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Comments
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Stability of Coupled Hybrid Oscillators*

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Abstract

We describe a method for the decentralized phase regulation of two coupled hybrid oscillators. In particular, we prove that the application of this synchronization method to two hopping robots, each of which individually achieves only asymptotically stable hopping, results in an asymptotically stable limit cycle for the coupled system exhibiting the desired phase difference. This extends our previous work wherein the application of the method to two individually deadbeat-stabilized oscillators (paddle juggling mechanisms) was shown to yield the desired result. Central to this method is the idea that cyclic systems may be composed into a larger, aggregate, cyclic system. Its application entails moving from physical coordinates (for example, the position and velocity of each constituent mechanism) to the coordinates of phase and phase velocity. Within this canonical coordinate system we construct a model dynamical system, called a reference field, which encodes the desired behavior of each cyclic system as well as the phase relationships between them. We then force the actual composite system to behave like the model.

1 Introduction

Dynamic, cyclic systems abound in robotics and automation. A legged robot has a cyclic gait composed of single-leg cyclic behaviors [24] each of which manages in part the balance of total kinetic and potential energy. A juggling robot must control a number of balls in a cyclic pattern by mirroring the desired pattern with its actuators [4]. A factory robot on an assembly line repeats an assembly task over and over in synchronization with other robots performing other, complementary tasks in such a manner that the ultimate goal of the factory is realized [16, 13]. Coordinating subsystems of independently cyclic components — whether legs, arms, or assembly line work stations — requires a coupling mechanism sensitive to both the constituent dynamics as well as the coherence of the working composite system.

Ideally, one would wish to treat the problem of coupling cyclic systems as a matter of formal *composition* incorporating a compositional semantics similar to that found in computer programming languages. We have pursued this idea, for example, in previous work on composing factory designs and programs from information about how to assemble a product [16, 13]. Those results, however, presuppose a static world model wherein "wait modes" may by employed by factory robots at any time in their cycles. The present work makes no such assumptions. In fact, quite the opposite, a falling ball or leg presents a robot with impending events of increasing urgency that, due to the intrinsic momentum involved, cannot be stopped. In general, we believe that the real criterion of merit for any synthesis methodology in robotics and automation arises from its relevance to problems involving the exchange of energy with an environment — the capability to program physical work.

A formal composition technique for assembling coherent dynamical systems from modular dynamical constituents allows for distributed control. As robotic systems become more complex and modular, centralized control becomes less feasible because of bandwidth limitations as well as programming complexity. The coupling mechanisms we describe require only that each constituent robot accomplish a specified (sub) task itself and communicate some aspect of its state to neighboring robots. Thus, from a practical point of view, a robot program may consist simply of a set of locally correct behaviors and communication channels where the global state is used only for purposes of analysis. Conversely, coupled dynamical behaviors are hard enough to get right that no such composition technique can ever hold a claim to practical utility in the absence of analytical guarantees of correctness.

This paper advances our goal of a practical method-

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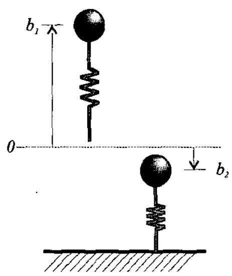


Figure 1: The physical model examined in this paper. Two hopping robots, controllable via adjustable spring stiffnesses, are forced to a desired hopping height and phase separation.

ology for dynamically valid composition systems by demonstrating for a special case of particular interest — an intermittently actuated hybrid oscillator — that the constituent subsystems may be correctly composed in a more or less formulaic fashion. Following a prescription we first outlined in [17], we show that two Raibert style vertical hopping robots [25] can be synchronized (see Figure 1) using a method we first applied to juggling [17] and which we consider to be a way-point between juggling and running. The method involves changing coordinates from the position and velocity of each constituent system, to the canonical coordinates of *phase* and *phase velocity*. Within this canonical coordinate system, we construct a model dynamical system, called a *reference field*, which encodes the desired behavior of each of the cyclic subsystems as well as the desired phase relationships between them. We then force the actual coupled system to behave like the model. Section 2.2 is a review, from [17], of the application of this method to juggling two balls with a paddle. Then, Section 3, comprising the main contribution of this paper, applies the method to synchronized hopping. The technical advance this new example represents may be appreciated by noting the difference in the presumed control authority over the oscillators to be composed. In juggling, the paddle can regulate a ball to a desired apex within one hit. In the hopping example, a leg can only asymptotically approach the desired hopping height after the stiffness of its leg-spring is adjusted. It is not obvious that the method we used to couple two bouncing balls should apply to synchronizing two hopping legs. However, this is exactly what we show to be the case.

