Elasticity of a filamentous kagome lattice

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Abstract
The diluted kagome lattice, in which bonds are randomly removed with probability 1−p, consists of straight lines that intersect at points with a maximum coordination number of 4. If lines are treated as semiflexible polymers and crossing points are treated as cross-links, this lattice provides a simple model for two-dimensional filamentous networks. Lattice-based effective-medium theories and numerical simulations for filaments modeled as elastic rods, with stretching modulus μ and bending modulus κ, are used to study the elasticity of this lattice as functions of p and κ. At p=1, elastic response is purely affine, and the macroscopic elastic modulus G is independent of κ. When κ=0, the lattice undergoes a first-order rigidity-percolation transition at p=1. When κ>0, G decreases continuously as p decreases below one, reaching zero at a continuous rigidity-percolation transition at p=p_b≈0.605 that is the same for all nonzero values of κ. The effective-medium theories predict scaling forms for G, which exhibit crossover from bending-dominated response at small κ/μ to stretching-dominated response at large κ/μ near both p=1 and p_b, that match simulations with no adjustable parameters near p=1. The affine response as p→1 is identified with the approach to a state with sample-crossing straight filaments treated as elastic rods.

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Physical Sciences and Mathematics | Physics

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Elasticity of a filamentous kagome lattice

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(Received 4 January 2013; published 9 April 2013)

The diluted kagome lattice, in which bonds are randomly removed with probability \(1-p\), consists of straight lines that intersect at points with a maximum coordination number of 4. If lines are treated as semiflexible polymers and crossing points are treated as cross-links, this lattice provides a simple model for two-dimensional filamentous networks. Lattice-based effective-medium theories and numerical simulations for filaments modeled as elastic rods, with stretching modulus \(\mu\) and bending modulus \(\kappa\), are used to study the elasticity of this lattice as functions of \(p\) and \(\kappa\). At \(p=1\), elastic response is purely affine, and the macroscopic elastic modulus \(G\) is independent of \(\kappa\). When \(\kappa=0\), the lattice undergoes a first-order rigidity-percolation transition at \(p = p_b\). When \(\kappa > 0\), \(G\) decreases continuously as \(p\) decreases below one, reaching zero at a continuous rigidity-percolation transition at \(p = p_b \approx 0.605\) that is the same for all nonzero values of \(\kappa\). The effective-medium theories predict scaling forms for \(G\), which exhibit crossover from bending-dominated response at small \(\kappa/\mu\) to stretching-dominated response at large \(\kappa/\mu\) near both \(p = 1\) and \(p_b\), that match simulations with no adjustable parameters near \(p = 1\).

The affine response as \(p \to 1\) is identified with the approach to a state with sample-crossing straight filaments treated as elastic rods.

DOI: 10.1103/PhysRevE.87.042602 PACS number(s): 61.25.H–, 87.16.Ka, 62.20.de, 05.70.Jk

I. INTRODUCTION

Filamentous networks [1] are an important class of materials in which the interplay between stretching and bending energies and temperature leads to unique mechanical properties such as strong nonlinear response and strain stiffening [2–5], negative normal stress [5,6], nonaffine response [4,7–9], crossover from nonaffine to affine response [10,11], and power-law frequency dependence of the storage and loss moduli [12]. They are a part of such important components of living matter [13–16] as the cytoskeleton, the intercellular matrix, and clotted blood and of industrial materials like paper [17,18]. Here we introduce the diluted kagome lattice, shown in Fig. 1, in which elastic rods on nearest-neighbor (NN) bonds are removed with probability \(1-p\), as a model for filamentous networks in two dimensions. Lines of contiguous and collinear occupied bonds are identified as filaments, which are modeled, as in previous work [7–11,19], as elastic rods with one-dimensional stretching modulus \(\mu\) and bending modulus \(\kappa\). Because the diluted kagome lattice, like real filamentous networks, has a maximum coordination number of 4, it provides a better model for the real networks than do more highly coordinated lattices like the triangular and fcc lattices.

Effective-medium theories (EMTs) [20–22] have proven to be a powerful tool for the study of random systems. We use recently derived EMTs [23–26] that treat bending as well as stretching forces in elastic networks to calculate the shear modulus \(G\) as a function of \(\mu\), \(\kappa\), and \(p\) and derive scaling forms for its behavior near \(p = 1\) and near the \(\kappa > 0\) rigidity threshold at \(p = p_b\). We also calculate \(G\) using numerical simulations. The simulations and EMTs are in qualitative agreement over the entire range of \(p\) and in quantitative agreement near \(p = 1\). Our results are also in general agreement with those for the Mikado model [4,10,19], including, in particular, a crossover from bending-dominated nonaffine response at low density (small \(p\)) to stretching-dominated affine response at high density (\(p \approx 1\)). Our model, however, highlights special properties of networks of straight filaments, including scaling collapse, described analytically by our EMTs, of the region near \(p = 1\) where filament length \(L\) approaches infinity, and the underlying origins of affine response in this limit.

The building blocks of filamentous networks are typically semiflexible polymers, characterized by a bending modulus \(\kappa = l_p k_B T/2\) (in two dimensions), where \(l_p\) is the persistence length, \(k_B\) is the Boltzmann constant, and \(T\) is the temperature. Where filaments intersect, physical or chemical cross-links bind them together but do not inhibit their relative rotation. Since only two filaments cross at a point, each cross-link is connected to at most four others. The average distance \(l_c\) between cross-links is less than \(l_p\). A remarkably faithful description of experimentally measured linear [27,28] and nonlinear [3] elastic responses is provided by a model in which elastic response is assumed to be affine and filamentous sections between cross-links are treated as nonlinear central-force springs with a force-extension curve determined by the wormlike chain model [3,27,29] augmented by an enthalpic stretching energy. This simple model, however, misses some important properties of filamentous networks, including the existence of a rigidity-percolation threshold [30,31], nonaffine response [4,7–9], and crossover from bending-dominated nonaffine response at small \(L/l_c\) or low frequency [32], to almost affine, stretching-dominated response at large \(L/l_c\) or high frequency.

The Mikado model [11,19] provides additional insight into the physics of filamentous networks. In this model, straight filaments of length \(L\) are deposited with random position and orientation on a two-dimensional flat surface and cross-linked at their points of intersection. Filaments are treated as elastic rods with one-dimensional flat surface and cross-linked at their points of intersection. Filaments are treated as elastic rods with one-dimensional flat surface and cross-linked at their points of intersection. Numerical simulations on this model reveal a rigidity-percolation transition from a floppy network to one with nonvanishing shear and bulk...
moduli and nonaffine bending-dominated elastic response. As $L/l_c$ is increased, response becomes more affine and stretching dominated, reaching almost perfectly affine response in the $L/l_c \to \infty$ limit. Our EMTs and numerical simulations yield similar results for the diluted kagome model.

Though the Mikado and the diluted kagome lattice are quite similar with cross-links of maximum coordination 4 and variable values of the ratio $L/l_c$, they model slightly different parts of the phase space of possible two-dimensional filamentous networks. In particular, since the starting point of the diluted kagome lattice is the full lattice with all bonds occupied, it necessarily deals directly with the $L \to \infty$ limit, which is not generally accessed in studies of the Mikado model in which $L$ is restricted to be less than the sample width $W$. The Mikado model is an off-lattice model, whereas the kagome model is lattice based. The latter property of the kagome model facilitates the application of lattice-based EMTs [20–22] (modified to include bending) for all values of $p$. Finally, in the Mikado model, the distance between between cross-links on a single filament follows a Poisson distribution $p$, whereas in the kagome lattice, this cross-links on a single filament follows a Poisson distribution $p$.

As detailed in Appendices B and C, our EMTs clearly show that there are three critical points (or fixed points in the renormalization-group sense): the trivial empty lattice point at $p = p_b$, the $p = 1$ full lattice point. They also provide analytic scaling relations for the shear modulus $G$ as a function of $\mu$, $\kappa$, and $p$ in the $W \to \infty$ limit near both $p = p_b$ and 1. In general, the ratio $G/G_0$ of the shear modulus to its value $G_0$ at $p = 1$ is a function of $\hat{k} = \kappa/(\mu a^2)$ and $p$:

$$\frac{G}{G_0} = g(p, \hat{k}),$$

where $g(p, \hat{k})$ has a well defined $\hat{k} \to \infty$ limit $g(p, \infty)$ and is proportional to $\hat{k}$ for $\hat{k} \ll 1$. The properties of $g(p, \hat{k})$ guarantee a crossover from bending-dominated (and non-affine) response at small $\hat{k}$ to stretching-dominated (and affine) response at large $\hat{k}$. The crossover occurs at approximately the value of $\hat{k}$ at which $g(p, \hat{k}) = g(p, \infty)/2$.

