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Heegaard Floer Invariants and Cabling

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Abstract
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ABSTRACT

HEEGAARD FLOER INVARIANTS AND CABLING

Jennifer Hom

Paul Melvin, Advisor

A natural question in knot theory is to ask how certain properties of a knot behave under satellite operations. We will focus on the satellite operation of cabling, and on Heegaard Floer-theoretic properties. In particular, we will give a formula for the Ozsváth-Szabó concordance invariant τ of iterated cables of a knot K in terms of the cabling parameters, $\tau(K)$, and a new concordance invariant, $\varepsilon(K)$. We show that, in many cases, ε gives better bounds on the 4-ball genus of a knot that τ alone, and discuss further applications of ε. We will also completely classify when the iterated cable of a knot admits a positive L-space surgery.

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Chapter 1

Introduction

In this thesis, we address the question of how certain properties of knots behave under the satellite operation of cabling. The (p, q)-cable of a knot K, denoted $K_{p,q}$, is the satellite knot with pattern the (p, q)-torus knot $T_{p,q}$ (where p indicates the longitudinal winding and q indicates the meridional winding) and companion K. We will assume throughout that $p > 1$. (This assumption does not cause any loss of generality, since $K_{-p,-q} = \mathrm{r}K_{p,q}$, where $\mathrm{r}K_{p,q}$ denotes $K_{p,q}$ with the opposite orientation, and since $K_{1,q} = K$.)

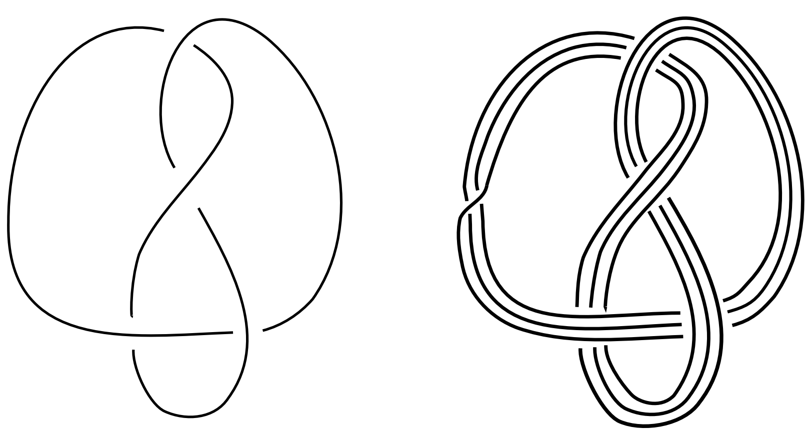


Figure 1.1: The figure 8 knot, and its $(3, 1)$-cable.

It is well-known that the Alexander polynomial of the (p, q)-cable of a knot K is completely determined by p, q, and the Alexander polynomial of K, $\Delta_K(t)$, in the following manner:

$$\Delta_{K_{p,q}}(t) = \Delta_K(t^p) \cdot \Delta_{T_{p,q}}(t),$$

where

$$\Delta_{T_{p,q}}(t) = \frac{(t^{pq} - 1)(t - 1)}{(t^p - 1)(t^q - 1)}.$$

In this thesis, we focus on the behavior of the Ozsváth-Szabó concordance invariant τ under cabling.

Two knots $K_0, K_1 \subset S^3$ are called *concordant*, denoted $K_0 \sim K_1$, if there exists a smooth, properly embedded cylinder in $S^3 \times [0, 1]$ such that one end of the cylinder is $K_0 \times \{0\}$ and the other is $K_1 \times \{1\}$. This gives us an equivalence relation on the set of knots. A knot K is called *slice* if K is concordant to the unknot. The set $\{K\}/\sim$ forms the *concordance group* $\mathcal{C}$, where the operation is induced by connected sum. The class of slice knots is the identity element, and the inverse of $[K]$ is $[-K]$, where $-K$ denotes the reverse of the mirror image of K. If we loosen the conditions and only require that the cylinder be locally flat, rather than smooth, we obtain the *topological concordance group*.

To a knot $K \subset S^3$, Ozsváth and Szabó [11], and independently Rasmussen [19], associate a $\mathbb{Z}\oplus\mathbb{Z}$-filtered chain complex $CFK^\infty(K)$, whose doubly filtered chain homotopy type is an invariant of K. Looking at just one of the filtrations (i.e., taking the degree zero summand of the associated graded object with respect to the other filtration) yields the $\mathbb{Z}$-filtered chain complex $\widehat{CFK}(K)$, and associated to this chain complex is the $\mathbb{Z}$-valued smooth concordance invariant $\tau(K)$; see [9].

In [9], Ozsváth and Szabó show that the concordance invariant τ has the following properties:

1. $\tau : \mathcal{C} \to \mathbb{Z}$ is a surjective homomorphism.

2. $|\tau(K)| \leq g_4(K)$, where $g_4(K)$ denotes the smooth 4-ball genus of K.

3. $|\tau(T_{p,q})| = g(T_{p,q})$, where $g(K)$ denotes the Seifert genus of K.

We completely describe the behavior of τ under cabling, generalizing work of Hedden [2], Van Cott [20], and Petkova [18]. As one might expect, $\tau(K_{p,q})$ is closely related to $\tau(T_{p,q})$. However, it depends on strictly more than just $\tau(K)$, p, and q; we also need to know the value of $\varepsilon(K)$, a new $\{-1,0,1\}$-valued concordance invariant associated to the knot Floer complex $CFK^\infty(K)$.

Theorem 1. *The behavior of $\tau(K_{p,q})$ is completely determined by p, q, $\tau(K)$, and $\varepsilon(K)$. In particular,*

1. *If $\varepsilon(K)=0$, then $\tau(K)=0$ and* $\tau(K_{p,q})=\tau(T_{p,q})=\begin{cases}\frac{(p-1)(q+1)}{2} & \text{if } q<0\\ \frac{(p-1)(q-1)}{2} & \text{if } q>0.\end{cases}$
2. *If $\varepsilon(K)\neq 0$, then*
$$\tau(K_{p,q})=p\tau(K)+\frac{(p-1)(q-\varepsilon(K))}{2}.$$

One consequence of Theorems 1 and 2 is that the only additional concordance information about K coming from τ of iterated cables of K is the invariant ε. Conversely, knowing τ of just two cables of K, one positive and one negative, is sufficient to determine $\varepsilon(K)$. In particular, knowing information about the $\mathbb{Z}$-filtered chain complex $\widehat{CFK}(K_{p,q})$, namely $\tau(K_{p,q})$, can tell us information about the $\mathbb{Z}\oplus\mathbb{Z}$-filtered chain complex $CFK^\infty(K)$, i.e., $\varepsilon(K)$.

Since $\tau(K_{p,q})$ depends on both $\tau(K)$ and $\varepsilon(K)$, we would also like to know the behavior of ε under cabling in order to compute τ of iterated cables.

Theorem 2. *The invariant ε behaves in the following manner under cabling:*

1. *If $\varepsilon(K)=0$, then* $\varepsilon(K_{p,q})=\varepsilon(T_{p,q})=\begin{cases}-1 & \text{if } q<-1\\ 0 & \text{if } |q|=1\\ 1 & \text{if } q>1.\end{cases}$
2. *If $\varepsilon(K)\neq 0$, then $\varepsilon(K_{p,q})=\varepsilon(K)$ for all p and q.*

We see that $\tau(K_{p,q})$ depends on strictly more than just $\tau(K)$, so it is natural to ask if there exist knots K and K' with $\tau(K)=\tau(K')$ but $\tau(K_{p,q})\neq\tau(K'_{p,q})$. We answer this question in the affirmative:

Corollary 3. *For any integer n, there exists knots K and K' with $\tau(K) = \tau(K') = n$, such that $\tau(K_{p,q}) \neq \tau(K'_{p,q})$, for all p and q, $p \neq 1$.*

Recall that the absolute value of $\tau(K)$ gives a lower bound on the 4-ball genus of a knot; that is, $g_4(K) \geq |\tau(K)|$. By looking at both τ and ε, we can give stronger bounds in many cases. The following corollary was suggested to me by Livingston:

Corollary 4 (Livingston)**.** *If $\varepsilon(K) \neq \operatorname{sgn} \tau(K)$, then $g_4(K) \geq |\tau(K)| + 1$.*

If two knots are concordant, it follows that their cables are also concordant. Thus, it follows from Therorem 1 that ε is a concordance invariant. We also prove the following properties about ε:

- If K is slice, then $\varepsilon(K) = 0$.
- $\varepsilon(-K) = -\varepsilon(K)$.
- If $\varepsilon(K) = 0$, then $\tau(K) = 0$.
- There exist knots K with $\tau(K) = 0$ but $\varepsilon(K) \neq 0$; that is, $\varepsilon(K)$ is *strictly stronger* than $\tau(K)$ at obstructing sliceness.
- Let $g(K)$ denote the genus of K. If $|\tau(K)| = g(K)$, then $\varepsilon(K) = \operatorname{sgn} \tau(K)$.
- If K is *homologically thin* (meaning $\widehat{HFK}(K)$ is supported on a single diagonal with respect to its bigrading), then $\varepsilon(K) = \operatorname{sgn} \tau(K)$.
- If $\varepsilon(K) = \varepsilon(K')$, then $\varepsilon(K\#K') = \varepsilon(K) = \varepsilon(K')$. If $\varepsilon(K) = 0$, then $\varepsilon(K\#K') = \varepsilon(K')$.

Moreover, the invariant ε can be used to define a new concordance homomorphism, as discussed in [4].

In the latter part of this thesis, we will consider a special kind of 3-manifold, called an L-space, and knots on which some positive integral surgery yields an L-space. Such a knot is called an *L-space knot*.

Let $g(K)$ denote the Seifert genus of K. In Theorem 1.10 of [2], Hedden proves that if K is an L-space knot and $q/p \geq 2g(K) - 1$, then $K_{p,q}$ is an L-space knot. We will prove the converse:

Theorem 5. *The (p, q)-cable of a knot $K \subset S^3$ is an L-space knot if and only if K is an L-space knot and $q/p \geq 2g(K) - 1$.*

Thus, we see that whether or not the cable of K is an L-space knot depends only on the cabling parameters, and whether or not K is an L-space knot. The proof of this theorem relies on classical low-dimensional techniques, as well as various properties of Ozsváth-Szabó invariants.

Chapter 2

A quick trip through bordered Heegaard Floer homology

We begin with a few algebraic preliminaries, before proceding to a brief overview of bordered Heegaard Floer homology and knot Floer homology.

2.1 Algebraic preliminaries

For the reader unfamiliar with the algebraic structures involved in bordered Heegaard Floer homology, such as $\mathcal{A}_\infty$-modules, the Type D structures of [7], and the "box" tensor product, we recount the definitions below. For a more detailed description, we refer the reader to [7, Section 2].

Let $\mathcal{A}$ be a unital graded algebra over $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$ with an orthogonal basis $\{\iota_i\}$ for the subalgebra of idempotents, $\mathcal{I} \subset \mathcal{A}$, such that $\sum \iota_i = 1 \in \mathcal{A}$. In what follows, all of the tensor products are over $\mathcal{I}$. We suppress grading shifts for ease of exposition.

A *(right unital)* $\mathcal{A}_\infty$*-module* is an $\mathbb{F}$-vector space M equipped with a right $\mathcal{I}$-action such that

$$M = \bigoplus_i M\iota_i,$$

and a family of maps

$$m_i : M \otimes \mathcal{A}^{\otimes i-1} \to M$$

satisfying the $\mathcal{A}_\infty$ conditions

$$\begin{aligned} 0 = &\sum_{i=1}^{n-1} m_{n-i+1}(m_i(x \otimes a_1 \otimes \ldots \otimes a_{i-1}) \otimes \ldots \otimes a_{n-1}) \\ &+ \sum_{i=1}^{n-2} m_{n-1}(x \otimes a_1 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots a_{n-1}) \end{aligned}$$

and the unital conditions

$$m_2(x, 1) = x$$

$$m_i(x, \ldots, 1, \ldots) = 0, \quad i > 2.$$

We say that M is *bounded* if there exists an integer n such that $m_i = 0$ for all $i > n$.

A *Type D structure over* $\mathcal{A}$ is an $\mathbb{F}$-vector space N equipped with a left $\mathcal{I}$-action such that

$$N = \bigoplus_i \iota_i N,$$

and a map

$$\delta_1 : N \to \mathcal{A} \otimes N$$

satisfying the Type D condition

$$(\mu \otimes \mathbb{I}_N) \circ (\mathbb{I}_{\mathcal{A}} \otimes \delta_1) \circ \delta_1 = 0,$$

where $\mu : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ denotes the multiplication on $\mathcal{A}$.

On the Type D structure N, we define maps

$$\delta_k : N \to \mathcal{A}^{\otimes k} \otimes N$$

inductively by

$$\delta_0 = \mathbb{I}_N$$

$$\delta_i = (\mathbb{I}_{\mathcal{A}}{}^{\otimes i-1} \otimes \delta_1) \circ \delta_{i-1}.$$

We say that N is *bounded* if there exists an integer n such that $\delta_i = 0$ for all $i > n$.

The *box tensor product* $M \boxtimes N$ is the $\mathbb{F}$-vector space

$$M \otimes_{\mathcal{I}} N,$$

endowed with the differential

$$\partial^{\boxtimes}(x \otimes y) = \sum_{k=0}^{\infty} (m_{k+1} \otimes \mathbb{I}_N)(x \otimes \delta_k(y)).$$

If at least one of M or N is bounded, then the above sum is guaranteed to be finite.

The above definitions can be suitably modified if one would like to work over a *differential* graded algebra instead of merely a graded algebra; see [7, Section 2] or [6, Section 2.1].

2.2 Bordered Heegaard Floer homology

We assume the reader is familiar with Heegaard Floer homology for closed 3-manifolds, and with the filtration induced on this invariant by a knot K in the 3-manifold. See, for example, the expository overview [14]. We begin with an overview of the invariants associated to 3-manifolds with parameterized boundary, as defined by Lipshitz, Ozsváth and Thurston in [7]. Let Y be a closed 3-manifold. Decompose Y along a closed surface F into pieces Y_1 and Y_2 such that $\partial Y_1 = -\partial Y_2 = F$. In particular, this gives us an orientation preserving diffeomorphism from F to ∂Y_1, and an orientation reversing diffeomorphism from F to ∂Y_2. A 3-manifold with a diffeomorphism (up to isotopy) from a standard surface to its boundary is called a *bordered* 3-*manifold*, and we call this isotopy class of diffeomorphisms a *marking* of the boundary. To the closed surface F, we associate a differential graded algebra $\mathcal{A}(F)$. To Y_1, we associate the invariant,

$\widehat{CFA}(Y_1)$, which will be a right $\mathcal{A}_\infty$-module over the algebra $\mathcal{A}(F)$, while to Y_2, we associate the invariant, $\widehat{CFD}(Y_2)$, which will be a Type D structure. To a knot K in Y_1, we may associate either $\widehat{CFA}(Y_1, K)$, a filtered $\mathcal{A}_\infty$-module, or $CFA^-(Y_1, K)$, an $\mathcal{A}_\infty$-module over the ground ring $\mathbb{F}[U]$.

The pairing theorems of [7, Theorems 1.3 and 10.12] state that there exists a quasi-isomorphism between $\widehat{CF}(Y)$ and the box tensor product of $\widehat{CFA}(Y_1)$ and $\widehat{CFD}(Y_2)$:

$$\widehat{CF}(Y) \simeq \widehat{CFA}(Y_1) \boxtimes \widehat{CFD}(Y_2).$$

We may also consider the case where we have a knot $K_1 \subset Y_1$, in which case we have the following quasi-isomorphism of $\mathbb{Z}$-filtered chain complexes:

$$\widehat{CFK}(Y, K) \simeq \widehat{CFA}(Y_1, K_1) \boxtimes \widehat{CFD}(Y_2),$$

and the following quasi-isomorphism of $\mathbb{F}[U]$-modules:

$$CFK^-(Y, K) \simeq CFA^-(Y_1, K_1) \boxtimes \widehat{CFD}(Y_2).$$

Note that the information contained in the $\mathbb{Z}$-filtered chain complex $\widehat{CFK}(Y, K)$ is equivalent to that in the $\mathbb{F}[U]$-module $CFK^-(Y, K)$. Similar pairing theorems hold when we have a knot $K_2 \subset Y_2$.

In this thesis, we will use these tools to study cabling. Thus, we will restrict ourselves to the case where F is a torus. To use the bordered Heegaard Floer package to study the $(p, pn+1)$-cable of a knot K, we will let Y_1 be a solid torus equipped with a $(p, 1)$-torus knot, and let Y_2 be the knot complement $S^3 - \text{nbd } K$ with framing n; that is, the marking specifies a meridian of the knot and a n-framed longitude.

We will now describe the algebra $\mathcal{A}(F)$, the modules $\widehat{CFA}(Y_1)$ and $\widehat{CFD}(Y_2)$, and the box tensor product, all in the case of $F = T^2$. When F is a torus, $\mathcal{A}(F)$ is merely a graded algebra, while when $g(F) \geq 2$, it is a *differential* graded algebra. At the end of this section, we note the modifications needed in the more general case.

To specify the identification of T^2 with ∂Y_1 and $-\partial Y_2$, we need to identify a meridian and a longitude of the torus. One way to do this is to specify a handle-decomposition for the surface; that is, a disk with two 1-handles attached such that the resulting boundary is connected and can be capped off with a disk. For technical reasons, we also place a basepoint somewhere along the boundary of the disk.

Schematically, we can represent this information by a *pointed matched circle*, which we think of as the boundary of the disk with markings at the feet of the 1-handles. In this case, the pointed matched circle $\mathcal{Z}$ consists of a circle with five marked points: a_1, a_2, a_3, a_4, and z, in that order as we traverse the circle in the clockwise direction, where the arc α_1^a has endpoints at a_1 and a_3, and the arc α_2^a has endpoints at a_2 and a_4.

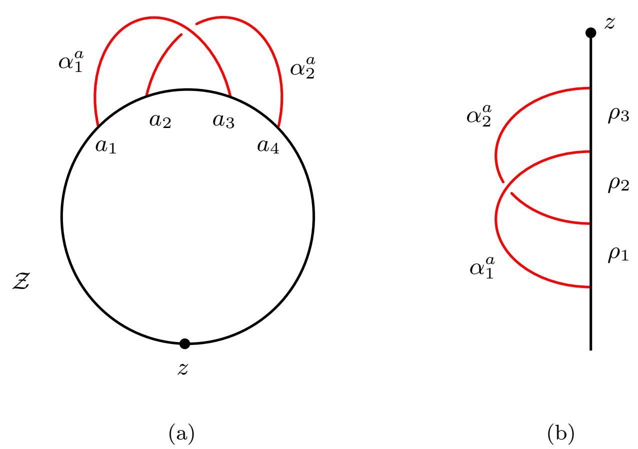


Figure 2.1: Above left, the pointed matched circle for the surface T^2. Above right, the same pointed matched circle cut open at z.

To the surface T^2 parametrized by the pointed matched circle $\mathcal{Z}$, we associate a graded algebra, $\mathcal{A}(T^2)$. The algebra $\mathcal{A}(T^2)$ is generated over $\mathbb{F}$ by the two idempotents

$$\iota_1 \quad \text{and} \quad \iota_2,$$

and the six "Reeb" elements

$$\rho_1,\ \rho_2,\ \rho_3,\ \rho_{12},\ \rho_{23},\ \rho_{123}.$$

These algebra elements may be understood pictorially, as in Figure 2.2, where multiplication is understood to correspond to concatenation.

