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Khoury, J., Miller, G. E. J., & Tolley, A. J. (2012). Spatially Covariant Theories of a Transverse, Traceless Graviton: Formalism. *Physical Review D, 85*(8), 084002. doi: 10.1103/PhysRevD.85.084002
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Disciplines
Physical Sciences and Mathematics | Physics

Comments
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PHYSICAL REVIEW D **85,** 084002 (2012)

Spatially covariant theories of a transverse, traceless graviton: Formalism

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(Received 12 August 2011; published 2 April 2012)

General relativity is a generally covariant, locally Lorentz covariant theory of two transverse, traceless graviton degrees of freedom. According to a theorem of Hojman, Kuchař, and Teitelboim, modifications of general relativity must either introduce new degrees of freedom or violate the principle of local Lorentz covariance. In this paper, we explore modifications of general relativity that retain the same graviton degrees of freedom, and therefore explicitly break Lorentz covariance. Motivated by cosmology, the modifications of interest maintain explicit spatial covariance. In spatially covariant theories of the graviton, the physical Hamiltonian density obeys an analogue of the renormalization group equation which encodes invariance under flow through the space of conformally equivalent spatial metrics. This paper is dedicated to setting up the formalism of our approach and applying it to a realistic class of theories. Forthcoming work will apply the formalism more generally.

DOI: 10.1103/PhysRevD.85.084002 PACS numbers: 04.50.Kd

I. INTRODUCTION

For nearly a century, general relativity has been the most successful paradigm for interpreting and understanding classical gravitational phenomena. To this day, despite ongoing experimental efforts, there have been no unequivocal refutations of general relativity. Nonetheless, there are compelling reasons to study alternative gravitational theories.

Perhaps the most obvious reason to consider alternative gravitational theories is to explain *empirical* anomalies, most notably the observed magnitude of the cosmic acceleration. The ΛCDM concordance model achieves a parsimonious fit to cosmological observations by invoking a cosmological constant corresponding to a vacuum energy density $\rho_\Lambda \approx (\mathrm{meV})^4$. However, known quantum corrections to ρ_Λ are of order $M_{\mathrm{Pl}}^4 \approx 10^{120}\ (\mathrm{meV})^4$, so ΛCDM suffers from a serious fine-tuning problem [1]. Since we have no handle on the microphysics responsible for the magnitude of vacuum energy, it is an outstanding theoretical challenge to determine what physical degrees of freedom are associated with late-time acceleration. Theories of dynamical dark energy or modified gravity typically introduce new scalar degrees of freedom, but to date there is no unambiguous evidence for cosmologically relevant scalars [2]. One motivation for this paper is the possibility that cosmic acceleration might be directly associated with the transverse, traceless graviton degrees of freedom.

Apart from any attempt to understand empirical anomalies, there remains a compelling *theoretical* reason to study alternatives to general relativity: to determine which of its features are essential to its experimental success, and which features are merely incidental. To analyze the theory in this manner, we must know what freedom we have to modify the theory while retaining its explanatory power. The two transverse, traceless graviton degrees of freedom are a key feature of general relativity. Though graviton exchange has never been measured and gravitational waves have never been detected, there is substantial indirect evidence for these two graviton degrees of freedom [3]. It is therefore natural to ask whether and how we can modify the behavior of the known graviton degrees of freedom.

In this paper, we construct manifestly consistent modifications of general relativity that retain the same local degrees of freedom. Since general relativity is the unique *Lorentz covariant* theory of a massless spin-2 particle [4–7], our theories must break Lorentz covariance *explicitly*. Theories in which Lorentz symmetry is only broken spontaneously necessarily rely on additional local degrees of freedom, which appear in the broken phase as massless Goldstone modes; an example of such a theory is ghost condensation [8].

General relativity as formulated by Einstein and Hilbert is also a *generally covariant* theory, which means that the equations of motion for the spacetime metric $g_{\mu\nu}$ take the same form in any coordinate system. Unfortunately, invariance under coordinate transformations implies that the theory contains a great deal of gauge arbitrariness, and the true dynamical degrees of freedom of the theory have proven difficult to isolate. The inaccessibility of the physical graviton degrees of freedom is a significant obstacle to modifying their behavior. In fact, the notorious elusiveness of the physical degrees of freedom is also an obstacle to the canonical quantization of general relativity [9].

This gauge arbitrariness can be understood most clearly by treating general relativity as a *constrained field theory.*

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By writing the spacetime metric $g_{\mu\nu}$ in Arnowitt-Deser-Misner (ADM) form[1] and discarding a boundary term, the Einstein-Hilbert action can be rewritten in canonical form as a theory of a spatial metric h_{ij} and a conjugate momentum tensor π^{ij} subject to four constraints $\mathcal{H}_\mu$ [10,11]. Though h_{ij} and π^{ij} are not themselves generally covariant objects, the general covariance of the theory follows from the *first class* character of the $\mathcal{H}_\mu$'s, which generate gauge transformations corresponding to spacetime diffeomorphisms [12,13]. By representing gauge symmetries as constraints on phase space, it becomes straightforward to count degrees of freedom. According to the standard counting prescription, the presence of four first class constraints $\mathcal{H}_\mu$ in a theory of six canonical coordinates h_{ij} ensures that general relativity contains two local degrees of freedom; schematically,

$$6 \cdot h_{ij}{}'s - 4 \cdot \mathcal{H}_\mu{}'s = 2 \text{ degrees of freedom.} \quad (1)$$

See Sec. II for more detail. In the passage to quantum theory, these transverse, traceless degrees of freedom become the two polarizations of the graviton.