The EMTs provide analytic forms for the function $g(p, \hat{k})$ near $p = 1$ and $p_b$. Near $p = 1$, $g(p, \hat{k})$ becomes a function $g_1(\tau)$ of the single scaling variable

$$\tau = \frac{\kappa}{(1 - p)^2 \mu a^2} \sim \frac{\hat{L}_c^2 (L_c^2)^2}{l_c^4} = \tau \cdot \kappa (1 - \kappa) \frac{(L_c^2)^2}{l_c^4},$$

where $\hat{L}_c^2 = \kappa/\mu$ is the bending length, $\langle L \rangle = a/(1 - p)$ is the average filament length, and $l_c \approx a$ is the average spacing between cross-links along a filament. As $\tau \to \infty$, $G$ approaches $G_0 [g_1(\tau) \to 1]$, but as $\tau \to 0$, there is crossover to a bending-dominated nonaffine regime in which

$$G \propto \kappa \langle L \rangle^2 \frac{L_c^2}{l_c^4}.$$

This behavior, which appears in our simulations, has also been observed in simulations in diluted three-dimensional lattices of fourfold-coordinated filaments [33,34]. In Sec. V we speculate about the reasons for the same behavior appearing in both two and three dimensions. Huessinger et al. [7,8] developed an off-lattice EMT for filamentous networks that predicts a bending-dominated nonaffine regime with a different power of $L/l_c$, $G \sim (L_c/l_c)^4$, than the one we predict. The origin of this different scaling is the absence of a lower cutoff in the Poisson distribution of the distance between cross-links in the Mikado model compared to the fixed cutoff equal to the lattice spacing in the kagome model. The analysis of Refs. [7,8] yields the kagome lattice scaling law if the probability distribution for distances is replaced by one with a fixed lower cutoff.

Near the rigidity-percolation threshold at $p = p_b$, our EMTs predict a different scaling form for $g(p, \hat{k})$:

$$g_2(p, \hat{k}) = (p - p_b)^t \frac{c_2 \hat{k}}{c_1 + \hat{k}},$$

where $t = 1$ and $c_1$ and $c_2$ are constants. As required, this form exhibits a crossover from bending-dominated to stretching-dominated response with increasing $\hat{k}$. The regimes predicted by the EMTs are summarized in Fig. 2.

Our numerical simulations follow the EMT prediction very closely with no adjustable parameters near $p = 1$ if $\kappa$ is not too large. For really small values of $\kappa/(\mu a^2)$, finite-size effects become important and simulations can be fit with a combination of the exactly calculable finite-size results at $\kappa = 0$ and the EMT scaling form valid at infinite $W$. Near $p = p_b$, simulations are consistent with a scaling function of the form of Eq. (1.4) but with $t$ approximately 0.2 rather than 1 and with different values of $c_1$ and $c_2$.

Under periodic boundary conditions, the kagome lattice has exactly $z = 4 = 2d$ neighbors per site, where $d$ is the spatial
FIG. 2. (Color online) Schematic phase diagram for the diluted kagome lattice showing the bending-dominated region in which \( G \sim \kappa \) and the stretching-dominated region in which \( G \sim \mu \). The crossover curve from the bending-dominated to the stretching-dominated regime was calculated by approximating \( g'(p, \kappa) / g'(p, \infty) \) with the function \( h(p, \kappa) = (A/Z) g_z(\tau) + (B/Z) g_z(p, \kappa) \), where \( Z = A g_3(\infty) + B g_2(p, \infty) \), \( A = (p - p_b)/(1 - p_b) \), and \( B = (1 - p)^2/(1 - p_a)^2 \), which interpolates smoothly between \( g_z(\tau)/g_z(\infty) \) near \( p = 1 \) and \( g_z(p, \kappa)/g_z(p, \infty) \) near \( p = p_a \), and solving \( h(p, \kappa) = 1/2 \) for each \( p \).

We are ultimately interested in three-dimensional filamentous networks, which are subisostatic with \( N_0 = 3N - 2N_s = N \) because their maximum coordination number, like that of two-dimensional networks, is 4. If constituent filaments are straight, these networks have a state of self-stress for each filament in the undiluted limit and their elastic moduli are nonzero, proportional to the bond-spring constant, and independent of \( \kappa \). Associated with each state of self-stress there is an additional zero mode, but as in the kagome lattice, the addition of bending forces raises all but the three trivial zero modes of uniform translation to finite frequency and stabilizes the lattice. We will argue in Sec. V that these properties are the likely origin of the very similar behavior, seen in simulations, of the shear modulus in two- and three-dimensional networks with a maximum coordination of 4.

The elastic response of lattices with \( z > 2d \) is fundamentally different from that of isostatic lattices with \( z = 2d \). The former lattices, which include the triangular lattice in two dimensions and the fcc lattice in three dimensions, with nearest-neighbor central-force springs support both compressional and shear stresses upon removal of bonds down to a critical bond-occupation probability \( p_{CF} < 1 \). There is a central-force second-order rigidity-percolation threshold from a floppy state to a rigid state \([30, 31, 39]\). The addition of bending forces lowers the rigidity threshold \([40–42]\) to a new value \( p_b < p_{CF} \), which is independent of the bending modulus \( \kappa \), and introduces a scaling crossover region \([24, 25]\) in the vicinity of the central-force threshold in which the shear modulus scales as \( \mu^{1-x} \kappa^x \), where \( x = 1/2 \) in EMT and \( x = 0.5 \) and 0.4 in two- and three-dimensional simulations, respectively. In the kagome lattice, the central-force rigidity threshold is at \( p = 1 \), the rigidity transition is first order, and the intermediate crossover regime near \( p = p_{CF} \) does not exist.

This paper consists of five sections and three Appendices. Section II compares and contrasts the Mikado and kagome models and demonstrates how straight, sample-traversing filaments with the energy of an elastic beam produce an affine elastic response. Section II also defines the lattice model that we use. Section III describes EMT procedures and presents their results. Section IV presents the results of our numerical simulations and compares them with those of the EMTs. Section V reviews our results and speculates about the application of our two-dimensional calculations to three-dimensional systems. Appendix A discusses general properties of the EMT dynamical matrix. Appendices B and C present details of the solutions to the EMT equations near \( p = 1 \) and \( p_b \) for the bending EMTs described in Refs. [24, 25] and [23, 26], respectively.

II. FILAMENTOUS NETWORKS AND THE DILUTED KAGOME LATTICE

Networks composed of straight filaments with elastic-rod energies have special properties. In this section we will explore some general properties of these networks before setting up the energy for the kagome lattice.

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A. Elastic filamentous networks

The elastic energy of a one-dimensional elastic beam with stretching modulus $\mu$ and bending modulus $k$ is

$$E_{\text{fil}} = \frac{1}{2} \mu \int_0^L ds \left( \frac{du(s)}{ds} \right)^2 + \frac{1}{2} k \int_0^L ds \left( \frac{d\theta(s)}{ds} \right)^2,$$

where $u(s)$ is the local longitudinal displacement and $\theta(s)$ the local angle of the tangent to the filament at the point $s$. The usual practice is to treat the stretching modulus as independent of $\kappa$ even though at nonzero temperature the spring constant for the entire filament has an important entropic component that depends on $\kappa$.

The spring constant for a filament section of length $l$ is $k = \mu l^{-1}$. This leads to affine response in linear elasticity for any lattice in any dimension consisting of sample-traversing straight filaments with a sufficient number of orientations to ensure stability with respect to all strain [7,8,43]. To see this, consider a cross-link (node of the lattice) at the origin on a filament parallel to the unit vector $\mathbf{e}$ and let it be connected to two other cross-links on the same filament at respective positions $s_1 = l_1 \mathbf{e}$ and $s_2 = -l_2 \mathbf{e}$ relative to the origin. Under a uniform, affine deformation, the relative positions $s_i, a = 1, 2$, transform to $\Lambda \cdot s_i \equiv s_i + \eta_i \cdot s_i$, where $\Lambda = \mathbf{I} + \eta$ is the deformation tensor with components $\Lambda_{ij} = \delta_{ij} + \eta_{ij}$. The forces that cross-links 1 and 2 exert on the origin are then

$$\mathbf{f}_1 = k(l_1)(l_1 \eta_i \eta_j \mathbf{e}_i) \mathbf{e}_j, \quad \mathbf{f}_2 = -k(l_2)(l_2 \eta_i \eta_j \mathbf{e}_i) \mathbf{e}_j,$$

where $i$ and $j$ run over $x,y$ and the summation convention is understood. The sum of these forces is zero because $k(l) = \mu / l$. The same analysis applies to any site and filament in the lattice. Under affine distortions, filaments do not bend, so the energy of the affine distortion depends only on the central force and does not depend on $\kappa$. Thus, under affine distortions of sample-traversing filaments, the force on every intermediate cross-link is zero and nonaffine distortions are not introduced: The response is affine and independent $\kappa$. When the lattice is diluted, not every cross-link is connected to two others, the above cancellation does not occur, and the result is a nonaffine response with a bending component.

