 to local conformal flatness, the Cotton tensor:

$$C_{ijk} = \nabla_k R_{ij} - \nabla_j R_{ik} + \frac{1}{2(n-1)}[\nabla_j R g_{ik} - \nabla_k R g_{ij}] \tag{2.2.34}$$

where R_{ij} is the Ricci curvature, and R the scalar curvature. A study of the Liouville theorem $\mathbb{R}^3 \to \mathbb{R}^3$ in situation of nontrivial Cotton tensor seems like an interesting project in its own right.

As we saw earlier, on S^3, a manifold which is LCF, we know that a multiholomorphic endomorphism is a covering transformation.

Overdeterminedness

The linearization of the MCR equation as calculated in the analytic setup with section is

$$\begin{aligned}
&(D_u \eth)\xi = \eta \mapsto \\
&\mapsto \frac{n-1}{(n+1)^{(n-1)/2}}|du|^{(n-3)} < u^*\nabla^M \xi, du > du(\eta) + \frac{1}{(n+1)^{(n-1)/2}}|du|^{n-1} u^* \nabla^M_\eta \xi - \\
&\qquad - \frac{(-1)^n}{n-1} J_u(\Lambda^{n-1} du \wedge u^* \nabla^M \xi \circ k)(\eta) + u^* \nabla^M_\xi J(\Lambda^n du \circ k)(\eta)
\end{aligned} \tag{2.2.35}$$

From this we can see clearly the overdeterminedness of the equation as this is *prima facie* m equations on $n+1$ unknowns even with the condition of parallel vector cross product

J. Suppose that ξ is a normal deformation up to first order. That is,

$$< \xi, du(X) >= 0 \forall X \in TX, \quad \text{and} \quad < \nabla\xi, du >= 0, \quad \text{everywhere.} \tag{2.2.36}$$

Then $D_u \eth \xi = 0$ simplifies to

$$\frac{1}{(n+1)^{(n-1)/2}} |du|^{n-1} u^* \nabla^M_\eta \xi - - \frac{(-1)^n}{n-1} J_u(\Lambda^{n-1} du \wedge u^* \nabla^M \xi \circ k)(\eta) + u^* \nabla^M_\xi J(\Lambda^n du \circ k)(\eta) \tag{2.2.37}$$

In the simplest case of $u : \mathbb{R}^3 \to \mathbb{R}^7$, where u is the inclusion of $Im\mathbb{H} \to Im\mathbb{H} \oplus \mathbb{H}$ as the 3-plane $< i, j, k >$, then ξ is a first order normal deformation if

$$\partial_0 \xi = \quad -j \cdot Im(\partial_1 u \xi_2) + k \cdot Im(\partial_2 u \cdot \xi_1) \tag{2.2.38}$$

$$\partial_1 \xi = \quad i \cdot Im(\partial_2 u \xi_0) - k \cdot Im(\partial_0 u \cdot \xi_2) \tag{2.2.39}$$

$$\partial_2 \xi = \quad -i \cdot Im(\partial_0 u \xi_1) + j \cdot Im(\partial_1 u \cdot \xi_0) \tag{2.2.40}$$

But one can readily see that multliplication by i, j, k permutes these equations, all of which are equivalent to the Dirac equation on $\mathbb{R}^3$ on spinor fields

$$D\xi := [i \cdot \partial_0 + j \cdot \partial_1 + k \cdot \partial_2]\xi = 0 \tag{2.2.41}$$

This is a determined elliptic equation. Indeed this matter has been known for some time, most-notably in the paper of McLean [McL]. There it is shown that any normal (1st order) deformation ξ can be identified with a harmonic spinor over $im(u)$. This done by virtue of an identification of the normal bundle to an embedded associative to a twisted spinor bundle over the associative (this is described in the bounded case in the section on the deformation index). Unfortunately, in this situation the obstructions to full integrability of deformations lie the cokernel of the Dirac operator, which is nontrivial whenever its kernel is. (The index of a differential operator on a closed 3-manifold is 0.)