To isolate the physical graviton degrees of freedom, one would have to solve the four constraints $\mathcal{H}_\mu$. One could then modify the behavior of the graviton in a straightforward manner. By taking the configuration space for the spatial metric to be Wheeler's superspace, it is possible to solve the three momentum constraints $\mathcal{H}_i$ by fiat, but the Hamiltonian constraint $\mathcal{H}_0$ has thus far defied solution in general. Unless the Hamiltonian constraint can be solved, the gauge arbitrariness of general relativity cannot be eliminated. Fortunately, though no general solution to the Hamiltonian constraint has been found, it can be solved in certain circumstances by imposing an appropriate gauge-fixing condition.

We have established that modifying the behavior of the graviton without new degrees of freedom will force us to break Lorentz covariance explicitly, but it is enlightening to see how this conclusion arises in the canonical picture. Under the action of the Poisson bracket, the $\mathcal{H}_\mu$'s of general relativity obey the *Dirac algebra* [14,15], which encodes the local Lorentz covariance of a generally covariant theory [16]. In 1974, Hojman, Kuchař, and Teitelboim (HKT) proved that general relativity is the *unique minimal representation* of the Dirac algebra [17,18]. It follows immediately that Lorentz covariant modifications of general relativity introduce additional degrees of freedom beyond the two graviton degrees of freedom in general relativity [19]. To modify general relativity, one must either introduce new degrees of freedom or violate the principle of local Lorentz covariance.

[1] i.e., in terms of a spatial metric h_{ij}, a lapse $N \equiv N^0$, and a shift N^i.

We wish our theories to retain the same local degrees of freedom as general relativity, so in accordance with the theorem of HKT, our theories cannot be Lorentz covariant. This aspect of our approach is not necessarily a defect. Since we do not observe exact spacetime symmetry in our universe, this property of general relativity is not necessarily key to the success of the theory. Simply put, on cosmological scales there is a strong asymmetry between the past and the future, and the observable universe has a preferred rest frame; these observations are conventionally understood as a result of spontaneous symmetry breaking, but *explicit* symmetry breaking is another logical possibility.

That being said, on cosmological scales in the cosmological rest frame there is substantial evidence for spatial homogeneity and isotropy. To maximize the verisimilitude of our treatment, the theories we consider will retain explicit covariance under *spatial* diffeomorphisms. To summarize, we will attempt to modify general relativity while preserving (1) the number of graviton degrees of freedom, and (2) explicit spatial covariance. In this paper, we develop a general framework within which to explore the freedom we have to modify general relativity while retaining these two desirable properties.

Concretely, we will begin by recasting general relativity in spatially covariant form, by solving the Hamiltonian constraint (which generates local time reparametrizations) while preserving the three momentum constraints (which generate spatial diffeomorphisms). We will solve the Hamiltonian constraint by choosing a cosmologically motivated gauge: we will take the determinant of the spatial metric to be the measure of time. This operation destroys the manifest diffeomorphism covariance and local Lorentz covariance of the theory. We emphasize that this gauge breaks down in the general case when the determinant of the spatial metric is allowed to evolve nonmonotonically, but it is a natural choice when considering perturbative corrections to Friedmann-Robertson-Walker (FRW) spacetime. By solving the Hamiltonian constraint, the determinant of the spatial metric and the trace of the momentum tensor drop out of the phase space of the theory. We thereby obtain general relativity as a theory of a unit-determinant metric $\tilde{h}_{ij}$ and a traceless conjugate momentum tensor $\tilde{\pi}^{ij}$ subject to three first class momentum constraints $\tilde{\mathcal{H}}_i$, which act as the generators of spatial diffeomorphisms. By the standard counting prescription, the presence of three first class constraints $\tilde{\mathcal{H}}_i$ in a theory of five canonical coordinates $\tilde{h}_{ij}$ guarantees that spatially covariant general relativity contains two degrees of freedom, as it should; schematically,

$$5 \cdot \tilde{h}'_{ij}s - 3 \cdot \tilde{\mathcal{H}}'_i s = 2 \text{ degrees of freedom.} \quad (2)$$

See Sec. III D for more detail.

Our strategy for modifying general relativity relies on the fact that *any* theory of five canonical coordinates subject to three first class constraints contains two degrees of

freedom. To modify general relativity, we will modify the functional form of the physical Hamiltonian density on the reduced phase space $(\tilde{h}_{ij}, \tilde{\pi}^{ij})$, subject to the condition that the momentum constraints $\tilde{\mathcal{H}}_i$ remain first class; to ensure the consistency of the modification, we will also demand that the constraints $\tilde{\mathcal{H}}_i$ remain preserved by the equations of motion. Any theory that satisfies these two restrictions will retain manifest spatial covariance, and by the counting prescription will necessarily contain two graviton degrees of freedom. In this paper, we introduce the formalism necessary to pursue this program of modification and apply the formalism to a class of realistic theories. Forthcoming work will apply the formalism developed here to the goal of constructing viable alternatives to general relativity [20]. In particular, these alternatives stand a good chance of being consistent with binary pulsar constraints, which principally constrain the number of gravitational degrees of freedom.