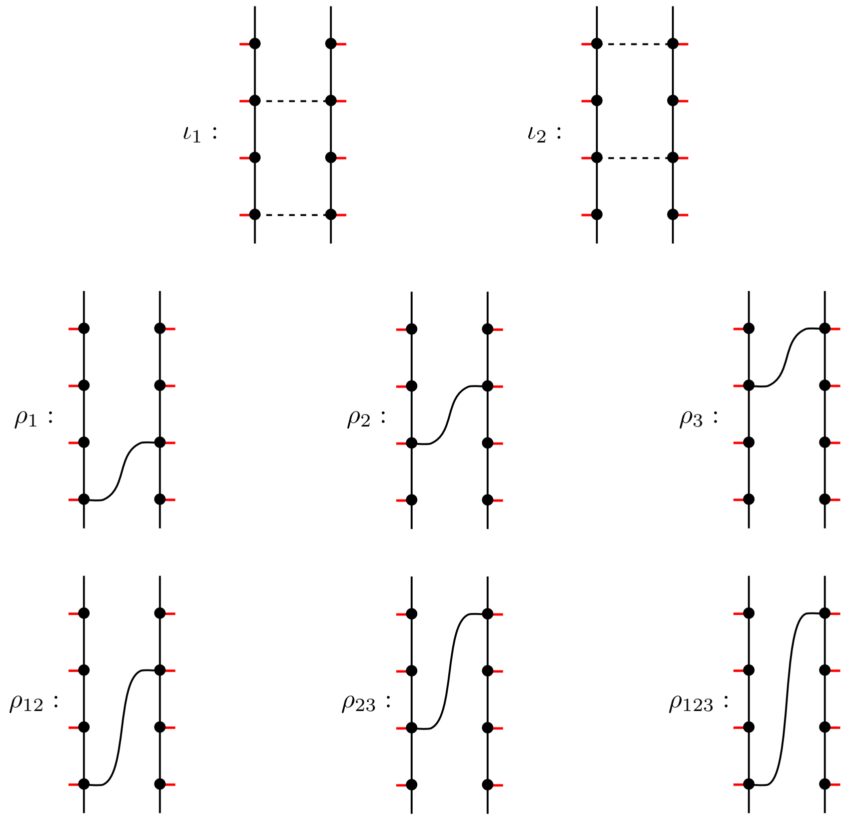


Figure 2.2: The idempotents and algebra elements.

The idempotents correspond to α_1^a and α_2^a, respectively. We will often need to consider the *ring of idempotents*,

$$\mathcal{I} = \mathbb{F}\langle\iota_1\rangle \oplus \mathbb{F}\langle\iota_2\rangle.$$

We have the following compatibility conditions with the idempotents:

$$\rho_1 = \iota_1\rho_1 = \rho_1\iota_2 \qquad \rho_2 = \iota_2\rho_2 = \rho_2\iota_1 \qquad \rho_3 = \iota_1\rho_3 = \rho_3\iota_2$$

$$\rho_{12} = \iota_1\rho_{12} = \rho_{12}\iota_1 \qquad \rho_{23} = \iota_2\rho_{23} = \rho_{23}\iota_2 \qquad \rho_{123} = \iota_1\rho_{123} = \rho_{123}\iota_2,$$

and the following non-zero products:

$$\rho_1\rho_2 = \rho_{12} \qquad \rho_2\rho_3 = \rho_{23} \qquad \rho_1\rho_{23} = \rho_{12}\rho_3 = \rho_{123}.$$

We will let ρ_1 refer to the arc of $\mathcal{Z} - \{z\}$ between a_1 and a_2, ρ_2 the arc between a_2 and a_3, and ρ_3 the arc between a_3 and a_4. Similarly, ρ_{12}, ρ_{23} and ρ_{123} will refer to the appropriate concatenations.

This completes the description of the algebra $\mathcal{A}(T^2)$. See Section 3 of [7] for the full description of the algebra in the general case.

A *bordered Heegaard diagram* for a 3-manifold Y with $\partial Y = T^2$ is a tuple $(\overline{\Sigma}, \boldsymbol{\alpha}^c, \boldsymbol{\alpha}^a, \boldsymbol{\beta}, z)$ consisting of the following:

- a compact, oriented surface $\overline{\Sigma}$ of genus g with a single boundary, $\partial\overline{\Sigma}$
- a $(g-1)$-tuple of pairwise disjoint circles $\boldsymbol{\alpha}^c = (\alpha_1^c, \dots, \alpha_{g-1}^c)$ in the interior of $\overline{\Sigma}$
- a pair of disjoint arcs $\boldsymbol{\alpha}^a = (\alpha_1^a, \alpha_2^a)$ in $\overline{\Sigma}\backslash\boldsymbol{\alpha}^c$ with endpoints on $\partial\overline{\Sigma}$
- a g-tuple of pairwise disjoint circles $\boldsymbol{\beta} = (\beta_1, \dots, \beta_g)$ in the interior of $\overline{\Sigma}$
- a basepoint z on $\mathcal{Z}\backslash\partial\boldsymbol{\alpha}^a$

Let $\boldsymbol{\alpha}$ denote $\boldsymbol{\alpha}^c \cup \boldsymbol{\alpha}^a$. We further require that all intersections be transverse and that $\overline{\Sigma}\backslash\boldsymbol{\alpha}$ and $\overline{\Sigma}\backslash\boldsymbol{\beta}$ are connected. The data $\{\partial\overline{\Sigma},\ z,\ \partial\alpha_1^a,\ \partial\alpha_2^a\}$ describes a pointed matched circle.

For bordered Heegaard diagrams, there are two types of periodic domains, and hence two notions of admissibility. Consider closed domains $\mathcal{P}$ in $\overline{\Sigma}$ whose interiors consist of linear combinations of connected components in $\overline{\Sigma}\backslash(\boldsymbol{\alpha}^a, \boldsymbol{\alpha}^c, \boldsymbol{\beta})$. We call $\mathcal{P}$ a *periodic domain* if $\partial\mathcal{P}$ consists of a collection of α-arcs, α-circles, β-circles, and arcs in $\partial\overline{\Sigma}$, with $z \notin \partial\mathcal{P}$. We call $\mathcal{P}$ a *provincial periodic domain* if $\partial\mathcal{P}$ consists of a collection of full α-circles and β-circles, with $z \notin \partial\mathcal{P}$. Notice that this implies that $\mathcal{P}$ is not adjacent to $\partial\overline{\Sigma}$.

We say a bordered Heegaard diagram is *provincially admissible* if every provincial periodic domain has both positive and negative multiplicities. A bordered Heegaard diagram is *admissible* if every periodic domain has both positive and negative multiplicities. Provincial admissibility is sufficient for the bordered invariants to be well-defined, and admissibility is sufficient for the bordered invariants to be bounded. We will return to this point in more detail later in this section.

To construct a 3-manifold with parameterized boundary, Y, from the data $(\overline{\Sigma}, \boldsymbol{\alpha}^c, \boldsymbol{\alpha}^a, \boldsymbol{\beta}, z)$, we attach 2-handles to $\overline{\Sigma} \times [0,1]$ along $\boldsymbol{\alpha}^c \times \{0\}$ and $\boldsymbol{\beta} \times \{1\}$. The parametrization of the boundary of Y is given by the identification of $\{\partial\overline{\Sigma},\ z,\ \partial\alpha_1^a,\ \partial\alpha_2^a\} \times \{0\}$ with the pointed matched circle $\mathcal{Z}$.

Let $\mathcal{H}$ be a bordered Heegaard diagram for Y. We will now describe the invariants $\widehat{CFD}(\mathcal{H})$ and $\widehat{CFA}(\mathcal{H})$. The set of generators, $\mathfrak{S}(\mathcal{H})$, are unordered g-tuples of intersection points of α- and β-curves such that

- each β-circle is occupied exactly once
- each α-circle is occupied exactly once
- each α-arc is occupied at most once.

In the case we are considering, where $\partial Y = T^2$, notice that these conditions imply that exactly one of the α-arcs is occupied.

We now define the Type D structure $\widehat{CFD}(\mathcal{H})$. We identify $\{\partial\overline{\Sigma},\ z,\ \partial\alpha_1^a,\ \partial\alpha_2^a\}$ with $-\mathcal{Z}$. Let $\widehat{CFD}(\mathcal{H})$, or simply $\widehat{CFD}$, denote the $\mathbb{F}$-vector space generated by $\mathfrak{S}(\mathcal{H})$, with the left $\mathcal{I}$-action on $\mathbf{x} \in \mathfrak{S}(\mathcal{H})$ defined to be

$$\iota_1\mathbf{x} = \begin{cases} \mathbf{x} & \text{if } \mathbf{x} \text{ does } \textit{not} \text{ occupy the arc } \alpha_1^a \\ 0 & \text{otherwise} \end{cases}$$

$$\iota_2\mathbf{x} = \begin{cases} \mathbf{x} & \text{if } \mathbf{x} \text{ does } \textit{not} \text{ occupy the arc } \alpha_2^a \\ 0 & \text{otherwise.} \end{cases}$$

We will define maps

$$\delta_1 : \widehat{CFD} \to \mathcal{A} \otimes \widehat{CFD}$$

by counting certain pseudo-holomorphic curves. Note that the above tensor product is over the ring of idempotents, $\mathcal{I}$, as are the rest of the tensor products in this section. Let Σ denote Int $\overline{\Sigma}$. Define a *decorated source* $S^{\triangleright}$ to be a topological type of smooth surface S with boundary and a finite number of boundary punctures endowed with

- a labeling of each puncture by one of $-$, $+$, or e
- a labeling of each e puncture of S by a Reeb chord ρ.

Consider the 4-manifold $\Sigma \times [0,1] \times \mathbb{R}$, with the following projection maps:

$$\pi_\Sigma : \Sigma \times [0,1] \times \mathbb{R} \to \Sigma$$

$$\pi_D : \Sigma \times [0,1] \times \mathbb{R} \to [0,1] \times \mathbb{R}$$

$$\pi_I : \Sigma \times [0,1] \times \mathbb{R} \to [0,1]$$

$$\pi_\mathbb{R} : \Sigma \times [0,1] \times \mathbb{R} \to \mathbb{R}.$$

Let $\Sigma_{\overline{e}}$ denote Σ with its puncture filled in. Similarly, let $S_{\overline{e}}$ denote S with its e punctures filled in.

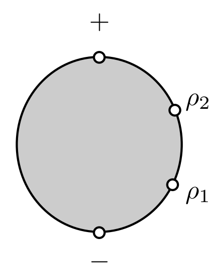


Figure 2.3: A decorated source, $S^\triangleright$.

We are interested in proper maps

$$u : (S, \partial S) \to \big(\Sigma \times [0,1] \times \mathbb{R}, (\boldsymbol{\alpha} \times \{1\} \times \mathbb{R}) \cup (\boldsymbol{\beta} \times \{0\} \times \mathbb{R})\big)$$

such that

- The map $\pi_D \circ u$ is a g-fold branched cover.
- At each $-$-puncture q of S, $\lim_{z \to q} (\pi_\mathbb{R} \circ u)(z) = -\infty$.
- At each $+$-puncture q of S, $\lim_{z \to q} (\pi_\mathbb{R} \circ u)(z) = +\infty$.
- At each e puncture q of S, $\lim_{z \to q} (\pi_\Sigma \circ u)(z)$ is the Reeb chord ρ labeling q.

- $\pi_\Sigma \circ u$ does not cover the region of Σ adjacent to z.

- The map u extends to a proper map $u_{\overline{e}} : S_{\overline{e}} \to \Sigma_{\overline{e}} \times [0,1] \times \mathbb{R}$.

- For each $t \in \mathbb{R}$ and each $i = 1, \ldots, g$, there is exactly one point in $u^{-1}(\beta_i \times \{0\} \times \{t\})$. Similarly, for each $t \in \mathbb{R}$ and each $i = 1, \ldots, g-1$, there is exactly one point in $u^{-1}(\alpha_i^c \times \{0\} \times \{t\})$.

We also require the map u to be J-holomorphic and of finite energy in the appropriate sense. See Section 5 of [7].

The map $\pi_{\mathbb{R}} \circ u_{\overline{e}}$ gives an ordering on the e punctures, and their respective labels; this is induced by the $\mathbb{R}$-coordinate of their images. We denote the resulting sequence of Reeb chords by $\overrightarrow{\boldsymbol{\rho}} = (\rho_{i_1}, \ldots, \rho_{i_n})$.

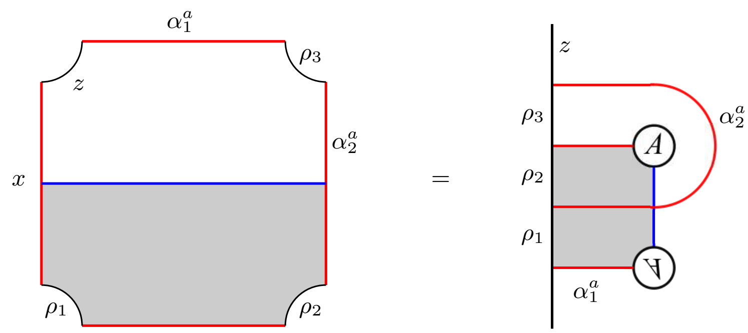
We let $\big(\mathcal{M}^B(\mathbf{x}, \mathbf{y}, \overrightarrow{\boldsymbol{\rho}})\big)$ denote a certain reduced moduli space. Roughly, this moduli space consists of curves from a decorated source $S^{\triangleright}$ with asymptotics corresponding to $\overrightarrow{\boldsymbol{\rho}}$ and in the homology class $B \in \pi_2(\mathbf{x}, \mathbf{y})$, where $\pi_2(\mathbf{x}, \mathbf{y})$ is the set of homology classes of curves connecting $\mathbf{x}$ to $\mathbf{y}$. The index $\operatorname{ind}(B, \overrightarrow{\boldsymbol{\rho}})$ is equal to the expected dimension of $\big(\mathcal{M}^B(\mathbf{x}, \mathbf{y}, \overrightarrow{\boldsymbol{\rho}})\big)$ plus one.

The map δ_1 is defined as:

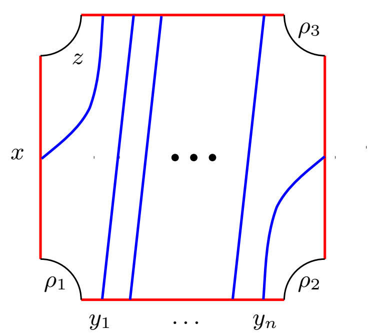
$$\delta_1(\mathbf{x}) := \sum_{\mathbf{y} \in \mathfrak{S}(\mathcal{H})} \sum_{\substack{B \in \pi_2(\mathbf{x}, \mathbf{y}) \\ \{\overrightarrow{\boldsymbol{\rho}} \mid \operatorname{ind}(B, \overrightarrow{\boldsymbol{\rho}}) = 1\}}} \#\big(\mathcal{M}^B(\mathbf{x}, \mathbf{y}, \overrightarrow{\boldsymbol{\rho}})\big) \rho_{i_1} \cdot \ldots \cdot \rho_{i_n} \otimes \mathbf{y},$$

where $\#\big(\mathcal{M}^B(\mathbf{x}, \mathbf{y}, \overrightarrow{\boldsymbol{\rho}})\big)$ is the number of points, modulo two, in the zero-dimensional moduli space $\mathcal{M}^B(\mathbf{x}, \mathbf{y}, \overrightarrow{\boldsymbol{\rho}})$. Provincial admissibility implies that the sum is well-defined.

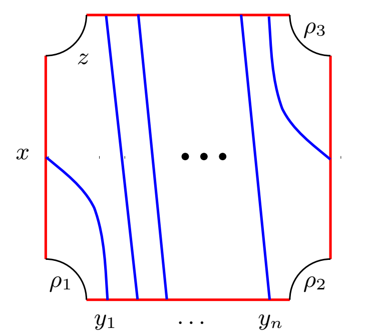
As in the theory for closed 3-manifolds, there is a combinatorial formula to compute the index of a map, in terms of the Euler measure and local multiplicities of the associated domain on $\overline{\Sigma}$ (that is, the image of $\pi_\Sigma \circ u$), and the manner in which the domain abuts $\partial\overline{\Sigma}$. See Definition 5.46 of [7].



(a) Two different ways of viewing the Heegaard diagram $\mathcal{H}_0$. The shaded region contributes to the map $\delta_1(x) = \rho_{12}x$.



(b) The Heegaard diagram $\mathcal{H}_n$.



(c) The Heegaard diagram $\mathcal{H}_{-n}$.

Figure 2.4: Three bordered Heegaard diagrams for the solid torus, labeled to compute $\widehat{CFD}$, with different parametrizations of the boundary. In 2.4(a), we show two equivalent ways of drawing the Heegaard surface: first, as a square with opposite sides identified, and second, as a disk with a 1-handle attached.

Example 2.1. *The invariant* $\widehat{CFD}(\mathcal{H}_0)$ *associated to the bordered Heegaard diagram in Figure 2.4(a) has a single generator* x *in the idempotent* ι_1 *and the map*

$$\delta_1 : x \mapsto \rho_{12}x.$$

Example 2.2. *The invariant* $\widehat{CFD}(\mathcal{H}_n)$ *associated to the bordered Heegaard diagram in Figure 2.4(b) has one generator* x *in the idempotent* ι_1*,* n *generators* $y_1, \ldots, y_n$ *in the idempotent* ι_2 *and* *the maps*

$$\delta_1 : x \mapsto \rho_{123} y_1$$

$$\delta_1 : y_i \mapsto \rho_{23} y_{i+1}, \quad 1 \le i \le n-1$$

$$\delta_1 : y_n \mapsto \rho_2 x.$$

Example 2.3. *The invariant $\widehat{CFD}(\mathcal{H}_{-n})$ associated to the bordered Heegaard diagram in Figure 2.4(b) has one generator x in the idempotent ι_1, n generators $y_1, \ldots, y_n$ in the idempotent ι_2 and the maps*

$$\delta_1 : x \mapsto \rho_1 y_1$$

$$\delta_1 : y_i \mapsto \rho_{23} y_{i-1}, \quad 2 \le i \le n$$

$$\delta_1 : x \mapsto \rho_3 y_n.$$

To complete the definition of the Type D structure on $\widehat{CFD}$, we define the maps

$$\delta_k : \widehat{CFD} \to (\mathcal{A}^{\otimes k} \otimes \widehat{CFD})$$

inductively by

$$\delta_0 = \mathbb{I}_{\widehat{CFD}}$$

$$\delta_i = (\mathbb{I}_{\mathcal{A}}{}^{\otimes i-1} \otimes \delta_1) \circ \delta_{i-1}.$$

Recall that $\widehat{CFD}$ is *bounded* if there exists an integer N such that $\delta_i = 0$ for all $i > N$. Lemma 6.5 of [7] tells us that if $\mathcal{H}$ is admissible, then $\widehat{CFD}(\mathcal{H})$ is bounded.

We now define $\widehat{CFA}$. We identify $\{\partial\overline{\Sigma},\ z,\ \partial\alpha_1^a,\ \partial\alpha_2^a\}$ with $\mathcal{Z}$. As an $\mathbb{F}$-vector space, $\widehat{CFA}$ is generated by $\mathfrak{S}(\mathcal{H})$, with the right $\mathcal{I}$-action defined to be

$$\mathbf{x}\iota_1 = \begin{cases} \mathbf{x} & \text{if } \mathbf{x} \text{ occupies the arc } \alpha_1^a \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbf{x}\iota_2 = \begin{cases} \mathbf{x} & \text{if } \mathbf{x} \text{ occupies the arc } \alpha_2^a \\ 0 & \text{otherwise.} \end{cases}$$

The $\mathcal{A}_\infty$-structure on $\widehat{CFA}$ is defined by counting certain pseudoholomorphic curves, giving maps

$$m_{j+1} : \widehat{CFA} \otimes \mathcal{A}^{\otimes j} \to \widehat{CFA},$$

defined to be

$$m_{j+1}(\mathbf{x}, \rho_{i_1}, \ldots, \rho_{i_j}) = \sum_{\mathbf{y} \in \mathfrak{S}(\mathcal{H})} \sum_{\substack{B \in \pi_2(\mathbf{x},\mathbf{y}) \\ \mathrm{ind}(B, \overrightarrow{\boldsymbol{\rho}})=1}} \#(\mathcal{M}^B(\mathbf{x}, \mathbf{y}, \overrightarrow{\boldsymbol{\rho}}))\mathbf{y}$$

$$m_2(\mathbf{x}, 1) = \mathbf{x}$$

$$m_{j+1}(\mathbf{x}, \ldots, 1, \ldots) = 0, \quad j > 1.$$

As in the case of $\widehat{CFD}$, provincial admissibility of the Heegaard diagram guarantees that the above sum is well-defined. Recall that $\widehat{CFA}$ is *bounded* if there exists an integer N such that $m_j = 0$ for all $j > N$. If $\mathcal{H}$ is admissible, then $\widehat{CFA}$ is bounded.