In our case, of course we are looking at more data, namely deformations of maps. For instance the term

$$< \nabla\xi, du > |du|^{n-3} du \tag{2.2.42}$$

appears in the linearization. This is a scaled projection of $\nabla\xi$ onto the 1-form du. It is relatively unsatisfying at this point that we have not yet identified a situation in which there exist continuous moduli of normal deformations of multiholomorphic maps into a G_2-manifold which are emphnot given by applying a continuous family of global isometries of the target manifold to a particular MCR-map.

Integrability

In general we will require integrability conditions in order to specify the existence of solutions to the MCR equations. Almost always we will assume that the G_2-structure on M is parallel, $\nabla J = 0$. This means in particular that MCR maps are locally solutions to the homogeneous rather than inhomogeneous $(n+1)$-Laplace equation.

Theorem 2.2.9. *(Bryant) Any compact, real-analytic surface S in a G_2-manifold locally extends uniquely to an associative submanifold.*

This fact gives some picture of how deformations of associatives might be understood as arising from deformations of some special surfaces.

Examples Related to $SU(3)$-Geometry

There is a relatively trivial construction of a fibered G_2-manifold which works well as a conceptual reference point. It is ultimately an outworking of the fact that $SU(3)$ sits as a proper subgroup inside of G_2. Given a Calabi-Yau 3-fold Y, one can consider the cross product $Y \times S^1$ with the obvious S^1 action by translation. This Cartesian product picks up a G_2-structure via

$$\phi := Re\Omega + \omega \wedge dt \tag{2.2.43}$$

where Ω is the holomorphic volume form on Y, ω is the compatible symplectic form, and t is the coordinate along S^1. There are some simple means of constructing associative and coassociative submanifolds from the Kähler geometry. In particular, one easily sees that a holomorphic curve ψ in Y becomes an associative submanifold in M by crossing with S^1. Likewise, a special Lagrangian in Y with phase of $\frac{\pi}{2}$ becomes coassociative in M after crossing with S^1. By crossing with a point, a special Lagrangian in Y with phase 0, becomes associative in M, and the same construction with a holomorphic 2-cycle in Y produces a coassociative submanifold in M.

Although any deformation of a holomorphic surface in Y produces a deformation of associative submanifolds, the same will not be true at the level of maps. This ultimately arises from the fact that if ϕ is a holomorphic curve, $\phi \times I$ is likely not a multiholomorphic map from $\Sigma \times S^1$ to $Y \times S^1$. What is required in $Y \times S^1$ is a conformal deformation of $im\phi \times S^1$. And in fact, since this S^1-factor cannot by expanded or shrunken, then a conformal deformation of $im\phi \times S^1$ must be realized by a global symmetry of $Y \times S^1$ under which $im\phi \times S^1$ is moved isometrically. This approach does in fact yield some examples in simple cases, but it seems clear that actually the S^1 factor is rather limiting and in this case it is not remotely clear that anything can be gleaned from consideration of this fibered case.

2.2.5 Coassociative Boundary conditions for the G_2-MCR equation

As noted previously there are natural boundary conditions for associative submanifolds. The relevant boundary manifolds, or *branes*, are the coassociative submanifolds. These are the maximal-dimension isotropic submanifolds for the calibration ω. Interestingly they are also calibrated submanifolds for the dual form $\star\omega$. One can cite [McL] for their deformation theory, at least the compact ones. One might (following Lagrangian Floer theory) consider bounded associative submanifolds whose boundaries lie on particular coassociative submanifolds. The main analytic desideratum for this configuration is that the energy identity holds true relative to these boundary components. That is,

Proposition 2.2.10. *Suppose u_t is a smooth 1-parameter family of multiholomorphic maps $u_t : X \times I \to M$, where X has boundary components labeled $\partial X = \sqcup_i \partial X^i$. Each X_i is some closed surface. And furthermore suppose that these surfaces lie for all t inside some chosen coassociative submanifolds $\{C_i\}$, i.e. $u_t(X_i) \subset C_i$, $\forall t$. Then, the energy of u_t does not depend on t.*