The literature abounds with many and varied approaches to the pursuit of modified gravity theories, but generally covariant, locally Lorentz covariant modifications of general relativity that introduce additional degrees of freedom have been the most widely explored. The well-known method for finding such theories is to construct a scalar Lagrangian density out of manifestly covariant objects by contracting all free spacetime/Lorentz indices.[2] Using this technique, all manner of theories have been explored: scalar-tensor theories [21], theories with higher order curvature terms [22–24], theories of massive gravity [25–28], higher-dimensional gravity theories [29–31], Galileons [32–37], chameleons [38], symmetrons [39,40], cuscutons [41], etc. For a comprehensive review of Lorentz covariant massive gravity theories with detailed references, see [7]. For a comprehensive review of observational tests of modified gravity, see [2].

Approaches to gravity which do not assume general covariance and local Lorentz covariance at the outset have been tried as well. The natural procedure for constructing such a theory depends on which symmetries it is assumed to possess; more often than not, theories without spacetime symmetry are assumed to maintain explicit spatial symmetry. For example, in [15] the action of gravity is assumed to be invariant under spatial diffeomorphisms. In [42], Lorentz-violating massive graviton theories were classified by assuming the graviton mass to be invariant under the three-dimensional Euclidean group. A prominent recent example of a Lorentz-violating theory is Hořava-Lifshitz gravity [43–50].[3] Also of note is the work of Barbour, Koslowski, and collaborators on the theory of Shape Dynamics [51,52].

This paper is organized as follows. In Sec. II, we cover the basic concepts of constrained field theory in the context of analyzing the phase space and constraint structure of general relativity. In Sec. III, we show how to impose our cosmological gauge condition and solve the Hamiltonian constraint to obtain a consistent spatially covariant formulation of general relativity. In Sec. IV, we introduce the formalism of our approach to modifying gravity in the context of *ultralocal* theories of the graviton. In Sec. V, we apply our method to derive consistency relations for a class of realistic *local* theories which includes general relativity.

II. GENERAL RELATIVITY AS A CONSTRAINED FIELD THEORY

In this section, we will analyze general relativity by treating it as a constrained field theory. In particular, we will examine its phase space and constraint structure, and count its local degrees of freedom.

Our starting point is the Einstein-Hilbert action with a cosmological constant,

$$S = \int dt d^3x \sqrt{-g}(R^{(4)} - 2\Lambda). \tag{3}$$

From this action, the general covariance of the theory is manifest, but the counting of degrees of freedom is not. The metric tensor $g_{\mu\nu}$ has ten components, but the theory has only two independent local degrees of freedom. To facilitate the counting of degrees of freedom, it is conceptually simplest to rewrite the action in *canonical form*, which makes the counting manifest. To this end, the spacetime metric $g_{\mu\nu}$ must first be expressed in ADM form, in terms of a lapse N, a shift N^i, and a spatial metric h_{ij}:

$$\begin{aligned} ds^2 &= g_{\mu\nu}dx^\mu dx^\nu \\ &\equiv -N^2dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt). \end{aligned} \tag{4}$$

The lapse N is a three-scalar, the shift N^i is a three-vector, and the spatial metric h_{ij} is a three tensor. Up to a boundary term, the Einstein-Hilbert action is equivalent to the ADM action

$$S = \int dt d^3x \sqrt{h} N(K_{ij}K^{ij} - K^2 + R - 2\Lambda). \tag{5}$$

In this expression, indices are lowered with h_{ij} and raised with its inverse h^{ij}, $R \equiv R^{(3)}$ is the Ricci scalar of the metric h_{ij}, the extrinsic curvature tensor K_{ij} is defined by

$$K_{ij} \equiv \tfrac{1}{2}N^{-1}(\dot{h}_{ij} - \nabla_i N_j - \nabla_j N_i), \tag{6}$$

$K \equiv h^{ij}K_{ij}$, and $\nabla_i \equiv \nabla_i^{(3)}$ is the covariant spatial derivative with respect to the metric h_{ij}. From Eq. (6), it is clear that time derivatives in the action (5) act only on h_{ij}, not on

[2] In the presence of spinor fields, one must treat spacetime indices and Lorentz indices separately.

[3] The original incarnation [43] of Hořava-Lifshitz gravity struggled with consistency issues [46–48] which were resolved in [49] by imposing a consistent constraint algebra.

N and N^i, so the lapse and shift are essentially nondynamical. To obtain the canonical action, one must first define the momentum conjugate to the spatial metric,

$$\pi^{ij} \equiv \frac{\delta L}{\delta \dot{h}_{ij}} = \sqrt{h}(K^{ij} - Kh^{ij}); \tag{7}$$

the momentum π^{ij} is a three-tensor density of unit weight.[4] By inverting the relation between π^{ij} and K^{ij} and dropping a boundary term, one can rewrite the action of general relativity in canonical form as

$$S = \int dt d^3x(\pi^{ij}\dot{h}_{ij} - N^\mu \mathcal{H}_\mu), \tag{8}$$

where $N^0 \equiv N$ and

$$\mathcal{H}_0 \equiv -\sqrt{h}(R - 2\Lambda) + \frac{1}{\sqrt{h}}\left(\pi^{ij}\pi_{ij} - \frac{1}{2}(\pi^i_i)^2\right),$$
$$\mathcal{H}_i \equiv -2h_{ij}\nabla_k \pi^{jk}. \tag{9}$$