With these observations, we can calculate the shear modulus of any network of straight filaments with stretching energy only, i.e., $\kappa = 0$. For simplicity, we restrict our attention to two-dimensional systems in a rectangular or rhomboid box with base length $W_x$ and height $W_y$ as shown in Fig. 1 and consider only shear deformations in which the only nonvanishing component of $\eta$ is $\eta_{xy}$. In this case, only those filaments that cross from the bottom to the top of the sample will contribute to the shear modulus. The length of such a sample-traversing filament oriented along a unit vector $\mathbf{e}(\phi) = (\cos \phi, \sin \phi)$ making an angle $\phi$ with the $x$ axis is $L(\phi) = W_y / (a \sin \phi)$. Under a shear deformation $\eta$, its length will change by $\delta L = L(e_1(\phi)\eta_{xy} e_2(\phi))$ to lowest order in $\eta$. Since the spring constant of a filament of length $L$ is $\mu / L$, the elastic energy of the filament is

$$E_{\text{fil}}(\phi) = \frac{1}{2} \mu W_y \left[ \frac{e_1(\phi)\eta_{xy} e_2(\phi)}{\sin \phi} \right]^2,$$

and the total energy from all filaments is $E_{\text{fil}} = \frac{1}{2} N W_y \mu \left[ (e_1(\phi)\eta_{xy} e_2(\phi))^2 / \sin \phi \right]$, where $N$ is the total number of sample-traversing filaments and $\langle \cdots \rangle$ signifies an average over the orientation angles of the filaments.

The linearized energy density $E/W_x W_y$ is

$$f = \frac{1}{2} C_{ijkl} u_{ij} u_{kl},$$

where $C_{ijkl}$ are the components of the elastic constant or elastic modulus tensor and $u_{ij} = \frac{1}{2}(\eta_{ij} + \eta_{ji})$ are the components of the linearized strain tensor $\dot{u}$. In the isotropic limit, the energy density reduces to

$$f = \frac{1}{2} B \text{Tr} [\dot{u}]^2 + G \text{Tr} [\dot{u}]^2,$$

for the shear modulus of a network of elastic rods with stretching modulus $\mu$. Note that rods parallel to the $x$ axis ($\phi = 0$) do not contribute to $G$. In the kagome lattice, bottom-to-top traversing filaments all have length $L_y = W_y / (\sin(\pi/3))$ and their angles with respect to the $x$ axis are restricted to $\phi = \pi / 3, 2\pi / 3$ for which $\sin \phi = 2 \sin 2\phi = 3/\sqrt{3}$. There are $W_x/a$ sites along the $x$ axis from which a single filament aligned along either $\pi / 3$ or $2\pi / 3$ can emerge, so $N$ is simply $q_{\text{cross}} W_x/a$, where $q_{\text{cross}}(p)$ is the probability a filament emerging from a given site along the $x$ axis traverses the sample. For simplicity, we take $L_y = W$ in which $q_{\text{cross}}(p,W) = p W/a$. When $p = 1$, all bonds are occupied and the shear modulus of the undiluted kagome lattice is $G_0 = (\sqrt{3}/8) (\mu/a)$. When $p < 1$ and $\kappa = 0$, there is a nonzero stretching contribution to $G$ in finite samples:

$$G_{\text{in}}(\mu,\kappa,p,W) \equiv G(\mu,\kappa = 0, p,W) = q_{\text{cross}}(p,W) G_0.$$  

In the $W \to \infty$ limit, the probability of sample traversing filaments vanishes for any $p < 1$ and $G_{\text{in}}(\mu,\kappa,p) \to \lim_{W \to \infty} G(\mu,\kappa,p,W)$ is zero at $\kappa = 0$ for all $p < 1$. Thus, as $\kappa \to 0$, $G_{\text{in}}(\mu,\kappa,p)$ must become smaller than $G_{\text{in}}(\mu,p,W)$ and for sufficiently small $\kappa$ and $p$ near one, $G(\mu,\kappa,p,W) \approx G_{\text{in}}(\mu,\kappa,p,W)$ and, as we will show in Sec. IV, the simple interpolation formula

$$G(\mu,\kappa,p,W) \approx \max \{ G_{\text{in}}(\mu,\kappa,p), G_{\text{in}}(\mu,\kappa,p,W) \}$$

provides an excellent description of the simulation data with the EMT form for $G_{\text{in}}$ near $p = 1$ where the finite-size corrections are the most important.

B. Kagome lattice energy

The kagome lattice has three sites per unit cell, which we take to be the three sites, labeled 1, 2, and 3, on an elemental triangle in Fig. 3. All bonds in the lattice are parallel to one of the vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ specifying the direction of bonds in the unit elemental triangle. We label each site by an index $l = (1,\sigma)$, where $l = (l_1,l_2)$ labels the position of site 1 in the unit cell at dimensionless (i.e., taking $a = 1$) position $r_l = 2(l_1 \mathbf{e}_1 + l_2 \mathbf{e}_2)$, where the factor of 2 arises because the separation between unit cells is twice the bond length. The
Effective-medium theory [20–22] is a well-established approximation for calculating properties such as electronic band structure and phonon or magnon dispersions of random media. Various formulations of EMT exist, but all replace the random medium with a homogeneous one whose parameters (such as NN hopping strength or spring constant) are determined by a self-consistent equation. Here we use the formulation, presented in greater detail in Appendix A, in which self-consistency equations for the effective-medium parameters are determined by requiring that the average multiple-scattering potential or $T$ matrix associated with a single random bond (or group of bonds) in the homogeneous effective medium vanishes.

The development of an EMT for elastic networks with both stretching and bending forces presents challenges not encountered in networks with stretching forces only. In our system, stretching forces are associated with single NN bonds, but bending forces are associated with contiguous pairs of NN bonds that couple next-nearest-neighbor (NNN) sites in what we call phantom NNN bonds that exist only if both its constituent NN bonds are occupied. Thus the replacement of a single NN bond, which we will refer to as the replacement bond, in an effective medium affects not only the stretching energy of that bond but also the bending of the two NNN phantom bonds containing that bond. There are currently two versions [23–26] of EMT on lattices with bending energies, which we will refer to as EMT I and EMT II, respectively, that deal with this problem in different ways. In both methods, the effective medium is characterized by homogeneous stretching and bending moduli $\mu_m$ and $\kappa_m$, respectively.

In EMT I [24,25], the replacement bond has a stretching modulus $\mu_s$ and a bending modulus $\kappa_s$ that take on respective values $\mu$ and $\kappa$ if the bond is occupied and 0 if the bond is not occupied. Thus the probability distribution for $\mu_s$ and $\kappa_s$,

$$P(\mu_s, \kappa_s) = \rho \delta(\mu_s - \mu) \delta(\kappa_s - \kappa) + (1 - \rho) \delta(\mu_s) \delta(\kappa_s),$$

(3.1)

exhibits a strong correlation between the values of $\mu_s$ and $\kappa_s$. As discussed in detail in the preceding paper [25], the bending constant of the NNN phantom bonds containing the replacement bond is calculated by assuming it is composed of two elastic beams connected in series, one with the bending modulus $\kappa_m$ of the effective medium and one with the bending modulus $\kappa_s$ of the replacement bond. This leads to a bending constant $\kappa_c$ for both NNN bonds containing the replacement bond, that is a nonlinear function of $\kappa_s$ and $\kappa_m$:

$$\kappa_c(\kappa_s) = 2 \left( \frac{1}{\kappa_s} + \frac{1}{\kappa_m} \right)^{-1}.$$

(3.2)

The perturbation to the effective medium arising from the replacement bond then consists of a stretching energy on that bond with spring constant $(\mu_s - \mu_m)/a$ and bending constants on the two NNN bonds of $(\kappa_c - \kappa_m)/a^3$. It turns out, as discussed more fully in Refs. [24,25] and in Appendix B, that the EMT equations in method I do not close unless an additional term, with coupling constant $\lambda_m$, coupling the angles on neighboring NNN bonds along a single filament is added to...
the effective-medium energy. This leads to an additional term in the perturbation arising from the replacement bond, with coupling constant \(-\lambda_m/a^2\), that couples the two angles on the two NNN bonds containing the replacement bond. Thus this EMT is characterized by three parameters \(\mu_m, \kappa_m\), and \(\lambda_m\).

