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Modeling and Control of Color Xerographic Processes

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NOTE: At the time of publication, author Daniel Koditschek was affiliated with the University of Michigan. Currently, he is a faculty member in the Department of Electrical and Systems Engineering at the University of Pennsylvania.

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Comments
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Modeling and Control of Color Xerographic Processes*

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Abstract

The University of Michigan and Xerox's Wilson Research Center have been collaborating on problems in color management systems since 1996, supported in part by an NSF GOALI grant. This paper is divided into three sections. The first discusses the basics of xerography and areas where systems methodology can have a potential impact. The second section describes the authors' approach to the approximation of color space transformations using piecewise linear approximants and the graph intersection algorithm, with a brief review of some of the analytical and numerical results. The last section expounds on some of the benefits and difficulties of industry-university-government (IUG) collaboration.

1 Introduction

When Xerox introduced the 914 copier in 1959 it created an industry that quickly revolutionized the office. Many subsequent breakthroughs have spawned an almost $100B annual business and the industry is currently enjoying a new era of growth stimulated by the advent of high speed and high quality color printing and copying. At stake is the nearly four trillion page annual color printing market, much of which has yet to benefit from the laser printer's ability to produce customized documents on demand. Many xerographic products have already been introduced to this marketplace but significant quality issues threaten to impede growth. Customers are often surprised and disappointed to find that the colors on their monitors are not accurately reproduced on the prints.

A few years ago, we developed a collaborative research project aimed at modeling and control of color printing. This research collaboration has been supported in part by NSF under the GOALI program. The purpose of this paper is to give a brief overview of the research activities under this GOALI project. It is hoped that this will be of benefit to those interested in modeling and control issues in color management systems as well as to those interested in industry-university-government (IUG) collaborations.

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2 Control Problems in Xerography

In previous publications [1] we have given specific xerographic component process descriptions, and discussed at length the prospects for systems oriented solutions to the problems of this industry [2, 3]. We will very briefly review some of these issues here, before reporting on the specific technical direction our recent work has taken.

2.1 Xerography within a Color Management System

Xerography, the process invented by Chester Carlson in 1938, is the foundation of laser printing. In its modern realization [4], xerographic printing is a sequential imaging process involving the successive activation of numerous subsystems whose coordination is essential to the quality of the ultimate product - a printed page whose colors are perceptually indistinguishable from the original image. A photoconductor belt or drum is charged to a potential. A laser then selectively discharges the surface of the device according to image information received from a computer. This charged surface is then presented to the developer station, a device containing charged particles of colored plastic about 5-10 microns in diameter. These particles, called toner, are attracted to the photoreceptor in the areas where the image is to be colored. The toner is transferred to paper, heated between rollers causing the powder to adhere to the paper, and presented to the output tray for finishing. The process entails so many distinct physical modalities - optics, chemistry, thermal processes, mechanical transport - that a modern day mid-volume printer will typically incorporate on the order of 10 computers and microcontrollers along with on the order of 100 distinct sensory channels.

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Color may be specified in a 3-dimensional space [5]. Each point in the space describes a single color. One psycho-physically convenient representation, CIELab, uses coordinates L* to describe lightness, a* for red-green, and b* for blue-yellow. In contrast, color printing entails a device specific representation of color with respect to three or more primary colorants, typically Cyan, Magenta, and Yellow, (CMY) used in various combinations to produce all the colors desired. The total set of printable colors using the primaries is referred to as the color "gamut."* When a particular psycho-physically specified color is desired a "recipe book," typically taking the form of a look up table, is used to compute the required change of coordinates to the device specific representation. The formula for any specific color is obtained by interpolation into this table. Differences between colors are described as Euclidean distances between the points in the psycho-physical space and labeled ΔE. The human visual system is sensitive to ΔE values greater than or equal to 1. Typical color documents will vary by more than 10 in some areas and often by as much as 25.

