Field-Theoretic Approach to Biconnectedness in Percolating Systems

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Field-Theoretic Approach to Biconnectedness in Percolating Systems

Abstract
A general method is given whereby m-connectedness correlation functions can be studied in the percolation problem. The method for m=1 involves calculating the kth power of the correlation function $\langle \sigma(x)\sigma(x') \rangle$ for a randomly dilute Ising model at non-zero temperature and subsequently averaging over configurations. The final step is to take the limit $k \to 0$. This method is tested by reproducing the standard results for the percolation problem from an extension of the calculation of Stephen and Grest. For m>1 an additional "color" index is introduced and a Hamiltonian is constructed in which different colors repel one another, thereby giving an exact prescription for m-connectedness. Order parameters for m-connectedness are identified. The m=2 order parameter couples through a trilinear term to the m=1 order parameter. The main result is that $\beta(m)$ the exponent for m-connectedness is given by $\beta(m) = \beta + \nu \psi(m)$, where $\beta$ and $\nu$ are the usual exponents for percolation and $\psi(m)$ is a new crossover exponent which, to lowest order in $\epsilon=6-d$, is given by $\psi(m) = m(m-1)\epsilon^2/49$. This result implies that the fractal dimensionality of the biconnected part of the critically percolating cluster is given in terms of the percolation critical exponents as $\gamma/\nu - \psi(2)$. If "nodes" are defined as triconnected points, then $\beta(3)$ is the critical exponent associated with their density in the infinite cluster. We also discuss evidence that the "node-link" model of Skal-Shklovskii-de Gennes breaks down for d less than some critical value $d^*$. Numerically we adduce evidence that $d^*$ may be larger than 3.

Disciplines
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Comments
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Field-theoretic approach to biconnectedness in percolating systems

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A general method is given whereby \( m \)-connectedness correlation functions can be studied in the percolation problem. The method for \( m = 1 \) involves calculating the \( k \)th power of the correlation function \( \langle \sigma(x)\sigma(x') \rangle \) for a randomly dilute Ising model at nonzero temperature and subsequently averaging over configurations. The final step is to take the limit \( k \to 0 \). This method is tested by reproducing the standard results for the percolation problem from an extension of the calculation of StepheIl and Grest. For \( m > 1 \) an additional "color" index is introduced and a Hamiltonian is constructed in which different colors repel one another, thereby giving an exact prescription for \( m \)-connectedness. Order parameters for \( m \)-connectedness are identified. The \( m=2 \) order parameter couples through a trilinear term to the \( m=1 \) order parameter. The main result is that \( \beta^{(m)} \) the exponent for \( m \)-connectedness is given by \( \beta^{(m)} = m \beta + v \psi^{(m)} \), where \( \beta \) and \( v \) are the usual exponents for percolation and \( \psi^{(m)} \) is a new crossover exponent which, to lowest order in \( \epsilon = 6 - d \), is given by \( \psi^{(m)} = m (m-1) \epsilon^2 / 49 \). This result implies that the fractal dimensionality of the biconnected part of the critically percolating cluster is given in terms of the percolation critical exponents as \( \gamma / \nu - \psi^{(2)} \).

If "nodes" are defined as triconnected points, then \( \beta^{(3)} \) is the critical exponent associated with their density in the infinite cluster. We also discuss evidence that the "node-link" model of Skal-Shklovskii–de Gennes breaks down for \( d \) less than some critical value \( \hat{d} \). Numerically we adduce evidence that \( \hat{d} \) may be larger than 3.

I. INTRODUCTION

Recently much attention has been given to the problem of formulating the percolation problem in a field-theoretic framework within which modern techniques, such as the renormalization group, could be applied. This program was initiated by the work of Fortuin and Kasteleyn, who showed that the generating function for bond percolation was isomorphic to the partition function of the \( s \)-state Potts model in the limit \( s \to 1 \). The importance of this work was that it led to a field-theoretic formulation for bond percolation. The original suggestion of Toulouse that the upper critical dimension should be \( d_c = 6 \) was confirmed and an \( \epsilon \) expansion was given from which the values of the critical exponents for percolation near \( d_c \) could be determined. These results were consistent with numerical estimates based both on simulation and on series expansion techniques. Subsequently, the Potts model has been used to extend the field-theoretical formulation to include consideration of many properties not initially considered in connection with the percolation problem. Among these are the conductivity, various site and bond distribution functions, distribution functions for perimeters, and so forth.

In connection with studies of the conductivity there naturally arose the concept of the "backbone." Refs. 10–12 only a fraction of this loosely connected cluster plays any role in the conduction process. If one removes bonds through which no current flows, then the remaining cluster constitutes the "backbone." This construction is related to the picture advanced by Skal and Shklovskii \( ^{10} \) and de Gennes \( ^{10} \) in which the infinite cluster is viewed as nodes connected by tortuous strands of occupied bonds. This picture has recently been elaborated by Coniglio \(^ {14} \) to include consideration of the internal structure of the nodes. \( ^{7,15} \) This construction in the ordered phase where an infinite cluster exists has its counterpart in the disordered phase where only finite-sized clusters exist. For finite clusters one considers a generalization of the pair-connectedness function, which is the order-parameter susceptibility for percolation. If \( \chi(x,x') \) denotes the nonlocal susceptibility, one has

\[
\chi(x,x') = [\psi(x,x')]_p ,
\]

where \( \psi(x,x') \) is unity if the sites \( x \) and \( x' \) are connected by a path of occupied bonds in the configuration \( C \), and is zero otherwise. Here \( [ \cdot ]_p \) denotes an average over all configurations of occupied bonds. To discuss a susceptibility analogous to the backbone density, we consider the "biconnectedness" susceptibility, \( \chi^{(2)}(x,x') \), defined by

\[
\chi^{(2)}(x,x') = [\psi^{(2)}(x,x')]_p ,
\]

where \( \psi^{(2)}(x,x') \) is unity in the configuration \( C \) if it is possible to delineate two independent paths over occupied bonds connecting the sites \( x \) and \( x' \), and is zero otherwise. Here two paths are independent if and only if they have
no bonds in common. More generally we define an m-connectedness susceptibility $\chi^m(x,x')$ by

$$\chi^m(x,x') = \left[ \psi^m(x,x') \right]_p,$$  \hspace{1cm} (2b)

where $\psi^m(x,x')$ is unity in the configuration C if there exist $m$ independent paths over occupied bonds which connect the sites $x$ and $x'$ and is zero otherwise. To describe the ordered state, we classify sites as being $m$-connected in the configuration C if they are connected by $m$ independent paths to arbitrarily distant sites.

Up to now, essentially only numerical information has been available for the backbone exponents. The order-parameter exponent for the backbone $\beta$ has been estimated by numerical simulation and a real-space renormalization-group calculation. An exact solution for percolation of a family of fractal lattices gave values of $\beta$ and the conductivity exponent $\nu$. But how these results are related to those for real lattices is not clear, since it is known that different-shaped fractals with the same fractal dimension give different critical exponents. In any event, the fractals used in Refs. 17 and 19 do not bear an obvious resemblance to percolation clusters which possess a large number of dangling free ends.

In this paper we will construct a field theory which not only yields the ordinary percolation correlation functions, but also provides a prescription for evaluating the biconnectedness, and in fact the $m$-connectedness, susceptibilities. The techniques introduced here may have more general application in that they make it possible to construct projection operators by means of which the weight associated with different configurations in the partition function can be altered arbitrarily. The principle result of this work is that the $m$-connectedness susceptibility obeys

$$\chi^m(x,x') = \frac{1}{m!} \left[ \chi(x,x') \right]^m |x-x'|^{-2\psi^m}, \quad d < 6$$  \hspace{1cm} (3a)

where $\psi^m$ is a crossover exponent which is given to lowest order in $\epsilon = 6-d$ as

$$\psi^m = m(m-1)e^2/49 + O(\epsilon^3).$$  \hspace{1cm} (3b)

For $d > 6$ mean field is valid and $\psi^m = 0$, so that

$$\chi^m(x,x') = \frac{1}{m!} \left[ \chi(x,x') \right]^m, \quad d > 6.$$  \hspace{1cm} (3c)

Note that there are corrections to the mean-field result of Eq. (3c) for $d$ in the range $4 < d < 6$ in spite of the Gaussian property of self-avoiding walks in more than four spatial dimensions. However, the fact that the percolation problem itself is non-Gaussian for $d < 6$ already shows that the critical clusters cannot really be viewed as consisting of links made of self-avoiding walks.

The result of Eqs. (3) implies that the critical indices for biconnectedness are related to those for single-connectedness by

$$\beta^m = m\beta + \nu\psi^m,$$  \hspace{1cm} (4a)

$$\gamma^m = d\nu - 2m\beta - 2\nu\psi^m,$$  \hspace{1cm} (4b)

where $\beta$ and $\nu$ are the usual exponents for percolation, describing the singly-connectedness properties, and $\beta^m$ and $\gamma^m$ are the order-parameter exponent and the susceptibility exponent, respectively, for $m$-connectedness. Kirkpatrick has mentioned that Eq. (4a) with no correction (in $\psi$) is a plausible guess for $m = 2$. For $d < d_c = 6$ the $m$-connectedness exponents satisfy the usual scaling relation,

$$2\beta^m + \gamma^m = d\nu.$$  \hspace{1cm} (5a)

For $d > d_c = 6$ the $m$-connectedness exponents assume their mean-field values:

$$\beta^m = m, \quad \gamma^m = 3 - 2m,$$  \hspace{1cm} (5b)

as is verified by an analysis (in Appendix C) of the properties of percolating clusters on the Cayley tree.