In EMT II [23,26], the phantom NNN bonds carrying the bending energy are elevated to the status of real bonds that exist whether or not the NN bonds of which they are composed are occupied. This leads to the great simplification that stretching and bending are effectively decoupled and there is no necessity of introducing \(\lambda_m\). There are separate probability distributions for the stretching modulus of the NN replacement bond and the bending modulus of the NNN replacement bond. However, since the bending modulus of a NNN bond is zero unless both NN bonds comprising it are occupied, the probability that the NNN is present with a bending modulus \(\kappa\) is chosen to be \(p^2\), the probability that any two given NN bonds are occupied:

\[
P(\mu_s) = p\delta(\mu_s - \mu) + (1 - p)\delta(\mu_s).
\]

The predictions of these two methods are qualitatively similar: They both yield a rigidity-percolation threshold, below which the lattice loses rigidity, and a smoothly varying shear modulus approaching the pure lattice value of \(G_0\) as \(p \to 1\).

Figure 4 plots values of \(G(\kappa, \kappa)\) calculated using both EMT methods for different values of \(\kappa/(\mu a^2)\) and \(p\) along with the results of simulations. The \(\kappa = 0\) simulation curves follows the finite-size result of Eq. (2.7). The \(\kappa/(\mu a^2) = 10^{-6}\) curve follows the EMT curve, which corresponds to infinite \(W\), with increasing \(p\) and then the \(\kappa = 0\) finite-size curve after the two cross in accord with Eq. (2.8). As is the case for the triangular lattice [26], EMT II predicts a value of \(p_b\) that is close to that measured in simulations, whereas EMT I predicts a considerably larger value. In addition, for values of \(\kappa/(\mu a^2)\) beyond 0.1, the EMT I numerical solution ceases to exist for small \(p\) and as a result the \(\kappa/(\mu a^2) = 1\) curve is not included for EMT I in Fig. 4. For \(p \geq 0.7\) and \(\kappa/(\mu a^2) > 10^{-6}\), the two EMT curves and the simulation curves are essentially indistinguishable and near \(p = 1\), they become analytically identical to the effective-medium modulus \(\mu_m\) satisfying a scaling form

\[
\mu_m = \mu \Psi(\tau),
\]

where the scaling variable is

\[
\tau = \frac{\kappa/(\mu a^2)}{(1 - p)^2} = \frac{\kappa}{\mu a^2} (L) - \frac{1}{2}
\]

and the scaling function is

\[
\Psi(\tau) = -1 + \sqrt{1 + A \tau^2}/(A \tau),
\]

with the constant \(A = [40(1 - \sqrt{7}/3)]^2 \approx 53.9\). This scaling function can be expanded in the limits

\[
\Psi(\tau) \approx \begin{cases} 
1 & \text{if } \tau \gg 1, \\
A \tau/4 & \text{if } \tau \ll 1.
\end{cases}
\]

Therefore, for the case of \(\tau \gg 1\) corresponding to \(p \ll 1\) so that \(\kappa/(\mu a^2) \gg (1 - p)^2\), the shear modulus is approximately

\[
G = \frac{\sqrt{3} \mu_m}{8 a} \approx \frac{\sqrt{3} \mu}{8 a} = G_0,
\]

indicating that the macroscopic elastic response of the network is dominated by the stretching stiffness of the filaments and reaches the affine pure lattice limit as \(p \to 1\); we call this the “stretching-dominated” elastic regime.

In the other limit \(\tau \ll 1\) corresponding to smaller \(\kappa\) or larger \(1 - p\) so that \(\kappa/(\mu a^2) \ll (1 - p)^2\), we have the shear modulus

\[
G \approx A \sqrt{3} \frac{\kappa}{32 a^3 (1 - p)^2} = A \frac{\sqrt{3} \kappa L^2}{32 a^3 a^2},
\]

indicating that the macroscopic elastic response of the network is dominated by the bending stiffness of the filaments; we call this the “bending-dominated” elastic regime. The EMT solution for \(p\) near one is plotted in terms of the scaling variable \(\tau\) together with the asymptotic scaling function \(\Psi(\tau)\) in Fig. 5. Because the asymptotic solution (3.6) assumes small \(1 - p\), it requires very small \(\kappa/(\mu a^2)\) to make the regime \(\Psi(\tau) \approx A \tau/4\)
visible. We conclude from this that the elasticity of the network is bending dominant as long as

$$\kappa/(\mu a^2) \ll (1 - p)^2. \quad (3.10)$$

As discussed in Sec. I, near the rigidity threshold, $G(p, \kappa)$ vanishes as $\mu(p - p_0)^t$, where $t = 1$ and $p_0 = 0.6920$ for EMT I and $p_0 = 0.6180$ for EMT II, times a scaling function of $b_m = \kappa/(\mu a^2)$ that is a constant at large $b_m$ and proportional to $b_m$ at small $b_m$. Thus, at sufficiently small $b_m$ for all $p < 1$, response is bending dominated with $G \propto \kappa$; at sufficiently large $b_m$, response is stretching dominated with $G \propto \mu$ as shown in the phase diagram of Fig. 2.

IV. NUMERICAL SIMULATIONS

In the numerical portion of our work, we study the elasticity of the filamentous kagome lattice by generating diluted lattice conformations on a computer and then calculating their mechanical response numerically. Practically this is done via deforming the network by imposing a certain $\eta$ and then minimizing the elastic energy (2.9) over the nonaffine displacements $\delta u_i = u_i - \eta r_i$ of the sites using a conjugate gradient algorithm. To explore the response to shear, e.g., we set $n_{ijkl} = \gamma(\delta_{ij} \delta_{jk} + \delta_{ij} \delta_{jk})$, where $\gamma$ specifies the magnitude of the applied deformation. We use the same small magnitude $\gamma = 0.01$ for all deformations. For a range of $p$ values, we generate up to $M = 200$ random conformations, calculate several measurable quantities for multiple $\kappa$ values, and average arithmetically over all conformations. The quantities that we calculate are the elastic moduli and the corresponding fractions of rigid conformations $n_{ijkl}$ and nonaffinity parameters $\Gamma_{ijkl}$. Here $n_{ijkl}$ is defined as the number of conformations with nonzero $C_{ijkl}$ (nonzero meaning larger than a small numerical threshold, here $10^{-8}$) divided by $M$. The nonaffinity parameters [44]

$$\Gamma_{ijkl} = \frac{1}{N \gamma^2} \sum_{\ell} (\delta u_i^0)^2, \quad (4.1)$$

where $N$ is the total number of sites and $\delta u_i^0$ is the equilibrium nonaffine displacement of site $\ell$ in the presence of the $\eta$ that leads to $C_{ijkl}$, measure the deviation from a homogeneous strain field. To mitigate boundary effects, we apply periodic boundary conditions on all boundaries. We simulate system sizes ranging from $6^2$ to $200^2$ unit cells, which corresponds to $N = 108$ and 120 000 sites, respectively.

Figures 4 and 6 show log-linear plots of the shear modulus $G$ and the bulk modulus $B$, respectively, as functions of the occupation probability $p$ for $48^2$ unit cells. The EMT predicts the bulk modulus to be proportional to $\mu$, and hence both $\mu$ and $B$ should follow the same curves when they are normalized by their respective affine values $G_0$ and $B_0$ at $p = 1$. Within the numerical errors, this prediction is indeed borne out by Figs. 4(a) and 4(b). Moreover, these figures are consistent with the EMT prediction that the rigidity percolation threshold $p_b$ is the same for all $\kappa > 0$. Numerically, we find $p_b = 0.605 \pm 0.005$, which is only slightly smaller than the EMT II prediction $p_b \approx 0.616$ and about 15% smaller than the EMT I prediction $p_b \approx 0.692$. Previous EMT predictions for elastic networks have produced somewhat larger values for the rigidity threshold than found in corresponding numerical studies (see, e.g., Refs. [24,25]) and this apparent trend is continued here. For $\kappa = 0$, the EMT predicts a first-order rigidity-percolation transition at $p = 1$. The curves for $\kappa = 0$ in Figs. 4(a) and 4(b) rise from zero with finite slope below $p = 1$, a clear finite-size effect. Below we will analyze this finite-size effect systematically and find that our data are indeed consistent with a first-order transition at $p = 1$.