Example 2.4. *The invariant $\widehat{CFA}(\mathcal{H})$ associated to the bordered Heegaard diagram in Figure 2.5(a) has a single generator b in the idempotent ι_2, with the algebra relations*

$$m_{3+i}(b, \rho_2, \overbrace{\rho_{12}, \ldots, \rho_{12}}^{i}, \rho_1) = b, \quad i \geq 0.$$

Example 2.5. *The invariant $\widehat{CFA}(\mathcal{H}')$ associated to the bordered Heegaard diagram in Figure 2.5(b) has two generators a_1 and a_2 in the idempotent ι_1 and a single generator b in the idempotent ι_2, with the algebra relations*

$$m_1(a_1) = a_2$$

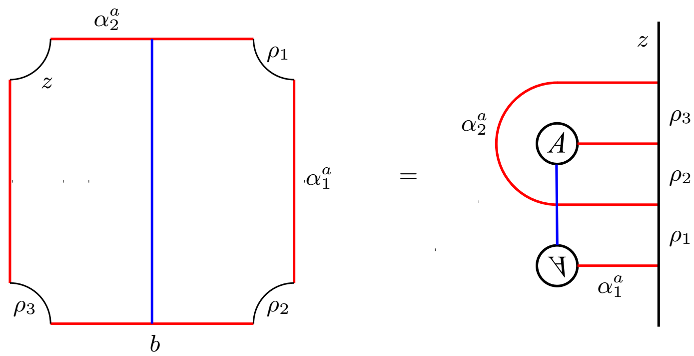
$$m_2(a_1, \rho_{12}) = a_2$$

$$m_2(a_1, \rho_1) = b$$

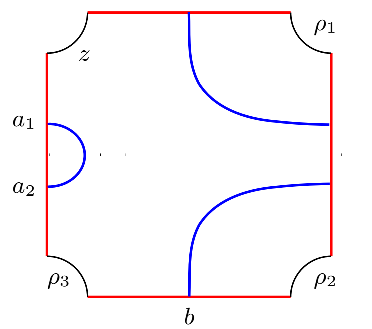
$$m_2(b, \rho_2) = a_2$$

The tensor product $\widehat{CFA} \boxtimes \widehat{CFD}$ is the $\mathbb{F}$-vector space

$$\widehat{CFA} \otimes_{\mathcal{I}} \widehat{CFD},$$



(a) The Heegaard diagram $\mathcal{H}$.



(b) The admissible Heegaard diagram $\mathcal{H}'$.

Figure 2.5: A bordered Heegaard diagram for the framed solid torus, labeled to compute the invariant $\widehat{CFA}$. In 2.5(a), we show the Heegaard surface first as a square with opposite sides identified, and second as disk with a 1-handle attached. In 2.5(b), we have isotoped the β-circle so that the diagram is admissible.

equipped with the differential

$$\partial^{\boxtimes}(\mathbf{x} \otimes \mathbf{y}) = \sum_{k=0}^{\infty} (m_{k+1} \otimes \mathbb{I}_{\widehat{CFD}})(\mathbf{x} \otimes \delta_k(\mathbf{y})).$$

If at least one of $\widehat{CFA}$ and $\widehat{CFD}$ is bounded, then the above sum is guaranteed to be finite.

The following description of the tensor product in terms of a basis for $\mathcal{A}$ is often useful for calculations. Define $\rho_\emptyset$ to be $\iota_1 + \iota_2$. Then we can rewrite δ_1 as

$$\delta_1 = \sum_i \rho_i \otimes D_i,$$

where the sum is taken over $i \in \{\emptyset, 1, 2, 3, 12, 23, 123\}$, and the D_i are coefficient maps

$$D_i : \widehat{CFD} \to \widehat{CFD}.$$

The tensor product $\widehat{CFA} \boxtimes \widehat{CFD}$ is still the $\mathbb{F}$-vector space

$$\widehat{CFA} \otimes_{\mathcal{I}} \widehat{CFD},$$

with the differential now given by

$$\partial^{\boxtimes}(\mathbf{x} \otimes \mathbf{y}) = \sum m_{k+1}(\mathbf{x}, \rho_{i_1}, \ldots, \rho_{i_k}) D_{i_k} \circ \ldots \circ D_{i_1}(\mathbf{y}),$$

where the sum is taken over all k-element sequences $i_1, \ldots, i_k$ (including the empty sequence when $k = 0$) of elements in $\{\emptyset, 1, 2, 3, 12, 23, 123\}$.

Example 2.6. *Using Examples 2.2 and 2.5 above, we compute the tensor product* $\widehat{CFA}(\mathcal{H}') \boxtimes \widehat{CFD}(\mathcal{H}_n) \simeq \widehat{CF}(L_{n,1})$. *The generators in the tensor product are*

$$a_1x,\ a_2x,\ by_1,\ \ldots,\ by_n,$$

with the differential

$$\partial(a_1x) = a_2x.$$

Example 2.7. *Using Examples 2.1 and 2.5 above, we compute the tensor product* $\widehat{CFA}(\mathcal{H}') \boxtimes \widehat{CFD}(\mathcal{H}_0) \simeq \widehat{CF}(S^1 \times S^2)$. *Note that the only nontrivial coefficient map on* $\widehat{CFD}(\mathcal{H}_0)$ *is*

$$D_{12}(x) = x.$$

The generators in the tensor product are a_1x *and* a_2x, *with trivial differential since*

$$\partial(a_1x) = m_1(a_1) \otimes x + m_2(a_1, \rho_{12}) \otimes D_{12}(x) = a_2x + a_2x = 0 \quad \text{and} \quad \partial(a_2x) = 0.$$

We conclude this section by highlighting a few of the differences that occur when ∂Y has genus ≥ 2.

A bordered Heegaard diagram for a 3-manifold Y with parameterized boundary F of genus k consists of a punctured surface $\overline{\Sigma}$ of genus g, a g-tuple of β-circles, a $(g-k)$-tuple of α-circles, a $2k$-tuple of α-arcs, and a basepoint z on $\partial\overline{\Sigma}$, such that $\overline{\Sigma}\backslash\boldsymbol{\alpha}$ and $\overline{\Sigma}\backslash\boldsymbol{\beta}$ are connected (where $\boldsymbol{\alpha}$ denotes the collection of α-arcs and -circles, and $\boldsymbol{\beta}$ denotes the collection of β-circles). Notice that when $g(F) \geq 2$, there is a not a unique parametrization (i.e., handle-decomposition) of the diffeomorphism type of F, hence there is not a unique algebra associated to the diffeomorphism type of the surface F. In other words, for different pointed matched circle describing a surface of genus $g(F)$, we get a different algebra.

Furthermore, while in the case of torus boundary, $\mathcal{A}(T^2)$ is a simply a graded algebra, the algebra associated to a parameterized surface of genus 2 or higher is a differential graded algebra. See Section 3 of [7] for details.

In the case of torus boundary, each generator $\mathbf{x}$ occupies exactly one α-arc, hence for a map u, we have at most one Reeb chord occurring at any given time (i.e., the $\mathbb{R}$-coordinate of its image). Thus, the map u allows us to consider a sequence of Reeb chords. In the general case, it is possible for multiple Reeb chords to occur at the same time, so u induces a sequence of *sets* of Reeb chords instead. See Section 5 of [7] for a complete description, or Section 5 of Zarev [21] for a description of the analogous construction in the bordered sutured case.

2.3 The knot Floer complex

We now review some basic facts about the various flavors of the knot Floer complex, defined by Ozsváth and Szabó in [11] and independently by Rasmussen in [19]. For an expository overview of these invariants, we again refer the reader to [14]. We specify a knot $K \subset S^3$ by a doubly pointed Heegaard diagram, $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w, z)$, where w and z are each basepoints in the complement of the α- and β-circles. The chain complex $CFK^-(K)$ is freely generated over $\mathbb{F}[U]$ by the set of g-tuples of intersection points between the α- and β-circles, where each α- and each β-circle are used exactly once, and g is the genus of the surface Σ. The differential is defined as

$$\partial \mathbf{x} := \sum_{\mathbf{y} \in \mathfrak{S}(\mathcal{H})} \sum_{\substack{\phi \in \pi_2(\mathbf{x},\mathbf{y}) \\ \mathrm{ind}(\phi)=1}} \#\widehat{\mathcal{M}}(\phi)\ U^{n_w(\phi)} \cdot \mathbf{y}.$$

This complex has a homological $\mathbb{Z}$-grading, called the *Maslov grading* M, as well as a $\mathbb{Z}$-filtration, called the *Alexander filtration* A. The relative Maslov and Alexander gradings are defined as

$$M(\mathbf{x}) - M(\mathbf{y}) = \mathrm{ind}(\phi) - 2n_w(\phi) \qquad \text{and} \qquad A(\mathbf{x}) - A(\mathbf{y}) = n_z(\phi) - n_w(\phi),$$

for $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$. The differential, ∂, decreases the Maslov grading by one, and respects the Alexander filtration; that is,

$$M(\partial\mathbf{x}) = M(\mathbf{x}) - 1 \qquad \text{and} \qquad A(\partial\mathbf{x}) \leq A(\mathbf{x}).$$

Multiplication by U shifts the Maslov grading and respects the Alexander filtration as follows:

$$M(U \cdot \mathbf{x}) = M(\mathbf{x}) - 2 \qquad \text{and} \qquad A(U \cdot \mathbf{x}) = A(\mathbf{x}) - 1.$$

Setting $U = 0$, we obtain the filtered chain complex $\widehat{CFK}(K) = CFK^-(K)/(U = 0)$. The total homology of $\widehat{CFK}(K)$ is isomorphic to $\widehat{HF}(S^3) \cong \mathbb{F}$. The normalization for the Maslov grading is chosen so that the generator for $\widehat{HF}(S^3)$ lies in Maslov grading zero. We denote the homology of the associated graded object of $\widehat{CFK}(K)$ by

$$\widehat{HFK}(K) = \bigoplus_s \widehat{HFK}(K, s),$$

where s indicates the Alexander grading induced by the filtration. Similarly, we denote the homology of the associated graded object of $CFK^-(K)$ by

$$HFK^-(K) = \bigoplus_s HFK^-(K, s).$$

We normalize the Alexander grading so that

$$\min\{s \mid \widehat{HFK}(K,s) \neq 0\} = -\max\{s \mid \widehat{HFK}(K,s) \neq 0\}.$$

Equivalently, we can define the absolute Alexander grading of a generator $\mathbf{x}$ to be

$$A(\mathbf{x}) = \tfrac{1}{2}\langle c_1(\underline{\mathfrak{s}}(\mathbf{x})), [\widehat{F}]\rangle,$$

where $\widehat{F}$ is a Seifert surface for K capped off in the 0-surgery, $\underline{\mathfrak{s}}(\mathbf{x}) \in \mathrm{Spin}^c(S^3_0(K))$ denotes the Spin^c structure over $S^3_0(K)$ associated to the generator $\mathbf{x}$ by the basepoints w and z, and $c_1(\underline{\mathfrak{s}}(\mathbf{x}))$ is the relative first Chern class of $\underline{\mathfrak{s}}(\mathbf{x})$; see [11, Section 3].

At times, we will consider the closely related complex

$$CFK^\infty(K) := CFK^-(K) \otimes_{\mathbb{F}[U]} \mathbb{F}[U, U^{-1}],$$

which is naturally a $\mathbb{Z} \oplus \mathbb{Z}$-filtered chain complex, with one filtration induced by the Alexander filtration and the other by $-(U\text{-exponent})$. It is often convenient to view $CFK^\infty(K)$ and $CFK^-(K)$ graphically in the (i,j)-plane, suppressing the homological grading from the picture, where the i-coordinate corresponds to $-(U\text{-exponent})$, and the j-coordinate corresponds to the Alexander grading. An element of the form $U^i \cdot x$ is plotted at the coordinate $(-i, A(U^i \cdot x))$, or equivalently, $(-i, A(x) - i)$. In particular, the complex $CFK^-(K)$ is contained in the part of the (i,j)-plane with $i \leq 0$, and a generator x of $CFK^-(K)$ has coordinates $(0, A(x))$.

We may denote the differential by arrows which will necessarily point non-strictly downwards and to the left. We say that $(i',j') \leq (i,j)$ if $i' \leq i$ and $j' \leq j$. Given $S \subset \mathbb{Z} \oplus \mathbb{Z}$, we let $C\{S\}$ denote the set of elements in $CFK^\infty(K)$ whose (i,j)-coordinates are in S. If S has the property that $(i,j) \in S$ implies that $(i',j') \in S$ for all $(i',j') \leq (i,j)$, then $C\{S\}$ inherits the structure of a subcomplex. Similarly, for appropriate S, $C\{S\}$ may inherit the structure of a quotient complex, or of a subquotient complex. For example, $\widehat{CFK}(K)$ is the subquotient complex $C\{i = 0\}$.

The integer-valued smooth concordance invariant $\tau(K)$ is defined in [9] to be

$$\tau(K) := \min\{s \mid \iota : C\{i = 0, j \leq s\} \to C\{i = 0\} \text{ induces a non-trivial map on homology}\},$$

where ι is the natural inclusion of chain complexes. Alternatively, $\tau(K)$ may be defined in terms of the U-action on $HFK^-(K)$, as in [17, Appendix A]:

$$\tau(K) = -\max\{s \mid \exists\ [\mathbf{x}] \in HFK^-(K,s) \text{ such that } \forall\ d \geq 0,\ U^d[\mathbf{x}] \neq 0\}.$$

Recall that the complex $CFK^\infty(K)$ is doubly filtered, by the Alexander filtration, and by powers of U. Taking the degree 0 part of the associated graded object with respect to the Alexander filtration, we define the *horizontal complex*,

$$C^{\text{horz}} := C\{j = 0\},$$

equipped with a differential, ∂^{horz}. Graphically, this can be viewed as the subquotient complex of $CFK^\infty(K)$ consisting of elements with j-coordinate equal to zero, with the induced differential consisting of horizontal arrows pointing non-strictly to the left. The horizontal complex inherits the structure of a $\mathbb{Z}$-filtered chain complex, with the filtration induced by $-(U\text{-exponent})$. Similarly, we may consider the degree 0 part of the associated graded object with respect to the filtration by powers of U, and define the *vertical complex*,

$$C^{\text{vert}} := C\{i = 0\},$$

equipped with a differential, ∂^{vert}. Note that this is equivalent to $CFK^-(K)/(U \cdot CFK^-(K))$. In the vertical complex, the induced differential may be graphically depicted as vertical arrows pointing non-strictly downwards. The vertical complex inherits the structure of a $\mathbb{Z}$-filtered chain complex, with the filtration induced by the Alexander filtration.

Symmetry properties of $CFK^\infty(K)$ from [11, Section 3.5] show that both C^{horz} and C^{vert} are filtered chain homotopy equivalent to $\widehat{CFK}(K)$. (In fact, if we ignore grading and filtration shifts, any row or column is filtered chain homotopic to $\widehat{CFK}(K)$.) More generally, $CFK^\infty(K)$ is filtered chain homotopic to the complex obtained by reversing the roles of i and j. The filtered chain homotopy type of $\widehat{CFK}(K)$, $CFK^-(K)$ and $CFK^\infty(K)$ are all invariants of the knot K.

The chain complex $CFK^-(K)$ is called *reduced* if the differential ∂ strictly drops either the

Alexander filtration or the filtration by powers of U. Graphically, this means that each arrow points strictly downwards or to the left (or both). A filtered chain complex is always filtered chain homotopic to a reduced complex, i.e., it is filtered chain homotopic to the E_1 page of its associated spectral sequence.

A basis $\{x_i\}$ for a filtered chain complex (C, ∂) is called a *filtered basis* if the set $\{x_i \mid x_i \in C_S\}$ is a basis for C_S for all filtered subcomplexes $C_S \subset C$. Two filtered bases can be related by a filtered change of basis. For example, given a filtered basis $\{x_i\}$, replacing x_j with $x_j + x_k$, where the filtration level of x_k is less than or equal to that of x_j, is a filtered change of basis. More generally, we may consider a doubly filtered chain complex with two doubly filtered bases, related by a doubly filtered change of basis.

We say a filtered basis $\{x_i\}$ over $\mathbb{F}[U]$ for the reduced complex $CFK^-(K)$ is *vertically simplified* if for each basis element x_i, exactly one of the following holds:

- x_i is in the image of ∂^{vert} and there exists a unique basis element x_{i-1} such that $\partial^{\text{vert}} x_{i-1} = x_i$.
- x_i is in the kernel, but not the image, of ∂^{vert}.
- x_i is not in the kernel of ∂^{vert}, and $\partial^{\text{vert}} x_i = x_{i+1}$.

(In the statements above, we are considering the basis that $\{x_i\}$ naturally induces on C^{vert}; that is, $\{x_i \bmod \big(U \cdot CFK^-(K)\big)\}$. For ease of exposition, we suppress this from the notation.) When $\partial^{\text{vert}} x_i = x_{i+1}$, we say that there is a vertical arrow from x_i to x_{i+1}, and the length of this arrow is $A(x_i) - A(x_{i+1})$. Notice that upon taking homology, the differential ∂^{vert} cancels basis elements in pairs. Since $H_*(C^{\text{vert}}) \cong \mathbb{F}$, there is a distinguished element, which after reordering we denote x_0, with the property that it has no incoming or outgoing vertical arrows.

Similarly, we define what it means for a filtered basis $\{x_i\}$ over $\mathbb{F}[U]$ for the reduced complex $CFK^-(K)$ to be horizontally simplified. Notice that $\{U^{m_i} x_i\}$, where $m_i = A(x_i)$, naturally induces a basis on C^{horz}. We say the basis $\{x_i\}$ is *horizontally simplified* if for each basis element

x_i, exactly one of the following holds:

- $U^{m_i}x_i$ is in the image of ∂^{horz} and there exists a unique basis element x_{i-1} such that $\partial^{\text{horz}}U^{m_{i-1}}x_{i-1} = U^{m_i}x_i$.
- $U^{m_i}x_i$ is in the kernel, but not the image, of ∂^{horz}.
- $U^{m_i}x_i$ is not in the kernel of ∂^{horz}, and $\partial^{\text{horz}}U^{m_i}x_i = U^{m_{i+1}}x_{i+1}$.

When $\partial^{\text{horz}}U^{m_i}x_i = U^{m_{i+1}}x_{i+1}$, we say that there is a horizontal arrow from x_i to x_{i+1}, and the length of this arrow is $A(x_i) - A(x_{i+1})$. Notice that upon taking homology, the differential ∂^{horz} cancels basis elements in pairs. Since $H_*(C^{\text{horz}}) \cong \mathbb{F}$, there is a distinguished element, which after reordering we denote x_0, with the property that it has no incoming or outgoing horizontal arrows.