The results obtained here were reported briefly previously. These results can also be obtained by an alternative method given by Lubensky which involves the direct enumeration of polymer conformations. Each method has its virtues and drawbacks. In the replica method used here, the analytic continuations can become very involved, and some ambiguity is possible. However, in the present method an order parameter for biconnectedness appears naturally. As a result mean-field theory can be formulated for the $m$-connectedness functions. In addition, it is convenient to implement the momentum-shell recursion relations of the renormalization group to obtain a systematic expansion for $\psi^m$. In the polymer approach the reverse situation occurs: The order parameters for $m$-connectedness are seemingly difficult to display. However, calculations of the $m$-connectedness susceptibility are unambiguous. Abstracting the best features of each approach yields a coherent understanding of $m$-connectedness.

Briefly, this paper is organized as follows. In Sec. II we show how the formalism of Stephen and Grest can be used at finite temperature to obtain the pair-connectedness susceptibility from the dilute Ising model. This formulation is extended in Sec. III to discuss the general $m$-connectedness susceptibilities, and a mean-field treatment for $m = 2$ is presented. Section IV contains the renormalization-group analysis of the model introduced in Sec. III. Conclusions and a discussion of the validity of the “node-link” picture are given in Sec. V. The mean-field values of the $m$-connectedness exponents are evaluated for the Cayley tree in Appendix C.

II. FORMULATION IN TERMS OF THE DILUTE ISING MODEL

In this section we illustrate one aspect of our formulation by generalizing the treatment of Stephen and Grest. They presented a treatment of the dilute Ising model in which the fixed point in the Ising spin variables at zero temperature reduced to that for percolation. What we will show here is that by consideration of suitable analytic continuation, we can construct an Ising-model correlation function which is controlled by the percolation fixed point at nonzero temperature. To facilitate comparison with the work of Stephen and Grest, we use their notation.
throughout this section.

We start from the \(n\)-replica formulation\(^24\)

\[
Z(n) = \text{Tr} \prod_{NN} \left[ 1 - p + p \exp \left( \beta J \sum_{\alpha=1}^{n} (\sigma_\alpha(x)\sigma_\alpha(x') - 1) \right) \right]
\]

\[
= \text{Tr} \exp(-\beta H_{\text{eff}}),
\]

where \(p\) is the probability for bond occupation, the Ising spin variable \(\sigma_\alpha(x)\) assumes the values \(\pm 1\), the product is over pairs of nearest-neighboring (NN) sites on a hypercubic lattice, and \(\text{Tr}\) indicates a normalized sum over all Ising spin variables, so that for any such variable the only nonzero sum is \(\text{Tr} \sigma^{2g} = 1\). In the limit \(n \to 0\) this partition function describes the properties of a quenched random system in the following sense. If \(\langle Q \rangle_{\beta,C}\) denotes the thermal average of the operator \(Q\) at inverse temperature \(\beta\) in the configuration \(C\), then we have

\[
G^{(1)}(x,x') = \lim_{n \to 0} \langle \sigma(x)\sigma(x') \rangle_{\beta,C} = \lim_{n \to 0} \left[ \text{Tr} \exp(-\beta H_{\text{eff}}) \sigma_\alpha(x)\sigma_\alpha(x') \right].
\]

Thus \(G^{(1)}\) is expressed as an average with respect to \(H_{\text{eff}}\). For reasons that will become apparent in what follows we also introduce, following Stephen and Grest\(^23\), the higher-order correlation functions

\[
G^{(k)}(x,x') = \lim_{n \to 0} \left[ \text{Tr} \exp(-\beta H_{\text{eff}}) \left( \prod_{m=1}^{k} \sigma_\alpha^m(x)\sigma_\alpha^m(x') \right) \right].
\]

In the last line we require that all the indices \(\alpha^m\) be different. As Stephen and Grest point out, the correlations in \(G^{(k)}\) for different \(k\) will not be simultaneously critical at finite temperature. In that case, \(G^{(1)}\) will become critical first (as the temperature is lowered). When \(G^{(1)}\) becomes critical, the other variables are integrated out, and one recovers the usual theory for the dilute Ising model. At zero temperature, as Stephen and Grest note, all the \(G^{(k)}\)s are simultaneously critical. To see this we use their result for \(H_{\text{eff}}\):

\[
-\beta H_{\text{eff}} = \sum_{\alpha} \left[ K_1 \sigma_\alpha(x)\sigma_\alpha(x') + K_2 \sum_{\alpha < \beta} \sigma_\alpha(x)\sigma_\beta(x)\sigma_\alpha(x')\sigma_\beta(x') \right],
\]

where

\[
2cK_k = \ln [1/(1-p)] + (n - 2k)[p/(1-p)] \exp(-2BJ).
\]

At zero temperature all the coupling constants \(K_k\) are equal, all the corresponding correlation functions are equally long ranged, and they reproduce the pair-connectedness correlation function of percolation. The observation we make is that in the limit \(k \to 0\), \(G^{(k)}\) performs the same function at finite temperatures. At finite temperature the correlation function \(\langle \sigma(x)\sigma(x') \rangle_{\beta,C}\) in the configuration \(C\) is zero if the sites \(x\) and \(x'\) are disconnected and is nonzero if they are connected. In the former case raising it to the \(k\)th power still gives zero. In the latter case, raising it to the \(k\)th power gives unity, in the limit \(k \to 0\). Thus the \(k\)th power of the Ising-model correlation function in the limit \(k \to 0\) is the pair-connectedness correlation function in the percolation problem. This interpretation is supported by the form of \(K_k\) in Eq. (10). In the limit \(n \to 0, k \to 0\) we recover the zero-temperature value of \(K_k\) which is known to describe percolation.

To implement this idea we proceed as follows. We write down the recursion relation for the various vertices we expect to be relevant in the \(k \to 0\) limit. For general \(k\) these recursion relations involve noncritical quantities. However, these perturbative expressions in principle do enable one to calculate \(G^{(k)}\) for any \(k\). It is these relations which we then analytically continue to \(k = 0\). In so doing, we obtain the desired result: \(G^{(0)}\) reproduces the percolation correlation function.

We start from the Hamiltonian of Stephen and Grest\(^23\) in their Eq. (10):

\[
\beta H_E = \frac{1}{2} \sum_{k=1,2,3,\ldots} (r_k + q^2) \sum_{a_1,a_2,\ldots,a_k} \left| Q_{a_1,a_2,\ldots,a_k}(q) \right|^2
\]

\[
- \frac{1}{6N^{1/2}} \sum_{|\alpha|,|\beta|,q_1,q_2} \sum_{k,l,m} u(k,l,m)Q_{|\alpha|k}(q_1)Q_{|\beta|l}(q_2)Q_{a|\alpha\times\beta|m}(-q_1-q_2),
\]

where the \(Q\)'s are the fields conjugate to the various products of \(\sigma\)'s introduced in constructing the Hubbard-Stratonovich transformation to Gaussian integrals, the bare coupling constants \(u(k,l,m)\) are of order unity, and the \(r_k\) are given by

\[
r_k = \ln(1-p) + z^{-1} + (n - 2k)[p/(1-p)] \exp(-2BJ),
\]

where \(z\) is the coordination number of the lattice. Note that for \(n \to 0\), setting \(r_0 = 0\) gives the mean-field value of the critical percolation concentration, \(p_c\), independent of \(\beta\). In the last term of Eq. (11), \(|\alpha|_k\) denotes a set of \(k\) indices and \(|\alpha \times \beta|_m\) the set of \(m\) indices formed by multiplying indices as if they obeyed \(\alpha^2 = 0\): Thus the set \(|\alpha \times \beta|_m\) is formed by deleting from the union of the two sets \(|\alpha|_k\) and \(|\beta|_l\) all common elements. In the sum over \(|\alpha|_k\) and \(|\beta|_l\) we again only consider ordered sets of different integers.

The renormalization of the propagator comes from the diagram shown in Fig. 1 which is labeled as follows. The incoming line in general has \(k\) indices, so we decompose the totality of \(n\) possible indices into a group of \(k\) incident
ones and \( n-k \) others. By \((m_1, m_2)\) we indicate a set of indices of which \( m_1 \) are taken from the set of \( k \) incident ones and \( m_2 \) from the remaining set of \( n-k \) indices. In this notation the incident set of indices is just \((k, 0)\). The reason for introducing this notation is, of course, that all diagrams labeled in the same way give the same contribution. The only point to remember is that in Fig. 1 there is no diagram for which \( p = t = 0 \) or for which \( p = k \) and \( t = 0 \). Thus the recursion relation for \( a_{2,k}(q) \), the coefficient of the quadratic term in the Hamiltonian, is

\[
a'_{2,k}(q) = b^{2-\eta} \left[ a_{2,k}(q/b) + \sum_{p=0}^{k} \sum_{t=0}^{n-k} \frac{k!(n-k)!}{p!(n-k-t)!} F(p+t,k-p+t) - F(0,k) - F(k,0) \right].
\]

Here \( F(l,m) \) represents the contribution to \( a'_{2,k}(q)/b^{2-\eta} \) from the diagram in which the propagator \((r_1+q^2)^{-1}\) is inserted in one leg and \((r_m+q^2)^{-1}\) is inserted in the other leg and the vertex potentials are \( u(k,p+t,k-p+t) \) as is evident from Fig. 1. The sum in Eq. (13) is constructed by choosing \( p \) out of the \( k \) incident indices for the upper leg and therefore putting the other \( k-p \) incident indices on the lower leg. Out of the remaining \( n-k \) indices one can choose \( t \) to place on both legs. The point is that \( F(m_1,m_2) \) is an analytic function of \( m_1 \) and \( m_2 \) which is easily continued to zero values. This fact is a consequence of the fact that \( r_m \) can be written as a polynomial in \( m \).