Figure 7 shows the nonaffinity functions $\Gamma_{xyxy}$ and $\Gamma_{xxxx}$ extracted from the simulation runs producing the results for $\mu$ and $B$ discussed in the previous paragraph. For $p = 1$, the network is affine and the nonaffinity functions are zero. The curves for $\kappa > 0$ are expected to have their maxima at $p_b$ and our data are consistent with that expectation. The data for the $\kappa = 0$ curves for $G$ and $B$ should show no indication of the existence of $p_b$ and indeed they do not.

Next we turn to the fractions of rigid conformations. These quantities are convenient for the purposes of finite-size analysis, in particular for the expected first-order transition for
\begin{equation}
\frac{\Delta M}{\Delta p} = \frac{\Delta \mu}{\Delta p}
\end{equation}

FIG. 9. (Color online) Scaling behavior of the shear modulus near \( p_b \). The plot shows both our simulation results and our numerical solutions of the EMT I equations relative to the asymptotic scaling form near \( p_b \) predicted by the EMT [see Eq. (1.4)]. The data points stem from our simulations for 482 unit cells and we use \( p_b = 0.605 \), \( c_1 = 0.3 \), \( c_2 = 0.007 \), and \( t = 0.2 \). The lower branches correspond to values of \( p \) below \( p_b \) and are a typical finite-size effect. The continuous curves represent the numerical solutions to the EMT I equations, which reduce to Eqs. (B22)–(B24) as \( \Delta p \) tends to zero with \( c_1 = 0.038 \), \( c_2 = 4.697 \), and \( p_b = 0.692 \).

Let us look at the elastic response near \( p = 1 \). Figure 5 shows the shear modulus and \( \mu_m \) as functions of the scaling variable \( t \) defined in Eq. (3.5). We find remarkably good agreement between the simulation data, the numerical solution of the EMT equations, and the EMT prediction for the asymptotic scaling function \( M(\tau) \) including the predicted value for the constant \( A \). Note that this agreement is obtained without adjusting any fit parameters.

Finally, Fig. 9 displays the scaling behavior near the rigidity-percolation threshold \( p_b \). Our numerical data collapse in full qualitative agreement with the EMT scaling form (1.4), albeit with an exponent \( t = 0.2 \) that is significantly smaller than the \( t = 1 \) predicted by the EMT. The value \( c_1 = 0.25 \), in contrast, is in semiquantitative agreement with the EMT.

V. DISCUSSION

We have introduced the diluted kagome lattice in two dimensions as a model for networks of semiflexible polymers whose maximum coordination number is 4. We identify straight lines of contiguous occupied bonds as filaments that, following previous work [7–11,19], we endow with the energy of an elastic beam characterized by a stretching (Young) modulus \( \mu \) and a bending modulus \( \kappa \). We contrast this model, which has filaments of arbitrary length, with the Mikado lattice whose filaments are of finite (and usually fixed) length. We show that when \( \kappa = 0 \), this kagome model has affine response and nonvanishing elastic moduli so long as there are sample-traversing filaments, which is the case for samples with finite lengths and widths \( W \), even for bond-occupation probabilities \( p \) less than one. As \( W \) increases, the probability of sample-traversing filaments decreases and in the \( W \to \infty \) limit, elastic moduli, which are nonzero for undiluted \( p = 1 \) lattice, fall precipitously to zero for \( p = 1^- \) in a first-order transition. The undiluted lattice has \( \kappa \)-independent, and thus affine, elastic moduli. The addition of bending forces restores rigidity for any \( \kappa \) for \( p \) greater than \( p_b \), the rigidity-percolation

\begin{equation}
\frac{\Delta M}{\Delta p} = \frac{\Delta \mu}{\Delta p}
\end{equation}

FIG. 8. (Color online) Fraction of rigid conformations for (a) 482 unit cells and various values of \( \kappa \) (with the same color code as in Figs. 4 and 6) and (b) \( \kappa = 0 \) and system sizes of 32 unit cells with \( S = 6,12,24,48,200 \) (increasing from left to right). The bold lines correspond to the theoretical result (4.3).

\[ \kappa = 0 \] at \( p = 1 \). In the following we will with focus on \( n = n_{xy} \). Figure 8(a) displays \( n \) for 482 unit cells for \( \kappa = 0 \) and several values of \( \kappa > 0 \). The curves for \( \kappa > 0 \) are consistent with the above statement that the rigidity-percolation threshold for all nonvanishing values of \( \kappa \) is at \( p_b = 0.605 \pm 0.005 \). Figure 8(b) shows \( n \) for \( \kappa = 0 \) for a variety of system sizes of 32 unit cells. Along with the data points, the figure follows predictions for \( n(\kappa = 0, p, S) \) that follow from the elementary combinatorics (see Sec. II A). Consider a system such as that shown in Fig. 1 with \( N_x = 2S \) bonds along its two sides. Only those filaments, which consist of \( N_x \) bonds, that extend from the top to the bottom of the sample will contribute to \( G \) and \( G \) will be zero unless at least one of the filaments starting on bottom reaches the top, which occurs with probability

\begin{equation}
P_{\text{aff}}(p, N_x) = 1 - (1 - p^{N_x})^{N_x}.
\end{equation}

This function becomes a step function at \( p = 1 \) when \( N_x \to \infty \). Thus we expect the data for \( \kappa = 0 \) to be fit by the function

\begin{equation}
n(\kappa = 0, p, S) = P_{\text{aff}}(p, 2S).
\end{equation}

Figure 8(b) reveals that the data is fit by this prediction remarkably well even though there is not a single adjustable fit parameter involved. Now we are in position to discuss the infinite-size limit. For \( S \to \infty \), \( n(\kappa = 0, p, S) \) approaches a unit-step function at \( p = 1 \). This establishes that the rigidity-percolation transition for \( \kappa = 0 \) is a first-order transition at \( p = 1 \) where the shear modulus jumps discontinuously from zero to its affine value.
threshold, and elastic moduli approach the nonzero affine values of the \( p = 1 \) lattice as \( p \to 1 \). We argue that this is the underlying cause for the affine limit found in the large-\( l / l_c \) limit (\( p \) near one) in the Mikado model.

We use two recently introduced lattice-based effective-medium theories to calculate elastic moduli of the diluted kagome lattice as a function of \( \mu, \kappa, \) and \( p \) and calculate scaling forms for the shear modulus near both \( p = 1 \) and \( p_0 \).

Both forms are a function of the unitless ratio \( b_m = \kappa / (\mu a^2) \), where \( a \) is the lattice spacing, and yield a crossover for all \( p_0 < p < 1 \) between bending-dominated response at small \( b_m \) and stretching-dominated response at large \( b_m \). We supplement our EMTs with numerical simulations of the shear modulus and other functions such as that measuring the degree of nonaffine response. The results of these simulations agree qualitatively with EMT predictions for all \( p \) and quantitatively with them near \( p = 1 \).

Our study of the kagome lattice provides insight into the behavior of three-dimensional filamentous lattices with a maximum coordination number of 4. With stretching forces only, these lattices are subisostatic with at least one zero mode per site and one might therefore conjecture that their elastic moduli should vanish when \( \kappa = 0 \). However, simulations on two model 3d lattices consisting of straight filaments with coordination number 4 when undiluted yield curves of elastic moduli as a function of \( p \) that look very similar to those of the kagome lattice presented here with an approach to affine \( \kappa \)-independent response as \( p \to 1 \). One [34] of these lattices is constructed from an fcc lattice by placing cross-links at each lattice site that randomly connects only two of the three filaments passing through it and the other [33] is constructed by stacking planes of 2d kagome lattices in orthogonal directions in such a way that each crossing point is connected to only four others. The underlying cause of these results is that straight sample-traversing filaments in any dimension give rise to nonvanishing macroscopic elastic moduli. Indeed, exact calculations [33] of one of the 3d lattices with elastic-beam energies show that all elastic moduli are nonzero when \( \kappa = 0 \) and \( p = 1 \) and that response is affine. Near \( p = 1 \), simulations on both of the 3d lattices show a crossover from a bending-dominated regime with \( G \sim (\kappa / a^4) (L/a)^2 \) to a stretching-dominated regime in which \( G \sim \mu a \) in agreement with the predictions of our kagome-based lattice EMT. In fact, the results of one of the simulations [33] track the scaling function of our kagome EMT. We conjecture that the important common feature of the 2d and 3d lattices is that they both consist of sample-traversing straight filaments in the undiluted limit. Away from \( p = 1 \), bending forces stabilize the lattices in a process that is not so sensitive to lattice dimension, even though the central-force 3d lattice is subisostatic. Our EMTs provide some support for this conjecture. The characteristic properties of the kagome EMT solutions, such as the form of the scaling function near \( p = 1 \), are a direct consequence of the existence of lines of zeros in the phonon spectrum at \( \kappa = 0 \) that get raised to nonzero frequency for \( \kappa > 0 \) as shown in Fig. 10. This feature causes the integral \( H_{zz} \) [Eqs. (B14)–(B17)] to diverge as \( \sqrt{b_m} \) as \( b_m = \kappa / (\mu a^2) \to 0 \). In three dimensions, the lines of zero modes in the phonon spectrum become lines of zero modes and we believe that the 3d version of \( H_{zz} \) will have a form analogous to Eq. (B17) but with \( g(q_y) \) replaced by a similar function of \( q_z \).