In an imaging system comprising a network of scanners, video monitors, printers, and other color imaging devices, each device communicates with the others in terms of a device independent color space. Typically the psycho-physical (L^*, a^*, b^*) is transformed to the printer specific color separated space, (C, M, Y), by using multidimensional color correction tables containing the inverse of the printer characteristics inside the image processing system. The color separated image is then converted to half-tones delivered to the digital image path electronics where the electronics perform the function of generating in real time the raster page images on the photoreceptor of a xerographic printer. These page images are then printed according to the xerographic process described above.

In summary, the problems of regulating the component processes within a given device are compounded by the larger issues of color systems management that require the tightly integrated functioning of geographically distinct scanners, video monitors and printers. Since human color perception is so discriminating, the tolerance for error in the internal regulation and integration of the constituent devices and in the coordination between devices is very low.

2.2 Systems Approaches to the Color Systems Management Problem

Systems and control theory can impact color systems management by addressing the problems of:

- **Color Consistency:** The relationship between color specification (the psycho-physical coordinates) and color production (device specific coordinates) is variable, eventually leading to inaccurate color reproduction. Despite recent advances in the control of xerographic marking engines [6], fielded printing subsystems are generally inadequately controlled, making this variability a significant performance issue. Even under the best circumstances, the transformation between color space representations is non-linear and dynamic. As they drift, these transformations must be tracked via color calibration of printing devices using a fielded spectrophotometer. In the absence of parsimonious representations of the color space transformations, calibrating look up tables requires that many patches with specified values of CMY be printed and then read. In practice, this tedious process is performed during machine design and only infrequently thereafter. While more efficiently parametrized representations of the color space mapping are available, none presently capture in computationally effective form the invertibility of the process - for each combination of CMY within the color gamut on a specific device there results one and only one L*a*b* value. Yet printing consistently the device independent color specification across the networked family of devices benefits significantly from a closed form formula for the map and its inverse.

- **Hybrid Diagnostics:** A hierarchy of diagnostic levels is required in complex print engines to increase reliability and reduce service cost. In this hierarchy, some levels might diagnose active failures, some might initiate failure recovery algorithms and some might simply report trend data to track impending machine failures, component wear, machine usage, and so on. Since there is so tight a coupling between discrete logical machine states and the component physical process variables the diagnostic tool sets necessarily take on a correspondingly hybrid character. Because of the multiple physical modalities and numerous interlocking subsystems, it is a challenge to insure that these diagnostic components operate automatically and with sufficient performance guarantees as to afford effective service warranties.

- **Media Handling:** Xerographic engines must be capable of printing on a range of media, including various types of paper, films, etc. These media differ in properties such as stiffness, strength, coefficient of friction, and moisture content. The media transport system must move media through the machine's "paper path," from the intake tray through the xerographic engine to the output tray, without the occurrence of jams, smearing, or misregistration of images. To ensure reliable operation of the machine, the transport process must be robust to varying media properties as well as environmental conditions. Increasing the speed of media throughput and decreasing the amount down-

*In modern printers black is often added (CMYK)to expand the color gamut. The printing industry is also exploring the use of six or more colors to further expand the range of reproducible colors. For ease of exposition we will confine the discussion in this paper to three dimensional (CMY) colorant space.

time due to mishandling can significantly increase the efficiency of the reproduction system.

• **Integrated Subsystem Design:** Materials – chiefly toner/carrier and photoreceptors – represent the most difficult and crucial components of xerographic engines. Typically, these elements are the last items to be decided during product design and they occasionally change even after product launch. The nature and parameters of the control algorithms depend strongly on the behavior of the materials and thus, absent a rapid algorithm design capability, product release or upgrade is delayed. Typically, products have a fixed terminus to their marketplace acceptance window: the number of machine placements diminishes significantly with setbacks in product release date. Thus, delays in initial machine offering have a strongly deleterious effect on product lifetime and profitability. These considerations motivate corporate interest in controls almost as powerfully as the perceived opportunities for intrinsic product improvement. Furthermore product performance and cost effectiveness can be enhanced through simultaneous, cooperative design of the control system and plant, but in order to include feedback controllers in the evolution of the product, it will become necessary to embed physically meaningful parameters in the controller itself. In sum, a well understood and rational control system development methodology can have a major impact on time to market.