Also the sums over \( p \) and \( t \) can be allowed to run from 0 to \( \infty \), because the factorial functions restrict these sums properly. Thus we see that in Eq. (13) we may not only take the \( n \to 0 \) limit as Stephen and Grest have done, but we may further let \( k \to 0 \). In this limit the factorial functions force all the indices to be zero, so that the sum over \( p \) and \( t \) reduces to the single term \( F(0,0) \). Thus, in the notation of Stephen and Grest, we obtain

\[
r_0 = b^{2-\eta} r_0 \{ 1 - 2[u(0,0,0)]^2 n \},
\]

which is the same recursion relation as the \( s \to 1 \) state Potts model.

We label the three point interaction \( u(0,0,0) \) to emphasize that it indeed is the coupling constant for three \( Q_0 \)'s. Here \( Q_0 \) is a "ghost" operator in that it is the analytic continuation of the operator \( Q_{a_1,a_2,\ldots,a_n} \) in the limit \( k \to 0 \). [This object in itself, of course, is not defined. We know it only through its propagator \((r_0+q^2)^{-1}\).]

The above operations may seem completely trivial. However, it should be realized that the following plausible procedure does not work. At nonzero temperature consider the situation as \( p \) is increased from 0. The first variable to become critical is clearly \( Q_0 \), because the Ising transition at \( p(T_c) \) occurs at higher \( p \) than \( p_c \). One is therefore tempted to reason that the \( Q_k \) for \( k > 0 \) should be integrated out as one would do for noncritical variables. However, this procedure is not well defined here, because the nature of the bubble diagram for \( k = 0 \) is completely unclear. The only way to make any sense out of it, is to imagine doing a real calculation of \( Q_k \) and letting \( k \) go to zero only at the final step.\(^{25}\)

We may analyze the three-point vertex in the same way. Here we group the indices into four groups. The first two groups, having respectively \( k_1 \) and \( k_3 \), members, are used to describe the incident lines. The fourth group, having three members, is denoted \( (j_1,j_2,j_3) \). In order to accommodate the constraint that indices occur in pairs, we label the incident lines as in Fig. 2. The internal lines are represented by the labels given by

\[
\begin{align*}
|a| &= (k_1-j,k_2-l,m,t), \\
|b| &= (j,l,m,t), \\
|c| &= (j,k_2-l,k_3-m,t).
\end{align*}
\]

For fixed sets of incident indices we must sum over the sets of \( [j] \), \( [l] \), \( [m] \), and \( [t] \) which describe the internal indices, subject to the constraint that each internal line must have at least one index. Thus the three cases,

\[
\begin{align*}
j - k_1 &= l - k_2 = m = t = 0, \\
j &= l = m = t = 0,
\end{align*}
\]

and

\[
j - l = k_2 = m - k_3 = t = 0
\]

are not allowed. Accordingly we write

\[
\begin{align*}
\text{FIG. 1. Perturbation contribution to Eq. (13).}
\end{align*}
\]

\[
\text{FIG. 2. Perturbation contribution to the renormalization of the three-point interaction. The indices indicated by a, b, and c are given explicitly in Eq. (15).}
\]
\[ u'(k_1,k_2,k_3) = b^{(q-3)/2} \left[ u(k_1,k_2,k_3) + \sum_{j,l,m,t} a_{j,l,m,t} H(j,l,m,t) - H(k_1,k_2,0,0) - H(0,0,0,0) - H(0,k_2,3,0) \right], \]  

where

\[ a_{j,l,m,t} = \begin{bmatrix} k_1 \\ j \\ k_2 \\ l \\ k_3 \\ m \\ n - k_1 - k_2 - k_3 \\ t \end{bmatrix}, \]

where \( (m^n) = m!/[ (m-n)!n! ] \) and \( H \) is the contribution to the recursion relation with given internal indices. Here again, we are in a position to take the analytic continuation to \( n, k_1, k_2, \) and \( k_3 \) equal zero. Again, in this limit, all the internal indices are forced to zero by the factorial functions and the sum reduces to the single term \( H(0,0,0,0) \). Thereby we recover the result

\[ u'(0,0,0) = u(0,0,0) b^{(q-3)/2} \left[ 1 - 4 u(0,0,0)^3 \ln b \right]. \]

As in the case of Eq. (14), this recursion relation is the same as that for the \( s \rightarrow 1 \) state Potts model. As before, this result cannot be obtained by directly setting the \( k \)'s to zero and integrating out the noncritical variables.

A final check of the procedure used here is that the recursion relations for the ghost vertices should only involve ghost vertices and, in particular, should not involve the coupling constant \( \exp(-2\beta J) \). This clearly should be the case, since purely percolative properties can be calculated without reference to the thermodynamic variables. Equation (13) is clearly consistent with this criterion. Equation (17) is also consistent with this criterion in the following sense. We assume that couplings like \( u(0,0,0) \) and \( u(0,k_1,k_2) \) vanish when \( k_1 \) and \( k_2 \) are nonzero. One can then verify that the new coupling constants \( u' \) also vanish for these cases. Thus the recursion relation of Eq. (17) never introduces coupling between the ghost vertices and the real ones.

For future reference we note a general feature of the analytic continuation \( n, k \rightarrow 0 \) used here. In Eqs. (13) and (17) there occur "internal partition variables" which describe the way the incoming indices are partitioned into subsets. In Eq. (13) these internal partition variables are \( p \) and \( t \), and in Eq. (17) they are \( j, l, m \) and \( n, k_1, k_2 \). Such partition variables will always be forced to vanish in the analytic continuation made here. This may be viewed as being the result of the relation

\[ \lim_{p \rightarrow 0} \frac{p!}{(p-m)!m!} = \delta_{m,0}. \]

This result will be frequently used below, where we refer to it as "the principle of nul partitions."

The reason for treating the dilute Ising model in some detail is twofold. Firstly, in its own right it is interesting to show that appropriate handling of the Ising-model correlation functions still reproduces the percolation transition even though the thermodynamics at nonzero temperature are no longer sensitive to this transition. Secondly, the technique needed to make the analytic continuation, which is validated by reproducing the percolation limit as expected, would be harder to see with more indices present. The additional indices are needed to distinguish different degrees of connectedness. The treatment of the dilute Ising model contains the motivation for the procedure we will use below.

### III. HIGHER CONNECTEDNESS CORRELATIONS: MEAN-FIELD THEORY

We now present a formulation in which \( m \)-connectedness correlation functions can be evaluated. Consider the "Hamiltonian" defined by

\[ e^{-B H} = \prod_{\alpha=1}^{n} \left[ 1 - p + p \sum_{\mu=1}^{b} \sigma_{\mu}^{h}(x) \sigma_{\mu}^{h}(x') \right], \]

where \( \sigma_{\mu}^{h}(x) \) is an Ising variable. The upper index \( a \) is a replica index and the lower index \( \mu \) may be thought of as a color index within each replica. The number of replicas \( n \) will be continued to zero, but the number of colors \( b \) is arbitrary. The coupling constant \( \lambda \) will play no role in the result after all the analytic continuations are taken. We introduce the set of correlations

\[ G_{\alpha}^{(m)}(x,x') = \left( \prod_{j=1}^{k} \sigma_{\mu}^{h}(x) \sigma_{\mu}^{h}(x') \right), \]

\[ G_{\alpha}^{(m)}(x,x') = \left( \prod_{j=1}^{k} \sigma_{\mu}^{h}(x) \sigma_{\mu}^{h}(x) \sigma_{\mu}^{h}(x') \sigma_{\mu}^{h}(x') \right), \]

etc. The correlation function \( G_{\alpha}^{(m)} \) will consist of a product over \( k \) replicas in each of which \( m \) different colors appear. We use \( G_{\alpha}^{(m)}(x,x') \) to discuss \( m \)-connectedness.

Consider now the meaning of averages taken with weight \( e^{-B H} \). To do this we imagine expanding this quantity in powers of \( p \) and \( 1 - p \). Bonds associated with factors of \( p \) are occupied and those associated with \( 1 - p \) are unoccupied. Therefore, we write this expansion as a sum over configurations \( C \):

\[ e^{-B H} = \sum_{C} P(C) e^{-B h(C)}, \]

where \( P(C) \) is the probability of realizing configuration \( C \) and

\[ e^{-B h(C)} = \prod_{\alpha=1}^{n} \left[ 1 + \lambda \sum_{\mu=1}^{b} \sigma_{\mu}^{h}(x) \sigma_{\mu}^{h}(x') \right], \]

\[ \equiv \prod_{\alpha=1}^{n} e^{-B h_{\alpha}(C)}. \]

where "\( \alpha \)" indicates that the product is over occupied bonds. Because \( h(C) \) is a sum of independent replica Hamiltonians, one has

\[ G_{\alpha}^{(m)}(x,x') = [\phi_{\alpha}(x,x')^{k}]_{p}, \]

where
\[
\phi_1(x,x') = \frac{\text{Tr} \sigma_2^x(x')e^{-\beta H(x)}}{\text{Tr} e^{-\beta H(x)}}
\]
and so forth. Thus \(\phi_m(x,x')\) is a sort of \(m\)-loop polygon generating function, depending both on the coupling constant \(\lambda\) and on the number of available colors \(b\). The exact analysis of \(\phi_m(x,x')\) is not needed here. However, as mentioned below Eq. (10), the \(k\)th power of \(\phi_1\) does reduce to the ordinary connectedness function in the limit \(k \to 0\). Thus we have

\[
\lim_{k \to 0} G^{[k]}(x,x') = \chi(x,x')
\]

Next consider \(G^{[k]}(x,x')\). The corresponding function \(\phi_2(x,x')\) will be nonzero in the configuration \(C\) if and only if we can connect the points \(x\) and \(x'\) simultaneously with paths of different colors. However, for a given replica only one color at a time can appear on any given bond. This is the result of having a sum rather than, say, a product over \(\mu\) in Eq. (21). It is this hard-core repulsion of colors that allows us to discuss \(m\)-connectedness. Since a bond can only support one color at a time in a given replica at a time, we see that \(\phi_2(x,x')\) is nonzero if and only if the points \(x\) and \(x'\) are biconnected. Again, the correct weighting for \(\chi^{12}(x,x')\) is only achieved in the limit \(k \to 0\):

\[
\lim_{k \to 0} G^{[k]}(x,x') = \chi^{12}(x,x')
\]

Clearly this procedure generalizes to higher-order connectedness.