![FIG. 10. (Color online) (a) Dispersion relation of the floppy mode taking \( \mu_m = 1, \kappa_m = 0.0005, \) and \( \lambda_m = 0 \), where the frequency \( \omega_{\lambda} \) is the square root of the lowest eigenvalue of \( D(\mu_m, \kappa_m, \lambda_m) \). (b) Floppy mode branch of the eigenvalues of the dynamical matrix along the direction of \( \pi / 2 \) (which is \( q_y = 0 \)) taking \( \mu_m = 1, \kappa_m = 10^{-5}, \) and \( \lambda_m = 0 \). The red solid curve shows the actual eigenvalues calculated numerically from the dynamical matrix and the blue dashed line shows the approximation we used in Eq. (B12) that \( \omega_{\lambda}^2 = q_y^2 / 16 \) if \( q_y < Q_y \) and \( \omega_{\lambda}^2 = 16b_m g(q_y) \) otherwise [only the \( q_y > Q_y \) part is included in Eq. (B12) because other contributions are much smaller]. (c) Comparison between the numerical (blue points connected by a line) and the asymptotic form (B19) (red line) of \( 1 - H_{zz}(b_m, 0) \).

\[
H_{zz}(b_m, 0) \sim \int d^3 q \frac{\left| B_{\lambda}^\dagger \right| q_z^2}{c q_z^2 + b_m g(q_y, q_z)} \sim b_m^{-1/2}. \tag{5.1}
\]

When \( b_m = 0 \), the denominator vanishes for all \( q_y \) and \( q_z \) when \( q_y = 0 \) and the integral diverges. Nonzero \( b_m \) yields a \( b_m^{-1/2} \) divergence just as in the 2d kagome case. Unfortunately, the 3d lattices are quite complicated, with a 54-site unit cell in one case [33] and complicated crossing configurations in
the other [34], and we have not been able to set up a 3d EMT to verify this conjecture.

Much of the work on the elastic properties of filamentous networks (with the notable exceptions of [9,32,45]) have focused on models, such as the Mikado or kagome lattice presented here, consisting of straight filaments. This work has also largely focused on what are really mechanical models with beam energies assigned to the filaments and the effects of thermal fluctuations have been ignored. As our analysis shows, these are exceptional lattices because they can support stress with central forces only, even when their coordination number $z$ is substantially less the Maxwell stability limit of $z = 2d$. It also shows that even when filaments are straight, an affine response is guaranteed only if either (i) all spring lengths are the same or (ii) spring constants scale linearly with the inverse spring length. Transmission electron microscopy images (see, for example, Ref. [3]) of a range of networks of filamentous semiflexible polymers reveal filamentous morphologies that are rarely straight. These networks are formed under complex conditions that are not fully understood, but at least in some case their morphology is a product of splitting of filamentous bundles. These bent structures are most probable under stress, and more affine as $\kappa \rightarrow 0$ limit, but it has also been observed in a more realistic model of a filamentous lattice [45]. If $\kappa$ is needed to stabilize the system, the elastic response to external stresses will involve bending and thus be nonaffine. Thus it would seem that the straight-filament models are not so good for real systems, though they do provide us with valuable insight into how network architecture influences elastic response. One of the predictions of the straight-filament models is that response becomes more stretching dominated and more affine as $\kappa$ increases. Thus it is plausible that if $\kappa$ is large enough, the elastic response of even those lattices whose shear moduli vanish if $\kappa = 0$ will exhibit stretching-dominated nearly affine response for sufficiently large $\kappa$.

ACKNOWLEDGMENTS

We are grateful for helpful discussions with Fred MacKintosh and Chase Broedersz. This work was supported in part by the National Science Foundation under Grant No. DMR-1104707 and under the Materials Research Science and Engineering Center Grant No. DMR11-20901.

APPENDIX A: EFFECTIVE-MEDIUM THEORY

GENERALITIES AND THE EFFECTIVE-MEDIUM DYNAMICAL MATRIX

To implement the $T$-matrix version of EMT, we deal with the dynamical matrix $D^m$ of the effective medium (EM), its associated Green’s function $G^m = -(D^m)^{-1}$ (we consider only zero frequency), the perturbation $V$ associated with bond replacement, the dynamical matrix $D = D^m + V$, and its associated Green’s function $G = -(D)^{-1}$. (For more details of this formalism, see Refs. [25,46].) Because there are three sites per unit cell, all of these matrices are $6N \times 6N$ matrices, to be detailed further below, where $N$ is the number of unit cells in the lattice. We will specify $G^m$ more completely below. With these definitions,

$$G = G^m + G^m \cdot T \cdot G^m = (G^m - V)^{-1}, \quad (A1)$$

where

$$T = V \cdot (I - G^m \cdot V)^{-1}. \quad (A2)$$

The EMT self-consistency equation requires that the disorder average over the probability distribution of Eq. (3.1) for EMT I or Eq. (3.3a) for EMT II of the perturbation vanishes

$$\langle T(\mu_s, \kappa_s) \rangle = 0. \quad (A3)$$

All of the matrices discussed here can be expressed in a position or wave-number representation. The six independent displacements in a unit cell can be expressed as a six-dimensional vector $U_i = (u_{1i}, u_{2i}, u_{3i})$ for each $i$ or as its Fourier transform $U_q = \frac{1}{N} \sum_{l} e^{-iql} U(l) = (u_{1q}, u_{2q}, u_{3q})$. The energy of the bond-replaced system is then

$$E = \frac{1}{2} \sum_{i,i'} U_i D_{ii'} U_{i'} = \frac{1}{2N^2} \sum_{q,q'} U_{-q} D_{qq'} U_q. \quad (A4)$$

where $D_{ii'}$ and $D_{qq'}$ are $(6 \times 6)$-dimensional matrices for each pair $(i, i')$ and $(q, q')$. The effective medium is translationally invariant and

$$D^m_{qq'} = N \delta_{qq'} D^m_{ll}. \quad (A5)$$

The energy of the effective medium is constructed by occupying all bonds with identical beams with stretching and bending moduli $\mu_m$ and $\kappa_m$ and adding (for EMT I) an additional coupling of strength $\lambda_m$ coupling angles in neighboring NNN bonds along a filament (see Ref. [25] for details). The EM energy is then

$$E^m = E_s(\mu_m) + E_b(\kappa_m) + E^m_{bb}(\lambda_m) \quad (A6)$$

$$= \frac{1}{2N} \sum_{q} U_{q} \cdot D^m_{qq} \cdot U_{-q}. \quad (A7)$$

where $E_s(\mu_m)$ and $E_b(\kappa_m)$ are evaluated at $g_{\ell, \ell'} = 1$ for all bonds $(\ell, \ell')$, where

$$E^m_{bb}(\lambda_m) = \frac{\lambda_m}{a^3} \sum_{\ell, \ell', \ell_2, \ell_3, \ell_4} g_{\ell, \ell_2} g_{\ell_2, \ell_3} g_{\ell_3, \ell_4}. \quad (A8)$$

where it is understood that $\ell_1, \ell_2, \ell_3$, and $\ell_4$ are all contiguous sites along a single filament. Note that $G^m$ can be constructed from the Fourier transforms of the stretching and bending energies on NN and phantom NNN bonds. For example, the stretching energy of bond 5 in Fig. 3, connecting site $\ell$ to $l$, is $E_s = (l, 3) = (l, 1, 3)$ at position $\mathbf{r}_l = \mathbf{e}_l$ with site $\ell' = (l, 2)$ at position $\mathbf{r}_l + 2\mathbf{e}_2 + \mathbf{e}_1$ such that $\mathbf{r}_l - r_{\ell'} = \mathbf{e}_2$, is $\frac{1}{2}(\mu_m/\alpha)\left[(\mathbf{e}_2 - \mathbf{e}_l) \cdot \mathbf{e}_2\right]^2$. The Fourier transform of $(\mathbf{e}_2 - \mathbf{e}_l) \cdot \mathbf{e}_2$ is