1.1 Background and Related Research

Coupled oscillators have long been used to model complex physical and biological settings wherein phase regulation of cyclic behaviors is paramount [11]. The biological reality of neural central pattern generators (CPGs) — oscillatory signals that arise spontaneously from appropriate intercommunication between neurons — seems to have been conclusively demonstrated in organisms ranging from insects [23, 9] to lampreys [7]. Mathematical models proposed to explain the manner in which families of coupled dynamical systems can stimulate a sustained oscillation and stably entrain a desired phase relationship have become progressively more biologically detailed [6, 10, 12]. However, while we are intrigued by the capabilities of purely "clock driven" systems [28, 27, 21], it seems clear that no significant level of autonomy can be developed in the absence of perceptual feedback. The present investigation cleaves to the opposite (i.e., perceptually driven) end of the sensory spectrum in requiring that the state of a controlled system be sensed or reported at least intermittently. In this sense, the present work bears a closer relationship to the biological literature concerned with reflex modulated phase regulation [8].

Many tasks in robotics and automation entail a cyclic exchange of energy between a machine and its environment. This is evidently the case for legged locomotion systems as well as for many less obvious examples wherein a mechanism repeatedly changes its local "shape" so as to effect some global "progress" [22]. When viewed from an appropriate geometric perspective, the recourse to repetitive self-motion may be interpreted as a means of "rectification" — exercising indirectly the unactuated degrees of freedom through the influence of the actuated degrees of freedom arising from an interaction between symmetries and constraints [1]. Because our notion of a task is so completely bound up with the requirement to perform work — tuning the closed loop dynamical interaction between the robot and its environment — this invaluable geometric control perspective provides no solution but merely a complete account of the (open loop) setting within which our search for stabilizing feedback controllers can begin. Since the dynamics in question are inevitably nonlinear, the relation between open loop controllability properties and feedback stabilizability properties is far from clear.

2 Controlling Phase

In this section we describe in general our method of phase regulation. Portions of this section appeared previously in [17] and are repeated here to provide the context for the present result.

Let $f^t : \mathbb{R} \times X \to X$ be a flow on X. We are concerned with flows that are cyclic in the sense that a global cross section Σ can be found. For any point

$x \in X$, define the **time to return** of x to be

$$t^+(x) = min\{t > 0 \mid f^t(x) \in \Sigma\} \qquad (1)$$

and define the **time since return** of x to be

$$t^-(x) = min\{t \geq 0 \mid f^{-t}(x) \in \Sigma\} \, . \qquad (2)$$

The first return map, $P : \Sigma \to \Sigma$, is the discrete, real valued map given by $P(x) = f^{t^+}(x)$.

Let $s(x) = t^+(x) + t^-(x)$ be the time it takes the system starting at the point $f^{t^-}(x) \in \Sigma$ to reach Σ again, also know as the **period** of x. Now, define the **phase** of a point x by

$$\phi(x) = \frac{t^-(x)}{s(x)} \in [0, 1]. \qquad (3)$$

Notice that the rate of change of phase, $\dot{\phi}$, is equal to $1/s$. Therefore, $\dot{\phi}$ is constant or piecewise constant, changing only as the state passes through Σ.

In this paper, we give examples (juggling and hopping) where $h : X \to Y$, defined by $h(x, \dot{x}) = (\phi, \dot{\phi})$, is actually a change of coordinates where $Y = S^1 \times \mathbb{R}^+$. We use the section $\Sigma \in X$ which corresponds to the set of states where the robot may contact (and thereby actuate) the system. The image of this section $h(\Sigma)$ will be given by the set $\mathcal{C} = \{(0, \dot{\phi}) \mid \dot{\phi} \in \mathbb{R}^+\}$. Because we consider intermittent control situations, it is only in this section that $\dot{\phi}$ may be altered by the control input u. That is, we change $\dot{\phi}$ according to a control policy u to get the return map $P' : \mathcal{C} \to \mathcal{C}$ given by $P'(0, \dot{\phi}) = (0, u(\dot{\phi}))$ in the case of deadbeat control and $(0, g(u(\dot{\phi}), \dot{\phi}))$ in the case of asymptotic control. Here g encodes the lag between the assertion of the control and its effect.