At this point it is helpful to introduce more compact notation. We set

\[
T(|\alpha,\mu|;x) = \prod_{\alpha,\mu} \sigma_{\alpha,\mu}^x(x)
\]

where the product is over pairs of indices \(\alpha,\mu\) which are members of the set of pairs of indices \([\alpha,\mu]\). For instance,

\[
T((\alpha_1,\mu_1),(\alpha_2,\mu_2)) = \sigma_{\alpha_1,\mu_1} \sigma_{\alpha_2,\mu_2}^x(x)
\]

and so forth. In this notation we have

\[
-\beta H = \frac{1}{2} \sum_{\alpha,\mu} \sum_{x,x'} \gamma(x,x') K(|\alpha,\mu|) T(|\alpha,\mu|;x) T(|\alpha,\mu|;x')
\]

Clearly this procedure generalizes to higher-order connectedness.

At this point it is helpful to introduce more compact notation. We set

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\]

and so forth. In this notation we have

\[
-\beta H = \frac{1}{2} \sum_{\alpha,\mu} \sum_{x,x'} \gamma(x,x') K(|\alpha,\mu|) T(|\alpha,\mu|;x) T(|\alpha,\mu|;x')
\]

where \(\gamma(x,x')\) is unity if sites \(x\) and \(x'\) are nearest neighbors and is zero otherwise. The results of Appendix A are

\[
\lim_{k \to 0} K_{1,k} = -\ln(1-p) - p,
\]

\[
\lim_{k \to 0} K_{2,k} = -\ln(1-p) - \frac{p}{1-p} \sim - \frac{p^2}{2},
\]

where \(1, k\) refers to the set \((\alpha_1,\mu_1), (\alpha_2,\mu_2), \ldots, (\alpha_k,\mu_k)\) and \(2, k\) to the set \((\alpha_1,\mu_1), (\alpha_1,\nu_1), (\alpha_2,\mu_2), (\alpha_2,\nu_2), \ldots, (\alpha_k,\mu_k), (\alpha_k,\nu_k)\) where \(\alpha_1 < \alpha_2 < \cdots < \alpha_k\) and \(\mu_i < \nu_i\) for all \(i\).

We now transform to fields conjugate to the \(T\)'s:
\[ \rho = \prod_x \rho(x), \]

where
\[ \rho(x) = 1 + \sum_{|\alpha, \mu|} X(\{\alpha, \mu\}, T(\{\alpha, \mu\}, x)). \]

The corresponding trial free energy is
\[ \beta F = \text{Tr}(\beta p H + \rho \ln \rho), \]

where \( H \) is given in Eq. (31). Keeping only the variables corresponding to the sets 1, \( k \) and 2, \( k \) we find to order \( X^1 \) that
\[
\beta F = \sum_{k=1}^{\infty} \frac{z}{2} \left[ r_{1,k}[X(1,k)]^2 + r_{2,k}[X(2,k)]^2 \left( \frac{b-1}{2} \right)^k \right] + \frac{1}{2} \sum_{k=1}^{\infty} \sum_{p,t=0}^{\infty} \frac{(1-\delta_{p+t,0})(1-\delta_{p,k+t})}{(n-k-t)!(k-p)!p!t!} X(1,k)X(1,p+t)X(1,k+t-p)
\]
\[
- \frac{1}{2} \sum_{k=1}^{\infty} \sum_{k=1}^{\infty} \frac{n!b^k}{k!} X(2,k)[X(1,k+t]^2).
\]

We may now solve these equations in the \( k \to 0 \) limit. From Eq. (40b) we find
\[ X(2,0) = \frac{[X(1,0)]^2}{2z r_{2,0}}, \]
and from Eq. (40a)

\[ X(1,0) = -2zr_{1,0} . \]  

The latter equation agrees with Stephen and Grest's original result. (For later convenience \( zr_{k,0} \) is defined to coincide with Stephen and Grest's \( r \).) Equation (41) indicates that \( \beta^2 = 2\beta \) within mean-field theory. One should note that the trilinear coupling of order \( X(2,k)|X(1,k)|^2 \) gives \( X(2,0) \sim |X(1,0)|^2 \), but has no effect on the equation of state for \( X(1,0) \). This is a very satisfactory result since percolation can be completely described by the connectedness function \( G_1^{(0)}(x,x') \).

### IV. HIGHER CONNECTEDNESS CORRELATIONS: RENORMALIZATION-GROUP ANALYSIS

We now give an analysis of \( H_{\text{eff}} \) defined by Eq. (33). To simplify the arguments we will immediately discard most of the variables which remain noncritical in the analytic continuations we will perform. In particular, we will keep only variables corresponding to the sets of indices \( (1,k) \) and \( (2,k) \). These are the variables we know to describe single-connectedness and biconnectedness. From Eq. (33) we obtain

\[
\beta H_{\text{eff}}(Q) = \frac{1}{2} \sum_{x,x'} \gamma^{-1}(x,x') \left[ \sum_{i,k} Q(1,k,x)Q(1,k,x') + \sum_{j,k} Q(2,k,x)Q(2,k,x') \right] \\
- \frac{1}{2} \sum_{x} \left[ \sum_{1,k} K(1,k)[Q(1,k,x)]^2 + \sum_{2,k} K(2,k)[Q(2,k,x)]^2 \right. \\
+ \sum_{j,k,m} [K(1,j)K(1,k)K(1,m)]^{1/2} Q(1,j,x)Q(1,k,x)Q(1,m,x) \\
\left. + \sum_{j,k,m} [K(1,j)K(1,k)K(2,m)]^{1/2} Q(1,j,x)Q(1,k,x)Q(2,m,x) \right],
\]

(43)

where the sums over indices in the cubic terms are dictated by the algebra of the \( T \) operators and, in the interest of simplicity, will not be specified in detail other than to say that they are similar to that in Eq. (11). Here and below, the sum over \( 1,k \) and \( 2,k \) is over the sets described after Eq. (32b). To develop recursion relation for \( H_{\text{eff}}(Q) \) we consider the Hamiltonian \( H(Q) \):

\[
\beta H(Q) = \int d\vec{r} \left[ \frac{1}{2} \sum_{1,k} r_{1,k}[Q(1,k,x)]^2 + | \vec{\nabla} Q(1,k,x) |^2 \right. \\
+ \sum_{k,l,m} w_{1}(k,l,m)Q(1,k,x)Q(1,l,x)Q(1,m,x) \\
\left. + \frac{1}{2} \sum_{2,k} r_{2,k}[Q(2,k,x)]^2 + | \vec{\nabla} Q(2,k,x) |^2 \right. \\
+ \sum_{j,k,m} w_{2}(j,k,m)Q(1,j,x)Q(1,k,x)Q(2,m,x) \right].
\]

(44)

We will adopt the convention that in the sums over third-order terms, a given product occurs only once, as we indicate by summing over \( j < k < m \). In principle, we can develop recursion relations analogous to those in Eqs. (13) and (17) by partitioning the indices into group-containing equivalent indices. In each such term there will appear combinatorial factors involving factorial functions which force these internal partition variables to vanish in conformity with the principle of null partitions. Proceeding in this fashion we would recover recursion relations like Eqs. (13) and (17) for the singly-connected potentials. The recursion relations for both singly-connected and biconnected potentials are written explicitly in Ref. 21. Here we give a direct evaluation of \( G_{2}^{(k)} \) following the method described by Ma\textsuperscript{26} for the calculation of the energy-density correlation function of the \( \phi^4 \) model. According to Eq. (34) we have

\[ G_{2}^{(k)}(Q) = [K(2,k)z^2 r_{2,k}]^{-1} . \]

(45)