These four areas are discussed at some length in Reference [1]. In this paper we focus on the problem of color consistency. The goals of this aspect of our GOALI project are to develop an accurate but efficiently parametrized representation of color space transformations that preserves in closed form the intrinsic invertibility of the mapping from CMY to L*a*b* coordinates, thereby promoting the universality of the device independent color representation across the geographically distributed color device network.

3 Approximating Color Space Transformations

In the color space transformation problem discussed above, we wish to find the inverse of the print engine map, T^{-1} : L*a*b* $\rightarrow$ CMY. We require a model for both T and T^{-1}, since any control applied to the printing engine will directly affect T, whereas T^{-1} is the function needed to precompute CMY commands for the printing engine. (See Figure 1.) The problem may be reduced to the estimation of an unknown function, from $\mathbb{R}^3$ to itself, from a limited set of data, since color experiments are both costly and time consuming. The estimated function should be approximated from a function family whose members are closed-form invertible, since both T and T^{-1} are of central importance in the practical problem setting. The approximants should admit a physically relevant parameterization so

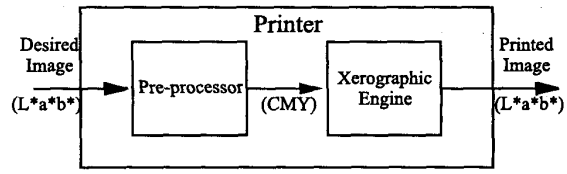

Figure 1: Decomposition of a xerographic printer into the print engine and pre-processor. The engine embodies the transformation T, while the pre-processor performs an approximation of the inverse $\tilde{T}^{-1}$, such that $T \circ \tilde{T}^{-1}$ is approximately the identity for within gamut colors.

that the functional form respects known constraints that generally reflect the influence of component technologies. For example, even though the print engine transformation is poorly characterized by first principles models, the gamut boundary is often known with some accuracy since it is "designed in" by the choice of toner composition.

These desiderata are not well served by any presently available function approximation methodology. There are many methods of estimating a function from data, and in recent years nonlinear-in-parameters function families, such as neural networks and radial basis functions, have become popular. However, these approximations generally do not preserve any invariant properties: invertibility and gamut boundaries being particularly relevant to the problem setting at hand. It is not surprising that the lookup table (LUT) has become the *de facto* industry standard for representing the print engine color space transformations. Since an LUT can be indexed with respect to either the domain or codomain entries, this functional representation can easily capture invertibility. Also, as *a priori* knowledge about the codomain values associated with particular regions of the domain may be locked into the table, enforcing the gamut boundary is straightforward. Of course, the central problem with the LUT representation is that it is badly overparametrized: reasonable accuracy can typically be achieved only at the expense of a huge number of entries. For example, to cover gamut volumes on the order of $(100\Delta E)^3$ units yet deliver accuracies on the order of $1\Delta E$ units requires at minimum one table entry per $10\Delta E$ units, a $10 \times 10 \times 10$ entry table, incurring 6000 parameters, typically acquired in a lengthy and tedious calibration exercise.