Suppose we wish to compose or interleave two such systems. That is, we suppose that we have the system $(x_1, \dot{x}_1, x_2, \dot{x}_2) \in X^2$ with corresponding phase coordinates $(\phi_1, \dot{\phi}_1, \phi_2, \dot{\phi}_2) \in Y^2$. As before, system i may only be actuated when $\phi_i = 0$. In the examples we will consider, we suppose that the systems can not or should not be actuated simultaneously. Thus the set of states where $\phi_1 = \phi_2 = 0$ should be repelling. We will design a controller such that the attracting limit cycle is given by

$$\begin{aligned} \mathcal{G} = \{(\phi_1, \dot{\phi}_1, \phi_2, \dot{\phi}_2) \quad | \quad & \phi_1 = \phi_2 + \frac{1}{2} \text{ (mod 1)} \\ \wedge \quad & \dot{\phi}_1 = \dot{\phi}_2 = \omega\} \, . \end{aligned} \qquad (4)$$

The constraint $\phi_1 = \phi_2 + \frac{1}{2}$ (mod 1) encodes our desire to have the pair of phases as far from the situation $\phi_1 = \phi_2 = 0$ as possible.

To analyze and control such a system, we restrict our attention to the sections $\Sigma_1 \subseteq Y^2$ and $\Sigma_2 \subseteq Y^2$ defined by $\phi_1 = 0$ and $\phi_2 = 0$ respectively. Suppose that the flow alternates between the two sections. Let $G^t = H \circ F^t \circ H^{-1}$ be the flow in Y^2 conjugate to the flow in X^2 where $F = (f, f)$ and $H = (h, h)$. Define τ_i by $\tau_i(w) = \min\{\tau > 0 \mid H \circ F^\tau \circ H^{-1}(w) \in \Sigma_{3-i}\}$. Start with a point $w \in \Sigma_1$. Let $w' = G^{\tau_1}(w)$ and $w'' = G^{\tau_2}(w')$. We have $w' \in \Sigma_2$ and $w'' \in \Sigma_1$, so we have defined the return map on Σ_1. Now since G is parameterized by the control inputs u_1 and u_2 we get

$$\begin{aligned} w = (0, \dot{\phi}_1, \phi_2, \dot{\phi}_2) \quad & \mapsto \quad w' = (\phi_1', u_1, 0, \dot{\phi}_2) \\ & \mapsto \quad w'' = (0, u_1, \phi_2', u_2) \, . \end{aligned}$$

Thus, the phase velocity updates $u_1(w)$ and $u_2(w')$ must be found so that (4) is achieved. This describes the problem of juggling. As a slight but important variant of this scheme, consider the situation wherein the control of $\dot{\phi}_i$ is not deadbeat but asymptotic instead. Thus, $w'' = (0, g(u_1, \dot{\phi}), \phi_2', g(u_2, \dot{\phi}))$. This is the situation we face in designing a controller for the synchronized hopping system. That we may apply essentially the same control scheme and still achieve stability is the main contribution of this paper.

2.1 Reference Fields

Notice that a single phase describes a circle S^1 and two phases describe a torus $\mathbb{T}^2 = S^1 \times S^1$. We now define a "reference" vector field on the k-dimensional $\mathbb{T}^k$ which encodes the ideal behavior of the system as though it were fully actuated.

We are concerned with regulating the two systems so that (1) the rate of change of each phase is some desired value (i.e. the first system oscillates A times for every B times the second does) and (2) the phases are maximally separated. That is, we require that

$$\begin{pmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \end{pmatrix} = \kappa_1 \begin{pmatrix} A \\ B \end{pmatrix} \quad \text{and} \quad A\phi_2 = B\phi_1 + \frac{1}{2} \text{ (mod 1)} \qquad (5)$$

where κ_1 scales the phase velocities A and B to values reasonable for the system (in this paper, $A = B = 1$.)