The following technical fact, proven at the end of this section, will be of use to us:

Lemma 2.8. *$CFK^-(K)$ is $\mathbb{Z} \oplus \mathbb{Z}$-filtered, $\mathbb{Z}$-graded homotopy equivalent to a chain complex C that is reduced. Moreover, one can find a vertically simplified basis over $\mathbb{F}[U]$ for C, or, if one would rather, a horizontally simplified basis over $\mathbb{F}[U]$ for C.*

2.4 From the knot Floer complex to the bordered invariant

Theorems 10.17 and 11.7 of [7] give an algorithm for computing $\widehat{CFD}(Y)$ for a framed knot complement $Y = S^3 - \text{nbd } K$ from $CFK^-(K)$. More precisely, we frame the knot complement by letting α_1^a correspond to an n-framed longitude, and α_2^a to a meridian. We recount the algorithm from CFK^- to $\widehat{CFD}$ here.

Let $\{x_i\}$ be a vertically simplified basis. Then $\{x_i\}$ is a basis for $\iota_1\widehat{CFD}(Y)$. To each arrow of length ℓ from x_i to x_{i+1} we introduce a string of basis elements $y_1^i, \ldots, y_\ell^i$ for $\iota_2\widehat{CFD}(Y)$ and differentials

$$x_i \xrightarrow{D_1} y_1^i \xleftarrow{D_{23}} \ldots \xleftarrow{D_{23}} y_k^i \xleftarrow{D_{23}} y_{k+1}^i \xleftarrow{D_{23}} \ldots \xleftarrow{D_{23}} y_\ell^i \xleftarrow{D_{123}} x_{i+1}.$$

Note the directions of the arrows. Similarly, let $\{x'_i\}$ be a horizontally simplified basis. Then $\{x'_i\}$ is also a basis for $\iota_1\widehat{CFD}(Y)$. To each arrow of length ℓ from x'_i to x'_{i+1} we introduce a string of basis elements $w^i_1, \dots, w^i_\ell$ for $\iota_2\widehat{CFD}(Y)$ and differentials

$$x'_i \xrightarrow{D_3} w^i_1 \xrightarrow{D_{23}} \dots \xrightarrow{D_{23}} w^i_k \xrightarrow{D_{23}} w^i_{k+1} \xrightarrow{D_{23}} \dots \xrightarrow{D_{23}} w^i_\ell \xrightarrow{D_2} x'_{i+1}.$$

Finally, there is the *unstable chain*, consisting of generators $z_1, \dots, z_m$ connecting x_0 and x'_0. The form of the unstable chain depends on the framing n relative to $2\tau(K)$. When $n < 2\tau(K)$, we introduce a string of basis elements $z_1, \dots, z_m$ for $\iota_2\widehat{CFD}(Y)$, where $m = 2\tau(K) - n$, and differentials

$$x_0 \xrightarrow{D_1} z_1 \xleftarrow{D_{23}} z_2 \xleftarrow{D_{23}} \dots \xleftarrow{D_{23}} z_m \xleftarrow{D_3} x'_0.$$

When $n = 2\tau(K)$, the unstable chain has the form

$$x_0 \xrightarrow{D_{12}} x'_0.$$

Lastly, when $n > 2\tau(K)$, the unstable chain has the form

$$x_0 \xrightarrow{D_{123}} z_1 \xrightarrow{D_{23}} z_2 \xrightarrow{D_{23}} \dots \xrightarrow{D_{23}} z_m \xrightarrow{D_2} x'_0,$$

where $m = n - 2\tau(K)$.

Remark 2.9. It is often possible to find a basis for CFK^- that is simultaneously vertically and horizontally simplified. It is an open question whether or not there always exists such a basis.

Remark 2.10. The examples in Figure 2.4 can be viewed as framed complements of the unknot, verifying the above algorithm in the special case where K is the unknot.

We conclude this section with the proof of Lemma 2.8.

Proof of Lemma 2.8. We need to show that we can find a basis over $\mathbb{F}[U]$ for $CFK^-(K)$ that is vertically simplified. What follows is essentially the well-known "cancellation lemma" for chain complexes in the filtered setting.

Let $\{x_i\}$ be a filtered basis (over $\mathbb{F}$) for $C^{\text{vert}} = C/U \cdot C$. For the remainder of the proof, we will let ∂ denote the differential on C^{vert}. Consider the set

$$B_n = \{x_i \mid A(\partial x_i) = A(x_i) - n\}.$$

We will prove the lemma by induction. Note that $B_{-1} = \emptyset$, since the differential ∂ respects the Alexander filtration. We say that B_n is *simplified with respect to the basis* $\{x_i\}$ if $\{B_n, \partial B_n\}$ is a direct summand of C^{vert} such that $\{x_i \mid x_i \in B_n\} \cup \{\partial x_i \mid x_i \in B_n\}$ form a simplified basis for $\{B_n, \partial B_n\}$.

Assume that $B_0, B_1 \ldots, B_{n-1}$ are simplified with respect to $\{x_i\}$. We will find a change of basis from $\{x_i\}$ to $\{x_i'\}$ so that B_n is simplified as well. If $\partial x_i = \sum c_j x_j$, then define

$$\partial_j x_i := c_j.$$

For $x_j \in B_n$, we would like to perform a change of basis such that x_j' and $\partial x_j'$ are elements in the new basis and form a direct summand. We begin by noticing that

$$x_j \in B_n \text{ implies } \exists\, k \text{ such that } \partial_k x_j = 1 \text{ and } A(x_k) = A(x_j) - n.$$

We now choose a new filtered basis $\{x_i'\}$ as follows:

$$\begin{aligned} x_j' &= x_j \\ x_k' &= \partial x_j \\ x_\ell' &= x_\ell + (\partial_k x_\ell) x_j, \quad \ell \neq j, k. \end{aligned}$$

This is a filtered change of basis. Indeed, we have that $A(\partial x_j) = A(x_k)$ by construction. Whenever $\partial_k x_\ell \neq 0$, we have that $A(x_\ell) \geq A(x_k) + n$, by the assumption that $B_0, \ldots, B_{n-1}$ are simplified with respect to $\{x_i\}$; then $A(x_\ell) \geq A(x_j)$, since $A(x_k) = A(x_j) - n$.

Now if $\partial x_i' = \sum c_j' x_j'$, then similarly define $\partial_j' x_i' = c_j'$, and we notice that

$$\begin{aligned} \partial_j' x_\ell' &= 0 \quad \forall\, \ell \\ \partial_k' x_\ell' &= 0 \quad \forall\, \ell \neq j. \end{aligned}$$

Indeed, that first equation follows from the facts that

$$\partial x_j' = x_k'$$

$$\partial_k' x_m' = 0 \ \forall \ m \neq j$$

$$\partial^2 = 0.$$

The second equation is true by construction of the basis $\{x_i'\}$. Notice that in this process, we have left $B_0, \ldots, B_{n-1}$ unchanged, and that we have not increased the size of B_n.

Now $\{x_j', x_k'\}$ splits as a direct summand as desired. Iterating this process, we can continue to change bases until B_n is simplified with respect to our new basis. By induction, we can construct a simplified basis for all of C^{vert}. This basis is also a basis for C.

Similarly, we may find a simplified basis for C^{horz}. This, too, will be a basis for C. $\square$

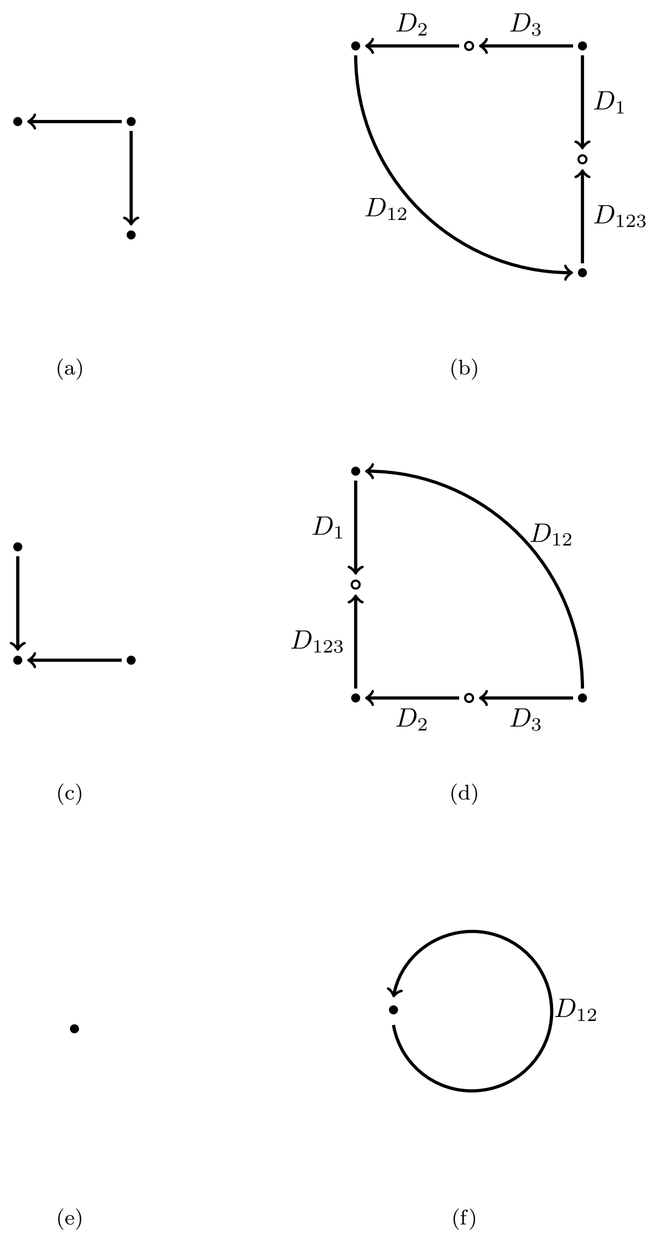


Figure 2.6: CFK^∞ and $\widehat{CFD}$ for different knots and their framed complements, respectively. (To be precise, $CFK^\infty(K)$ is the above diagram tensored with $\mathbb{F}[U, U^{-1}]$.) Top, K is the right-handed trefoil, and the framing on the complement is $2\tau(K) = 2$. Middle, K is the left-handed trefoil, and the framing on the complement is $2\tau(K) = -2$. Bottom, K is the unknot, and the framing is $2\tau(K) = 0$.

Chapter 3

Definition and properties of $\varepsilon(K)$

We will begin by defining the invariant $\varepsilon(K)$, in terms of $\tau(K)$ and the invariant $\nu(K)$, defined by Ozsváth and Szabó in [16, Definition 9.1]. Let (A_s, ∂_s) and (A'_s, ∂'_s) be the following sub-quotient complexes of $CFK^\infty(K)$:

$$A_s = C\{\max(i, j - s) = 0\}$$

$$A'_s = C\{\min(i, j - s) = 0\},$$

with induced differentials ∂_s and ∂'_s, respectively. We consider both A_s and A'_s as complexes over $\mathbb{F}$. Notice that if $\{x_i\}$ is a basis for $CFK^-(K)$, then $\{U^{n_i}x_i\}$ is a basis for A_s, where $n_i = \max(0, A(x_i) - s)$. Similarly, $\{U^{m_i}x_i\}$ will be a basis for A'_s, where $m_i = \min(0, A(x_i) - s)$. We have a map $v_s : A_s \to \widehat{CF}(S^3)$ defined as the following composition:

$$A_s \to C\{i = 0, j \leq s\} \to C\{i = 0\} \simeq \widehat{CF}(S^3)$$

where the first map is projection and the second is inclusion. Similarly, we have a map v'_s : $\widehat{CF}(S^3) \to A'_s$ defined as the following composition:

$$\widehat{CF}(S^3) \simeq C\{i = 0\} \to C\{i = 0, j \geq s\} \to A'_s$$

where the first map is projection and the second is inclusion. These definitions have geometric significance. Recall from [11, Section 4] that for $N \in \mathbb{N}$ sufficiently large,

$$A_s \simeq \widehat{CF}(S^3_N(K), [s])$$

where $S^3_N(K)$ denotes N-surgery along K, $|s| \leq N/2$, and $[s]$ denotes the Spin^c structure $\mathfrak{s}_s$ that extends over the corresponding 2-handle cobordism with the property that

$$\langle c_1(\mathfrak{s}_s), [\widehat{F}] \rangle + N = 2s,$$

where $\widehat{F}$ denotes the capped off Seifert surface in the 4-manifold. The map

$$v_s : \widehat{CF}(S^3_N(K), [s]) \to \widehat{CF}(S^3)$$

is induced by the cobordism from $S^3_N(K)$ to S^3 endowed with the Spin^c structure above. Similarly,

$$A'_s \simeq \widehat{CF}(S^3_{-N}(K), [s]),$$

and

$$v'_s : \widehat{CF}(S^3) \to \widehat{CF}(S^3_{-N}(K), [s])$$

corresponds to the 2-handle cobordism from S^3 to $S^3_{-N}(K)$.

Definition 3.1. *Define $\nu(K)$ by*

$$\nu(K) = \min\{s \in \mathbb{Z} \mid v_s : A_s \to \widehat{CF}(S^3) \text{ induces a non-trivial map on homology}\}.$$

Similarly, define $\nu'(K)$ by

$$\nu'(K) = \max\{s \in \mathbb{Z} \mid v'_s : \widehat{CF}(S^3) \to A'_s \text{ induces a non-trivial map on homology}\}.$$

For ease of notation, we will often write τ for $\tau(K)$ when the meaning is clear from context. Recall from [9, Proposition 3.1] that $\nu'(K)$ is equal to either $\tau - 1$ or τ. The idea is that if $s > \tau$, then v'_s induces the trivial map on homology, because quotienting by $C\{i = 0, j < s\}$ gives the zero map, and if $s < \tau$, then v'_s induces a non-trivial map on homology, because a generator x for $\widehat{HF}(S^3)$

must be supported in the (i, j)-coordinate $(0, \tau)$, thus x is not a boundary in A'_s. Thus, we should focus on the map v'_τ. In particular,

- $\nu'(K) = \tau - 1$ if and only if v'_τ is trivial on homology
- $\nu'(K) = \tau$ if and only if v'_τ is non-trivial on homology.

A similar argument shows that $\nu(K)$ is equal to either $\tau(K)$ or $\tau(K) + 1$; in particular,

- $\nu(K) = \tau$ if and only if v_τ is non-trivial on homology
- $\nu(K) = \tau + 1$ if and only if v_τ is trivial on homology.

We now proceed to show that v_τ and v'_τ cannot both be trivial on homology. Roughly, the idea is that v'_τ is trivial on homology when the class $[x]$ generating $\widehat{HF}(S^3) \cong H_*(C\{i = 0\})$ is in the image of the horizontal differential. Thus, $[x]$ must also be in the kernel of the horizontal differential, implying that v_τ is non-trivial. Similarly, v_τ is trivial on homology when $[x]$ is not in the kernel of the horizontal differential, hence $[x]$ is not in the image of the horizontal differential, implying that v'_τ is non-trivial.

The following lemma, and its proof, make this precise:

Lemma 3.2. *If $\nu'(K) = \tau(K) - 1$, then $\nu(K) = \tau(K)$, and there exists a horizontally simplified basis $\{x_i\}$ for $CFK^-(K)$ such that, after possible reordering, there is a pair of basis elements, x_1 and x_2, with the property that:*

1. *x_2 is the distinguished element of some vertically simplified basis.*
2. *$\partial^{\text{horz}} x_1 = x_2$.*

Similarly, if $\nu(K) = \tau(K) + 1$, then $\nu'(K) = \tau(K)$, and there exists a horizontally simplified basis $\{x_i\}$ for $CFK^-(K)$ such that, after possible reordering, there is a pair of basis elements, x_1 and x_2, with the property that:

1. *x_1 is the distinguished element of some vertically simplified basis.*

2. $\partial^{\text{horz}} x_1 = x_2$.

Proof. Let x_V be the distinguished element of a vertically simplified basis, and let ∂_s^{horz} be the differential on $C\{j = s\} \simeq C^{\text{horz}}$.

Since $\nu'(K) = \tau - 1$, for any element x that generates $H_*(C\{i = 0\})$, we have that $[x] = 0 \in H_*(A'_\tau)$. Thus, there exists $x' \in A'_\tau$ such that $\partial'_\tau x' = x_V$. Moreover, we may choose x' to be the element of minimal i-filtration such that $\partial'_\tau x' = x_V$; this will be convenient to us later. (Note that the complex A'_s inherits a natural i-filtration as a subquotient complex of CFK^∞.)

We can write x' as the sum of chains $x_{i=0}$ and $x_{i>0}$, where $x_{i=0} \in C\{i = 0, j > \tau\}$ and $x_{i>0} \in C\{i > 0, j = \tau\}$. Furthermore, $x_{i=0}$ can be taken to be a sum of vertically simplified basis elements that are not in the kernel of the vertical differential. Hence,

- $x_V + \partial^{\text{vert}} x_{i=0}$ generates $H_*(C\{i = 0\})$
- $\partial'_s x_{i>0} = x_V + \partial^{\text{vert}} x_{i=0}$.

We notice that $[\partial_\tau^{\text{horz}} x_{i>0}]$ is non-zero in both $H_*(A_\tau)$ and $H_*(C\{i = 0\})$. Therefore, $\nu(K) = \tau(K)$.

We now need to find an appropriate horizontally simplified basis. Replace x_V with $\partial_\tau^{\text{horz}} x_{i>0}$. This is a filtered change of basis and this new basis element is still the distinguished element of a vertically simplified basis, since elements in $\partial_\tau^{\text{horz}} x_{i>0} + x_V$ either have i-coordinate < 0, or have i-coordinate equal to zero and are in the image of ∂^{vert}.

Now apply the algorithm in Lemma 2.8, splitting off the arrow of length, say n, from $x_{i>0}$ to the new x_V first when simplifying B_n. This will yield a horizontally simplified basis with the desired property.

The proof in the case that $\nu(K) = \tau(K) + 1$ follows similarly. $\square$

When both v_τ and v'_τ are non-trivial on homology, the class $[x]$ generating $\widehat{HF}(S^3)$, which we identify with $H_*(C\{i = 0\})$, is in the kernel of the horizontal differential but not in the image. We make this more precise in the following lemma:

Lemma 3.3. *If $\nu(K) = \nu'(K)$, then there exists a vertically simplified basis $\{x_i\}$ for $CFK^-(K)$ such that the distinguished element, x_0, is also the distinguished element of a horizontally simplified basis for $CFK^-(K)$.*

Proof. Note that $\nu(K) = \nu'(K)$ implies that both are equal to $\tau(K)$. Let $\{x_i\}$ be a vertically simplified basis $\{x_i\}$ for $CFK^-(K)$ with distinguished element x_0. We have the following series of implications:

1. $\nu(K) = \tau(K)$ implies there exists a chain $x_H \in A_\tau$ such that $v_{\tau *}([x_H]) = [x_0]$.

2. Hence, $[x_H] \neq 0 \in H_*(A_\tau)$, and so $\partial_\tau x_H = 0$.

3. $\partial_\tau^{\text{horz}} x_H = 0$ as well, since $\partial_\tau^{\text{horz}} x_H = \partial_\tau x_H / C\{i = 0, j < \tau\}$.

Notice that since $v_{\tau *}([x_H]) = [x_0]$, it follows that x_H must be equal to x_0 plus possibly some basis elements in the image of ∂^{vert} and some elements in $C\{i < 0, j = \tau\}$. Hence, we may replace our distinguished vertical element x_0 with x_H.