It is natural, then, to wonder whether a more sophisticated lookup table incorporating a less uniform grid might achieve similar accuracy with far fewer parameters. The question now arises as to how to choose the sampling distribution — that is, how to partition the color space into regions just small enough to keep the approximation error low; and how to manage the interpolation between such unstructured regions in such a fashion that the resulting approximation is at least continuous. Such questions lead very naturally to the

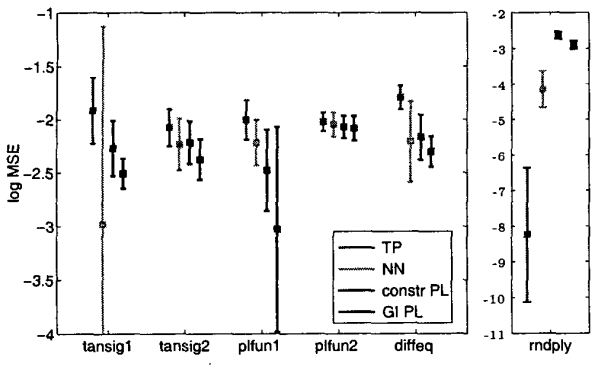

Figure 2: Mean and standard deviation bars for the Log Mean Squared Error on the six function families. The approximation techniques are Taylor Polynomial(TP), Neural Network(NN), PL using constrained optimization (constr PL) and PL using GI (GI PL)

Table 1: Mean computational cost (in megaflops) of the approximation techniques on the six function families.

	TP	NN	PL con	PL GI
tansig1	0.187	10.2	1.89	0.168
tansig2	0.187	9.38	1.68	0.240
plfun1	0.187	9.71	1.93	0.185
plfun2	0.187	9.74	2.12	0.538
rndply	0.187	10.7	1.02	0.157
diffeq	0.187	9.71	2.06	0.195

notion of a continuous piecewise linear representation.

A PL approximant partitions the domain into simplices (tetrahedra for $\mathbb{R}^3$), and provides an affine function over each partition cell. Due to constraints in the problem, we are only interested in continuous piecewise linear approximants, as opposed to the possibly discontinuous approximations typically studied in the spline community [7, 8, 9]. Under the condition of continuity, a PL function is parsimoniously parameterized by its "knots," domain-codomain pairs consisting of an extreme point of a simplex in the domain along with the codomain point to which it maps. If the codomain points form a valid triangulation of the codomain, then the approximation is invertible, and the inverse may be computed by simply switching the roles of the domain and codomain for each knot.

Piecewise linear approximations are a locally nonlinear-in-parameters approximation technique. That is, a change in a single parameter affects only a limited area of the domain. The graph intersection algorithm, our proposed method for computing PL approximations for one dimensional functions, takes advantage of this structure [3]. A single iteration of the graph intersection algorithm consists of two steps. The first step keeps the partition fixed, and the approximation within each cell is computed as the least squares linear fit, typically resulting in discontinuities at cell boundaries. The second step fixes the linear approximant in each cell and adjusts the partition in order to regain continuity, moving the knot shared by two cells to the intersection point of the cells' graphs. (Under special conditions, the function as well as the partition must be adjusted in the second step) The steps are iterated until convergence.

In order to gauge the potential of PL approximants and the GI algorithm, we studied the approximation of homeomorphisms of the interval $[0, 1]$ into itself from noiseless data. We compared the performance of PL approximations against neural networks(NN) and Taylor polynomials(TP) representing truncated Taylor series. Each approximation was given same number of parameters, 16, corresponding to a NN with 5 "neurons", a degree 15 polynomial, and a PL with 9 line segments. Six families of homeomorphisms were chosen. Families tansig1 and tansig2 are superpositions of hyperbolic tangents, and families plfun1 and plfun2 are piecewise linear. Families tansig1 and plfun1 lie within the representational power of the neural network and the piecewise linear approximant, respectively, whereas the approximations are "underparameterized" on tansig2 and plfun2. The remaining two families were polynomials, rndply, and time one maps from differential equations, diffeq. Functions were randomly sampled from each of these families. All code was implemented in MATLAB. TP uses linear projection and NN uses Levenberg-Marquardt descent. Results for two different forms of PL training are shown. The first technique uses constrained optimization on the squared error. The second PL training technique is the graph intersection algorithm discussed above. Figure 2 summarizes the approximation error for the study. In general the nonlinear-in-parameters families are performing better than TP, and the NN and PL approximations perform similarly in terms of mean squared error on the various families. An exception is the polynomial family, on which NN performs notably better than PL, but otherwise PL is competitive in terms of error. It is also interesting to note that the Graph Intersection algorithm outperforms the constrained optimization training method. This is in part due to the GI algorithm's restart mechanism when it finds a local minimum. Table 1 shows training cost in terms of floating point operations (flops) for the algorithms on various function families. The results show that GI is very fast computationally, sometimes using less flops than the linear-in-parameters approximation, TP, with the same number of parameters. Even with restarts, PL GI uses fewer flops than PL constr and much less than NN. Since PL GI are piecewise linear approximations, they also benefit from being closed form invertible.