Variation of the action (8) with respect to h_{ij} and π^{ij} yields Hamilton's equations,

$$\dot{h}_{ij}(x) = \frac{\delta H}{\delta \pi^{ij}(x)}, \qquad \dot{\pi}^{ij}(x) = -\frac{\delta H}{\delta h_{ij}(x)}, \tag{10}$$

where the Hamiltonian H is

$$H = \int d^3x N^\mu \mathcal{H}_\mu. \tag{11}$$

To evaluate the above variational derivatives, one must use the relations

$$\frac{\delta h_{ij}(x)}{\delta h_{kl}(y)} = \frac{\delta \pi^{kl}(x)}{\delta \pi^{ij}(y)} = \delta^{kl}_{ij}\delta^3(x - y), \tag{12}$$

where

$$\delta^{kl}_{ij} \equiv \tfrac{1}{2}(\delta^k_i\delta^l_j + \delta^l_i\delta^k_j). \tag{13}$$

Defining the Poisson bracket

$$\{A, B\} \equiv \int d^3z\left(\frac{\delta A}{\delta h_{mn}(z)}\frac{\delta B}{\delta \pi^{mn}(z)} - \frac{\delta A}{\delta \pi^{mn}(z)}\frac{\delta B}{\delta h_{mn}(z)}\right), \tag{14}$$

the equation of motion for any quantity $A(h_{ij}, \pi^{ij}, t)$ can be written as

$$\dot{A} = \frac{\partial A}{\partial t} + \{A, H\} = \frac{\partial A}{\partial t} + \int d^3y N^\nu(y)\{A, \mathcal{H}_\nu(y)\}. \tag{15}$$

If A has no explicit dependence on time, its evolution is *generated* by its Poisson bracket with the $\mathcal{H}_\mu$'s.

Variation of the action with respect to N^μ yields the four constraints

$$\mathcal{H}_\mu \sim 0. \tag{16}$$

The symbol $\sim$ denotes *weak equality*, or equality after the constraints $\mathcal{H}_\mu \sim 0$ have been enforced. For example, if $X = Y + \lambda^\mu \mathcal{H}_\mu$, then $X \sim Y$. Since the constraints define a surface in phase space, weak equality is also termed equality *on the constraint surface*. As an aside, it follows from (11) and (16) that $H \sim 0$; the vanishing of the Hamiltonian on the constraint surface is a feature common to generally covariant theories whose coordinates and momenta transform as scalars under time reparametrizations [12].

There is no $\pi_\mu \dot{N}^\mu$ term that would allow us to compute a variational expression for $\dot{N}^\mu$, so the time evolution of N^μ is unconstrained by the action. The four functions N^μ are thus arbitrary until and unless we gauge-fix them.

Constraint properties and degrees of freedom

Before examining the constraints more closely, we pause to review some terminology first introduced by Dirac for describing constrained theories [14]. A quantity whose Poisson bracket with each of the constraints vanishes (identically or weakly) is termed *first class*; a quantity whose Poisson bracket fails to vanish weakly with at least one constraint is termed *second class*. A *first class constraint* has vanishing Poisson bracket with all constraints, while a *second class constraint* has nonvanishing Poisson bracket with at least one other constraint. In most cases of interest, first class constraints generate *gauge symmetries* under the action of the Poisson bracket. Second class constraints can usually be solved, either implicitly by using the "Dirac bracket" [53], or explicitly by expressing some phase space variables in terms of others.[5]

By direct calculation—see Appendix A for details—it is possible to prove that the constraints $\mathcal{H}_\mu$ are first class, $\{\mathcal{H}_\mu(x), \mathcal{H}_\nu(y)\} \sim 0$. This means that the symmetry generators close under the action of the Poisson bracket, as they must in order to consistently represent a gauge symmetry. In particular,

$$\begin{aligned}
\{\mathcal{H}_0(x), \mathcal{H}_0(y)\} &= \mathcal{H}^i(x)\partial_{x^i}\delta^3(x - y) \\
&\quad - \mathcal{H}^i(y)\partial_{y^i}\delta^3(x - y), \\
\{\mathcal{H}_0(x), \mathcal{H}_i(y)\} &= \mathcal{H}_0(y)\partial_{x^i}\delta^3(x - y), \\
\{\mathcal{H}_i(x), \mathcal{H}_j(y)\} &= \mathcal{H}_j(x)\partial_{x^i}\delta^3(x - y) \\
&\quad - \mathcal{H}_i(y)\partial_{y^j}\delta^3(x - y).
\end{aligned} \tag{17}$$

This is the *Dirac algebra*, first discovered by Dirac in the context of parametrized field theories in flat spacetime [14,15]. The gauge symmetry corresponding to this first

[4] According to the standard convention, the weight of a tensor density is the number of times $\sqrt{h}$ multiplies the underlying tensor.

[5] See [12] for a pedagogical treatment of the general theory of constrained systems; see [13] for an in-depth analysis of several interesting constrained systems, including electromagnetism and general relativity.

class algebra is *general covariance*, and the constraints $\mathcal{H}_\mu$ generate spacetime diffeomorphisms.

The geometrical significance of the Dirac algebra was determined by Teitelboim in [16]: it is the algebra of the deformations of a spacelike hypersurface embedded in a Lorentzian spacetime manifold. When the $\mathcal{H}_\mu$'s satisfy (17), $\mathcal{H}_0$ generates deformations normal to the surface, while the $\mathcal{H}_i$'s generate deformations parallel to the surface. The Dirac algebra thus encodes the local Lorentz covariance of a generally covariant system. In fact, four first class constraints obeying the Dirac algebra are guaranteed to arise in any generally covariant field theory which satisfies the principle of local Lorentz covariance.