The noninteracting contribution to the Fourier transform of the correlation function \( \langle QQ \rangle Q \) yields

\[ \text{FIG. 3. Dominant contribution to } G_{2}^{(k)}(Q). \quad \text{The incident indices are shown. The indices indicated by } a \text{ are written explicitly in Eq. (48). The incident indices which are not listed in Eq. (48) are indicated by } b \text{ in this figure. Here and in Fig. 4 we do not label the right-hand external line, since it has the same indices as the left-hand one.} \]
the limit $k \to 0$. We then need only consider the possible ways to distribute the $2k$ incident indices over the two internal legs of the diagram. From the $2^{2k}$ ways of partitioning the $2k$ indices into two sets, we must subtract the two partitions which leave no indices in one of the sets. Thereby we obtain the result for the spatial Fourier transformed function as

$$G_2^{(k)}(q=0) = \sum_{j,l,m} \frac{k!}{(k-j-l-m)!m!j!} F(k,j,l,m) - F(k,0,k,0) - F(k,k,0,0), \quad (47)$$

where $F(k,j,l,m)$ is the contribution to $G_2^{(k)}$ when the upper leg has for $m$ of the incident replica (upper) indices the corresponding $\mu$, for $k-j-l-m$ of the incident replica indices the corresponding $v$, for $j$ incident replica indices both the corresponding $\mu$ and the corresponding $v$, and for $l$ of the incident replica indices, neither the corresponding $\mu$ nor the corresponding $v$. To be explicit, the upper leg is described by the indices,

$$\begin{bmatrix} \alpha_1 \alpha_2 \cdots \alpha_m \\ \alpha_{m+1} \alpha_{m+2} \alpha_{m+3} \cdots \alpha_{m+j} \\ \beta_1 \beta_2 \cdots \beta_m \end{bmatrix},$$

$$\begin{bmatrix} \alpha_{m+1} \alpha_{m+2} \alpha_{m+3} \cdots \alpha_{m+j} \\ \alpha_{m+1} \alpha_{m+2} \alpha_{m+3} \cdots \alpha_{m+j} \\ \beta_{m+1} \beta_{m+2} \beta_{m+3} \cdots \beta_{m+j} \end{bmatrix}, \quad (48)$$

All incident indices which are not in the above list appear on the lower leg labeled $b$. Now consider Eq. (47) in the limit $k \to 0$. The terms in the sum with nonzero values of the indices $j, l,$ or $m$ vanish in this limit. We are then left with

$$G_2^{(k)}(q=0) = F(k,0,0,0) - 2F(k,k,0,0). \quad (49)$$

These two terms are shown in Fig. 4, where we see that $F(k,k,0,0)$ involves the noncritical propagator of $G_2^{(k)}$. The first term in Eq. (48), however, involves putting the propagator for $G_1^{(k)}$ on both legs. Therefore, the diagram of Fig. 4 gives

$$|G_2^{(k)}(x,x')| = \lim_{k \to 0} |K_{2,k} z^2 - 1| 2^{k-1} w^2 \times \left\{ \langle Q(2,k,x)Q(2,k,x') \rangle_0 \right\} \times \left\{ \langle Q(2,x)Q(1,k,x') \rangle_{k_1} \right\}. \quad (50a)$$

The factor $2^{k-1}$ represents the number of ways the two colors in each replica can be assigned to each indistinguishable line. To evaluate this expression we set $w^2 = K_{2,0} K_{1,0}, K_{1,0} \sim z^{-1}$, and $\langle Q(2,0,x)Q(2,0,x') \rangle \sim r_{2,0} \sim z$. Then we find that

$$|G_2^{(k)}(x,x')| = \lim_{k \to 0} \frac{1}{1} |G_1^{(k)}(x,x')|^2 \quad (50b)$$

The Fourier transformed version of this equation is

$$\chi^{(2)}(q=0) = \frac{1}{2K_{1,0} z^4} \int [g(\bar{p})]^2 d\bar{p} \quad (50c)$$

FIG. 4. Labeling for $F(k,0,0,0)$ (top) and for $F(k,k,0,0)$ (bottom). Only $F(k,0,0,0)$ involves critical propagators with one color per replica. By definition $F(k,0,k,0) = F(k,k,0,0)$.

where $g(\bar{p}) = r + |\bar{p}|^2$. Here we will eventually set $r = r_{1,0} \sim (p_x - p)^2$. Thus the crossover exponent $\psi^{(2)}$ (and more generally, $\psi^{(m)}$) vanishes at this level of the calculation. Also we see that the negative contribution in Eq. (45) represents a short-ranged repulsion between the two loops in the biconnected diagram.

Next we consider the higher-order corrections to $\langle Q(2,k,x)Q(2,k,x') \rangle_0$ from diagrams like that shown in Fig. 5. Counting powers of momenta shows that diagrams for $G_2^{(k)}$ made using $w_{1,j}$ vertices and with $G_{i,1}$'s on all internal lines are all equally singular and hence lead to potential modifications of Eq. (50) into the form written in Eq. (3). In particular, singular corrections to Eq. (50) will be generated from Fig. 5 if it is possible to distribute the indices such that each of the five internal propagators involves a $G_1^{(i)}(q)$. In fact, there is no way to label the diagram to fulfill this condition. To see this let us start by forcing the four internal lines which intersect the two external vertices to be $G_1^{(i)}(q)$ propagators. For each upper (i.e., replica) index there are two colors emanating from each external vertex. One of these colors must appear on the bottom line, the other on the top line. Thus for each replica the color indices will appear either as in Fig. 6(a) or as in Fig. 6(b). In Fig. 6(a) the cross line car-
(a) \[ \mu \] \hspace{1cm} \[ \mu \] 

(b) \[ \mu \] \hspace{1cm} \[ \nu \]

None

\[ \nu \] \hspace{1cm} \[ \nu \]

\[ \mu \nu \]

FIG. 6. Possible ways the cross line in Fig. 5 can be labeled with color indices for a given replica.

There is no color for the replica in question. In Fig. 6(b) the cross line carries two colors for the replica in question. But a \( G^{(k)}(q) \) cannot carry two colors from the same replica. On the other hand, if all replicas were labeled as in Fig. 6(a), then the cross line would not exist at all. Therefore, we conclude that the diagram of Fig. 5 cannot exist with all \( G^{(k)}(q) \) propagators, and hence is not singular at \( d = 6 \) dimensions. The conclusion is then that Eq. (50) has no corrections from a bubble with ladder insertions.

The leading correction to Eq. (50) comes from the diagram in Fig. 7. Since this diagram is of order \( w^4 \), it will give an \( \epsilon^2 \) contribution to \( \psi^{(2)} \). To evaluate this diagram it suffices to consider only the \( 2^{k-1} \) ways the two colors in each replica can be distributed, one to the one line and one to the other line. (Remember that one cannot distinguish "lines.") Then the cross links allow the interchange of colors without having any line be a \( G_2 \) (i.e., carry more than one color per replica). Over the cross links, \( 1, 2, \ldots, k-1 \) colors can be exchanged. Exchange of either 0 or \( k \) colors is not allowed because such a case corresponds to one of the cross links being absent. Therefore, there are \( 2^{k-1}-1 \) possible color interchanges. (Again, there is a factor of \( \frac{1}{2} \) for indistinguishable lines.) Thus the contribution \( \delta G \) from Fig. 7 is

\[
\lim_{k \to 0} \frac{\delta G^{(k)}_2(\mathbf{k} = 0)}{\partial \mathbf{r}} = \lim_{k \to 0} \frac{1}{\mathbf{K}_{1,0}^d z^4} \frac{w^4}{(2k-1)(2k-1)} \int d^4 \mathbf{q} d^4 \mathbf{p} d^4 \mathbf{s} g(\mathbf{q})^2 g(\mathbf{p})^2 g(\mathbf{s})^2 \Omega(q),
\]

(51)

In view of Eq. (4b) we are led to consider \( \delta G_2^{(k)}(0)/\partial r \). Power counting shows that we only need differentiate the propagators associated with lines connected to the external vertices, all four of which are equivalent. Thereby we have

\[
\lim_{k \to 0} \frac{\delta G^{(k)}_2(0)}{\partial r} = \frac{w^4}{K_{1,0}^d z^4} \int d^4 \mathbf{q} [g(\mathbf{q})]^2 \Omega(q),
\]

(52)

where

\[
\Omega(q) = \int d^4 \mathbf{p} d^4 \mathbf{s} g(\mathbf{q} - \mathbf{s}) g(\mathbf{p} - \mathbf{s}) g(\mathbf{q} - \mathbf{p}) \times g(\mathbf{q} - \mathbf{s}) [g(\mathbf{p} - \mathbf{s})]^2.
\]

(53)

The leading contribution in Eq. (52) of order \( w^4 \ln r \) is found to be

\[
\lim_{k \to 0} \frac{\delta G^{(k)}_2(0)}{\partial r} = \frac{w^4}{8K_{1,0}^d z^4} K_{1,0}^d (\ln r)^2,
\]

(54)

where \( K_d \) is the usual phase space factor in \( d \) dimensions.

As \( M = \frac{\pi^2}{6} \) explains, we replace \( K_d w_1^2 \) by its "matching" value, which to first order in \( \epsilon \) is simply its value at the fixed point, viz., \( K_d w_1^2 = 2\epsilon/7 \). (See Ref. 2, where \( w \) is equivalent to \( 6w_1 \) used here.) The result of Eq. (50) can be expressed in terms of \( \epsilon \) and \( \eta \). Combining that version of Eq. (50) with Eq. (54) yields

\[
\lim_{k \to 0} \frac{\delta G^{(k)}_2(0)}{\partial r} = A + B \ln r \left[ 1 - \frac{1}{2} \left( 1 + \frac{\chi^{(2)}}{\gamma} \right) \ln r \right],
\]

(55)

where \( A \) and \( B \) are unimportant constants. Comparing Eqs. (55) and (56) leads to the results quoted in Eq. (4) for \( m = 2 \).