In addition to the numerical studies, we have proven

a local convergence result for the one-dimensional approximation (as opposed to estimation) version of the algorithm, where the entire underlying function is available (rather than just data). The result shows that when the underlying function is also piecewise linear with the same number of cells, then there is a neighborhood of the parameter values within which the GI algorithm's approximation will converge to the underlying function in the L_∞ sense. Extensive coverage of the convergence proof and the numerical studies is presently in preparation [10].

The generalization of the graph intersection to higher dimensions is of critical importance in the color space problem, since the domain of any color space is at least three. The difficulty is that in higher dimensions, each knot is shared by several partition cells, rather than just two, so generically there is no unique intersection point of graphs of all cells which share a knot. The rigorous theoretical results in 1-dimension and the successes of the numerical work encouraged us to pursue an experimental extension of the Graph Intersection algorithm into higher dimensions. We have generalized the GI algorithm to maps in $\mathbb{R}^3$. (There is no straightforward generalization – we have taken one of many possible generalizations.) We have also written software to implement this generalization. Piecewise linear datasets with both single and multiple interior vertices were accurately fit for cases of both small and large distortion of the domain with starting points distant from the values for which the dataset was calculated. Figure 3 shows the domain partition of an approximation with two interior points. Favorable results also have been obtained in the presence of noise and "incorrect" triangulations. Thus we are confident that the multidimensional GI algorithm serves to approximate effectively the domain-codomain transformation.

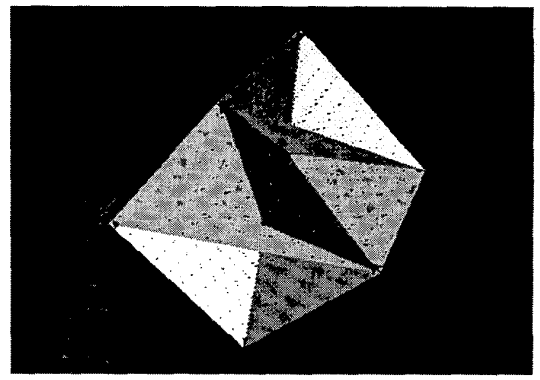

Figure 3: A partition of a three dimensional cubic domain. The eight exterior vertices of the cube are fixed, and the two interior vertices move freely, giving 18 cells in the partition.

4 Industry-University-Government Collaboration

In developing and shepherding this GOALI project, we faced and resolved a number of issues generic to successful industry-university-government (IUG) collaboration. This section highlights some of the critical lessons learned with the hope that these may benefit other groups pursuing similar collaborations.

It is critical that the University and the Corporation identify their mutual goals relative to the disposition of Intellectual Property. The purpose of joint industry-university projects is to construct and develop ideas that have both academic interest and industrial application. Naturally, the corporate partner desires access to the application for their products from which they will acquire revenue. The academic partners' contributions have monetary as well as intellectual value and as such deserve compensation. With increased marketplace competition and scarcity of research funding from traditional sources the problem of compensation is complex because of the Corporate desire to keep costs low and achieve competitive advantage and the University desire to find creative sources of revenue.