We construct a reference vector field on $\mathbb{T}^2$ with this circle as a limit cycle such that $(\dot{\phi}_1, \dot{\phi}_2) = \kappa_1(A, B)$ along the cycle. This field encodes the ideal behavior of the system as though it were fully actuated. Let V be defined by

$$V(\phi_1, \phi_2) \quad = \quad \cos(2\pi[A\phi_2 - B\phi_1]) \, . \qquad (6)$$

Then the field is

$$\mathcal{R}(\phi_1, \phi_2)^T = \kappa_1 \begin{pmatrix} A \\ B \end{pmatrix} - \kappa_2 \nabla V(\phi_1, \phi_2). \qquad (7)$$

Here κ_2 is an adjustable gain which controls the rate of convergence to the limit cycle. The lines $A\phi_2 = B\phi_1$

and $A\phi_2 = B\phi_1 + \frac{1}{2}$ are equilibrium orbits. The first is unstable, the second is stable. In designing the potential function V, we have imitated the idea in [26] of defining the goal as the configuration of least energy and the obstacles as the configuration of greatest energy – except that here we have extended this idea beyond point configurations to limit cycles.

2.2 Example: Juggling

As an example of this method, the details of which can be found in [17], consider a system wherein a paddle with position p controls two balls with positions b_1 and b_2 to bounce so that one is hit exactly when the other is at its highest point. We suppose the paddle always strikes ball i at $p = b_i = 0$ and instantaneously changes its velocity according to the rule

$$\dot{b}_{i,new} = -\alpha\dot{b}_i + (1+\alpha)\dot{p} \tag{8}$$

where α is the coefficient of restitution in a simple ball and paddle collision model. The paddle may thus, in deadbeat fashion, control the ball to have any desired rebound velocity (as long as the paddle's motors can provide enough torque).

First we define the phase of a single ball according to the discussion in Section 2 supposing that ϕ is 0 just after an impact and 1 just before the next impact. The change of coordinates $h : (\mathbb{R}^+ \times \mathbb{R}) - (0,0) \to S^1 \times \mathbb{R}^+$ from ball coordinates to phase coordinates is then given by $h(b,\dot{b}) = (\phi,\dot{\phi})$ where, following the recipe (3), we take

$$\phi = \frac{t^-}{s} = \frac{\dot{b}_0 - \dot{b}}{2\dot{b}_0} \text{ and } \dot{\phi} = \frac{g}{2\dot{b}_0}\,. \tag{9}$$

In this manner, for a two ball system with ball positions b_1 and b_2, we obtain two phases ϕ_1 and ϕ_2. The velocity $\dot{\phi}_i$ is reset instantaneously upon collisions, corresponding to the update rule (8).

We next take advantage of the fact that the flow $G^t = H \circ F^t \circ H^{-1}$, described in Section 2 and instantiated here, has the very simple form $(y_1, \dot{y}_1, y_2, \dot{y}_2) \mapsto (y_1 + \dot{y}_1 t, \dot{y}_1, y_2 + \dot{y}_2 t, \dot{y}_2)$ between collisions.

For each ball a *mirror law* [4], controlling the motion of the paddle with respect to the ball, can be defined so that after collision with the paddle, the new phase velocity of the ball is $\dot{\phi}_{i,new} = \mathcal{R}(\phi_j)$, where $\mathcal{R}(\phi)$ is defined by $\mathcal{R}(0,\phi) = \mathcal{R}(\phi, 0)$. The details can be found in [17]. The mirror laws for each ball can be combined using an *attention function* which controls the paddle to use the mirror law of ball that will next strick the paddle, as described in [14].

Now let Σ be the Poincaré section defined by $\phi_1 = 0$ and suppose that adjustments to the phase velocities alternate between the two phases (i.e. the system is near the limiting behavior). It can be shown that the map from Σ into Σ is

$$\begin{aligned} x' &= \frac{\mathcal{R}\left[\frac{\mathcal{R}(x)}{z}(1-x)\right]}{\mathcal{R}(x)}\left[1 - \frac{\mathcal{R}(x)}{z}(1-x)\right] \\ y' &= \mathcal{R}(x) \\ z' &= \mathcal{R}\left[\frac{\mathcal{R}(x)}{z}(1-x)\right] \end{aligned} \tag{10}$$

where $(x,y,z) = (\phi_2, \dot{\phi}_1, \dot{\phi}_2)$. Since the x and z advance functions are not functions of y, we can treat y as an output of this system. Thus, analytically, it suffices to treat (10) as an iterated map of the the variables $(x,z) \in S^1 \times \mathbb{R}^+$ given by $F(x,z) = (x', z')$.