With x_H is as above, $\nu'(K) = \tau(K)$ implies that $v'_{\tau *}([x_H]) \neq 0 \in H_*(A'_\tau)$. Thus, $x_H \notin \text{Im } \partial'_\tau$. Since $x_H \notin \text{Im } \partial'_\tau$, x_H is not homologous in $C\{j = \tau\} \simeq C^{\text{horz}}$ to anything of strictly lower filtration level. Moreover, $x_H \notin \text{Im } \partial_\tau^{\text{horz}}$. Therefore, there exists a horizontally simplified basis for $CFK^-(K)$ with distinguished element x_H, which is also the distinguished element of a vertically simplified basis. □

We see that there are three different possibilities for the values of the pair $(\nu(K), \nu'(K))$: $(\tau(K), \tau(K) - 1)$, $(\tau(K), \tau(K))$ or $(\tau(K) + 1, \tau(K))$. This motivates the following definition:

Definition 3.4. *Define $\varepsilon(K)$ to be*

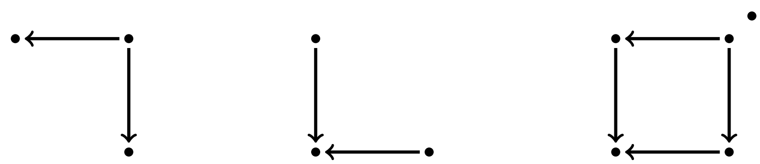
$$\varepsilon(K) := 2\tau(K) - \nu(K) - \nu'(K).$$

In particular, $\varepsilon(K)$ can take on the values -1, 0, or 1.

Remark 3.5. By various symmetry properties of $CFK^\infty(K)$ (see [11, Section 3.5]), we may equivalently define $\varepsilon(K)$ as

$$\varepsilon(K) = \big(\tau(K) - \nu(K)\big) - \big(\tau(\overline{K}) - \nu(\overline{K})\big),$$

where $\overline{K}$ denotes the mirror of K.



(a) $\varepsilon(K) = 1$ (b) $\varepsilon(K) = -1$ (c) $\varepsilon(K) = 0$

Figure 3.1: $CFK^\infty(K)$, for different knots K. Above left, K is the right-handed trefoil, which has $\varepsilon(K) = 1$, and the unique generator with no incoming or outgoing vertical arrows lies at the head of a horizontal arrow. Center, K is the left-handed trefoil, which has $\varepsilon(K) = -1$, and the unique generator with no incoming or outgoing vertical arrows lies at the tail of a horizontal arrow. Right, K is the figure 8 knot, which has $\varepsilon(K) = 0$, and the unique generator with no incoming or outgoing vertical arrows also has no incoming or outgoing horizontal arrows.

See Figure 3.1 for examples of knots K with different values of $\varepsilon(K)$. Recall that a knot K is called *homologically thin* if $\widehat{HFK}(K)$ is supported on a single diagonal in the plane whose axes correspond to Alexander and Maslov gradings of the groups.

Proposition 3.6. *The following are properties of $\varepsilon(K)$:*

1. *If K is smoothly slice, then $\varepsilon(K) = 0$.*

2. *If $\varepsilon(K) = 0$, then $\tau(K) = 0$.*

3. $\varepsilon(\overline{K}) = -\varepsilon(K)$.

4. If $|\tau(K)| = g(K)$, then $\varepsilon(K) = \operatorname{sgn} \tau(K)$.

5. If K is homologically thin, then $\varepsilon(K) = \operatorname{sgn} \tau(K)$.

6. (a) If $\varepsilon(K) = \varepsilon(K')$, then $\varepsilon(K\#K') = \varepsilon(K) = \varepsilon(K')$.

(b) If $\varepsilon(K) = 0$, then $\varepsilon(K\#K') = \varepsilon(K')$.

Proof of (1). If K is slice, then the surgery correction terms defined in [8] vanish (i.e, agree with those of the unknot), and the maps

$$\widehat{HF}(S^3_N(K), [0]) \to \widehat{HF}(S^3) \quad \text{and} \quad \widehat{HF}(S^3) \to \widehat{HF}(S^3_{-N}(K), [0])$$

are non-trivial. Indeed, the surgery corrections terms can be defined in terms of the maps

$$HF^+(S^3) \to HF^+(S^3_{-N}(K), [s])$$

and we have the commutative diagram

$$\begin{array}{ccc} \widehat{HF}(S^3) & \xrightarrow{v'_s} & \widehat{HF}(S^3_{-N}(K), [s]) \\ {\scriptstyle \iota}\big\downarrow & & {\scriptstyle \iota_{-N,s}}\big\downarrow \\ HF^+(S^3) & \xrightarrow{u'_s} & HF^+(S^3_{-N}(K), [s]). \end{array}$$

Let $N \gg 0$. If the surgery corrections terms vanish and $s = 0$, then u'_s is an injection and so the composition $\iota \circ u'_s$ is non-trivial. By commutativity of the diagram, it follows that v'_s must be non-trivial. A similar diagram in the case of large positive surgery shows that v_s must be non-trivial as well. Thus, $\nu(K)$ and $\nu'(K)$ both equal $\tau(K) = 0$, and so $\varepsilon(K) = 0$. $\square$

Proof of (2). If $\varepsilon(K) = 0$, then by Lemma 3.3, there exists an element x_0 that is the distinguished element of both a vertically and horizontally simplified basis. If $A(x_0)$ is the Alexander grading of x_0 viewed in $C^{\text{vert}} \simeq \widehat{CFK}(K)$, then $-A(x_0)$ is the Alexander grading of x_0 viewed in $C^{\text{horz}} \simeq \widehat{CFK}(K)$. This implies that $\tau(K) = -\tau(K)$, hence $\tau(K) = 0$. $\square$

Proof of (3). The symmetry properties of $CFK^\infty(K)$ imply that $\tau(\overline{K}) = -\tau(K)$ and $\nu(\overline{K}) = -\nu'(K)$. Hence, $\varepsilon(\overline{K}) = -\varepsilon(K)$. $\square$

Proof of (4). Without loss of generality, we can consider the case $\tau(K) > 0$. By the adjunction inequality for knot Floer homology [11, Theorem 5.1],

$$H_*(C\{i < 0, j = g(K)\}) = 0.$$

Hence by considering the short exact sequence

$$0 \to C\{i < 0, j = g(K)\} \to A_{g(K)} \to C\{i = 0, j \leq g(K)\} \to 0$$

and the fact that the inclusion $C\{i = 0, j \leq g(K)\} \hookrightarrow C\{i = 0\}$ is a quasi-isomorphism (again, by the adjunction inequality), we see that the map $v_{g(K)} : A_{g(K)} \to C\{i = 0\}$ induces an isomorphism on homology. Thus $\nu(K) = \tau(K)$, so $\varepsilon(K)$ is equal to 0 or 1. Since $\tau(K) > 0$, it follows that $\varepsilon(K)$ is not equal to zero, and hence is equal to 1. □

Proof of (5). In [18, Lemma 5], Petkova constructs model complexes for $CFK^-(K)$ of homologically thin knots, from which the values of $\tau(K)$ and $\varepsilon(K)$ follow readily. □

Proof of (6). Recall from [11, Theorem 7.1] that $CFK^-(K\#K') \simeq CFK^-(K) \otimes_{\mathbb{F}[U]} CFK^-(K')$. We first consider the case where $\varepsilon(K) = \varepsilon(K') = 1$. Then by Lemma 3.2, there exists a horizontally simplified basis $\{x_i\}$ for $CFK^-(K)$ such that

1. x_2 is the distinguished element of a vertically simplified basis

2. $\partial^{\text{horz}} x_1 = U^m x_2$,

and similarly, a horizontally simplified basis $\{y_i\}$ for $CFK^-(K')$ with y_2 the distinguished element of a vertically simplified basis, and $\partial^{\text{horz}} y_1 = U^n y_2$. Note that $A(x_2) = \tau(K)$ and $A(y_2) = \tau(K')$. Then $A(x_2 \otimes y_2) = \tau(K) + \tau(K') = \tau(K\#K')$, and $[x_2 \otimes y_2] \neq 0 \in H_*(C^{\text{vert}}(K\#K'))$. Recall the map

$$v'_s : C^{\text{vert}} \to A'_s,$$

and notice that $[x_2 \otimes y_2] = 0 \in H_*(A'_{\tau(K\#K')}(K\#K'))$ since $\partial'_{\tau(K\#K')}(U^{-m} x_1 \otimes y_2) = x_2 \otimes y_2$. Hence $\nu'(K\#K') = \tau(K\#K') - 1$, implying that $\varepsilon(K\#K') = 1$. The case where $\varepsilon(K) = \varepsilon(K') = -1$ follows similarly.

Finally, if $\varepsilon(K) = 0$, then by Lemma 3.3, there exists a basis $\{x_i\}$ for $CFK^-(K)$ such that the element x_0 is the distinguished element of both a horizontally simplified basis and a vertically simplified basis. Then to determine $\nu(K\#K')$ and $\nu'(K\#K')$, it is sufficient to consider just $\{x_0\} \otimes CFK^-(K')$, in which case, $\varepsilon(K\#K') = \varepsilon(K')$. $\square$

Chapter 4

Computation of τ for $(p, pn+1)$-cables

We will first consider $(p, pn+1)$-cables, whose Heegaard diagrams are easier to work with, and prove the following version of Theorem 1:

Theorem 4.1. *$\tau(K_{p,pn+1})$ behaves in one of three ways. If $\varepsilon(K) = 1$, then*

$$\tau(K_{p,pn+1}) = p\tau(K) + \frac{pn(p-1)}{2}.$$

If $\varepsilon(K) = -1$, then

$$\tau(K_{p,pn+1}) = p\tau(K) + \frac{pn(p-1)}{2} + p - 1.$$

Finally, if $\varepsilon(K) = 0$, then

$$\tau(K_{p,pn+1}) = \begin{cases} \frac{pn(p-1)}{2} + p - 1 & \text{if } n < 0 \\ \frac{pn(p-1)}{2} & \text{if } n \geq 0. \end{cases}$$

The proof will proceed as follows. We will determine that only a certain small piece of the Type D bordered invariant associated to the framed knot complement is necessary to determine a suitable generator for $\widehat{HF}(S^3)$. The form of this piece of $\widehat{CFD}$ depends only on the framing parameter relative to $2\tau(K)$, and on $\varepsilon(K)$. We will then determine the absolute Alexander grading of this generator in terms of combinatorial data associated to the Heegaard diagrams for the pattern and companion knots.

4.1 The case $\varepsilon(K) = 1$

We first consider the case $\varepsilon(K) = 1$. By Lemma 3.2 and the symmetry properties of $CFK^\infty(K)$, we can find a *vertically* simplified basis $\{x_i\}$ over $\mathbb{F}[U]$ for $CFK^-(K)$ with the following properties, after possible reordering:

1. x_2 is the distinguished element of a *horizontally* simplified basis.

2. $\partial^{\text{vert}} x_1 = x_2$.

Let $Y_{K,n}$ be the 3-manifold $S^3 - \text{nbd } K$ with framing n. We will use Theorems 10.17 and 11.7 of [7] to compute $\widehat{CFD}(Y_{K,n})$ from $CFK^\infty(K)$. Consider the basis $\{x_i\}$ as above. Then if $n < 2\tau(K)$, there is a portion of $\widehat{CFD}(Y_{K,n})$ (consisting of the unstable chain and an additional generator y from a vertical chain) of the form

$$x_0 \xrightarrow{D_1} z_1 \xleftarrow{D_{23}} z_2 \xleftarrow{D_{23}} \dots \xleftarrow{D_{23}} z_m \xleftarrow{D_3} x_2 \xrightarrow{D_{123}} y, \tag{4.1}$$

where $m = 2\tau(K) - n$. If $n = 2\tau(K)$, there is a portion of $\widehat{CFD}(Y_{K,n})$ of the form

$$x_0 \xrightarrow{D_{12}} x_2 \xrightarrow{D_{123}} y. \tag{4.2}$$

Finally, if $n > 2\tau(K)$, there is a portion of $\widehat{CFD}(Y_{K,n})$ of the form

$$x_0 \xrightarrow{D_{123}} z_1 \xrightarrow{D_{23}} z_2 \xrightarrow{D_{23}} \dots \xrightarrow{D_{23}} z_m \xrightarrow{D_2} x_2 \xrightarrow{D_{123}} y, \tag{4.3}$$

where $m = n - 2\tau(K)$. The generators x_0 and x_2 are in the idempotent ι_1, while the generators $z_1, \dots, z_m$ and y are in the idempotent ι_2.

Let $\widehat{CFA}(p, 1)$ be the bordered invariant associated to the pattern knot in the solid torus that winds p times longitudinally and once meridionally. See Figure 4.1. We will use the pairing theorem for bordered Heegaard Floer homology of [7, Theorem 10.12] to compute $\tau(K_{p,pn+1})$ by studying $\widehat{CFA}(p,1) \boxtimes \widehat{CFD}(Y_{K,n})$, which is filtered quasi-isomorphic to $\widehat{CFK}(K_{p,pn+1})$.

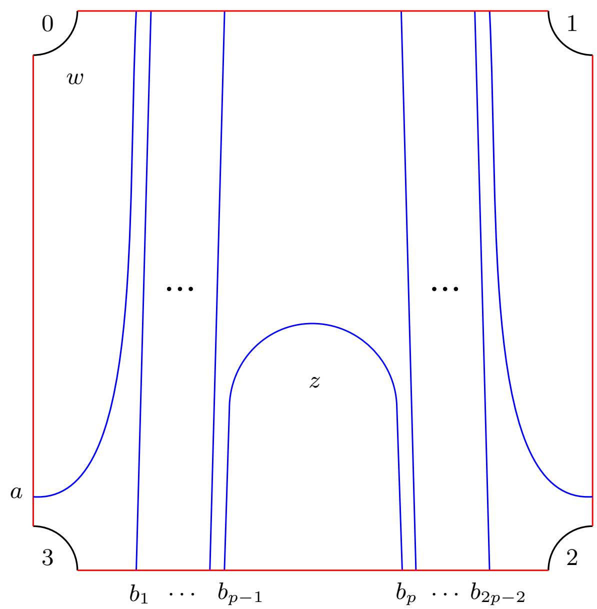


Figure 4.1: A genus one bordered Heegaard diagram $\mathcal{H}(p,1)$ for the $(p,1)$-cable in the solid torus.

Remark 4.2. We remark here on the basepoint conventions used henceforth in this thesis. We prefer to work with filtered chain complexes, rather than $\mathbb{F}[U]$-modules, and thus compute the filtered chain complex $\widehat{CFK}$, rather than the $\mathbb{F}[U]$-module CFK^-. This convention requires that we interchange the roles of w and z, i.e., we now place the basepoint w on $\partial\overline{\Sigma}$.

Each algebra relation m_i in $\widehat{CFA}(p,1)$ contributes a relative filtration shift, denoted Δ_A, which is equal to the number of times that the domain inducing m_i crosses the basepoint z. (Each domain must miss the basepoint w completely.) This relative filtration shift naturally extends to the tensor product.

Since switching the roles of w and z induces a chain homotopy equivalence on $CFK^\infty(K)$, it does not change the homotopy type of $\widehat{CFD}$ of the framed knot complement.

We see that we have the following algebra relations on $\widehat{CFA}(p,1)$, where Δ_A records the relative filtration shift, i.e., the number of times that the associated domain crosses the basepoint z:

$$\begin{array}{lll}
m_{3+i}(a,\rho_3,\overbrace{\rho_{23},\ldots,\rho_{23}}^{i},\rho_2)=a, & \Delta_A = pi+p, & i\geq 0\\
m_{4+i+j}(a,\rho_3,\overbrace{\rho_{23},\ldots,\rho_{23}}^{i},\rho_2,\overbrace{\rho_{12},\ldots,\rho_{12}}^{j},\rho_1)=b_{j+1}, & \Delta_A = pi+j+1, & 0\leq j\leq p-2\\
 & & i\geq 0\\
m_{2+j}(a,\overbrace{\rho_{12},\ldots,\rho_{12}}^{j},\rho_1)=b_{2p-j-2}, & \Delta_A=0, & 0\leq j\leq p-2
\end{array}$$

$$\begin{array}{lll}
m_1(b_j)=b_{2p-j-1}, & \Delta_A = p-j, & 1\leq j\leq p-1\\
m_{3+i}(b_j,\rho_2,\overbrace{\rho_{12},\ldots,\rho_{12}}^{i},\rho_1)=b_{j+i+1}, & \Delta_A = i+1, & 1\leq j\leq p-2\\
 & & 0\leq i\leq p-j-2\\
m_{3+i}(b_j,\rho_2,\overbrace{\rho_{12},\ldots,\rho_{12}}^{i},\rho_1)=b_{j-i-1}, & \Delta_A = 0, & p+1\leq j\leq 2p-2\\
 & & 0\leq i\leq j-p-1
\end{array}$$

The generator a is in the idempotent ι_1 and the generators $b_1,\ldots,b_{2p-2}$ are in the idempotent ι_2. By the pairing theorem for bordered Floer homology [7, Theorem 10.12], we have the filtered quasi-isomorphism

$$\widehat{CFK}(K_{p,pn+1})\simeq\widehat{CFA}(p,1)\boxtimes\widehat{CFD}(Y_{K,n}).$$

The following lemma identifies the generator of $\widehat{HF}(S^3)$ in the tensor product:

Lemma 4.3. *When $\varepsilon(K)=1$, the element in the tensor product*

$$\widehat{CFK}(K_{p,pn+1})\simeq\widehat{CFA}(p,1)\boxtimes\widehat{CFD}(Y_{K,n})$$

that survives to generator $\widehat{HF}(S^3)$ is ax_2, regardless of n, the framing on $Y_{K,n}$.

Proof. When we tensor $\widehat{CFA}(p,1)$ with the portion of $\widehat{CFD}(Y_{K,n})$ in Equation 4.1, we see that ax_2 has no incoming or outgoing differentials in the tensor product. This can be seen by noticing that a has no m_1 algebra relations, nor any algebra relations beginning with ρ_{123}, nor any algebra relations of the form $m_{2+i}(a, \rho_3, \rho_{23}, \ldots, \rho_{23})$. Hence, ax_2 represents a generator for $\widehat{HF}(S^3)$ of minimal Alexander grading. Similarly, we see that tensoring $\widehat{CFA}(p, 1)$ with either of the pieces of $\widehat{CFD}(Y_{K,n})$ in Equations 4.2 or 4.3 also gives us ax_2 as the generator for $\widehat{HF}(S^3)$. $\square$

We now need to compute the absolute Alexander grading of the generator ax_2. Recall that one way to define the absolute Alexander grading is

$$A(\mathbf{x}) = \tfrac{1}{2}\langle c_1(\underline{\mathfrak{s}}(\mathbf{x})), [\widehat{F}]\rangle.$$

Also recall that $\langle c_1(\underline{\mathfrak{s}}(\mathbf{x})), [\widehat{F}]\rangle$ can be computed in terms of combinatorial data associated to the Heegaard diagram for S^3 compatible with the knot K. More precisely, replace the α-circle representing a meridian of K with a 0-framed longitude λ. We refer to this local region of the Heegaard diagram as the winding region. Then we have the following formula [11, Equation 9]:

$$\langle c_1(\underline{\mathfrak{s}}(\mathbf{x})), [\widehat{F}]\rangle = \chi(\mathcal{P}) + 2n_{\mathbf{x}}(\mathcal{P}),$$

where $\mathcal{P}$ is a periodic domain representing $[\widehat{F}]$, $\chi(\mathcal{P})$ is the Euler measure of $\mathcal{P}$ and $n_{\mathbf{x}}(\mathcal{P})$ is the local multiplicity of $\mathbf{x}'$ at $\mathcal{P}$, where $\mathbf{x}'$ is obtained from $\mathbf{x}$ by moving the support of $\mathbf{x}$ on the meridian to the longitude, as in Figure 4.2. We will use this formula to compute the Alexander grading of ax_2.