$$e^{-i\mathbf{q} \cdot \mathbf{r}} (e^{-i\mathbf{q} \cdot \mathbf{e}_2} u_{2, q} - u_{3, q}) \cdot \mathbf{e}_2 = e^{-i\mathbf{q} \cdot \mathbf{r}} B^2_{3, 2} \cdot U_q. \quad (A9)$$
where $B_{5,q}^2$ is specified in detail in Eq. (A11) below. Similar procedures apply to all stretching and bending bonds and the EM dynamical matrix can be expressed as

$$D_q^b(\mu_m,\kappa_m,\lambda_m) = \frac{\mu_m}{a} \sum_{n=1}^{6} B_{n,q}^a B_{n,-q}^a + \frac{\kappa_m}{a^2} \sum_{n=1}^{6} B_{n,q}^b B_{n,-q}^b$$

$$+ \frac{\lambda_m}{a^3} \sum_{n=1}^{6} 2 \cos(q \cdot e_0) B_{n,q}^b B_{n,-q}^b$$

(A10)

where

$$B_{1,q} = \{-e_1, e_1, 0, 0\}, \quad B_{2,q} = \{0, 0, -e_2, e_2\} , \quad$$

$$B_{3,q} = \{e_3, 0, 0, -e_3\}, \quad B_{4,q} = \{e_1, e_1, e_2, -e_2\}, \quad B_{5,q} = \{0, 0, e^{-i Q} e_2, -e_2\} , \quad B_{6,q} = \{-e_3, 0, 0, e^{-i Q} e_3\}$$

(A11)

and

$$B_{1,q}^b = 2\{2 e_1, -(1 + e^{-i Q} e_2) | e_1^0, 0, 0\}, \quad$$

$$B_{2,q}^b = 2\{0, 0, 2 e_2, -(1 + e^{-i Q} e_1) | e_2^0, 0, 0\}, \quad$$

$$B_{3,q}^b = 2\{-(1 + e^{-i Q} e_1) e_1^0, 0, 0, 2 e_1\}, \quad$$

$$B_{4,q}^b = 2\{-(1 + e^{-i Q} e_2) e_2^0, 0, 0, 2 e_2\}, \quad$$

$$B_{5,q}^b = 2\{0, 0, -(1 + e^{-i Q} e_2) e_2^0, 2 e_2\}, \quad$$

$$B_{6,q}^b = 2\{2 e_2^0, 0, 0, -(1 + e^{-i Q} e_2) \}$$

(A12)

where $e_1^0$ are the unit vectors perpendicular to the bonds, $e_0$ should be understood as $(e_1, e_2, e_3)$, and the same for $e_2^0$, so all these vectors are six dimensional. The factor of 2 is from the fact that the length of the bonds is 1/2.

The scattering $V$ can also be expressed in this form. We assume that the changed bond is bond 1 in the unit cell at $\vec{r} = 0$ as marked in Fig. 3. Then we have

$$V_{q,q'} = \frac{\mu_m}{a} B_{1,q}^b B_{1,-q'}^b$$

$$+ \frac{\kappa_m}{a^2} (B_{1,q}^b B_{1,-q'}^b + B_{1,q}^b B_{1,-q'}^b)$$

$$- \frac{\lambda_m}{a^3} (B_{1,q}^b B_{1,-q'}^b + B_{1,q}^b B_{1,-q'}^b).$$

(A13)

**APPENDIX B: ASYMPTOTIC SOLUTIONS OF THE EMT I EQUATIONS**

Asymptotic solutions for the EMT I self-consistency equation (A3) were developed in the preceding paper [25] in the limit of small $\kappa / \mu$. In this section we review this asymptotic solution and apply it to the case of the kagome lattice. For convenience we define the notation

$$b_m \equiv \frac{\kappa_m}{(\mu_m a^2)}, \quad l_m \equiv \frac{\lambda_m}{(\mu_m a^2)}, \quad b \equiv \frac{\kappa}{(\mu_m a^2)},$$

(B1)

and

$$H(b_m, l_m) \equiv -\frac{\mu_m}{a} G(\mu_m, \kappa_m, \lambda_m) = -G(1, b_m, l_m).$$

(B2)

where the second line is derived from the definition of the dynamical matrix. The self-consistency equation (A3) can be solved by projection to the space $[B_1^b, B_2^b, B_3^b]$ that spans $V$. In this basis we can rewrite the EMT matrix equation as three independent equations

$$\mu_m = \frac{\mu}{H_1(b_m, l_m)} - H_1(b_m, l_m),$$

(B3a)

$$\left(1 - \frac{b_m}{b_m} \right)^{-1} \left[ \frac{1}{2} - \frac{1}{b_m} \right] b_m H_2 - l_m H_3$$

(B3b)

$$- l_m - 2 \left( \frac{1}{b_m} \right)^{-1} (l_m H_2 + b_m H_3)$$

(B3c)

where $H_1$ and $H_3$ are the projections of the Green’s function defined as

$$H_1(b, l) \equiv \{B_1^b | H(b, l) | B_1^b\},$$

(B4)

$$H_2(b, l) \equiv \{B_1^b | H(b, l) | B_2^b\},$$

(B4)

$$H_3(b, l) \equiv \{B_1^b | H(b, l) | B_3^b\},$$

with the inner product defined as

$$\langle B_1^b | H(b, l) | B_1^b \rangle = \frac{1}{N} \sum_\mathbf{q} B_{1,-q}^b H(b, l) \cdot B_{1,q}^b$$

(B5)

where the sum is over all $N$ vectors in the first Brillouin zone of the kagome lattice and the trace is understood to include the sum over these vectors (along with the factor of $1/N$) in addition to the sum over the six-dimensional space of $B_{1,-q}^b$. These equations are exactly equivalent to the matrix equation (A3). It is clear from their form that $\mu_m / \mu$, $b_m$, and $l_m$ are all functions of $p$ and $k = \kappa / (\mu_m a^2)$, implying that $G / G_0$ is a function of $p$ and $k$.

For the special case of $\kappa = 0$, Eqs. (B3a)-(B3c) simplify and give

$$\mu_m = \frac{\mu}{H_1(0,0)} - H_1(0,0),$$

(B6)

$$b_m = 0,$n

(B7)

$$l_m = 0.$$n

(B8)

As discussed in the preceding paper [25], it is straightforward from the definition $D$ and the fact that the central force undiluted kagome lattice is isostatic with $z = 2d$ to derive the relation

$$H_1(0,0) = 1.$$n

(B9)

The effective-medium filament stretching stiffness is then $\mu_m = 0$ for $p < 1$ and $\mu_m = \mu$ for $p = 1$. Therefore, this $\kappa = 0$ EMT solution indicates a first-order rigidity transition at $p = 1$. In the following we solve these equations asymptotically at small $\kappa$ near the two critical points $p = 1$ and $p_b$.

1. **Asymptotic solution near $p = 1$**

The EMT solution for small $\kappa > 0$ can be calculated perturbatively from the $\kappa = 0$ solution (B6). Near $p = 1$, as discussed in the preceding paper [25], we can make simplifications to the self-consistency equation [(B3b) and (B3c)] using
the fact that $\kappa \ll 1$ so that
\[
\kappa_m = \kappa(2p - 1), \quad \lambda_m = 0
\]
(B10)
(which we shall verify later) and therefore we only need to solve Eq. (B3a) using perturbation.

In contrast to the perturbative calculation in the triangular lattice, the kagome lattice effective medium is isostatic as $\kappa_m \rightarrow 0$ and thus the phonon Green’s functions exhibit singularities. These singularities correspond to the zero-frequency floppy modes of the dynamical matrix and make diverging contributions to $\mathbf{H}$.