In [17] we show that the fixed points of this system are $(0, \kappa_1)$ and $(1/2, \kappa_1)$ — corresponding to in-phase and out-of-phase juggling respectively — and we prove the latter fixed point is locally asymptotically stable under certain assumptions on the values of κ_1 and κ_2. Numerical simulations suggest a large domain of attraction for this behavior.

3 Synchronized Hopping

We now apply this method of coupling to a somewhat different system: a pair of hopping robots (see Figure 1). First, we describe a single, controlled hopping robot reminiscent of Raibert's one legged hopper [25] and examined analytically in [18]. We have altered the model of the system slightly so that it is more amenable to the analysis of the compositional treatment we apply. Then we show how to apply the phase regulation method described above to two such hopping robots, keeping in mind that here, the control of each cyclic system is asymptotic instead of deadbeat. Figure 2 shows a simulation of this system illustrative of the task at hand: to simultaneously and in a decentralized fashion, control the hopping height of the robots and the phase separation. We discuss our simulations and numerical results at the end of this section.

3.1 A Single Hopping Robot

We model a single, vertical hopping leg, a mass $m = 1$ attached to a massless spring leg, by a dynamical system with three discrete modes: flight, compression and decompression. These latter modes each have the dynamics of a linear, damped spring. Flight mode is entered again once the leg has reached its full extension. The equations of motion are

$$\ddot{x} = \begin{cases} -g \\ -\omega^2(1+\beta^2)x - 2\omega\beta\dot{x} \\ -\omega_2^2(1+\beta_2^2)x - 2\omega_2\beta_2\dot{x} \end{cases}$$

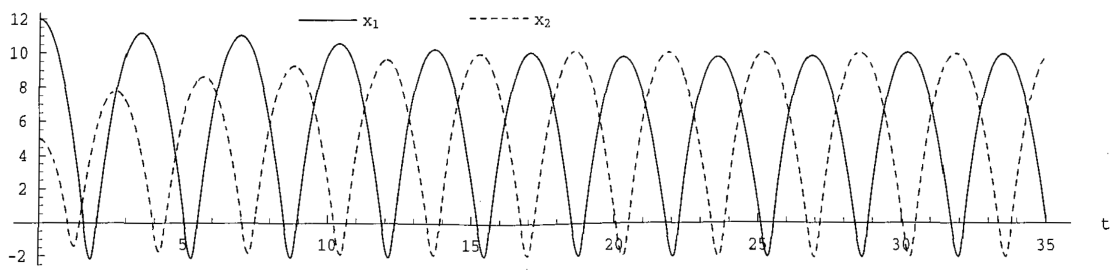


Figure 2: A simulation of the synchronized hopper system with $\omega = 6$, $\beta = 0.1$, $k_b = -2$ and $k_s = 0.1$.

$$\begin{array}{lll} \text{if } x > 0 & \text{flight} & \\ \text{if } x < 0 \wedge \dot{x} < 0 & \text{compression} & (11) \\ \text{if } x < 0 \wedge \dot{x} > 0 & \text{decompression} & \end{array}$$

where ω and β are parameters which determine the spring stiffness $\omega^2(1+\beta^2)$ and damping $2\omega\beta$ during compression. The constant $\gamma \approx 9.81$ is the gravitational constant. This model is similar to that studied in [18] where a period of thrust at the beginning of decompression was used to stabilize the hopper. We abstract the dynamics of thrust and suppose that, during decompression, thrust simply results in a change in spring stiffness and damping. Thus, ω_2 and β_2 are control inputs in our model.

We choose appropriate values for ω_2 and β_2 so that the system stabilizes at a desired hopping height. To this end, we derive the return map of the system, taking as a cross section of the cyclic system the set $\Sigma \doteq \{(x,\dot{x}) \mid x < 0 \wedge \dot{x} = 0\}$. A point in Σ is of the form $(x_b, 0)$ and represents the lowest point that the hopping leg reaches in a particular cycle. We will construct the function $f : \Sigma \to \Sigma$ which gives the lowest point of the next hop as a function of the lowest point of the current hop.