Lemma 4.4. *The Alexander grading of ax_2 is*

$$A(ax_2) = p\tau(K) + \tfrac{pn(p-1)}{2}.$$

Proof. We will construct a domain $\mathcal{P}$ that may be decomposed into a domain $\mathcal{P}_A$ on $\mathcal{H}(p, 1)$ and a domain $\mathcal{P}_D$ on $\mathcal{H}(Y_{K,n})$, whose multiplicities agree in the four regions surrounding the puncture on each surface. Then

$$\langle c_1(\underline{\mathfrak{s}}(ax_2)), [\widehat{F}]\rangle = \chi(\mathcal{P}_A) + 2n_a(\mathcal{P}_A) + \chi(\mathcal{P}_D) + 2n_{x_2}(\mathcal{P}_D)$$

since Euler measure and local multiplicity are both additive under disjoint union.

We first consider the domain $\mathcal{P}_D$ in $\mathcal{H}(Y_{K,n})$. Recall that x_2 is the preferred element of a horizontally simplified basis, and it corresponds to some linear combination of generators in the diagram $\mathcal{H}(Y_{K,n})$. Choose an element in that linear combination of maximal Alexander grading. For ease of notation, we will also denote this generator by x_2. Our conventions in this thesis for the base points in $\mathcal{H}(Y_{K,n})$ are the opposite of those in [7]; that is, we have switched the roles of w and z. (This was done so that we could compute the tensor product as a filtered chain complex, rather than as a $\mathbb{F}[U]$-module.) With our conventions, x_2, the preferred element of a horizontally simplified basis, will have Alexander grading $\tau(K)$ in $\widehat{CFK}(K)$. However, we are interested in the quantity $\chi(\mathcal{P}_D) + 2n_{x_2}(\mathcal{P}_D)$, which we claim is equal to

$$\chi(\mathcal{P}_D) + 2n_{x_2}(\mathcal{P}_D) = 2\tau(K) - \tfrac{n}{2} - \tfrac{1}{2}.$$

This can be seen from the fact that the domain $\mathcal{P}'$ (representing a Seifert surface for K) used to compute the Alexander grading of x_2 in $\widehat{CFK}(K)$ has multiplicities in the winding region as shown in Figure 4.2. Winding the longitude (that is, changing the framing) does not change the quantity $\chi(\mathcal{P}')+2n_{x_2}(\mathcal{P}') = 2\tau(K)$. However, $\mathcal{P}_D$ will differ from $\mathcal{P}'$ by the removal of a small disk around the intersection of the longitude and the meridian, which implies that $\chi(\mathcal{P}_D) = \chi(\mathcal{P}')+\frac{n}{2}+\frac{1}{2}$. See Figure 4.3. We have also moved the support of x_2 in the winding region from the intersection of the longitude with a β circle (denoted x_2' in Figure 4.2) to the unique intersection of the meridian with the same β circle (denoted x_2 in Figure 4.2), which implies that $n_{x_2}(\mathcal{P}_D) = n_{x_2}(\mathcal{P}') - \frac{n}{2} - \frac{1}{2}$. Hence, $\chi(\mathcal{P}_D) + 2n_{x_2}(\mathcal{P}_D)$ has the value claimed above.

We now consider the domain $\mathcal{P}_A$ in $\mathcal{H}(p,1)$. First, we stablilize the diagram to obtain a curve, β_2, that represents the meridian of the knot sitting in the solid torus. We replace the generator a with the generator a union the unique intersection of β_2 with an α circle; for ease of notation, we also denote this generator by a. We then add a closed curve, λ, to $\mathcal{H}(p,1)$, such that λ represents a 0-framed longitude for the knot $K_{p,pn+1}$ in S^3. See Figure 4.4. Note that λ, which is contained entirely in $\mathcal{H}(p,1)$, will depend on the framing parameter n of the knot complement $Y_{K,n}$. We

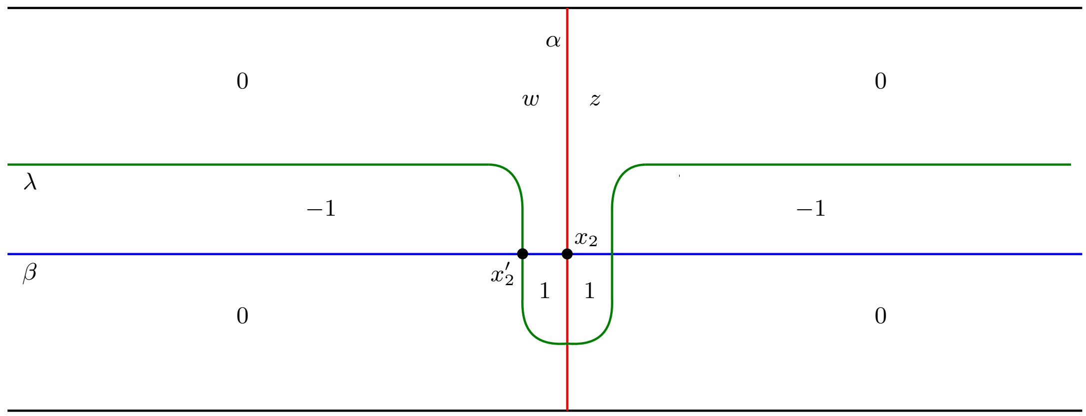


Figure 4.2: Winding region for a knot complement. The numbers indicate the multiplicities of $\mathcal{P}'$.

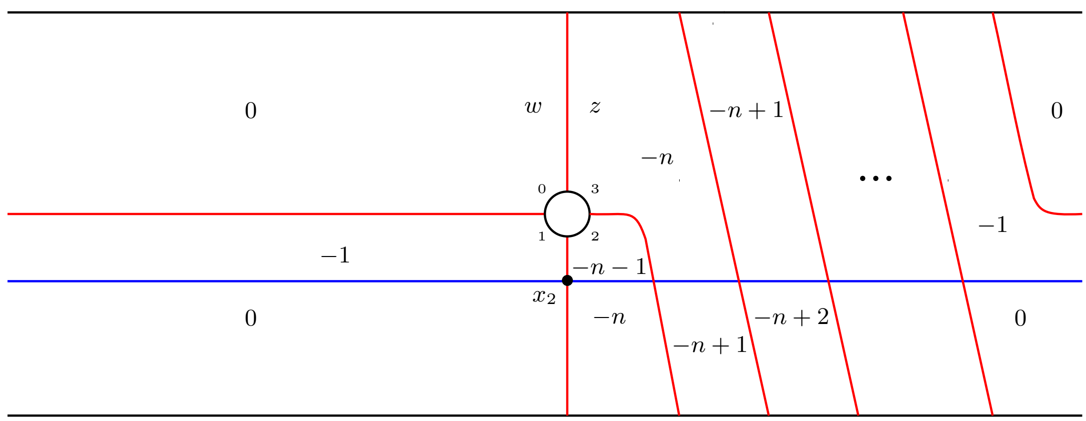


Figure 4.3: Winding region for a bordered knot complement with the multiplicities of $\mathcal{P}_D$ shown.

require $\partial\mathcal{P}_A$ to contain λ exactly once. Furthermore, we require the multiplicities of $\mathcal{P}_A$ in the regions 0, 1, 2 and 3 surrounding the puncture to be 0, $-p$, $-pn-p$ and $-pn$, respectively, in order to coincide with p (the winding number) times the multiplicities in the corresponding regions in $\mathcal{H}(Y_{K,n})$.

First consider the domain $\mathcal{P}_\mu$ shown in Figure 4.5. $\mathcal{P}_\mu$ has zero multiplicity in the regions 0 and 1 near the puncture, and multiplicity -1 in the regions 2 and 3. Furthermore, $\partial\mathcal{P}_\mu$ contains

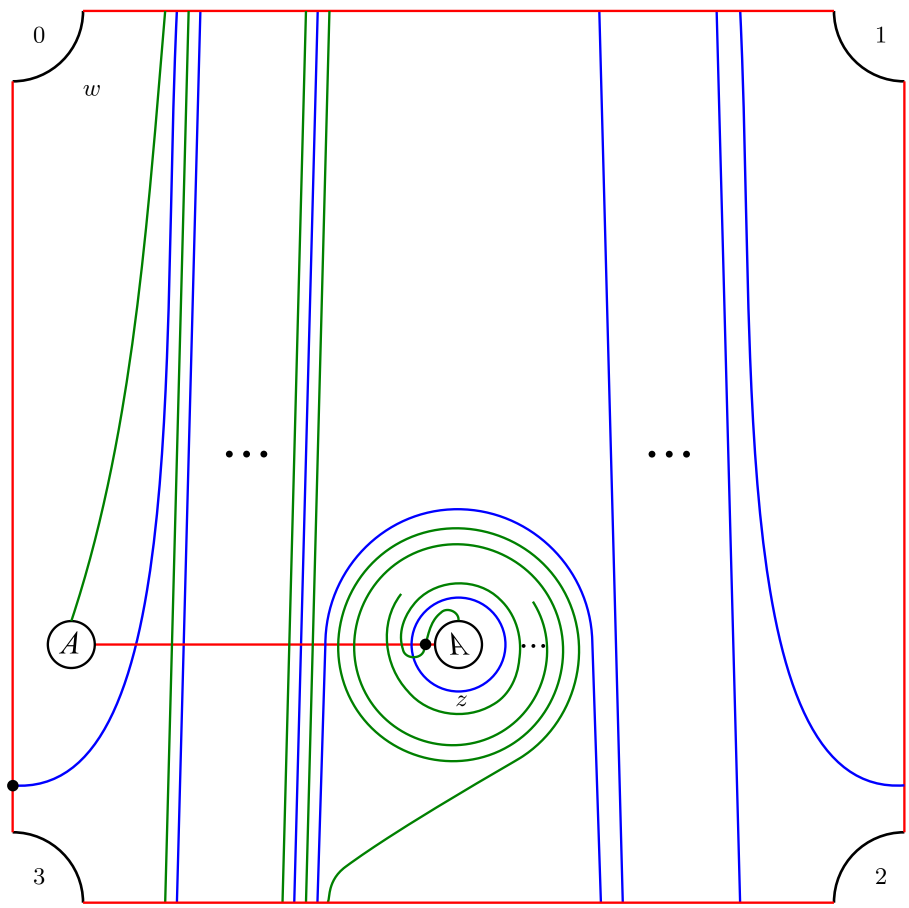


Figure 4.4: Stabilized bordered Heegaard diagram $\mathcal{H}(p,1)$ for the $(p,1)$-cable in the solid torus with the longitude λ shown in green. The pair of black dots indicate the generator a (with its support on the meridian moved to the longitude).

β_2 with multiplicity p (for an appropriate orientation of β_2). We see that $\chi(\mathcal{P}_\mu) = p + \frac{1}{2}$, and $n_a(\mathcal{P}_\mu) = -\frac{1}{2}$. Next, consider the domain $\mathcal{P}_\lambda$ shown in Figure 4.6. $\mathcal{P}_\lambda$ has zero multiplicity in regions 0 and 3, and multiplicity $-p$ in the regions 1 and 2. $\partial\mathcal{P}_\lambda$ contains the curve β_2 with multiplicity $-p^2 n$. We also have that $\chi(\mathcal{P}_\lambda) = \frac{3p}{2}$ and $n_a(\mathcal{P}_\lambda) = -\frac{p}{2}$. Let

$$\mathcal{P}_A = pn \cdot \mathcal{P}_\mu + \mathcal{P}_\lambda.$$

Notice that $\mathcal{P}_A$ has the desired multiplicities in the regions surrounding the puncture, and $\partial\mathcal{P}_A$

contains the longitude for the pattern knot exactly once. We have that $\chi(\mathcal{P}_A) = p^2 n + \frac{pn+3p}{2}$ and $n_a(\mathcal{P}_A) = \frac{-pn-p}{2}$.

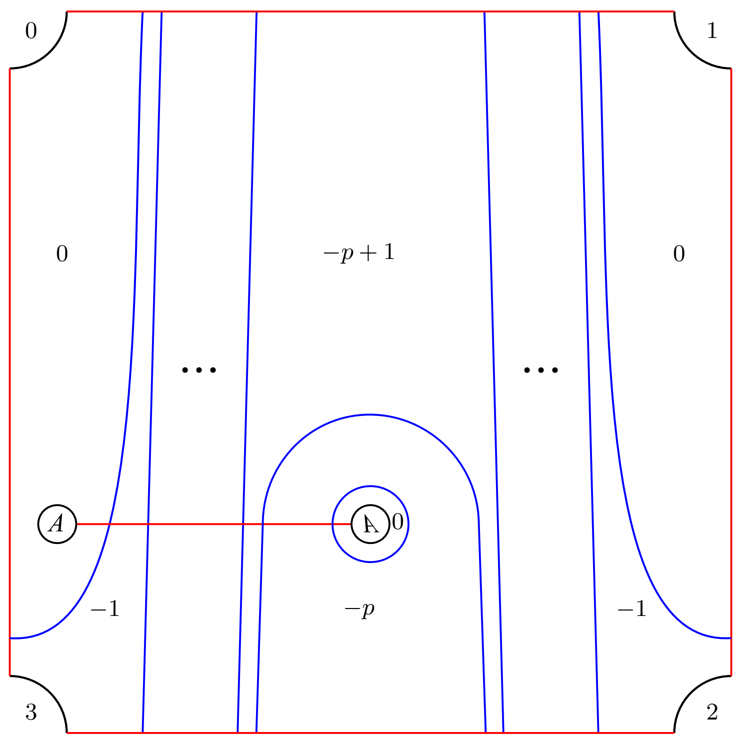


Figure 4.5: The periodic domain $\mathcal{P}_\mu$ in $\mathcal{H}(p,1)$.

The union of $\mathcal{P}_A$ and $p \cdot \mathcal{P}_D$ represents a Seifert surface for the cable knot $K_{p,pn+1}$, and we find that the Alexander grading of ax_2 is

$$\begin{aligned} A(ax_2) &= \tfrac{1}{2}\langle c_1(\underline{\mathfrak{s}}(ax_2)), [\widehat{F}]\rangle \\ &= \tfrac{1}{2}\big(\chi(\mathcal{P}_A) + 2n_a(\mathcal{P}_A) + p\chi(\mathcal{P}_D) + 2pn_{x_2}(\mathcal{P}_D)\big) \\ &= \tfrac{1}{2}\big(p^2 n + \tfrac{pn+3p}{2} - pn - p + 2p\tau(K) - \tfrac{pn+p}{2}\big) \\ &= p\tau(K) + \tfrac{pn(p-1)}{2}. \end{aligned}$$

$\square$

Combining this lemma with Lemma 4.3 yields the result that when $\varepsilon(K) = 1$,

$$\tau(K_{p,pn+1}) = p\tau(K) + \tfrac{pn(p-1)}{2}.$$

4.2 The case $\varepsilon(K) = -1$

We now consider the case $\varepsilon(K) = -1$, proceeding exactly as in the case $\varepsilon(K) = 1$ above, with the appropriate modifications.

By Lemma 3.2 and the symmetry properties of $CFK^\infty(K)$, we can find a *vertically* simplified basis $\{x_i\}$ over $\mathbb{F}[U]$ for $CFK^-(K)$ with the following properties, after possible reordering:

1. x_1 is the distinguished element of a *horizontally* simplified basis.

2. $\partial^{\text{vert}} x_1 = x_2$.

We again let $Y_{K,n}$ be the 3-manifold $S^3 - \text{nbd } K$ with framing n, and consider the basis $\{x_i\}$ as above. Now, if $n < 2\tau(K)$, there is a portion of $\widehat{CFD}(Y_{K,n})$ (consisting of the unstable chain and an additional generator y from a vertical chain) of the form

$$x_0 \xrightarrow{D_1} z_1 \xleftarrow{D_{23}} z_2 \xleftarrow{D_{23}} \dots \xleftarrow{D_{23}} z_m \xleftarrow{D_3} x_1 \xrightarrow{D_1} y, \tag{4.4}$$

where $m = 2\tau(K) - n$. If $n = 2\tau(K)$, there is a portion of $\widehat{CFD}(Y_{K,n})$ of the form

$$x_0 \xrightarrow{D_{12}} x_1 \xrightarrow{D_1} y. \tag{4.5}$$

Finally, if $n > 2\tau(K)$, there is a portion of $\widehat{CFD}(Y_{K,n})$ of the form

$$x_0 \xrightarrow{D_{123}} z_1 \xrightarrow{D_{23}} z_2 \xrightarrow{D_{23}} \dots \xrightarrow{D_{23}} z_m \xrightarrow{D_2} x_1 \xrightarrow{D_1} y, \tag{4.6}$$

where $m = n - 2\tau(K)$. In all of the cases above, y has an incoming arrow labeled either D_{23} or D_{123}, depending on the exact form of $CFK^\infty(K)$.

Lemma 4.5. *When $\varepsilon(K) = -1$, the element in the tensor product*

$$\widehat{CFK}(K_{p,pn+1}) \simeq \widehat{CFA}(p, 1) \boxtimes \widehat{CFD}(Y_{K,n})$$

that survives to generator $\widehat{HF}(S^3)$ is $b_1y + ax_2$, regardless of n, the framing on $Y_{K,n}$.

Proof. The proof of this lemma follows identically to the proof of Lemma 4.3. For example, tensoring $\widehat{CFA}(p,1)$ with the piece of $\widehat{CFD}(Y_{K,n})$ in Equation 4.4, we see that $\widehat{CFK}(K_{p,pn+1}) \simeq \widehat{CFA}(p,1) \boxtimes \widehat{CFD}(Y_{K,n})$ has a direct summand consisting of the three generators ax_1, b_1y and $b_{2p-2}y$ with a filtration-preserving differential $\partial(ax_1) = b_{2p-2}y$ and a differential $\partial(b_1y) = b_{2p-2}y$ that drops filtration level by $p-1$. There are no other differentials in this summand, since y has an incoming arrow labeled either D_{23} or D_{123}, neither of which can tensor non-trivially with any of the algebra relations in $\widehat{CFA}(p,1)$. Thus, $b_1y + ax_1$ generates $\widehat{HF}(S^3)$. The other cases follow similarly. $\square$

The Alexander grading of ax_1 is $p\tau(K) + \frac{pn(p-1)}{2}$, by Lemma 4.4, where now x_1, rather than x_2, is the distinguished element of a vertically simplified basis. By examining the grading shifts of the differentials in the subcomplex of $\widehat{CFK}(K_{p,pn+1})$ above, we see immediately that the Alexander grading of b_1y is $p\tau(K) + \frac{pn(p-1)}{2} + p - 1$, as desired.

4.3 The case $\varepsilon(K) = 0$

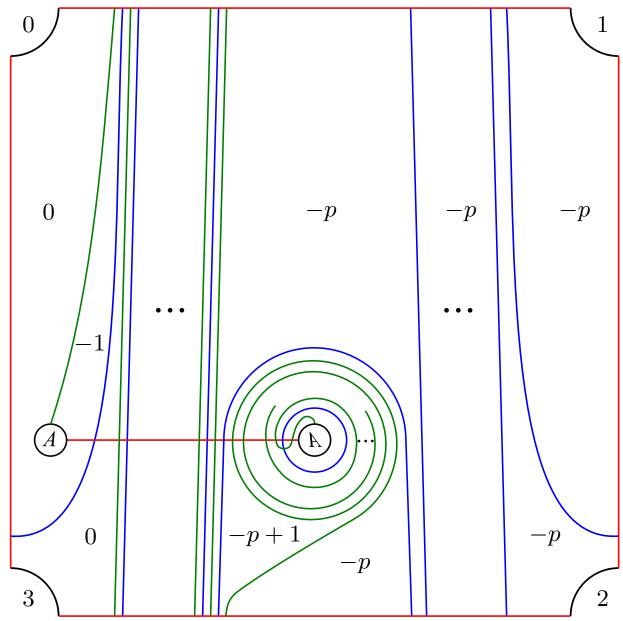
The values of $\tau(K_{p,pn+1})$ in the case $\varepsilon(K) = 0$ can be computed by considering the model calculation where K is the unknot. When $\varepsilon(K) = 0$, $\widehat{CFD}(Y_{K,n})$ has a direct summand that is isomorphic to $\widehat{CFD}(Y_{U,n})$, where U denotes the unknot. The tensor product splits along direct summands, so $\widehat{CFK}(K_{p,pn+1})$ has a direct summand that is filtered quasi-isomorphic to $\widehat{CFK}(T_{p,pn+1})$, where $T_{p,pn+1}$ is the $(p, pn+1)$-torus knot; that is, the $(p, pn+1)$-cable of the unknot.