For projects funded solely with public funds, one choice is to restrict the relationship to purely theoretical activities without being informed of the specific application. Here the academics involved are abstracted from the real problem and the corporate partners apply the results on their own. Intellectual property(IP) is publishable by the joint project members and application patents belong to the Corporate partner. The tendency in this situation is for the relationship to become more abstract and superficial while being conducted at arms length. The academic partners then frequently ask questions which the corporate partners respectfully decline to answer. The other extreme is that the project includes the application whose IP is either jointly held or enters the Public Domain. In this case the Corporate partner loses competitive advantage because the IP is jointly held and may be licensed for revenue by the University. There are many gradations which may be built into the documents defining the relationship but these two extrema bound the space.

We have addressed this very difficult problem by constructing a bi-modal relationship. In one aspect of the project, the academic players and the corporate partners collaborate on the construction of a generic theoretical framework with application to a broad class of problems. This aspect is funded through the NSF GOALI program and results in this mode may be freely published. If intellectual property of significant value is generated in this process, it is owned jointly by the university and the corporation. A second part of the relationship involves much more short term application focused work funded by corporate hire of the professors as consultants. This gives the academic partners access

to very sensitive proprietary corporate information. A strict division is in place between these two modes of the relationship. The corporate partners influence the activities of the student only in the GOALI program. The burden of maintaining this division lies with the academic partners as University employees. One disadvantage of this approach is that it places the graduate students in the uncomfortable position of being locked out of confidential discussions relating to the problems of short term interest and leaves them without a specific immediate context for their work until the patent applications are filed.

Once the relationship and its ground rules are established, sustaining the relationship is the next responsibility of the team. Commitment on the part of both parties is required but Corporate dedication is usually the determining factor in this aspect. The Corporate motivation is based on the prospect that the product line will be affected. Thus, semi annual reviews of the project by the Corporate partner together with the support of a senior management champion is useful. Progress reports and prototype demonstrations can sustain the interest of the Corporate management until product engagement is successful. The Corporate partners must have access to the engineering and manufacturing divisions of the Corporation to assure that there is a "path to the sea" for the work.

Academics are not accustomed to working under very short timelines. But time-to-market pressures are paramount in modern commerce, and industries generally adapt incremental product enhancement strategies in response. Hence, multiple engineering solutions are considered to solve a given technical problem. On most occasions, the winning solution is the one the industrial researcher understands best and can most clearly articulate within the industrial context, packaging it to emphasize the business value rather than technical elegance. Otherwise, the chances of ever getting ideas into products are limited at best. Successful collaboration requires of all partners a stretch beyond their customary mode of work but has the potential to repay richly intellectual rewards to the participating individuals and material rewards to the sponsoring institutions.

5 Future work

Having managed a workable path through the potentially vitiating institutional constraints, the UM-Xerox Collaboration on Control of Xerographic Imaging has now reached a mature and rewarding stage. Interesting new directions for the GOALI project continue to emerge from the short term and commercially focused consulting work, but we confine our discussion of near term future work to the technical points raised above.

A thorough examination of the graph intersection algorithm in higher dimensions is required to determine its potential value in the color reproduction problem. We plan to conduct a careful numerical study, similar to study in 1D, for the 3D case. Beyond this numerical study we will attempt to extend our theoretical insight to the algorithm in higher dimensions. One of the weaknesses of the PL approximant is the way the complexity of the partition explodes with increasing dimension. This is both a blessing and a burden. In higher dimensions relatively few knots give rise to a great number of partition cells, providing a parsimonious representation, but the "bookkeeping," determining in which cell a query lies, becomes expensive. The relationship between the approximation and the topological and computational nature of the partition will be critical to our understanding of the algorithm. Another possible direction is to consider total squared error (TSE) rather than mean squared error as our evaluation criterion.

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