Integrating the system starting at $x_b \in \Sigma$ until Σ is again reached results in the discrete, real-valued return map

$$f(x_b) = x_b \frac{\omega_2\sqrt{1+\beta_2^2}}{\omega\sqrt{1+\beta^2}} \mathrm{Exp}\left(\tan^{-1}(\frac{1}{\beta_2})(\beta_2 - \beta) - \beta_2\pi\right). \tag{12}$$

We choose $\beta_2 = \beta$ and $\omega_2 = \omega\tau$ where $\tau = \tau(x_b)$ is a thrust term that is a function of the lowest point in a particular cycle. Thus, τ is a constant during each decompression, but varies from one decompression to the next. Suppose we wish the lowest point in the steady state hopping behavior to be $k_b < 0$. Setting $\tau = (1-k_b)e^{\beta\pi}/(1-x_b)$ and simplifying (12) gives

$$f(x_b) = \frac{(1-k_b)x_b}{1-x_b}. \tag{13}$$

The return map f has the two fixed points 0 and k_b. Assuming $k_b < 0$, the derivative of f at 0 is $1 - k_b > 1$ and, thus, 0 is an unstable fixed point of f. At k_b, the derivative of f is $1/(1-k_b) < 1$ and, thus, k_b is a stable fixed point of f. Since there is a unique stable fixed point of f, there is a unique, closed stable orbit of the system given by (11) which passes through the point $(k_b, 0)$.

3.2 Hopping Height Related to Period

We ultimately wish to combine two hopping legs into a phase regulated system. Therefore, we need to understand the phase of a leg as it hops up and down. Thus, we derive the period, $s(x_b)$ of a cycle starting at $(x_b, 0) \in \Sigma$ as in Section 2. s is obtained by summing the decompression time t_d, the flight time t_f, and the compression time t_c. It can be shown that

$$\begin{aligned} s(x_b) &= t_d + t_f + t_c \\ &= \frac{(\pi - \theta_l)e^{\beta\pi}(1-x_b)}{1-k_b} \\ &\quad -\frac{2}{\gamma}\omega(1-k_b)\sqrt{1+\beta^2}e^{\beta\theta_l}\left(\frac{x_b}{1-x_b}\right) + \frac{\theta_l}{\omega} \end{aligned} \tag{14}$$

where $\theta_l \doteq \tan^{-1}(\frac{1}{\beta})$.

It can be shown that s is a diffeomorphism on $(-\infty, 0)$. We may, therefore, work equally well with the conjugate map,

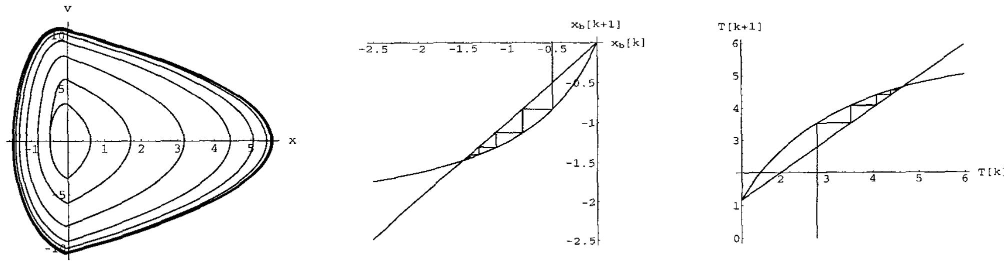
$$g(T) \doteq s \circ f \circ s^{-1}(T) \tag{15}$$

representing each orbit of the system (11) uniquely by it's period.

Given the period corresponding to a particular x_b, we define the phase of a hop $\phi(x_b) = t^-(x_b)/s(x_b)$. Figure 3 illustrates the relationship between the full dynamical system and its return map as well as the transformation from x_b-coordinates to period coordinates.

3.3 Synchronizing Two Hopping Robots

Now suppose we have two physically unconnected hoppers (as in Figure 1), operating simultaneously,



(a) (b) (c)

Figure 3: (a) A plot of a typical orbit of the system described in (11) in position-velocity space. (b) A plot of $f(x_b)$ showing iterations of the return map for the same system. (c) A plot of $g(T)$, the conjugate map, showing corresponding iterations. Note that the convergence to the set-point is asymptotic in this system. For these simulations, $\omega = 6$, $\beta = 0.1$, $k_b = -1.5$.

with states $(x_1, \dot{x}_1)$ and $(x_2, \dot{x}_2)$. We will show how to control both hoppers so that they are kept out of phase (one is at its highest point while the other is at its lowest point) and so that they stabilize at a desired hopping height x_b^* (or period T^*). We do this essentially by changing the set-points, now denoted $k_{b,i}$, for each hopper according to the phase of the other hopper. As was shown, this corresponds to changing the period and thus allows us to regulate the relative phase of the hoppers.