We remark that when $n \geq 2\tau(K)$ and $\varepsilon(K) = 0$, $\widehat{CFD}(Y_{K,n})$ is not bounded. However, by [7, Proposition 4.16], there exists an admissible diagram, and hence bounded $\widehat{CFA}$, for the $(p,1)$-torus knot in $S^1 \times D^2$, in which case the tensor products above will be well-defined.

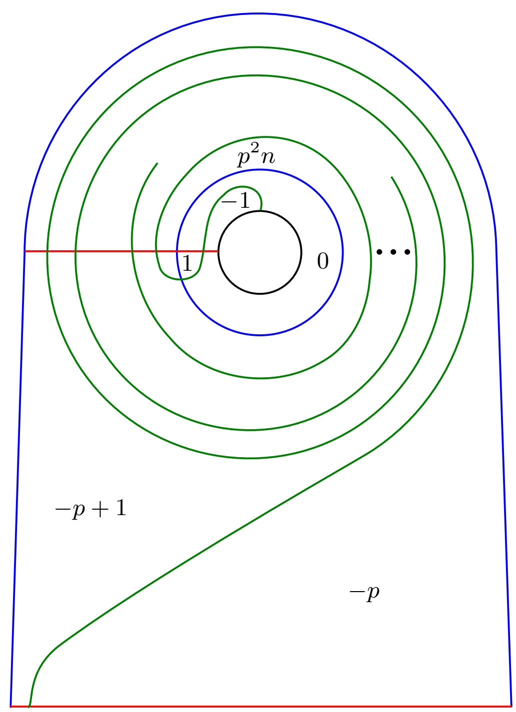
Hence, when $\varepsilon(K) = 0$, the results of [9] computing τ of torus knots tell us that

$$\tau(K_{p,pn+1}) = \begin{cases} \frac{pn(p-1)}{2} + p - 1 & \text{if } n < 0 \\ \frac{pn(p-1)}{2} & \text{if } n \geq 0. \end{cases}$$

This completes the proof of Theorem 4.1.



(a)



(b)

Figure 4.6: The periodic domain $\mathcal{P}_\lambda$ in $\mathcal{H}(p,1)$, with detail in (b).

Chapter 5

Computation of τ for general (p, q)-cables

We will now extend our results for $(p, pn+1)$-cables to general (p, q)-cables. That is, we would like to prove the following restatement of Theorem 1:

Theorem 5.1. *$\tau(K_{p,q})$ behaves in one of three ways. If $\varepsilon(K) = 1$, then*

$$\tau(K_{p,q}) = p\tau(K) + \frac{(p-1)(q-1)}{2}.$$

If $\varepsilon(K) = -1$, then

$$\tau(K_{p,q}) = p\tau(K) + \frac{(p-1)(q+1)}{2}.$$

Finally, if $\varepsilon(K) = 0$, then

$$\tau(K_{p,q}) = \begin{cases} \frac{(p-1)(q+1)}{2} & \text{if } q < 0 \\ \frac{(p-1)(q-1)}{2} & \text{if } q > 0. \end{cases}$$

This could be done by considering patterns for (p, r)-cables, for all $0 < r < p$ with r relatively prime to p. However, Van Cott's results from [20] eliminate the need to consider these more complicated patterns. We summarize her results below.

We expect the behavior of $\tau(K_{p,q})$ to be somehow related to $\tau(T_{p,q})$. Recall that as a function of q, $\tau(K_{p,q})$ is linear of slope $\frac{p-1}{2}$ for fixed p and $q > 0$. This motivates the following definition from [20]:

Definition 5.2. *Fix an integer p and a knot $K \subset S^3$. For all integers q relatively prime to p, define $h(q)$ to be*

$$h(q) = \tau(K_{p,q}) - \tfrac{(p-1)q}{2}.$$

Van Cott proves the following theorem:

Theorem 5.3 ([20])**.** *The function $h(q)$ is a non-increasing $\frac{1}{2}\cdot\mathbb{Z}$-valued function which is bounded below. In particular, we have*

$$-(p-1) \leq h(q) - h(r) \leq 0$$

for all $q > r$, where both q and r are relatively prime to p.

She then extends Hedden's work on $(p, pn+1)$-cables in [2] to general (p,q)-cables:

Theorem 5.4 ([20])**.** *Let $K \subset S^3$ be a non-trivial knot. Then the following inequality holds for all pairs of relatively prime integers p and q:*

$$p\tau(K) + \tfrac{(p-1)(q-1)}{2} \leq \tau(K_{p,q}) \leq p\tau(K) + \tfrac{(p-1)(q-1)}{2} + p - 1.$$

When K satisfies $\tau(K) = g(K)$, we have $\tau(K_{p,q}) = p\tau(K) + \frac{(p-1)(q-1)}{2}$, whereas when $\tau(K) = -g(K)$, we have $\tau(K_{p,q}) = p\tau(K) + \frac{(p-1)(q-1)}{2} + p - 1$.

The same argument used to prove the final two statements in the above theorem can be used to extend our Theorem 4.1 for $(p, pn+1)$-cables to general (p,q)-cables. For completeness, we repeat the argument here.

Let $\varepsilon(K) = 1$. By Theorem 4.1, we know that $\tau(K_{p,pn+1}) = p\tau(K) + \frac{pn(p-1)}{2}$. Our goal is to prove the analogous statement for general (p,q)-cables; that is, $\tau(K_{p,q}) = p\tau(K) + \frac{(p-1)(q-1)}{2}$. We see that

$$h(pn+1) = \tau(K_{p,pn+1}) - \tfrac{(p-1)(pn+1)}{2}$$
$$= p\tau(K) - \tfrac{p-1}{2},$$

for all n. Since the function h is non-increasing, it follows that

$$h(q) = p\tau(K) - \tfrac{p-1}{2}$$

for all q. Hence

$$\tau(K_{p,q}) = h(q) + \tfrac{(p-1)q}{2}$$
$$= p\tau(K) + \tfrac{(p-1)(q-1)}{2},$$

as desired. A similar argument shows that in the case $\varepsilon(K) = -1$,

$$\tau(K_{p,q}) = p\tau(K) + \tfrac{(p-1)(q+1)}{2}.$$

We are left with the case $\varepsilon(K) = 0$. Let $\widehat{CFA}(p,q)$ denote the bordered invariant associated to a bordered Heegaard diagram compatible with the (p,q)-torus knot in $S^1 \times D^2$. (Such a diagram exists by [7, Section 10.4], and can be made admissible by [7, Proposition 4.16].) We again consider the tensor product of $\widehat{CFA}(p,q)$ with $\widehat{CFD}(Y_{K,0})$, i.e., $Y_{K,0} = S^3 - \text{nbd } K$ with the zero framing. Since $\varepsilon(K) = 0$, the tensor product $\widehat{CFA}(p,q) \boxtimes \widehat{CFD}(Y_{K,0})$ contains a summand that is filtered quasi-isomorphic to $\widehat{CFK}(T_{p,q})$. Therefore, $\tau(K_{p,q})$ agrees with $\tau(T_{p,q})$, and by [9], we have

$$\tau(K_{p,q}) = \begin{cases} \tfrac{(p-1)(q+1)}{2} & \text{if } q < 0 \\ \tfrac{(p-1)(q-1)}{2} & \text{if } q > 0. \end{cases}$$

This completes the proof of Theorem 5.1, and hence also the proof of Theorem 1.

Chapter 6

Computation of $\varepsilon(K_{p,pn+1})$ when $\varepsilon(K) = 1$

In this chapter, we will compute $\varepsilon(K_{p,pn+1})$ in the case where $\varepsilon(K) = 1$. We will do this by computing $\tau(K_{p,pn+1;2,-1})$, since Theorem 1 tells us that if $\tau(K_{p,pn+1;2,-1}) = 2\tau(K_{p,pn+1}) - 1$, then $\varepsilon(K_{p,pn+1}) = 1$.

Proposition 6.1. *If $\varepsilon(K)$=1, then $\varepsilon(K_{p,pn+1}) = 1$.*

We consider the pattern knot $T_{p,1;2,2m+1} \subset S^1 \times D^2$. See Figure 6.1 and denote the associated bordered invariant $\widehat{CFA}(p, 1; 2, 2m + 1)$. Letting Y be $S^3 - \text{nbd } K$ with framing n, we then have

$$\widehat{CFK}(K_{p,pn+1;2,2p^2n+2m+1}) \simeq \widehat{CFA}(p, 1; 2, 2m + 1) \boxtimes \widehat{CFD}(Y).$$

Thus, we need to consider the case when $m = -p^2n - 1$.

We will proceed as in Chapter 4, by computing a portion of $\widehat{CFA}(p, 1; 2, 2m + 1)$ that is sufficient to determine a generator for $\widehat{HF}(S^3)$, and then determining the Alexander grading of that generator. The remainder of this chapter consists of those detailed computations.

If $\varepsilon(K) = 1$, then by Lemma 3.2, we can find a vertically simplified basis $\{x_i\}$ over $\mathbb{F}[U]$ for

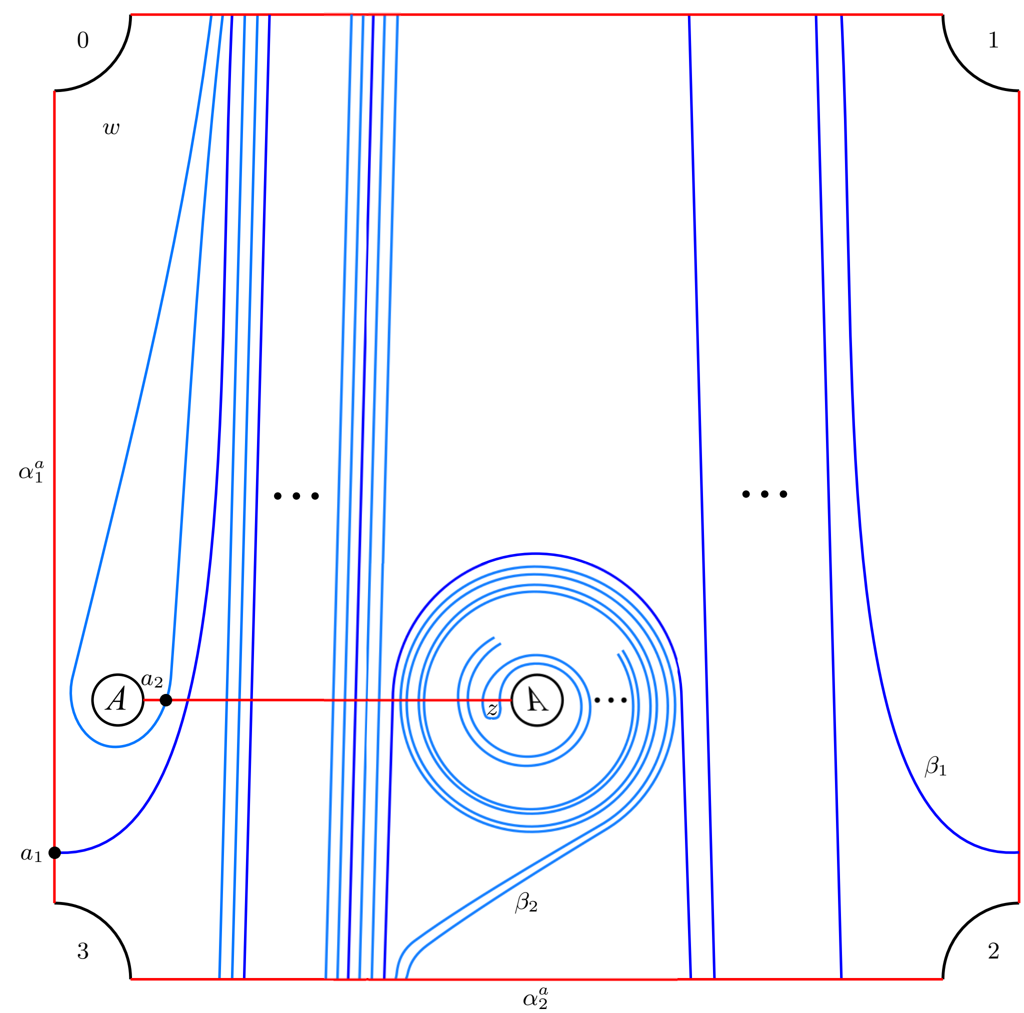


Figure 6.1: Bordered Heegaard diagram for the $(p,1;2,2m+1)$-iterated torus knot in the solid torus. The light blue circle, β_2, winds $p+m$ times.

$CFK^-(K)$ with the following properties, after possible reordering:

1. x_2 is the distinguished element of a horizontally simplified basis.

2. $\partial x_1 = x_2$.

Let $a = (a_1, a_2)$ in Figure 6.1. We claim that ax_2 will be a generator for $\widehat{HF}(S^3)$ in the tensor product $\widehat{CFK}(K_{p,pn+1;2,-1}) \simeq \widehat{CFA}(p,1;2,-2p^2n-1) \boxtimes \widehat{CFD}(Y)$.

Letting $Y_{K,n}$ be the 3-manifold $S^3 - \text{nbd } K$ with framing n, and considering the basis $\{x_i\}$

above, we again have the following pieces of $\widehat{CFD}(Y_{K,n})$:

- If $n < 2\tau(K)$, there is a portion of $\widehat{CFD}(Y_{K,n})$ of the form

$$x_0 \xrightarrow{D_1} z_1 \xleftarrow{D_{23}} z_2 \xleftarrow{D_{23}} \dots \xleftarrow{D_{23}} z_m \xleftarrow{D_3} x_2 \xrightarrow{D_{123}} y,$$

  where $m = 2\tau(K) - n$.

- If $n = 2\tau(K)$, there is a portion of $\widehat{CFD}(Y_{K,n})$ of the form

$$x_0 \xrightarrow{D_{12}} x_2 \xrightarrow{D_{123}} y.$$

- Finally, if $n > 2\tau(K)$, there is a portion of $\widehat{CFD}(Y_{K,n})$ of the form

$$x_0 \xrightarrow{D_{123}} z_1 \xrightarrow{D_{23}} z_2 \xrightarrow{D_{23}} \dots \xrightarrow{D_{23}} z_m \xrightarrow{D_2} x_2 \xrightarrow{D_{123}} y,$$

  where $m = n - 2\tau(K)$.

Recall that the generators x_0 and x_2 are in the idempotent ι_1, while the generators $z_1, \dots, z_m$ and y are in the idempotent ι_2. In all of the above cases, there is either an arrow labeled D_{23} leaving y, or an arrow labeled D_1 entering y.

Let us now consider $\widehat{CFA}(p, 1; 2, 2m+1)$. In particular, we would like to compute enough of $\widehat{CFA}$ to show that the generator ax_2 survives to generate $\widehat{HF}(S^3)$, so we look for algebra relations coming from domains entering or leaving $a = (a_1, a_2)$. We say that a domain from $a = (a_1, a_2)$ to $b = (b_1, b_2)$ *fixes* a_1 if one of b_1 or b_2 is equal to a_1. If a domain does not fix a_1, then we say that the domain *moves* a_1.

We first notice that no domains *from* a that fix a_2 contribute to arrows leaving ax_2 in the complex $\widehat{CFA} \boxtimes \widehat{CFD}$. Nor do any domains *to* a that fix a_2 contribute to arrows entering ax_2 in $\widehat{CFA} \boxtimes \widehat{CFD}$. Both of these statements follow from the computation in Section 4.1.

In light of the above observation, we must consider domains that move a_2. There are no domains to a that move a_2. This follows from the fact that there are only 3 distinct regions in $\overline{\Sigma} \backslash (\boldsymbol{\alpha} \cup \boldsymbol{\beta})$ adjacent to a_2, and that the basepoint w, which we must miss, is in one of them.

We now consider domains from a that move a_2. We claim that none of these domains will contribute to arrows leaving ax_2 in the complex $\widehat{CFA}\boxtimes\widehat{CFD}$. By inspection, there are no domains contributing to an algebra relation of the form

$$m_1(a).$$

Furthermore, there are no algebra relations in $\widehat{CFA}$ of the form

$$m_{2+i}(a,\rho_3,\overbrace{\rho_{23},\ldots,\rho_{23}}^{i}),\quad i\geq 0$$
$$m_{2+i}(a,\rho_{123},\overbrace{\rho_{23},\ldots,\rho_{23}}^{i}),\quad i\geq 0.$$

We can exclude relations of the from $m_{2+i}(a,\rho_3,\rho_{23},\ldots,\rho_{23})$ by attempting to find the boundary of the corresponding domain, or by using the techniques of [6, Section 2.3].

To exclude relations of the form $m_{2+i}(a,\rho_{123},\rho_{23},\ldots,\rho_{23})$, we will use $\mathcal{A}_\infty$-relations (Section 2.1) to reach a contradiction. For simplicity, we will consider the relation $m_2(a,\rho_{123})$; the other cases follow in an identical manner. We have the following $\mathcal{A}_\infty$-relation:

$$0=m_3(m_1(a),\rho_{12},\rho_3)+m_2(m_2(a,\rho_{12}),\rho_3)+m_2(a,\rho_{12}\cdot\rho_3)+m_1(m_3(a,\rho_{12},\rho_3))$$

The first term above is zero, since $m_1(a)=0$. Moreover, by attempting to find the boundary of the corresponding domain, we can also conclude that $m_2(a,\rho_{12})$ and $m_3(a,\rho_{12},\rho_3)$ are equal to zero. Hence, $m_2(a,\rho_{123})=0$ as well.

To compute the Alexander grading of ax_2, we again use the formula

$$\begin{aligned}A(ax_2)&=\tfrac{1}{2}\langle c_1(\underline{\mathfrak{s}}(ax_2)),[\widehat{F}]\rangle\\&=\tfrac{1}{2}\big(\chi(\mathcal{P})+2n_{ax_2}(\mathcal{P})\big),\end{aligned}$$

where now $\mathcal{P}=\mathcal{P}_A+2p\mathcal{P}_D$, and

$$\chi(\mathcal{P}_A)+2n_a(\mathcal{P}_A)=2p^2n-pn+p-2$$
$$\chi(\mathcal{P}_D)+2n_{x_2}(\mathcal{P}_D)=2\tau(K)-\tfrac{n}{2}-\tfrac{1}{2}.$$

(To compute these quantities, we again find it convenient to decompose the domain $\mathcal{P}_A$ as $\mathcal{P}_\lambda + 2pn\mathcal{P}_\mu$, with

$$\chi(\mathcal{P}_\lambda) = 3p - 4p^2 n$$

$$n_a(\mathcal{P}_\lambda) = -p + p^2 n - 1$$

$$\chi(\mathcal{P}_\mu) = 3p + \tfrac{1}{2}$$

$$n_a(\mathcal{P}_\mu) = -\tfrac{p}{2} - \tfrac{1}{2}.)$$

This implies that

$$\begin{aligned} A(ax_2) &= 2p\tau(K) + p^2 n - pn - 1 \\ &= 2\tau(K_{p,pn+1}) - 1, \end{aligned}$$

and so $\varepsilon(K_{p,pn+1}) = 1$, completing the proof of Proposition 6.1.

Chapter 7

Computation of ε for (p, q)-cables

In the previous chapter, we proved that if $\varepsilon(K) = 1$, then $\varepsilon(K_{p,pn+1}) = 1$. The goal of this chapter is to prove Theorem 2; that is, to describe the behavior of ε under cabling, for all values of ε and for all p and q.