Proof. W.l.o.g. we equate E_0 and E_1 where $E_t = \int_X u_t^* \omega$. Then,

$$0 = \int_{X\times I} u_t^*(d\omega) = \int_{X\times\{0\}} u_0^*\omega - \int_{X\times\{1\}} u_1^*\omega + \int_{\partial X\times I} u_t|_{\partial X}^*\omega = E_0 - E_1 \tag{2.2.44}$$

$\square$

So, as before, the moduli space of multiholomorphic maps with coassociative boundary conditions is divided into components by energy levels. In the next section we describe normal deformations of such configurations *of submanifolds*. This issue has been addressed by Gayet and Witt in [GW]. Our study of this proceeded without this knowledge at first, but was broken off after finding their prior work. We include a somewhat loose discussion of these results noting that our definition of the Maslov class may be divergent from theirs.

Deformation Index

We collect here some facts with are of central importance. Mostly these are from [McL] and [HL], although the former derives heavily from the essential [LM]. To start, it is useful to recall that $Sp_1 \cong Su_2 \cong Spin_3$, which may be used interchangeably based on the context. Suppose X is an associative submanifold embedded in M, a G_2-manifold.

There is a G_2-frame bundle on M which consists of orthonormal frames

$$\mathcal{F}_{G_2}(M)_p = \{\rho : T_pM \to \mathbb{R}^7, \rho^*\phi_0 = \phi\} \tag{2.2.45}$$

This is a principal right G_2-bundle. Given an associative manifold X in M, we pull back the G_2-frame bundle on M to X, and take the adapted orthonormal frame bundle $\mathcal{F}^{(1)}(X)$. This is the subbundle consisting of frames which respect the splitting $T_pM|_X = T_pX \oplus \nu_X$. $\mathcal{F}^{(1)}$ is a principal right $SO(4) \cong Sp(1) \times Sp(1)/\pm(1,1)$ bundle via the action

$$(p,q) \in Sp(1) \times Sp(1) : ((e_i) \in \mathcal{F}(Im\mathbb{H}), (v_j) \in \mathcal{F}(\mathbb{H})) \mapsto (q(e_i)\bar{q}, p(v_j)\bar{q}) \tag{2.2.46}$$

It can be seen that the tangent bundle to X is associated to $\mathcal{F}^{(1)}(X)$ via the representation

$$\rho : SO(4) \to GL(Im\mathbb{H}), \quad [p,q](x) = qx\bar{q} \tag{2.2.47}$$

And the normal bundle ν is associated via the representation

$$\tau : Sp(1) \times Sp(1)/\pm(1,1) \to GL(\mathbb{H}) \quad [p,q](y) = py\bar{q} \tag{2.2.48}$$

Now the spin structure on M determines a spin structure on X. Suppose $\mathcal{P}$ is the principal $Spin_7$-bundle over $\mathcal{F}_M$ determined by the G_2-structure on M. Then this determines a principal $Sp(1) \times Sp(1)$-bundle $\mathcal{P}^{(1)}$, which double-covers the adapted frame bundle. One can construct a principal $Spin_3$-bundle double covering $\mathcal{F}_X$ by taking a quotient of $\mathcal{F}^{(1)}$ by the image of $\{(1, Spin_3)\}$ under the quotient $Spin_3 \times Spin_3 \to Spin_3 \times Spin_3/\pm(1,1)$. However, all at once we can associate the spinor bundle to this $\mathcal{P}^{(1)}$. Namely, the spinor bundle $\mathbb{S}$ is associated to the principal bundle $\mathcal{P}^{(1)}$ by the representation

$$\sigma : Sp(1) \times Sp(1) \to GL(\mathbb{H}) \quad (p,q)(y) = y\bar{q} \tag{2.2.49}$$

ν is associated to this lift via the lifted representation

$$\tau' : Sp(1) \times Sp(1) \to GL(\mathbb{H}) \quad (p,q)(y) = py\bar{q} \tag{2.2.50}$$

Suppose we denote by E the bundle associated to $\mathcal{P}^{(1)}$ via $(p,q)(y) = py$. Then we have made the identification

$$\nu = \mathbb{S} \otimes_{\mathbb{H}} E \tag{2.2.51}$$

Thus ν is a rank-4 $G := Sp_1 \times Sp_1$-bundle over X.