To apply our phase regulation algorithm we reset the gains $k_{b,i}$, each time a leg reaches its lowest point, according to the reference field (7)

$$k_{b,i} \leftarrow \mathcal{R}(\phi_j) \doteq k_b - k_s \sin(2\pi\phi_j) \tag{16}$$

where $j = 3 - i$ and k_s is a gain about which we will have more to say later. The parameter k_b sets the desired lowest point in a cycle (which defines the hopping height). Recall that k_b corresponds to period. It appears in the first term of the phase regulation expression instead of phase velocity for convenience later. Using the fact that changing $x_{b,i}$ is equivalent to changing T_i, this amounts to a period adjustment scheme for each leg that pushes them out of phase with each other. However, a leg does not respond immediately to the reset because control is asymptotic and not deadbeat. It must, therefore, be shown that this simple method indeed achieves the desired result.

We have defined a system that may be described by the state vector $x = (\phi_1, \phi_2, T_1, T_2) \in \mathbb{T}^2 \times \mathbb{R}^+ \times \mathbb{R}^+$ which evolves as follows. We have $\dot{\phi}_i = 1/T_i$ until some ϕ_i becomes $1 \equiv 0$. At this point, its desired hopping height is changed according to (16) and the period is reset according to the assignment $T_i \leftarrow g_{k_{b,i}}(T_i)$. The system then continues similarly.

3.3.1 Derivation of the Return Map

As described in Section 2, analysis of the system requires a suitable cross section, which we define to be $\Sigma = \{x \mid \phi_1 = 0\}$. Assuming that resets of the legs alternate, we construct the return map $F : \Sigma \to \Sigma$. We begin with a point $(0, \phi_2, T_1, T_2)$ just before resetting the period of hopper one. This evolves until a reset of hopper two. If we suppose that C_1 is the phase of hopper one just before hopper two is reset, then, just after the reset we have the point $(C_1, 0, g(\mathcal{R}(\phi_2), T_1), T_2)$. This point evolves back to Σ so that the state just before hopper one is reset for a second time is $(0, C_2, g(\mathcal{R}(\phi_2), T_1), g(\mathcal{R}(C_1), T_2))$ where C_2 is the phase of the second hopper just before the second reset of the first hopper. (Note: by $g(k, x)$ we mean $g_k(x)$.) Calculating C_1 and C_2 we have

$$\begin{aligned} C_1 &= \frac{T_2}{g(\mathcal{R}(\phi_2), T_1)}(1 - \phi_2) \\ C_2 &= \frac{g(\mathcal{R}(\phi_2), T_1)}{g(\mathcal{R}(C_1), T_2)}(1 - C_1). \end{aligned}$$

Letting $x = T_1$, $y = \phi_2$ and $z = T_2$ we obtain

Proposition: The three dimensional, discrete, real-valued return map $F(x, y, z) = (x', y', z')$ corresponding to two coupled oscillating systems (11) is defined by

$$\begin{aligned} x' &= g(\mathcal{R}(y), x) \\ y' &= \frac{g(\mathcal{R}(y), x)}{g[\mathcal{R}(\frac{z}{g(\mathcal{R}(y),x)}(1 - y)), z]} \left[1 - \frac{z}{g(\mathcal{R}(y), x)}(1 - y)\right] \\ z' &= g\left[R(\frac{z}{g(\mathcal{R}(y), x)}(1 - y)), z\right]. \end{aligned} \tag{17}$$

It is instructive to compare these equations with the return map (10) for juggling — the difference being the appearance of g which accounts for the lag between the assertion of control and its effect.