What follows is a straightforward modification of Van Cott's work in [20]. Fix a knot K and integers p and m, and define the function

$$H(q) := \tau(K_{p,q;2,m}) - (p-1)q$$

for all q relatively prime to p.

Proposition 7.1. *The function H is non-increasing; that is,*

$$H(q) - H(r) \leq 0$$

for all $q > r$, where both q and r are relatively prime to p.

Proof. Recall our convention that $p > 1$. Let q and r be integers relatively prime to p with $q > r$. Consider the connect sum

$$K_{p,q;2,m} \# - (K_{p,r;2,m}).$$

Notice that $-(K_{p,r;2,m}) = (-K)_{p,-r;2,-m}$. Let k be the smallest positive integer such that $q-r-k$ is relatively prime to p. (Note that k may be equal to zero, and that $q-r-k>0$.) We leave it as an exercise for the reader to show that by performing $2p+2k(p-1)$ band moves, we can obtain the knot

$$(K\#-K)_{p,q-r-k;2,-1},$$

which is concordant to the iterated torus knot $T_{p,q-r-k;2,-1}$ since $K\#-K$ is slice. Thus, we have a genus $p+k(p-1)$ cobordism between $K_{p,q;2,m}\#-K_{p,r;2,m}$ and $T_{p,q-r-k;2,-1}$. Since $|\tau|$ is a lower-bound on the 4-ball genus, we have

$$\begin{aligned} |\tau(K_{p,q;2,m}\#-K_{p,r;2,m}\#-T_{p,q-r-k;2,-1})| &\leq p+k(p-1) \\ |\tau(K_{p,q;2,m})-\tau(K_{p,r;2,m})-\big((p-1)(q-r-k-1)-1\big)| &\leq p+k(p-1) \\ |H(q)-H(r)-(p-1)(-k-1)+1| &\leq p+k(p-1) \\ H(q)-H(r) &\leq 0, \end{aligned}$$

completing the proof of the proposition. □

For K with $\varepsilon(K)=1$, we have that

$$\begin{aligned} H(pn+1) &= \tau(K_{p,pn+1;2,m})-(p-1)(pn+1) \\ &= 2p\tau(K)+(p-1)pn+\tfrac{m-1}{2}-(p-1)(pn+1) \\ &= 2p\tau(K)+\tfrac{m-1}{2}-(p-1) \end{aligned}$$

for all n. But since the function H is non-increasing, this implies that $H(q)=2p\tau(K)+\frac{m-1}{2}-(p-1)$ for all q relatively prime to p. Hence,

$$\begin{aligned} \tau(K_{p,q;2,m}) &= 2p\tau(K)+\tfrac{m-1}{2}-(p-1)+(p-1)q \\ &= 2\big(p\tau(K)+\tfrac{(p-1)(q-1)}{2}\big)+\tfrac{m-1}{2} \\ &= 2\tau(K_{p,q})+\tfrac{m-1}{2} \end{aligned}$$

and so $\varepsilon(K_{p,q}) = 1$. Thus, we have shown that if $\varepsilon(K) = 1$, then $\varepsilon(K_{p,q}) = 1$ for all p and q.

Since $\varepsilon(-K) = -\varepsilon(K)$ and $(-K)_{p,q} = -K_{p,-q}$, we have that if $\varepsilon(K) = -1$, then $\varepsilon(K_{p,q}) = -\varepsilon(-K_{p,q}) = -\varepsilon((-K)_{p,-q}) = -1$; that is, if $\varepsilon(K) = -1$, then $\varepsilon(K_{p,q}) = -1$.

For the case $\varepsilon(K) = 0$, we again appeal to a model calculation; that is, if $\varepsilon(K) = 0$, then τ of any iterated cable agrees with τ of the underlying torus knot, implying that if $\varepsilon(K) = 0$, then

$$\varepsilon(K_{p,q}) = \varepsilon(T_{p,q}).$$

Thus, we have completely described the behavior of ε under cabling.

Chapter 8

Proof of Corollaries 3 and 4

In this chapter, we prove the two corollaries from the introduction.

Proof of Corollary 3. By Theorem 1, it is sufficient to find knots K_n^+ and K_n^- with $\tau(K_n^\pm) = n$ and $\varepsilon(K_n^\pm) = \pm 1$.

For the right-handed trefoil, which we will denote R, we have that $\tau(R) = \varepsilon(R) = 1$, and for the left-handed trefoil, we have that L, $\tau(L) = \varepsilon(L) = -1$. Hence, by Theorems 1 and 2:

$$\tau(R_{2,2m+1}) = 2 + m$$

$$\varepsilon(R_{2,2m+1}) = 1$$

$$\tau(L_{2,2m+1}) = 3 + m$$

$$\varepsilon(L_{2,2m+1}) = -1,$$

and so by taking a cable of a right- or left-handed trefoil, we can construct knots with arbitrary τ, and with ε equal to our choice of ± 1. (Note that this is one way to construct a knot K with $\tau(K) = 0$ but $\varepsilon(K) \neq 0$.) More precisely, let $K_n^+ = R_{2,2n-3}$ and let $K_n^- = L_{2,2n-5}$, and so $\tau(K_n^\pm) = n$ and $\varepsilon(K_n^\pm) = \pm 1$. This completes the proof of Corollary 3. $\square$

Proof of Corollary 4. This corollary was suggested to me by Livingston. We would like to prove that if $\varepsilon(K) \neq \operatorname{sgn} \tau(K)$, then $g_4(K) \geq |\tau(K)| + 1$. Recall that

- If $\varepsilon(K) = 0$, then $\tau(K) = 0$.
- $\tau(\overline{K}) = -\tau(K)$.
- $\varepsilon(\overline{K}) = -\varepsilon(K)$.

Hence, without loss of generality, we may assume that $\tau(K) \geq 0$ and that $\varepsilon(K) = -1$, in which case $\tau(K_{2,1}) = 2\tau(K) + 1$.

We can construct a slice surface for $K_{2,1}$ by taking two copies of the slice surface for K and connecting them with a single twisted strip, hence

$$g_4(K_{2,1}) \leq 2g_4(K).$$

We also have that $|\tau(K_{2,1})| \leq g_4(K)$, or

$$2\tau(K) + 1 \leq g_4(K_{2,1}),$$

so upon combining these two inequalities, we get

$$\tau(K) + \tfrac{1}{2} \leq g_4(K).$$

But $\tau(K)$ and $g_4(K)$ are both integers, hence

$$\tau(K) + 1 \leq g_4(K),$$

concluding the proof of Corollary 4 when $\tau(K) \geq 0$. The case $\tau(K) < 0$ follows by taking mirrors. $\square$

Chapter 9

Cabling and L-space Knots

In this chapter, we focus our attention on a class of 3-manifolds with particularly simple Heegaard Floer homology. For a rational homology sphere Y, Proposition 5.1 of [12] tells us that

$$\text{rk } \widehat{HF}(Y) \geq |H_1(Y, \mathbb{Z})|.$$

An *L-space* is a rational homology sphere Y for which the above bound is sharp. The name comes from the fact that lens spaces are L-spaces, which can be seen by examining the Heegaard Floer complex associated to a standard genus one Heegaard decomposition of a lens space.

We call a knot $K \subset S^3$ an *L-space knot* if there exists $n \in \mathbb{Z}$, $n > 0$, such that n surgery on K yields an L-space. We will denote the resulting 3-manifold by $S^3_n(K)$. Torus knots are a convenient source of L-space knots, since $pq \pm 1$ surgery on the (p, q)-torus knot yields a lens space. It was proved in [13, Theorem 1.2] that if a knot K is an L-space knot, then the knot Floer complex associated to K has a particularly simple form that can be deduced from the Alexander polynomial of K, $\Delta_K(t)$. Thus, knowing that a knot K admits a lens space (or L-space) surgery yields a remarkable amount of information about the Heegaard Floer invariants associated to both the knot K, and manifolds obtained by Dehn surgery on K. In particular, [13, Theorem 1.2] combined with [16, Theorem 1.1] allows one to compute the Heegaard Floer invariants of any

Dehn surgery on an L-space knot K from the Alexander polynomial of K.

Let $g(K)$ denote the Seifert genus of K. In [2, Theorem 1.10], Hedden proves that if K is an L-space knot and $q/p \geq 2g(K)-1$, then $K_{p,q}$ is an L-space knot. We will prove the converse:

Theorem 9.1. *The (p,q)-cable of a knot $K \subset S^3$ is an L-space knot if and only if K is an L-space knot and $q/p \geq 2g(K)-1$.*

It was already known that if $K_{p,q}$ is an L-space knot, then $q > 0$ and $\tau(K) = g(K)$ [20, Corollary 6]. We prove our theorem by methods similar to those used in [2, Theorem 1.10]. An interesting question to consider is whether there are other satellite constructions that also yield L-space knots.

An L-space Y can be thought of as rational homology sphere with the "smallest" possible Heegaard Floer invariants, i.e. $\text{rk } \widehat{HF}(Y) = |H_1(Y,\mathbb{Z})|$. In a similar spirit, an L-space knot K can be thought of as a knot with the "smallest" possible knot Floer invariants. For example, since

$$\Delta_K(t) = \sum_{m,s} (-1)^m \text{ rk } \widehat{HFK}_m(K,s) t^s,$$

so we see immediately that the total rank of $\widehat{HFK}(K)$ is bounded below by the sum of the absolute value of the coefficients of the Alexander polynomial of K, $\Delta_K(t)$. A necessary, but not sufficient, condition for a knot K to be an L-space knot is for this bound to be sharp; see [13, Theorem 1.2] for the complete statement. The spirit of our proof is that when either K is not an L-space knot, or $q/p < 2g(K)-1$, the knot Floer invariants of $K_{p,q}$ are not "small" enough for $K_{p,q}$ to be an L-space knot. We will determine this by looking at the rank of $\widehat{HF}(S^3_{pq}(K_{p,q}))$.

Proof. Recall that $\tau(K)$ is the integer-valued concordance invariant defined by Ozsváth and Szabó in [9]. Let $\mathcal{P}$ denote the set of all knots K for which $g(K) = \tau(K)$. We begin by assembling the following collection of facts.

1. If K is an L-space knot, then $K \in \mathcal{P}$. This follows from [13, Theorem 1.2] combined with the fact that knot Floer homology detects genus [10, Theorem 1.2]

2. Let

$$s_K = \sum_{s\in\mathbb{Z}} (\text{rk } H_*(\widehat{A}_s(K)) - 1),$$

where $\widehat{A}_s(K)$ is the sub-quotient complex of $CFK^\infty(K)$ defined in [15, Section 4.3]. We may think of $CFK^\infty(K)$ as generated over $\mathbb{F}[U,U^{-1}]$ by $\widehat{CFK}(K)$, in which case rk $\widehat{A}_s(K) =$ rk $\widehat{CFK}(K)$ for all s. Recall that rk $\widehat{CFK}(K)$ is always odd, since the graded Euler characteristic of $\widehat{CFK}(K)$ is the Alexander polynomial of K. Therefore, rk $H_*(\widehat{A}_s(K))$ is odd, hence greater than or equal to 1, and so s_K is always non-negative. Let

$$t_K^{a/b} = 2\ \max(0, (2g(K)-1)b - a),$$

for a pair of relatively prime integers a and b, $b > 0$. Notice that

$$t_K^{a/b} = 0 \text{ if and only if } a/b \geq 2g(K) - 1.$$

For $K \in \mathcal{P}$ and a, b as above,

$$\text{rk } \widehat{HF}(S^3_{a/b}(K)) = a + bs_K + t_K^{a/b}.$$

This is a special case of Proposition 9.5 of [16]. In particular, the term $\nu(K)$ appearing in Proposition 9.5 is bounded below by $\tau(K)$ [9, Proposition 3.1] and above by $g(K)$ [11, Theorem 5.1], so $K \in \mathcal{P}$ implies $\nu(K) = g(K)$. We notice that

$$K \text{ admits a positive } L\text{-space surgery if and only if } s_K = 0.$$

Indeed, if $s_K = 0$, then p surgery on K yields an L-space, for any integer $p \geq 2g(K) - 1$. Conversely, if K is an L-space knot, then there exists some integer $p > 0$ such that p surgery on K is an L-space, in which case s_K, which is always non-negative, must be 0.

3. Recall our convention that p, q are relatively prime integers, with $p > 1$. If $K_{p,q} \in \mathcal{P}$, then $K \in \mathcal{P}$, and if $K \in \mathcal{P}$, then $\tau(K_{p,q}) = p\tau(K) + \frac{1}{2}(p-1)(q-1)$. These facts are Corollaries

4 and 3, respectively, in [20]. Therefore, if $K_{p,q} \in \mathcal{P}$, we have

$$\begin{aligned}(2g(K)-1)p-q &= (2\tau(K)-1)p-q \\ &= 2\big(p\tau(K)+\tfrac{(p-1)(q-1)}{2}\big)-1-pq \\ &= 2\tau(K_{p,q})-1-pq \\ &= 2g(K_{p,q})-1-pq,\end{aligned}$$

or equivalently,

$$\text{if } K_{p,q} \in \mathcal{P}, \text{ then } t_K^{q/p} = t_{K_{p,q}}^{pq}.$$

4. It is well-known that pq surgery on $K_{p,q}$ is the manifold $L(p,q)\#S^3_{q/p}(K)$ (see [2, Proof of Theorem 1.10] for a nice proof of this fact). We also have from [**?**, Proposition 6.1] that

$$\text{rk } \widehat{HF}(Y_1 \# Y_2) = \text{rk } \widehat{HF}(Y_1) \cdot \text{rk } \widehat{HF}(Y_2).$$

Then

$$\begin{aligned}\text{rk } \widehat{HF}(S^3_{pq}(K_{p,q})) &= \text{rk } \widehat{HF}(L(p,q)) \cdot \text{rk } \widehat{HF}(S^3_{q/p}(K)) \\ &= p \cdot \text{rk } \widehat{HF}(S^3_{q/p}(K)).\end{aligned}$$

With these facts in place, we are ready to prove the theorem. Assume $K_{p,q}$ is an L-space knot. Then by (1) and (3), $K_{p,q} \in \mathcal{P}$ and $t_{K_{p,q}}^{pq} = t_K^{q/p}$, and by (2),

$$\text{rk } \widehat{HF}(S^3_{pq}(K_{p,q})) = pq + s_{K_{p,q}} + t_{K_{p,q}}^{pq} \quad \text{and} \quad \text{rk } \widehat{HF}(S^3_{q/p}(K)) = q + ps_K + t_K^{q/p}.$$

Then by (4), $\text{rk } \widehat{HF}(S^3_{pq}(K_{p,q})) = p \cdot \text{rk } \widehat{HF}(S^3_{q/p}(K))$, and $s_{K_{p,q}} = 0$, since $K_{p,q}$ is an L-space knot. So we find that

$$p^2 s_K + (p-1)t_K^{q/p} = 0.$$

Therefore, since $p > 1$, we have that s_K and $t_K^{q/p}$ must both be zero, or equivalently, K is an L-space knot and $q/p \geq 2g(K)-1$. This completes the proof of the theorem. $\square$

Remark 9.2. The contents of this chapter appears in [5].

Chapter 10

Future directions

The invariant ε defined in this thesis has applications beyond determining τ of iterated cables. These applications will be discussed in depth in [4]; we outline the main ideas below.

The goal is to use ε to define a new concordance homomorphism that is strong enough to detect linear independence in $\mathcal{C}$. We will turn the monoid of chain complexes $CFK^\infty(K)$ (under tensor product) into a group, which we will denote $\mathcal{F}$, in much the same way that the monoid of knots (under connected sum) can be made into the group $\mathcal{C}$ by quotienting by slice knots.

Definition 10.1. *Let $CFK^\infty(K)^*$ denote the dual of $CFK^\infty(K)$; that is,*

$$CFK^\infty(K)^* = \mathrm{Hom}_{\mathbb{F}[U,U^{-1}]}(CFK^\infty(K), \mathbb{F}).$$

Define the group $\mathcal{F}$ to be

$$\mathcal{F} := \big(\{CFK^\infty(K) \mid K \subset S^3\}, \otimes\big)/\sim$$

where

$$CFK^\infty(K_1) \sim CFK^\infty(K_2) \iff \varepsilon(CFK^\infty(K_1) \otimes CFK^\infty(K_2)^*) = 0.$$

Theorem 10.2 ([4])**.** *The map*

$$\mathcal{C} \to \mathcal{F},$$

sending a class in $\mathcal{C}$ represented by K to the class in $\mathcal{F}$ represented by $CFK^\infty(K)$ is a group homomorphism.

This group $\mathcal{F}$ has the advantage that it can be studied from an algebraic perspective. In particular, $\mathcal{F}$ is totally ordered, with an additional well-defined notion of "$\ll$". Moreover, we can use the relation $\ll$ to define a filtration on $\mathcal{F}$. We can also use spectral sequences to define a second, independent filtration.

The rich structure on $\mathcal{F}$ is powerful enough to detect linear independence in $\mathcal{C}$. Let $T_{p,q}$ denote the (p,q)-torus knot, $K_{p,q}$ the (p,q)-cable of K, and D the (positive, untwisted) Whitehead double of the right-handed trefoil.

Theorem 10.3 ([4]). *The topologically slice knots*

$$D_{p,p+1}\# - T_{p,p+1}, \quad p \geq 1$$

are independent in the smooth concordance group; that is, they freely generate a subgroup of infinite rank.

The first examples of an infinite family of smoothly independent, topologically slice knots was given by Endo [1]. His examples consist of certain pretzel knots. More recently, Hedden and Kirk [3] showed that an infinite family of (untwisted) Whitehead doubles of certain torus knots are smoothly independent. The structure of $\mathcal{F}$ shows that our examples (when $p \geq 2$) are smoothly independent from both of these earlier families.

Let $P(K)$ denote the satellite of K with pattern P; that is, P is a knot in $S^1 \times D^2$, which we then glue into the (zero framed) knot complement $S^3 - \text{nbd } K$. Recall that the map $P(-) : \mathcal{C} \to \mathcal{C}$ given by

$$[K] \mapsto [P(K)]$$

is well-defined, by "following" the concordance along the satellite.

We obtain a similar well-defined map on $\mathcal{F}$:

Proposition 10.4 ([4]). *The map $P(-) : \mathcal{F} \to \mathcal{F}$ given by*

$$[CFK^\infty(K)] \mapsto [CFK^\infty\big(P(K)\big)]$$

is well-defined.

By composing P with τ, we obtain a new concordance invariant

$$\tau_P(K) := \tau\big(P(K)\big),$$

since $K_1 \sim K_2$ implies that $P(K_1) \sim P(K_2)$. Let $\{\tau_P(K)\}_P$ denote the collection of τ_P for all patterns P in $S^1 \times D^2$. In the following theorem, we see that the information contained in $\{\tau_P(K)\}_P$ is exactly the information contained in $[CFK^\infty(K)]$.

Theorem 10.5 ([4]). $[CFK^\infty(K_1)] = [CFK^\infty(K_2)]$ *if and only if* $\{\tau_P(K_1)\}_P = \{\tau_P(K_2)\}_P$.

Does the map $P(-) : \mathcal{C} \to \mathcal{C}$ always take linearly independent collections of knots to linearly independent collections of knots? To address this question, we have the following theorem:

Theorem 10.6 ([4]). *For each $n \in \mathbb{N}$, there exists a collection of linearly independent knots*

$$\{K_i\}_{i=1}^n$$

and a pattern P such that

$$\{P(K_i)\}_{i=1}^n$$

are independent in $\mathcal{C}$.

We hope that the applications of ε, both in this thesis and in future work, convince the reader of the utility and power of this new concordance invariant. Our goal is to be able to give a complete algebraic description of the group $\mathcal{F}$, helping to shed light on the structure of the smooth concordance group.

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