We may extend the above calculation to obtain \( G^{(k)}_m \) for larger \( m \) in the limit \( k \to 0 \). We need to repeat the calculation for Fig. 3 for higher-order external vertices. Such diagrams lead us to evaluations of contributions like those shown in Fig. 8. As before, the most important contributions come when all propagators involve \( G^{(k)}_1(q) \), as shown in Fig. 8. The same argument leading to Eq. (50) yields the unperturbed result,

\[
\chi^{(m)}(x,x') = \frac{1}{m!} \chi(x,x')^m.
\]

(57)
Now we consider corrections to this result. As we have seen in Figs. 5 and 6, ladder insertions are not possible within the constraint that all internal lines be $G^{k}_{1}$ propagators. Thus diagrams like those shown in Fig. 9(a) are irrelevant. The leading correction to Eq. (57) comes from inserting a “cross” like that in Fig. 7, and such a diagram is shown in Fig. 9(b). Then the calculations leading to Eqs. (55) and (56) can be repeated for $\chi^{(m)}$. One can guess the result: There are $m(m-1)/2$ ways the cross links can be inserted between two of the $m$ propagators. Thus the $\varepsilon^{2}$ contribution to $\psi^{(m)}$ will be proportional to $m(m-1)$. We thus immediately have the result

$$\psi^{(m)} = m(m-1)\varepsilon^{2}/49 + O(\varepsilon^{3}).$$

One might object to these results as follows. For a hypercubic lattice in six dimensions the $m$-connectedness results for $m > 12$ clearly vanishes. This is clearly a result of short-wavelength behavior. For such a case our calculation should be viewed as describing the probability that $m$ closely spaced sites near $x$ are connected to an equal array of closely spaced points near $x'$. 

V. DISCUSSION

The perturbative approach we have used breaks down when one of the fourth-order vertices of the percolation problem becomes relevant. According to the naïve dimension of this operator, this would happen at $d = 4$ dimensions. However, this argument does not reflect the fact that now we must consider perturbation theory away from the cubic fixed point. Fucito and collaborators\textsuperscript{25} have considered this question and conclude that the cubic fixed point is unstable below a critical dimension $d$ which they estimate to lie between 2 and 3. Here we will estimate $d$ from numerical values of $\beta^{2}$\textsuperscript{21} and the conductivity exponents. For this purpose we list, in Table 1,\textsuperscript{25,73} numerical estimates for various conductive and percolative exponents as a function of spatial dimension $d$. Here $\beta$, $\beta^{2}$, and $\nu$ are the exponents for percolation already introduced. In addition, we have listed values for the conductivity exponents $t$ and $s$, which are defined as follows. In a random mixture consisting of a fraction $p$ of conducting elements and a fraction $1-p$ of insulating elements, the macroscopic conductivity $\Sigma$ varies for $p$ near the percolation concentration $p_c$ as\textsuperscript{74,75}

$$\Sigma \sim (p - p_c)^{\gamma_r}, \quad p \rightarrow p^+_c.$$  

(59a)

For a mixture consisting of a fraction $p$ of superconducting elements and a fraction $1-p$ of normally conducting elements, one writes\textsuperscript{87,68}

$$\Sigma \sim (p - p_c)^{-\gamma_s}, \quad p \rightarrow p_c^-.$$  

(59b)

In the heuristic picture of “nodes and links” advanced by Skal and Shklovskii\textsuperscript{10} and de Gennes,\textsuperscript{11} $t$ is given by

$$t = (d-2)\nu + \frac{\xi}{4},$$

(60)

where $\xi$ is an exponent which describes the divergence in the length $L$ (or more precisely, the resistance) of a link as $p \rightarrow p_c$.

$$L \sim (p - p_c)^{-\xi}.\quad (61)$$

de Gennes used the value $\xi = 1$, noting that this was the expected result for mean-field theory. The first direct numerical study of $\xi$ was made\textsuperscript{6,76} using a series estimate for a “resistive susceptibility” exponent $\gamma_s$, which was shown to satisfy

$$\xi = \gamma_s - \gamma,$$  

(62)

where $\gamma$ is the usual percolation exponent. Renormalization-group methods\textsuperscript{7} showed that $\xi$ could be regarded as a crossover exponent, and to first order in $\varepsilon = 6-d$, the result $\xi = 1$ was obtained.\textsuperscript{7,72} This was extended to all orders in the $\varepsilon$ expansion.\textsuperscript{73}

More recently Straley\textsuperscript{77} has presented a similar heuristic argument to give $s$ in which blobs of superconductor are separated by sheets of normal regions. In a sense, this picture is the dual of the “node-link” picture and yields

$$s = 2\nu - \frac{\xi}{4},$$

(63)

where $\xi$ is an exponent related to the surface-to-volume ratio of the superconducting blob. In fact, in previous work Stephen\textsuperscript{72} had obtained from a scaling argument a relation of the form of Eq. (63) with

$$\xi = \beta,$$  

(64)

where $\beta$ is the usual percolation exponent. Thus to all orders in $\varepsilon$ the predictions are

$$t = (d-2)\nu + 1,$$  

(65a)

$$s = 2\nu - \beta.$$  

(65b)
TABLE I. Critical exponents for biconnectedness and conductivity as a function of spatial dimension $d$. Here $v$ is the correlation length exponent, $\xi_{\text{ser}}$ is the crossover exponent of Eq. (62), as determined from series expansions, and $\xi_{\text{cond}}$ is the same crossover exponent as determined from $\beta_{\text{cond}}=t-(d-2)(d-2)$ using the values listed for $t$ and $v$. The quoted uncertainties reflect a subjective assessment of the data.

<table>
<thead>
<tr>
<th>$d$</th>
<th>$\beta^{2\lambda}$</th>
<th>$\beta$</th>
<th>$v$</th>
<th>$t^0$</th>
<th>$s^*$</th>
<th>$\xi_{\text{ser}}$</th>
<th>$\xi_{\text{cond}}$</th>
<th>$\beta^2$</th>
<th>$2v-s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$0.38 \pm 0.02^g$</td>
<td>$0.14^d$</td>
<td>$1.34^l$</td>
<td>$1.27 \pm 0.08^i$</td>
<td>$1.27 \pm 0.08^i$</td>
<td>$1.43 \pm 0.12$</td>
<td>$1.27 \pm 0.08$</td>
<td>$1.35 \pm 0.1$</td>
<td>$10.0 \pm 0.6$</td>
</tr>
<tr>
<td>3</td>
<td>$0.9 \pm 0.1^b$</td>
<td>$0.41 \pm 0.02^f$</td>
<td>$0.85 \pm 0.03^l$</td>
<td>$1.85 \pm 0.15^s$</td>
<td>$0.81 \pm 0.08^w$</td>
<td>$1.22 \pm 0.06$</td>
<td>$1.02 \pm 0.2$</td>
<td>$1.1 \pm 0.2$</td>
<td>$2.2 \pm 0.3$</td>
</tr>
<tr>
<td>4</td>
<td>$1.1 \pm 0.1^b$</td>
<td>$0.52 \pm 0.03^s$</td>
<td>$0.66 \pm 0.03^a$</td>
<td>$0.6 \pm 0.1^r$</td>
<td>$1.05 \pm 0.04$</td>
<td>$1.05 \pm 0.15$</td>
<td>$1.3 \pm 0.3$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>$0.79 \pm 0.08^h$</td>
<td>$0.57 \pm 0.02^m$</td>
<td>$3^r$</td>
<td>$0^w$</td>
<td>$1^y$</td>
<td>$1^y$</td>
<td>$1$</td>
<td>$1$</td>
<td></td>
</tr>
</tbody>
</table>

$^a$Values: $0.38 \pm 0.02$ (Ref. 16), $0.55 \pm 0.05$ (Ref. 28).

$^b$Reference 28.

$^c$Reference 18 and Appendix C.

$^d$Values: $0.148 \pm 0.004$ (Ref. 2), $0.138 \pm 0.007$ (Ref. 29), $0.139 \pm 0.003$ (Ref. 30), $0.146 \pm 0.02$ (Ref. 31).

$^e$Reference 32.

$^f$Values: $0.47 \pm 0.02$ (Ref. 33), $0.42 \pm 0.06$ (Ref. 34), $0.41 \pm 0.01$ (Ref. 35), $0.39 \pm 0.02$ (Ref. 36).

$^g$Reference 4.

$^h$Values: $0.66 \pm 0.05$ (Ref. 4). We used the value of $\gamma$ ($1.17 \pm 0.02$ from Ref. 6 and $1.18 \pm 0.07$ from Ref. 5) with the listed value of $\nu$ to deduce a value of $\beta$ using the scaling relation $\beta=\frac{\nu}{\nu-d}$. We prefer this rather indirect determination because the exponents so derived agree well with the renormalization-group results (Refs. 37 and 38) $\beta=1-\frac{\nu}{\nu-d}$ — and $\gamma=1+\frac{\nu}{2-\nu}$.

$^i$Reference 39.

$^j$Values: $1.34 \pm 0.02$ (Ref. 40), $1.33 \pm 0.04$ (Ref. 41), $1.35 \pm 0.05$ (Ref. 42), $1.25 \pm 0.05$ (Ref. 28), $1.35 \pm 0.015$ (Ref. 43), $1.343 \pm 0.015$ (Ref. 44), $1.7 \pm 0.15$ (Ref. 45), $1.22 \pm 0.15$ (Ref. 46), $1.34 \pm 0.02$ (Ref. 47), $1.35 \pm 0.06$ (Ref. 48), $1.333 \pm 0.034$ (Ref. 49), $1.23$ (Ref. 50), $1.33 \pm 0.05$ (Ref. 51), $1.380$ (Ref. 52), $1.25$ (Ref. 53), $1.43$ (Ref. 54).

$^k$Reference 55.

$^l$Values: $0.825 \pm 0.05$ (Ref. 40), $0.9 \pm 0.05$ (Ref. 41), $0.85 \pm 0.05$ (Ref. 35), $0.87$ (Ref. 50), $0.94$ (Ref. 46), $0.95 \pm 0.05$ (Ref. 56), $1.031$ (Ref. 52), $0.85 \pm 0.05$ (Ref. 51).