For the constraints to be consistent with the equations of motion, the constraints must be preserved by the equations of motion, i.e., $\dot{\mathcal{H}}_\mu \sim 0$. Since $\partial\mathcal{H}_\mu/\partial t = 0$, applying the equations of motion to $\mathcal{H}_\mu$ yields

$$\dot{\mathcal{H}}_\mu(x) = \int d^3y N^\nu(y)\{\mathcal{H}_\mu(x), \mathcal{H}_\nu(y)\}. \tag{18}$$

From the first class character of the constraints, it follows that $\dot{\mathcal{H}}_\mu \sim 0$, as desired.

The Hamiltonian formulation of GR is a theory of a spatial metric h_{ij} and its conjugate momentum π^{ij}, so the theory contains 12 canonical (or 6 real) variables. However, these variables are not independent. First, they are related by the four constraints $\mathcal{H}_\mu \sim 0$. Second, from Eqs. (10) and (11) it follows that the equations of motion for h_{ij} and π^{ij} depend on the four arbitrary functions N^μ; to gauge-fix N^μ would require imposing four gauge-fixing constraints [12].

$$\begin{aligned} &6\cdot h'_{ij}s + 6\cdot \pi^{ij\prime}s - 4\cdot \mathcal{H}'_\mu s - 4\cdot N^{\mu\prime}s \\ &\quad = 4\,\text{canonical DoF}. \end{aligned} \tag{19}$$

The theory therefore has four canonical (or two real) degrees of freedom.

III. SPATIALLY COVARIANT GENERAL RELATIVITY

We would like to depart from general relativity by modifying the equations of motion for the two graviton degrees of freedom. Ideally, we would like to solve all four gauge constraints, go down to the physical phase space, and modify the theory at that level. In this way, we would circumvent all the difficulties of consistently modifying a constrained field theory. Unfortunately, we do not know how to do this.

One possible approach is to modify the equations of motion for the phase space variables h_{ij} and π^{ij}. However, the counting of degrees of freedom in general relativity relies on the fact that the four constraints $\mathcal{H}_\mu$ satisfy a consistent first class algebra, namely, the Dirac algebra of Eq. (17), and we know from the HKT theorem that any modification of the action for h_{ij} and π^{ij} will destroy this algebra. If we modify the action for the phase space variables h_{ij} and π^{ij}, we must impose an alternative constraint structure that consistently constrains the phase space to the same degree as the covariance algebra; this is the approach taken in [15], as well as in [49,51,52]. Since we take the point of view that full spacetime covariance is a spurious symmetry, we do not wish our theory to contain a constraint structure that implies the same degree of redundancy as the Dirac algebra.

Though spacetime symmetry is manifestly broken on cosmological scales (whether spontaneously or explicitly), there is strong evidence for spatial homogeneity and isotropy, so we will attempt to modify general relativity while preserving the manifest spatial covariance of the theory. To obtain a spatially covariant formulation of general relativity to modify, we will solve the Hamiltonian constraint $\mathcal{H}_0$ while leaving the three momentum constraints $\mathcal{H}_i$ intact. The Hamiltonian constraint is famously hard to solve in general, but we are interested in using our theories in a cosmological context, so we will solve it using a gauge-fixing constraint which is well-defined on an expanding FRW background.

A. Metric decomposition

Before gauge-fixing, we decompose the metric h_{ij} into a conformal factor $\Omega \equiv h^{1/3}$ and a unit-determinant metric $\tilde{h}_{ij}$, i.e.,

$$h_{ij} = \Omega\tilde{h}_{ij}. \tag{20}$$

Note that $\Omega = (\sqrt{h})^{2/3}$ is a three-scalar density of weight $2/3$, while $\tilde{h}_{ij}$ is a three-tensor density of weight $-2/3$. The scalar density we will work with is not the conformal factor Ω, but the volume factor $\omega \equiv \sqrt{h} = \Omega^{3/2}$, which is a scalar density of unit weight. We choose ω because its conjugate momentum,

$$\pi_\omega \equiv \frac{\delta L}{\delta\dot{\omega}} = \frac{2\pi^i_i}{3\omega} = -\frac{4}{3}K, \tag{21}$$

is a three-scalar and hence invariant under spatial conformal transformations, which rescale Ω or ω; this fact will simplify matters in Secs. IV and V. The momentum conjugate to $\tilde{h}_{ij}$ is

$$\tilde{\pi}^{ij} \equiv \frac{\delta L}{\delta\dot{\tilde{h}}_{ij}} = \Omega\left(\pi^{ij} - \frac{1}{3}h^{ij}\pi^k{}_k\right) = \omega\Omega\left(K^{ij} - \frac{1}{3}Kh^{ij}\right), \tag{22}$$

which is a traceless three-tensor density of weight $5/3$; the quantity

$$\tilde{\pi}^{ij}_T \equiv \frac{\tilde{\pi}^{ij}}{\omega\Omega} \tag{23}$$

is the corresponding traceless three-tensor. By defining the traceless projection tensor $\tilde{\delta}^{kl}_{ij}$

$$\tilde{\delta}^{kl}_{ij} \equiv \delta^{kl}_{ij} - \tfrac{1}{3}\tilde{h}_{ij}\tilde{h}^{kl}, = \delta^{kl}_{ij} - \tfrac{1}{3}h_{ij}h^{kl}, \tag{24}$$

we can write $\tilde{\pi}^{ij}$ more compactly as

$$\tilde{\pi}^{ij} = \Omega\tilde{\delta}^{ij}_{kl}\pi^{kl} = \Omega\omega\tilde{\delta}^{ij}_{kl}K^{kl}. \tag{25}$$