3.3.2 Local Stability of the Return Map

It can be shown that the point $(T, 1/2, T)$ is a fixed point of this system, where $T = s_{k_b}(k_b)$ is the period corresponding to the set-point k_b. We now wish to show

Theorem: The point $(T, 1/2, T)$ is a stable fixed point of the system defined by (17) when the synchronization gain k_s is chosen to be

$$\frac{1}{b\pi k_b}(a + c - bk_b)\left[2k_b - 2 + \sqrt{1 - 4k_b + 3k_b^2}\right]. \quad (18)$$

Proof: We describe the salient points of the proof of this theorem. Essentially, we linearize F and show that the linearized system is stable at $(T, 1/2, T)$. To compute the Jacobian of the map F, first, define

$$\begin{aligned} a &\doteq (\pi - \theta_l)e^{\beta\pi} \\ b &\doteq \frac{1}{\gamma}2\omega e^{\beta\theta_l}\sqrt{1+\beta^2} \\ c &\doteq \theta/\omega \\ \delta &\doteq \frac{k_b k_s \pi b}{(1-k_b)(a - bk_b + c)} \end{aligned}$$

Straightforward computation of partial derivatives yields that the Jacobian evaluated at $J_{(T,1/2,T)}$ is equal to

$$\begin{pmatrix} \frac{1}{1-k_b} & 2T\delta & 0 \\ \frac{2+\delta}{2T(1-k_b)} & 1 + 3\delta + \delta^2 & \frac{2-k_b+\delta(1-k_b)}{2T(k_b-1)} \\ \frac{\delta}{k_b-1} & -2T\delta(1+\delta) & \frac{1}{1-k_b} + \delta \end{pmatrix}. \quad (19)$$

Finding the characteristic polynomial of (19) and substituting (18) for k_s gives

$$-\lambda^3 + \xi_1\lambda + \xi_0. \quad (20)$$

Where

$$\xi_0 = \frac{1}{(k_b - 1)^2} \text{ and}$$

$$\xi_1 = \frac{k_b}{(k_b - 1)^2}(6 - 7k - 4\sqrt{1 - 4k_b + 3k_b^2}).$$

Now suppose ρ_1, ρ_2 and ρ_3 are the roots of (20). Then

$$(\lambda - \rho_1)(\lambda - \rho_2)(\lambda - \rho_3) = \lambda^3 - \xi_1\lambda - \xi_0.$$

Using this condition and the properties of ξ_0 and ξ_1 it is straightforward to show that when $k_b < 0$, two of the roots are complex conjugates, the other is real and negative and all have magnitude less than one. We omit the details of this step.

Now, since the eigenvalues of (19) all have magnitudes less than one, we can conclude that $(T, 1/2, T)$ is a stable fixed point of the system (17). □

3.4 Numerical and Simulation Studies

Constraining the value of k_s to a function of k_b achieves analytical simplicity but is hardly necessary. Numerical simulations of the synchronized hopper system suggest a wide interval of k_s settings around the guaranteed values in (18) yield stability. In Figure 2, we show a simulation starting from arbitrarily chosen initial conditions which eventually stabilizes at the desired hopping height and phase relationship. In our simulations, with k_s suitably small, we could not find initial conditions that did not eventually stabilize — leading us to believe that the system is in fact globally asymptotically stable.

We also investigated the eigenvalues of (19) numerically without the simplification (18). For example, fixing k_s and changing k_b results in the following situation: when $k_b = 0$, the system is not stable (the eigenvalues have magnitude one) because the hopping height is zero and, therefore, the decompression phase never ends. As k_b decreases (resulting in a larger hopping height), the sizes of the eigenvalues decrease for a time and then one of them increases toward one as k_b approaches $-\infty$. If we instead fix k_b and vary k_s, the we observe the following: when $k_s = 0$, there is no coupling and the system is only neutrally stable at the fixed point. As k_s increases, the system stabilizes until a certain point, after which the magnitude of one eigenvalue exceeds one. In our simulations, values of k_s larger than the point at which two of the eigenvalues become imaginary resulted in significant overshoot of the fixed point and longer convergence time. We usually chose k_s to be such that the eigenvalues are all real, and this improved performance.

4 Present and Future Work

The most obvious extension to this research is to combine more than two oscillators in various combinations and connection schemes. We have successfully achieved this in simulation examples by phase regulating pairs of juggling robots. However, analysis of systems that have many more than two degrees of freedom and are coupled in highly nonlinear ways, may require a new perspective before a broader range of utilitarian behaviors is possible. This will involve an understanding of the topology of the systems, their stability properties in isolation and with respect to the effect of coupling mechanisms, and possibly a means of abstracting away underlying details yielding a correctness proof that relies only on the highest level characterizations of the systems so composed.

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