From another perspective one sees this structure as a $Spin_4 = Sp_1 \times Sp_1$-structure on ν, which has been shown to exist on the normal bundle of any 3-dimensional manifold inside a G_2-manifold. Given such a $Spin_4$-structure one can define $\mathbb{S}$ and E as above, realizing that in the case where the submanifold is associative, $\mathbb{S}$ becomes its spinor bundle [AS3], [Br1].

On the boundary ν has additional interesting structures.

Proposition 2.2.11. *A boundary component Σ has an outward-facing normal $\hat{n}$ which determines a complex structure on each boundary component. For $v \in \nu|_\Sigma$,*

$$Jv := J(\hat{n}, v) \tag{2.2.52}$$

Proof. Let x be a boundary point of X, with outward unit normal $\hat{n}$. For v, any unit vector in nu_{xX}, $(hatn, v, J(\hat{n}, v))$ is an associative, positively-oriented, orthonormal 3-frame. Considering

$$g(J(\hat{n},v)+v, J(\hat{n},v)+v) = \|v\|^2 + \|\hat{n} \wedge J(\hat{n},v)\|^2 - 2\omega(\hat{n}, v, J(\hat{n},v)) = 2-2 = 0 \tag{2.2.53}$$

yields $J(\hat{n}, \bullet)^2 = -I$. □

Thus the G_2 structure induces a holomorphic structure on each boundary component. We will be interested in deformations of such a configuration which have certain coassociative boundary conditions. In particular we are interested in sections of the normal bundle with boundary values inside a coassociative distribution in TM which intersects the normal bundle in a 2-*dimensional* space at each point of intersection. (Since the

2-dimensional boundary of X lies in the coassociative manifold, it's tangent space can intersect the normal direction in at most 2 dimensions.) This amounts to specifying a complex line bundle on each boundary component.

Indeed, suppose the coassociative plane bundle on Σ in TM intersects $\nu|_\Sigma$ in two-dimensions; then this two-dimensional subbundle is necessarily complex and vice-versa. A coassociative 4-plane always has an oriented, orthonormal frame $(e_1, e_2, e_3 \perp e_1e_2, e_1e_2e_3)$. We can choose e_1, e_2 to be the associative frame for TX such that $e_1e_2 = \hat{n}$. Then $(e_3, e_1e_2e_3$ an orthonormal complex frame since the complex structure is $J(\hat{n} = e_1e_2, \bullet) = e_1e_2\cdot\bullet$. On the other hand, any orthonormal complex frame (l_1, l_2) $(J(\hat{n}, l_1) = l_2, J(\hat{n}, l_2) = -l_1)$ in ν has $J(l_1, l_2) = \hat{n}$. Thus $< l_1 \wedge l_2 \wedge (\hat{n}^\perp) >$ is a coassociative 4-plane. Hence,

Proposition 2.2.12. *Given a bounded associative submanifold in M, its normal bundle restricted to the boundary gains a canonical complex structure. Furthermore, on any connected component Σ, there is a bijective correspondence between complex line-subbundles L of $\nu|_\Sigma$ and coassociative planes C in $TM|_\Sigma$ whose intersection with ν is 2-dimensional, and include the 2-plane $\hat{n}^\perp \subset TX$.*

Proof. □

One could go further and note that any non-vanishing, unit vector field on X yields a complex structure on ν. If this vector field is normal to the boundary components it may point inwards on some boundary components, hence inducing the opposite complex structure than the canonical one on those components.

In any case, a choice of such coassociative boundary conditions is equivalent to picking a particular complex line bundle inside ν along each boundary component. On a given boundary component Σ, such a choice is given by a section into the Grassmanian of complex 2-planes of ν. This is a bundle with fiber isomorphic to $\mathbb{CP}^2$:

$$\mathbb{CP}^2 \to \mathrm{Grass}(\nu, \mathbb{C}) \to X \tag{2.2.54}$$

which lifts to the complex frame bundle

$$SU_2 \to \mathrm{Frame}(\nu, \mathbb{C}) \to X \tag{2.2.55}$$

A principal SU_2-bundle over X.