The phase space variables h_{ij} and π^{ij} can thus be written as

$$h_{ij} = \omega^{2/3}\tilde{h}_{ij}, \qquad \pi^{ij} = \omega^{-2/3}\tilde{\pi}^{ij} + \tfrac{1}{2}\tilde{h}^{ij}\omega^{1/3}\pi_\omega. \tag{26}$$

The decomposition of the spatial metric into a volume factor and a unit-determinant metric is completely general. Though the corresponding conjugate momenta were derived by taking variational derivatives of the Einstein-Hilbert Lagrangian, the decomposition of the momentum tensor into its trace part and its traceless part is likewise completely general. Those familiar with the techniques of numerical relativity may be reminded of the York-Lichnerowicz conformal decomposition or the BSSNOK (Baumgarte, Shapiro, Shibata, Nakamura, Oohara, and Kojima) formalism [54,55].

B. Cosmological gauge

To solve the constraint $\mathcal{H}_0$, we must first gauge-fix the lapse N with a gauge-fixing constraint χ for which $\{\mathcal{H}_0, \chi\} \not\sim 0$; this renders $\mathcal{H}_0$ second class, and hence solvable. This process destroys manifest spacetime covariance. Since we wish to retain explicit spatial covariance, we wish our constraints $\mathcal{H}_i$ to remain first class.

In a cosmological context, it is natural to use the volume factor of the spatial metric as a clock, so that $t = t(\omega)$; we call this cosmological gauge. As mentioned in the introduction, cosmological gauge is only valid when the determinant of the spatial metric evolves monotonically, so this procedure is only valid when considering perturbative corrections to FRW spacetime. When the evolution of ω is monotonic, $t(\omega)$ is an invertible function, so this gauge is equivalent to taking the volume factor ω to be a function of time, i.e., $\omega = \omega(t)$.

For cosmological purposes, another good gauge choice would be to take π_ω to be a function of time. Since $\pi_\omega = -4K/3$, this is equivalent to the constant mean curvature gauge, in which the trace of the extrinsic curvature tensor $K = h_{ij}K^{ij}$ is chosen to be a function of time. This gauge will not be used in the present work, but the constant mean curvature gauge is used in [15,41] and mentioned in [54,55].

To impose cosmological gauge, we add to the canonical action of general relativity a gauge-fixing constraint

$$\chi \equiv \omega - \omega(t), \tag{27}$$

along with a corresponding Lagrange multiplier λ. The new gauge-fixed action is

$$S' = \int dt d^3x(\pi^{ij}\dot{h}_{ij} - N^\mu\mathcal{H}_\mu - \lambda\chi). \tag{28}$$

Varying the action with respect to λ then reproduces the constraint

$$\chi \sim 0. \tag{29}$$

By direct calculation—see Appendix B for details—one can verify that

$$\{\mathcal{H}_0(x), \chi(y)\} = \tfrac{1}{2}\pi^i_i(x)\delta^3(x-y); \tag{30}$$

the constraints $\mathcal{H}_0$ and χ are thus second class, so we expect to be able to solve them. The only wrinkle is that

$$\{\mathcal{H}_i(x), \chi(y)\} = \sqrt{h(x)}\partial_{x^i}\delta^3(x-y), \tag{31}$$

so the constraints $\mathcal{H}_i$ are also second class! By shuffling our constraints slightly, we can obtain a set of two second class constraints and three first class constraints, and thereby render explicit the spatial covariance of the gauge-fixed action. Indeed, since

$$\left\{2\sqrt{h(x)}\partial_{x^i}\left(\frac{\mathcal{H}_0(x)}{\pi^k_k(x)}\right), \chi(y)\right\} \sim \sqrt{h(x)}\partial_{x^i}\delta^3(x-y), \tag{32}$$

it follows that the combination

$$\tilde{\mathcal{H}}_i \equiv \mathcal{H}_i - 2\sqrt{h}\partial_i\left(\frac{\mathcal{H}_0}{\pi^k{}_k}\right) = \mathcal{H}_i - 2\sqrt{h}\nabla_i\left(\frac{\mathcal{H}_0}{\pi^k{}_k}\right) \tag{33}$$

obeys

$$\{\tilde{\mathcal{H}}_i(x), \tilde{\mathcal{H}}_j(y)\} \sim \{\tilde{\mathcal{H}}_i(x), \mathcal{H}_0(y)\} \sim \{\tilde{\mathcal{H}}_i(x), \chi(y)\} \sim 0. \tag{34}$$

The interpretation of this result is simple. The $\mathcal{H}_i$'s generate spatial diffeomorphisms, while $\mathcal{H}_0$ generates time translation. A generic spatial diffeomorphism will alter the conformal factor of the spatial metric. If the conformal factor is taken to be the measure of time, then the $\mathcal{H}_i$'s, by altering the conformal factor, will generate time translation, while $\mathcal{H}_0$, by generating time translation, will alter the conformal factor. The $\tilde{\mathcal{H}}_i$'s generate spatial diffeomorphisms and preserve the conformal factor, so they must differ from the $\mathcal{H}_i$'s by the gradient of a compensating time translation term.