Nevertheless, perturbation theory around the $\kappa = 0$ solution is still well defined. We shall see below that the $\mathbf{H}$ singularity is proportional to $b_m^{-1/2}$ and thus all the terms in the self-consistency equations are nonsingular. The expansion of $H_1$ at small $b_m$ can be calculated using the equality
\[
\begin{align*}
\langle \mathbf{B}_1^\dagger \mathbf{H}(b_m,0) \mathbf{B}_1 \rangle + b_m \langle \mathbf{B}_1^\dagger \mathbf{H}(b_m,0) \mathbf{B}_1^b \rangle \\
= \text{Tr} \mathbf{H}(b_m,0) \left( \langle \mathbf{B}_1^\dagger \mathbf{B}_1 \rangle + b_m \langle \mathbf{B}_1^\dagger \mathbf{B}_1^b \rangle \right) \\
= \frac{1}{6} \text{Tr} \mathbf{H}(b_m,0) \left[ \sum_{n=1}^{6} \left( \langle \mathbf{B}_n^\dagger \mathbf{B}_n \rangle + b_m \langle \mathbf{B}_n^b \rangle \right) \right] = 1.
\end{align*}
\]
(B11)
These relations follow because all of the six NN and six NNN bonds in a cell are, respectively, equivalent by symmetry, allowing the trace over one set of bonds to be replaced by $\frac{1}{6}$, the trace over the sum of the bonds. However, the quantity in square brackets is just the inverse of $\mathbf{H}(b_m,0)$ and the final result follows.

Employing the analysis of the phonon modes in Ref. [46], we find that $\langle \mathbf{B}_1^\dagger \rangle$ has a nonzero projection onto the floppy mode branch, which we call $|\nu_f\rangle$, whereas $\langle \mathbf{B}_1^b \rangle$ does not. The floppy mode branch has low frequencies that are proportional to $\sqrt{b_m}$ along symmetry directions at $(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2})$ in the first Brillouin zone, as shown in Fig. 10. In this calculation we shall use the direction $\pi/2$, which corresponds to $q_s = 0$ as an example. Near the $q_s = 0$ line the floppy mode branch phonon Green’s function takes the form
\[
G_{\text{floppy, } \pi/2, q_s} = \frac{-1}{4q_s^2 + 16b_m g(q_s)},
\]
(B12)
where
\[
g(q_s) = \frac{3Q_M - 2Q_S - q_s}{Q_M - Q_S},
\]
(B13)
with $Q_M = 2\pi/\sqrt{3}$ and $Q_S = 4(3b_m/2)^{1/2}$ and thus $g(q_s)$ takes a value between 2 and 3 in the first Brillouin zone. This form of the Green’s function is shown in Fig. 10. It was derived in Ref. [46], in which the weak additional interactions are NNN bonds rather than bending forces.

It is straightforward to calculate the singular part of $\langle \mathbf{B}_1^\dagger \mathbf{H}(b_m,0) \mathbf{B}_1^b \rangle$ along symmetry directions of the floppy modes, which involves the integral
\[
H_{2,\pi/2, q}(b_m,0) = v_0 \int_{\text{BZ}|q_s| \leq |q_s|/\sqrt{3}} \frac{d^2q}{(2\pi)^2} \frac{|\langle \mathbf{B}_{1q}^\dagger |\nu_f \rangle|^2}{q_s^2 + b_m g(q_s)},
\]
(B14)
where $v_0 = \sqrt{3}/2$ is the area of the kagome lattice unit cell and the condition $|q_s| \leq |q_s|/\sqrt{3}$ confines the integral to be around the floppy mode directions $\pi/2$ and $3\pi/2$ near which the expansion (B12) is valid. The total contribution involves the singular part of all the isostatic directions
\[
H_{2,\pi}(b_m,0) \simeq \frac{1}{(2\pi)^2/v_0} \int_{\pi}^{2\pi/\sqrt{3}} dq_s 4\pi |\langle \mathbf{B}_{1q}^\dagger |\nu_f \rangle|^2/\sqrt{3} \sqrt{b_m g(q_s)} \\
\simeq b_m^{-1/2},
\]
(B16)
indicating a leading-order divergence $b_m^{-1/2}$ of $H_2$ at small $\kappa_m$.

A calculation including all terms in Eq. (B15) yields
\[
H_{2,\pi, q}(b_m,0) \simeq \sqrt{\frac{A}{2}} b_m^{-1/2}.
\]
(B17)
where
\[
\sqrt{\frac{A}{2}} = 20(1 - \sqrt{2/3}).
\]
(B18)
From this we get
\[
H_1(b_m,0) \simeq 1 - \sqrt{\frac{A}{2}} b_m^{-1/2}
\]
(B19)
at small $\kappa_m$. Plugging this back into Eq. (B3a), we arrive at the solution
\[
\frac{\mu_m}{\mu_p} = \frac{(\Delta p)^2}{A(2p - 1)c/(\mu a^2)} \times \left[ -1 + \frac{A(2p - 1)c/(\mu a^2)}{(\Delta p)^2} \right]^2,
\]
(B20)
where $\Delta p \equiv 1 - \Delta p$; this solution is asymptotically accurate for $\kappa \ll 1$ and $\Delta p \ll 1$.

2. Asymptotic solution near $p = p_b$

The rigidity threshold $p_b$ can be obtained by plugging $\mu_m = 0$ into Eqs. (B3a)–(B3c), which leads to
\[
p_b \simeq 0.6920, \quad b_m \simeq 0.03788, \quad l_m \simeq 0.008032.
\]
(B21)
Following the calculation in Ref. [25], we arrive at the asymptotic solution for $\mu_m$,
\[
\mu_m = \mu_0 \Phi(\kappa/(\mu a^2))(p - p_b),
\]
(B22)
where
\[
\Phi(x) = \frac{c_2}{c_1 + x},
\]
(B23)
with
\[
c_1 \simeq 0.03802, \quad c_2 \simeq 4.697.
\]
(B24)
APPENDIX C: ASYMPTOTIC SOLUTIONS TO EMT II EQUATIONS

The self-consistency equation of the EMT II can be written as [23,26]
\[ \frac{\mu_m}{\mu} = \frac{p - a^*}{1 - a^*}, \quad \frac{\kappa_m}{\kappa} = \frac{p^2 - b^*}{1 - b^*}, \]  
(C1)
where the variables \( a^* \) and \( b^* \) correspond to the integrals we defined as
\[ a^* = H_1, \quad b^* = (\kappa_m/\mu_m)H_2, \]  
(C2)
as shown in the preceding paper [25]. Using the asymptotic forms we obtained in Eqs. (B17) and (B19), we have
\[ \frac{\mu_m}{\mu} = \frac{p - 1 + (\sqrt{\kappa}/2)(\kappa_m/\mu_m a^2)^{1/2}}{\sqrt{\kappa}/2(\kappa_m/\mu_m a^2)^{1/2}}, \]  
(C3)
\[ \frac{\kappa_m}{\kappa} = \frac{p^2 - (\sqrt{\kappa}/2)(\kappa_m/\mu_m a^2)^{1/2}}{1 - (\sqrt{\kappa}/2)(\kappa_m/\mu_m a^2)^{1/2}}. \]  
(C4)
Close to \( p = 1 \), if we take \( \kappa/\mu \ll 1 \), then \( (\sqrt{\kappa}/2)(\kappa_m/\mu_m a^2)^{1/2} \ll 1 \) and we have
\[ \frac{\kappa_m}{\kappa} \approx 1. \]  
(C5)
Plugging this back into Eq. (C3) we get
\[ \frac{\mu_m}{\mu} = \frac{p - 1 + (\sqrt{\kappa}/2)(\kappa_m/\mu_m a^2)^{1/2}}{\sqrt{\kappa}/2(\kappa_m/\mu_m a^2)^{1/2}}, \]  
(C6)
which is a quadratic equation in \( \mu_m^{1/2} \), and the solution to this leads to the same asymptotic solution as in Eqs. (B20) and (3.4).

Next we discuss asymptotic behaviors near the rigidity threshold \( p_{b,2} \) (the subscript 2 is used to distinguish it from the threshold \( p_b \) in EMT I). The value of \( p_{b,2} \) in EMT II can be readily obtained from
\[ 0 = p - a^*, \quad 0 = p^2 - b^*, \]  
(C7)
along with the condition that \( a^* + b^* = 2d/z \), as discussed in Appendix D of Ref. [25]. We then have
\[ p_{b,2} = \frac{1}{2} \left( -1 + \sqrt{1 + \frac{8d}{z}} \right) \simeq 0.6180. \]  
(C8)
The EMT II self-consistency equations (C1) can then be expanded at small
\[ \Delta p = p - p_{b,2}, \]  
(C9)
which leads to
\[ \mu_m = \mu \Phi_2(x/(\mu a^2))(p - p_{b,2}), \]  
(C10)
with
\[ \Phi_2(x) = \frac{(2p_{b,2} + 1)x}{p_{b,2}^2 + 1 - p_{b,2}x}. \]  
(C11)
This is in the same scaling form as in EMT I [Eqs. (1.4) and (B23)], apart from different constant factors.