So following the scheme set up in [McL], deformations of X with this type of coassociative boundary values are identified with sections of $\nu = E \otimes \mathbb{S}_X$ over X which lie inside $L \subset \nu$, a chosen line bundle on ∂X, which furthermore are in the kernel of the associated twisted Dirac operator D_E.

On the boundary components, it is not hard to see that the Dirac operator restricts to the Cauchy-Riemann operator, and the machinery of elliptic boundary value problems yields,[GW],

$$Ind[(D_E, L)] = Ind[\bar{\partial}] = \chi(\Sigma) + 2 < c_1(L), [\Sigma] > \tag{2.2.56}$$

This kind of theorem is quite similar in character to that of Freed [Fr], except that the line bundle L is not canonically given. The result of this calculation is that if one considers a bounded associative X as a bordism from some "inward-facing" to "outward-facing" boundary components (a question of whether or not a given orientation agrees with the one induced by the orientation of X), then we see that the index of such bordisms is the difference of the indices on the inward and outward ends.

The Maslov-type index

There is an obvious Maslov-type index in the above situation which is closely related to the above deformation index. Recall that given a manifold with boundary equipped with and a complex bundle, the *boundary Maslov index* [McDS] is an index which characterizes the topological class of a Lagrangian subbundle over the boundary. Suppose Σ is a Riemann surface with boundary, and E a rank-n complex bundle. Then E is trivial for topological reasons. Smoothly picking an n-dimensional totally real subspace over each boundary point corresponds to a map from $\partial\Sigma \to Grass_{Lag}(n, \mathbb{C})$ the Grassmanian of Lagrangian subspaces of $\mathbb{C}^n$. This space has a distinguished class in H^1 (the Maslov class) which can be paired with this path. The resulting index is the boundary Maslov index.

Completely analogously in our case, one can consider the normal bundle ν_X over $X, \partial X$ with a choice of a complex line bundle over each point in ∂X. Supposing first that $\nu_X|_{\partial X}$ is trivial, one recognizes that the coassociative boundary choice represents a map from ∂X to the Grassmanian of complex lines in $\mathbb{C}^2$, that is, $\mathbb{CP}^1$. Then we can regard the relevant Maslov-type index as the degree of

$$\partial X \to \mathbb{CP}^1 \tag{2.2.57}$$

If X is a 3-disk with boundary S^2 then in terms of a trivialization of ν_X we can regard the boundary as $\mathbb{CP}^1$ and hence define the normal forms for each k:

$$\Lambda_k : [z : w] \to [z^k ; w^k] \tag{2.2.58}$$

Then, we have

Proposition 2.2.13. *If X is a 3-disk as above with boundary map given by the normal form of degree k, then*

$$Ind[D_E, L] = 2 + \mu_{G_2}(X, L) = 2 + k \tag{2.2.59}$$

Proof. □

Furthermore, we can extend this Maslov class to any X. First suppose X is a handle body of genus g. Then the exact same identification as above goes through because

$Spin_3 \times Spin_3$-bundles over handle bodies are trivial, and complex line bundles over Riemann surfaces are classified by their first Chern classes. We get

$$Ind[D_E, L] = (2-2g) + \mu_{G_2}(X, L) = 2 - 2g + k \tag{2.2.60}$$

By a sum formula we can extend this to any 3-manifold, even closed ones. Every closed, compact 3-manifold admits a Heegaard splitting, which is a homeomorphism to a pair of handle bodies glued by virtue of a given isomorphism of the boundary surface. Thus a $G = Spin_3 \times Spin_3$-bundle on such a closed X with Heegaard decomposition $X = X_1 \cup_f X_2$ can be specified topologically by a clutching function-type specification. G is connected, so every G-bundle on a handle body (which is homology-equivalent to a bouquet of circles) is trivial. Then the specification of a clutching function on the overlap surface Σ_g is trivial as well since G has $\pi_2(G) = H_2(G) = 0$. One would expect then that the Maslov index on a closed 3-manifold would be 0. This would agree also with the fact that the relevant deformation problem has index zero.[McL]