From the definition of $\tilde{\mathcal{H}}_i$, it is apparent that demanding $\chi \sim 0$ and $\mathcal{H}_\mu \sim 0$ is equivalent to demanding $\chi \sim 0$, $\mathcal{H}_0 \sim 0$, and $\tilde{\mathcal{H}}_i \sim 0$. The latter set of constraints has the virtue that the $\tilde{\mathcal{H}}_i$ are first class, which makes manifest the presence of the remaining three gauge symmetries. We therefore take our five constraints to be the two second class constraints χ and $\mathcal{H}_0$ and the three first class constraints $\tilde{\mathcal{H}}_i$. Using $\mathcal{H}_i = \tilde{\mathcal{H}}_i + 2\sqrt{h}\nabla_i(\mathcal{H}_0/\pi^k_k)$, the gauge-fixed action can be rewritten in terms of $\tilde{\mathcal{H}}_i$ as

$$S' = \int dt d^3x\Big(\pi^{ij}\dot{h}_{ij} - N^0\mathcal{H}_0 - N^i\tilde{\mathcal{H}}_i - 2\sqrt{h}N^i\nabla_i\left(\frac{\mathcal{H}_0}{\pi^k_k}\right) - \lambda\chi\Big). \tag{35}$$

Upon integration by parts, the action becomes

$$S' = \int dt d^3x(\pi^{ij}\dot{h}_{ij} - \tilde{N}\mathcal{H}_0 - N^i\tilde{\mathcal{H}}_i - \lambda\chi), \quad (36)$$

where

$$\tilde{N} \equiv N - 2\frac{\sqrt{h}}{\pi^k{}_k}\nabla_i N^i. \quad (37)$$

Variation of the action S' with respect to h_{ij} and π^{ij} yields Hamilton's equations,

$$\dot{h}_{ij}(x) = \frac{\delta H'}{\delta\pi^{ij}(x)}, \qquad \dot{\pi}^{ij}(x) = -\frac{\delta H'}{\delta h_{ij}(x)}, \quad (38)$$

where the new Hamiltonian H' is

$$H' = \int d^3x(\tilde{N}\mathcal{H}_0 + N^i\tilde{\mathcal{H}}_i + \lambda\chi). \quad (39)$$

The equation of motion for any quantity $A(h_{ij}, \pi^{ij}, t)$ is therefore

$$\begin{aligned}\dot{A} &= \frac{\partial A}{\partial t} + \{A, H'\} \\ &= \frac{\partial A}{\partial t} + \int d^3y(\tilde{N}(y)\{A, \mathcal{H}_0(y)\} + N^i(y)\{A, \tilde{\mathcal{H}}_i(y)\} \\ &\quad + \lambda(y)\{A, \chi(y)\}), \end{aligned} \quad (40)$$

where the Poisson bracket is defined as in (14). Variation of the action S' with respect to $\tilde{N}$, λ, and N^i yields the five constraints

$$\mathcal{H}_0 \sim 0, \qquad \chi \sim 0, \qquad \tilde{\mathcal{H}}_i \sim 0. \quad (41)$$

The action does not contain time derivatives of the Lagrange multipliers, so at first their evolution appears unconstrained. Since the $\tilde{\mathcal{H}}_i$ are first class, the three functions N^i are indeed arbitrary until and unless we gauge-fix them. The evolution of $\tilde{N}$ and λ, however, will be determined by demanding the consistency of $\mathcal{H}_0$ and χ with the equations of motion.

For the constraints to be consistent with the equations of motion, they must be preserved by the equations of motion; we therefore demand that $\dot{\tilde{\mathcal{H}}}_i \sim 0$, $\dot{\mathcal{H}}_0 \sim 0$, and $\dot{\chi} \sim 0$. Since the $\tilde{\mathcal{H}}_i$ are first class and $\partial_t\tilde{\mathcal{H}}_i = 0$, it follows at once that $\dot{\tilde{\mathcal{H}}}_i \sim 0$. Since $\partial_t\mathcal{H}_0 = 0$, $\{\mathcal{H}_0(x), \mathcal{H}_0(y)\} \sim 0$, and $\{\mathcal{H}_0(x), \tilde{\mathcal{H}}_i(y)\} \sim 0$, it follows that

$$\dot{\mathcal{H}}_0(x) \sim \int d^3y\lambda(y)\{\mathcal{H}_0(x), \chi(y)\} \sim \frac{1}{2}\lambda(x)\pi^i{}_i(x). \quad (42)$$

On a flat FRW background,[6] $K = 3\dot{a}/a$ and hence $\pi^i{}_i = -2\omega K = -6\dot{a}a^2$. Since we are only considering gravity on an expanding background, we assume that $\pi^i{}_i(x) \not\sim 0$ more generally. The demand $\dot{\mathcal{H}}_0 \sim 0$ thus implies

$$\lambda \sim 0. \quad (43)$$

Since $\{\chi(x), \chi(y)\} = 0$, $\{\chi(x), \tilde{\mathcal{H}}_i(y)\} \sim 0$, and $\partial\chi/\partial t = -\dot{\omega}(t)$, it follows that

$$\begin{aligned}\dot{\chi}(x) &\sim -\dot{\omega}(t) + \int d^3y\tilde{N}(y)\{\chi(x), \mathcal{H}_0(y)\}, \\ &\sim -\dot{\omega}(t) - \frac{1}{2}\tilde{N}(x)\pi^i{}_i(x). \end{aligned} \quad (44)$$

Since $\pi^i{}_i \not\sim 0$, demanding $\dot{\chi} \sim 0$ allows us to solve for $\tilde{N}$,

$$\tilde{N} \sim \frac{-2\dot{\omega}(t)}{\pi^i{}_i}. \quad (45)$$

The functions $\tilde{N}$ and λ are thus not arbitrary. Since $N = \tilde{N} + 2\sqrt{h}(\nabla_i N^i)/\pi^k{}_k$, the lapse N has not been completely gauge-fixed, but its arbitrariness stems solely from its dependence on the three arbitrary functions N^i.