$^m$Reference 6.

$^n$Reference 3.

$^o$To get values of $t$ for $d$ equal 4 and 5 use the data for $v$ and $\xi$ in Eq. (60).

$^p$Values: $1.27 \pm 0.04$ (Ref. 16), $1.1 \pm 0.05$ (Ref. 57), $1.38 \pm 0.12$ (Ref. 58), $1.15 \pm 0.25$ (Ref. 59), $1.3 \pm 0.1$ (Ref. 45), $1.28 \pm 0.05$ (Ref. 46), $1.22 \pm 0.08$ (Ref. 48), $0.99 \pm 0.02$ (Ref. 60), $1.25 \pm 0.10$ (Ref. 61), $1.23$ (Ref. 50), $1.1 \pm 0.1$ (Ref. 62), $1.26$ (Ref. 54), $1.0 \pm 0.1$ (Ref. 63), $1 \pm 0.1$ (Ref. 64), $1.28 \pm 0.03$ (Ref. 65), $1.33 \pm 0.02$ (Ref. 52).

$^q$Values: $1.85 \pm 0.10$ (Ref. 61), $1.70 \pm 0.05$ (Ref. 57), $2.04$ (Ref. 46), $1.74$ (Ref. 50), $1.725 \pm 0.005$ (Ref. 66), $1.62 \pm 0.05$ (Ref. 28), $2.14 \pm 0.02$ (Ref. 52), $1.6 \pm 0.1$ (Refs. 63 and 64).

$^r$Reference 11.

$^s$To get values of $s$ for $d = 5$ use the data for $v$ and $\beta$ in Eq. (65b).

$^t$Values: $1.33 \pm 0.02$ (Ref. 52), $1.27 \pm 0.05$ (Ref. 46), $1.23$ (Ref. 50). We note that in two dimensions the special duality relation causes $s$ and $t$ to be equal (Refs. 67–69). This is discussed in detail in Ref. 57.

$^u$Values: $0.76 \pm 0.01$ (Ref. 52), $0.81$ (Ref. 46), $0.87$ (Ref. 50, using their equation $s = t/2$ with their value of $t$), $0.70 \pm 0.02$ (Ref. 70).

$^v$Reference 70.

$^w$References 71 and 72.

$^x$Reference 6. For $d = 4$ and 5 the values of $\xi$ would be closer to unity if confluent corrections were taken into account.

$^y$References 11, 7, 72, and 73.

Several authors (e.g., Refs. 7, 72, 77 and 78) have noted that these relations are not obeyed in low dimension, and in fact it was suggested some time ago that Eq. (65a) in particular, might break down for $d$ near 4. Here we analyze the existing data which show that Eqs. (65a) and (65b) do not hold for $d \leq 3$, and we identify this breakdown with the emergence of the relevant fourth-order potential at a critical dimension $d$ discussed in Ref. 27. At this critical dimension we also expect a possible anomaly in $\beta^{2\lambda}$. Accordingly, in Table I we record the value of $\xi$ as determined by series (we call this value $\xi_{\text{ser}}$) and $\xi_{\text{cond}}$ obtained by writing $\xi_{\text{cond}}=t-(d-2)v$. For $d > \tilde{d}$ these exponents will be unity. Also, using the numerical values of $s$ and $v$ we construct the value of $(2v-s)/\beta$, a quantity which is also unity for $d > \tilde{d}$. Finally, we list the values of $\beta^2/2\beta$, which should be close to unity for $d > \tilde{d}$.

It is clear that the most sensitive way to locate $\tilde{d}$ is to test Eq. (65b). The quantity $(2v-s)/\beta$ seems to definitely be larger than unity for $d = 3$, so we guess that $\tilde{d} > 3$. The value of $\xi_{\text{ser}}$ agrees with this conclusion. Clearly, an effort should be made to determine accurate values of $s$ in three
and four dimensions, to better locate \( \hat{d} \). Since the values of \( \beta^{(2)}/2\beta \) shown in the table do not vary appreciably away from unity for \( d \geq 3 \), determining \( \hat{d} \) from these values involves considerable guesswork. Values of \( \hat{d} \) near 3 would be plausible in view of the dependence of \( \beta^{(2)}/2\beta \) on \( d \). If \( \hat{d} \) is near 3, the analysis of numerical data concerning percolation at \( d = 3 \) will be hampered by confluent corrections.

The physical meaning of \( \hat{d} \) is that it is the marginal dimensionality between two regimes. In regime I, i.e., and for \( \hat{d} < \hat{d} \), the picture of nodes and links breaks down. This is due to the emergence of a relevant fourth-order repulsion between links. (This physical interpretation is more clearly seen in the polymer formulation \(^{21,22}\) What this means is that there are interconnections between links on all length scales and hence a meaningful decomposition of the cluster into nodes and links is no longer possible. Then we expect that Eq. (50) is no longer approximately valid and that Eqs. (65a) and (65b) no longer hold. The generalization of these relations to the regime \( d < \hat{d} \) thus requires a detailed understanding of the nature of the new fixed point which occurs when the fourth-order repulsion is relevant.\(^7\)

In regime II, i.e., for \( \hat{d} < d < d_c \) = 6, the picture of the critical percolation cluster as structureless nodes connected by tortuous self-avoiding stands is basically sound and the resulting relations, Eqs. (3), (65a), and (65b) are valid. However, even in this regime the node-link picture should not be taken too literally, as is discussed in Ref. 21. It would be interesting to test Eq. (3) numerically, not only by determining \( \beta^{(p)} \) and/or \( \gamma^{(p)} \) for \( p > 1 \), but also by checking the dependence of \( G^{(p)}(x,x') \) on \( |x-x'| \) \( \equiv r \) Here there are two regimes depending on how \( r \) compares with \( \xi \). For \( 1 < r < \xi \), one is in the power-law regime, so that

\[
G^{(m)}(x,x') \sim \tilde{a}^{(m)} r^{-m(d-2+\eta)-2\psi^{(m)}}.
\]

(66)

In this regime the \( m \)-connected cluster looks like a fractal with fractal dimension \( d_f^{(m)} \) which our results allow us to evaluate. This dimension is defined so that the number of \( m \)-connected vertices in a region of volume \( \xi^d \) is of order \( \xi^{d-d_f^{(m)}} \). We estimate the number of such vertices to be of order \( \xi^{d-d_f^{(m)} / \gamma} \), so that \( d_f^{(m)} = d - \beta^{(m)} / \gamma \). Using Eqs. (3b) and (4b), we find for \( m = 2 \)

\[
d_f^{(2)} = \gamma / \nu - \psi^{(2)}.
\]

(67)

It is interesting that some early misidentifications\(^79,80\) gave the result \( d_f = \gamma / \nu \) for the fractal dimension of the-single-connected part of the infinite cluster.

The behavior in regime III defined by \( d > d_c = 6 \) is worth noting. Here we find that

\[
\beta^{(m)} = m, \tag{68a}
\]

\[
\psi^{(m)} = 1 - 2m, \tag{68b}
\]

as shown in Appendix C. For hypercubic lattices of dimension \( d \) we have

\[
\chi^{(m)}(x,x') \sim \tilde{a}^{(m)} r^{-m(d-2)}. \tag{69}
\]

It would be interesting to numerically verify Eqs. (66) and (69).

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APPENDIX A: EVALUATION OF COUPLING CONSTANTS

We will evaluate \( K_k \) in the limit \( k \to 0 \), where

\[
K_k = \frac{\sum \sigma_m \chi^{(m)}(x,x') \ln \left( 1 - p + p \sum_{\mu} \left( 1 + \lambda \sum_{\alpha} \sigma_{m}^{\alpha} \right) \right)}{\sum_{\mu} \left( 1 + \lambda \sum_{\alpha} \sigma_{m}^{\alpha} \right)}, \tag{A1}
\]

where we have used \( \sigma_m^{\alpha} \) to function here as \( \sigma_m^{\alpha}(x) \sigma_m^{\alpha}(x') \). We evaluate \( K_0 \) by analyzing the expansion of the logarithm in powers of \( \lambda \). We first collect all terms where each \( \sigma_{m}^{\alpha}(x) \) occurs exactly once, and will call this contribution to \( K_k \), \( \delta^{(1)}K_k \):

\[
\delta^{(1)}K_k = -\lambda \sum_{r=1}^{\infty} \left( -p \right)^{r} \frac{1}{r} C_{r}^{(k)}, \tag{A2}
\]

where \( C_{r}^{(k)} \) is the number of ways of partitioning \( k \) objects into \( r \) sets, none of which is empty. In particular

\[
C_{r}^{(k)} = \sum_{j=0}^{r-1} \left( \begin{array}{c}
r \\
j
\end{array} \right) \left( -1 ight)^{j} \left( -r \right)^{j}, \tag{A3}
\]

In the limit \( k \to 0 \) we have \( C_{r}^{(k)} \to (1)^{r+1} \) and

\[
\lim_{k \to 0} \delta^{(1)}K_k = -\ln(1-p). \tag{A4}
\]

Now we consider other contributions to \( K_k \), obtained by expanding the logarithm in Eq. (A1) in powers of \( \lambda \). We classify these contributions according to the number of times \( N_{\mu} \) a \( \sigma_{m}^{\alpha} \) occurs irrespective of \( \gamma \). The value of a term can be associated with the collection of values \( |N_{\mu}| \). In fact, let \( \pi_1 \) be the number of \( N_{\mu} \)'s that are 1, \( \pi_2 \) the number of \( N_{\mu} \)'s that are 2, and generally \( \pi_\gamma \) is the number of \( N_{\mu} \)'s that are equal to \( j \). Then the remaining contribution to \( K_k \) is of the form

\[
\delta K_k = \sum_{\pi_1, \pi_2, \ldots} \Psi^{(k)}(\pi_1) \Psi^{(k)}(\pi_2) \cdots B_k, \tag{A5}
\]

where \( \Psi^{(k)}(\pi_1) \) is the contribution to \( \delta K_k \) from the set \( \pi_1 \) and the associated combinatorial factors are

\[
A_k = \frac{k!}{\pi_1! \pi_2! \pi_3! \cdots \left( k - \pi_1 - \pi_2 - \cdots \right)!}, \tag{A6}
\]

\[
B_k = \frac{(n - k)!}{\pi_1! \pi_2! \cdots \left( n - k - \pi_2 - \cdots \right)!}. \tag{A7}
\]
Now the principle of null partitions applies, so that \( \pi_j = 0 \) for \( j \geq 2 \). The only contribution to \( K_k \) in the limit \( k \to 0 \) is that for \( \pi_1 = k \) and \( \pi_0 = n - k \). This is the contribution labeled \( \frac{1}{k}\Delta K_k \) above. Therefore, the full result is

\[
\lim_{k \to 0} K_k = -\ln(1 - p) . \tag{A8}
\]

This result is expected in analogy with the work of Stephen and Grest.