Indeed, suppose that we have $X = X_0 \sqcup_I X_1$ with ∂X_0 and ∂X_1 covered by complex line bundles L_0, L_1 and I an isomorphism of bundles $L_0 \to L_1$ covering an orientation-reversing diffeomorphism between ∂X_0 and ∂X_1. Then,

$$\mu(X_0, L_0) + \mu(X_1, L_1) = \mu(X_0, L_0) + \mu(-X_0, L_0) = Ind[\bar{\partial}] + Ind[\partial] \tag{2.2.61}$$

firstly because the Maslov index is preserved by bundle isomorphisms over diffeomorphisms, and secondly because reversing the orientation on the gluing surface induces the opposite complex structure on the normal bundle over the surface. Hence we are looking at the sum of the indices of the Cauchy-Riemann and anti-Cauchy-Riemann operators on ∂X_0. But $-\star\bar{\partial}^*\star = \partial$ implies that these have opposite indices. Hence,

$$\mu(X, \partial X = 0) = \mu(X_0, L_0) + \mu(X_1, -L_0) = 0 \tag{2.2.62}$$

Similarly, a bounded 3-manifold admits a Heegaard-like decomposition in the sense that any 3-manifold can be written as the union of two compression bodies which have been glued along their respective positive boundary components according to a specified isomorphism.

Recall that a *compression body* is constructed from a surface Σ by attaching 1-handles $\{H_i\}$ to $\Sigma \times [0,1]$ on the outer surface $\Sigma \times \{1\}$. The result is a 3-manifold C with boundary divided into

$$\partial C = \partial_- C \sqcup \partial_+ C = \Sigma \times \{0\} \sqcup (\partial C - \partial_- C) \tag{2.2.63}$$

And so a 3-manifold with boundary X can be written $X = C_1 \cup_f C_2$. A compression body is homotopy equivalent to the surface Σ along with some bouquet of circles attached base point-to-base point. G-bundles (or more-basically SU_2-bundles) on surfaces are not necessarily trivial but are governed by the Chern classes of the boundary surfaces. Continuing in this direction is not essential for this thesis, but it represents a possibly interesting rabbit trail for future researcher.

2.2.6 Fukaya-like boundary conditions

One of the elements of Floer theory is the fact that a cylinder mapped holomorphically into a target symplectic manifold is guaranteed to close up smoothly at the end (converge unitformly in the parameter variable to a point) under the assumption that the energy of such a map is finite. This remarkable fact is another example of the strong constraints that the Cauchy-Riemann equation implies. This fact extends to the bounded situation in the Fukaya-categorical setting in which open strips are guaranteed to converge uniformly to intersection points of the bounding Lagrangians.

In our situation we have the following speculation about the behavior of these maps near asymptotically cylindrical ends.

Speculation 2.2.14. *Given a multiholomorphic map u from $\Sigma \times (0,1)$ into some G_2-manifold M, if $E(u) < \infty$ then u converges uniformly in the parameter variable to a single point, producing a cones on Σ as a singularity.*

This issue will not be tackled until further basic examples are worked out.

One could furthermore give some speculation on the form that a Fukaya-like theory would take, incorporating coassociative submanifolds as branes. This avenue immediately brings to mind many possibilities, all of which seem fraught with difficulty. One observation to make is that a generic intersection of two coassociatives is some finite number of circles. On the other hand based on the conjecture above, one would not expect some open multholomorphic map to extend to a 1-dimensional boundary such as one of these S^1-intersections. Another perspective would induce one to consider multiholomorphic maps of $S \times (0,1) \to M$ for which S is a 2-manifold with corners in which alternating boundary components are required to land on some chosen set of coassociatives, the others being free boundary components. This is merely too complicated to be worth studying at such an early phase of this putative mapping theory.

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