We also wish to know \( L_k \), where

\[
L_k = K_{1,2,1,2,1,2} \tag{A9}
\]

we have

\[
L_k = \operatorname{Tr}_{\sigma} \left[ k \sum_{j=1}^{k} \sigma_j \sigma_j \ln \left( 1 - p + \sum_{\mu} 1 + \lambda \sum_{\alpha} \sigma_\alpha^\mu \right) \right] . \tag{A10}
\]

Again using the principle of null partitions in analogy with the calculation of \( K_k \), we see that the limit \( k \to 0 \), \( L_k \) is found by collecting terms in the expansion of the logarithm which are of order

\[
\lambda^{2k} \sum_{j=1}^{k} \sigma_j \sigma_j . \tag{A11}
\]

Thus, for \( k \to 0 \)

\[
L_k = -\lambda^{2k} \sum_{r=1}^{k} (-p)^r \frac{1}{r} D_r(k) , \tag{A12}
\]

where \( D_r(k) \) is the number of ways \( 2k \) indices, \( \alpha_j, \beta_j \) for \( j = 1, 2, \ldots, k \) can be assigned to \( r \) subsets such that \( \alpha_j \) and \( \beta_j \) are in different subsets for all \( j \). Thus,

\[
D_1^{(k)} = 0 , \tag{A13}
\]

\[
D_r^{(k)} = \sum_{j=0}^{k} (-1)^j \binom{r-j}{j}^{(k)} \left[ \begin{array}{c} r \\ j \end{array} \right] . \tag{A14}
\]

Thus as \( k \to 0 \), \( D_r^{(k)} \to (-1)^{r+1}(1-r) \), so that

\[
\lim_{k \to 0} L_k = -\ln(1 - p) - \frac{p}{1-p} \sim -\frac{p^2}{2} . \tag{A15}
\]

This result shows that \( L_k < K_k \), so that the biconnectedness variables are noncritical at the percolation transition.

**APPENDIX B: ANALYTIC CONTINUATIONS**

We consider here some of the finer details in making analytic continuations. We start by “proving” that \( 2 = 1 \). Consider in the \( n \to 0 \) limit the quantity

\[
(1 + x)^n = \sum_{k=0}^{n} x^k \frac{n}{k} . \tag{B1}
\]

Clearly \( \binom{k}{n} \) vanishes as \( n \to 0 \) for \( k = 1, 2, \ldots \). However, for \( k = 0 \) or \( k = n \), \( \binom{k}{n} \); so perhaps we should write

\[
\lim_{n \to 0} (1 + x)^n = \lim_{n \to 0} (1 + x^n) = 2 . \tag{B2}
\]

Obviously, one must proceed more sensibly. One should allow \( n \) to go to zero through nonintegral values. In the above example, one can either write

\[
\lim_{n \to 0} \sum_{k=0}^{n} x^k \frac{n}{k} = \lim_{n \to 0} \sum_{k=0}^{n} x^k \frac{n}{k} = 1 \tag{B3a}
\]

or

\[
\lim_{n \to 0} \sum_{k=0}^{n} x^k \frac{n}{k} = \lim_{n \to 0} \sum_{k=0}^{n} x^k \frac{n}{k} = 1 . \tag{B3b}
\]

So you can keep either “end” of the series, but not both. Here it does not matter which end you keep. But if it did, then one should use Eq. (B3a) for small \( x \) and (B3b) for large \( x \).

A case where this problem is substantive concerns the calculation associated with Fig. 3. Suppose we write Eq. (42) as

\[
G_2^{(k)}(q = 0) = \sum_{j,l,m} \frac{k!}{(k-j-l-m)!m!j!} \times F(k,j-k-l-m,l,m)
\]

so that the upper leg has, for \( m \) of the incident replica indices the corresponding \( \alpha \), for \( j \) of the incident indices the corresponding \( \beta \), for \( k-j-l-m \) of the incident indices both the corresponding \( \alpha \) and the corresponding \( \beta \), and for \( l \) of the incident indices no corresponding indices. Using the principle of null partitions, we then have

\[
G_2^{(k)} = G(k,k,0,0) = 2F(k,k,0,0) \tag{B4a}
\]

\[
= -F(k,k,0,0) < < G_1^{(k)}(x,x')^2 . \tag{B4b}
\]

[To see the meaning of \( F(k,k,0,0) \) refer to Fig. 4.] The problem with this calculation is that we are choosing the wrong analytic continuation. A similar problem would arise with Stephen and Grest’s continuation for the antiferromagnetic Ising model. We know that changing the sign of the coupling constant \( J \) should not affect their results. But Eq. (10), above, does not reflect this. For negative \( J \) this is obviously the wrong analytic continuation. For the antiferromagnet this result corresponds to looking at the wrong end of the series. Reasoning similar to this also explains why the equilibrium value of the ordinary pair-connectedness order parameter \( X(1,0) \) is not dependent on \( X(2,0) \) in spite of the fact that the coupling of the last trilinear term in the free energy of Eq. (38) does cause the equilibrium value of \( X(2,0) \) to depend on \( X(1,0) \) as in Eqs. (40)—(42).

**APPENDIX C: BICONNECTEDNESS ON THE CAYLEY TREE**

In this appendix we derive the result \( \beta_m = m \) for the Cayley tree. For the Cayley tree we consider the probability \( p_T \) that a bond gave rise to a finite-sized branch of occupied bonds. Then the probability \( P^{(1)} \) that a site is in the infinite cluster is clearly

\[
P^{(1)} = 1 - (p_T)^{\sigma+1} . \tag{C1}
\]
The probability $P^{(2)}$ that a site is in the biconnected part of the infinite cluster is defined for the Cayley tree as the probability that from the site there emanate two infinitely large branches. Clearly, then

$$P^{(2)} = 1 - (p_f)^{a+1} - \frac{(a+1)(1-p_f)}{p_f}.$$  \hspace{1cm} (C2)

Here $p_f$ is the root of

$$p_f = 1 - p + p(p_f)^{\sigma},$$  \hspace{1cm} (C3)

which gives $p_f = 1$ for $p < \sigma^{-1}$ and $p_f < 1$ for $p > \sigma^{-1}$. For $p$ slightly larger than $\sigma^{-1}$ we have

$$p_f = 1 - \delta,$$  \hspace{1cm} (C4)

with $\delta$ small. From this we obtain

$$P^{(1)} = (a+1)\delta,$$  \hspace{1cm} (C5)

and

$$P^{(2)} \approx \frac{(a+1)!}{2} \delta^2,$$  \hspace{1cm} (C6)

so that $\beta^2 = 2\beta$. But it is easy to show that $\delta$ is proportional to $p - p_c$, where $p_c = \sigma^{-1}$. Therefore, $\beta^2 = 2$ as given in Table I. For the $m$-connectedness probability we find

$$P^{(m)} \approx \frac{(a+1)!}{(a+1-m)!m!} \delta^m,$$  \hspace{1cm} (C7)

so that $\beta^m = m\beta = m$ in mean-field theory.

To define the biconnectedness susceptibility on the Cayley tree, which has no loops, we proceed as follows.\(^{21}\) We say that two points are biconnected if they are connected to each other and also each have an infinitely large outgoing branch. Thus the two points are biconnected through the point at infinity, so to speak. If the points $i$ and $j$ are separated by a distance of $s$ bonds on the tree, we see that $\chi^{(2)}(i,j)$ is

$$\chi^{(2)}(i,j) = p^{(2)}(1-p_f)^2.$$  \hspace{1cm} (C8)

If we sum this over $j$, we get the susceptibility as

$$\chi^{(2)} = \frac{(a+1)p}{1-\sigma p} (1-p_f)^2.$$  \hspace{1cm} (C9)

Since $1-p_f$ and $1-\sigma p$ are both proportional to $p - p_c$, we obtain from Eq. (C9)

$$\chi^{(2)} \approx (p - p_c)^{1-\gamma^{(2)}},$$  \hspace{1cm} (C10)

with $\gamma^{(2)} = -1$. More generally, this kind of calculation gives

$$\gamma^{(m)} = 1 - 2m.$$  \hspace{1cm} (C11)


70J. P. Straley, in *Electrical Transport and Optical Properties of Inhomogeneous Media* (Ohio State University, 1977), Ref. 28, p. 118.


81This type of definition was suggested by B. G. Nickel.