As a check, let us revisit the counting of degrees of freedom in cosmological gauge. For these purposes, the only effect of gauge-fixing is to replace the first class constraint $\mathcal{H}_0 \sim 0$ and the arbitrary function N with the second class constraints $\mathcal{H}_0 \sim 0$ and $\chi \sim 0$. This modifies the left-hand side of Eq. (19), but does not change the final tally.

$$\begin{aligned}&6\cdot h_{ij}{}'s + 6\cdot\pi^{ij\prime}s - 1\cdot\mathcal{H}_0 - 1\cdot\chi - 3\cdot\tilde{\mathcal{H}}_i{}'s - 3\cdot N^{i\prime}s \\ &= 4 \text{ canonical DoF.} \end{aligned} \quad (46)$$

After gauge-fixing, the theory still has four canonical (or two real) degrees of freedom.

C. The action of spatially covariant general relativity

In this section, we will solve the constraints $\mathcal{H}_0$ and χ to obtain a spatially covariant formulation of general relativity as a theory of a unit-determinant metric $\tilde{h}_{ij}$ and its conjugate momentum $\tilde{\pi}^{ij}$. This will set the stage for modifying general relativity in Sec. IV.

Since χ and $\mathcal{H}_0$ are second class, they can be solved explicitly to yield expressions for ω and π_ω in terms of t, $\tilde{h}_{ij}$, $\tilde{\pi}^{ij}$, and spatial derivatives. "Solving" for ω is trivial: $\omega = \omega(t)$. Solving for π_ω requires us to take a square root and pick a sign, which amounts to picking either an expanding or a contracting background. We pause to emphasize once again that our procedure is only valid in a cosmological context, when the conformal factor of the spatial metric can be assumed to be evolving monotonically. To pick the sign corresponding to an expanding background, first recall that

$$\pi_\omega = -\tfrac{4}{3}K. \quad (47)$$

On a flat FRW background, $K = 3\dot{a}/a$ and hence $\pi_\omega = -4\dot{a}/a$. An expanding FRW background therefore corresponds to $\pi_\omega < 0$. Returning to the general case, we choose $\pi_\omega < 0$ to obtain

[6] A spatially-flat FRW spacetime corresponds to $N = 1$, $N_i = 0$, and $h_{ij} = a^2(t)\delta_{ij}$.

$$\pi_\omega = \pi_{\rm GR} \equiv -\sqrt{\frac{8}{3}}\sqrt{\frac{\tilde{\pi}_{ij}\tilde{\pi}^{ij}}{\omega^2} - \frac{\tilde{R}}{\omega^{2/3}} + 2\Lambda}, \tag{48}$$

where indices are raised and lowered with $\tilde{h}_{ij}$, and $\tilde{R}$ is the Ricci scalar for $\tilde{h}_{ij}$. Substituting these results for ω and π_ω back into the action S' yields the action of general relativity on the reduced phase space $(\tilde{h}_{ij}, \tilde{\pi}^{ij})$,

$$S'' = \int dt d^3x(\tilde{\pi}^{ij}\dot{\tilde{h}}_{ij} + \pi_\omega\dot{\omega} - N^i\tilde{\mathcal{H}}_i), \tag{49}$$

where

$$\tilde{\mathcal{H}}_i = -2\tilde{h}_{ij}\tilde{\nabla}_k\tilde{\pi}^{jk} - \omega\tilde{\nabla}_i\pi_\omega, \tag{50}$$

and $\tilde{\nabla}_i$ is the covariant derivative with respect to $\tilde{h}_{ij}$.[7] As discussed in Sec. III D, the constraints $\tilde{\mathcal{H}}_i$ remain first class and continue to represent spatial covariance, so (49) is the action of *spatially covariant general relativity.* This action yields the new Hamiltonian

$$H'' = \int d^3x(-\dot{\omega}\pi_\omega + N^i\tilde{\mathcal{H}}_i). \tag{51}$$

The term $\pi^{ij}\dot{h}_{ij}$ has split into the term $\tilde{\pi}^{ij}\dot{\tilde{h}}_{ij}$ and a contribution $-\dot{\omega}\pi_\omega$ to the physical Hamiltonian density. Variation of the action with respect to $\tilde{h}_{ij}$ and $\tilde{\pi}^{ij}$ yields Hamilton's equations,

$$\dot{\tilde{h}}_{ij}(x) = \frac{\delta H''}{\delta\tilde{\pi}^{ij}(x)}, \qquad \tilde{\delta}^{ij}_{ab}\dot{\tilde{\pi}}^{ab}(x) = -\frac{\delta H''}{\delta\tilde{h}_{ij}(x)}. \tag{52}$$

To evaluate these variational derivatives, one must use the relations

$$\frac{\delta\tilde{h}_{ij}(x)}{\delta\tilde{h}_{kl}(y)} = \tilde{\delta}^{kl}_{ij}\delta^3(x-y),$$
$$\frac{\delta\tilde{\pi}^{ij}(x)}{\delta\tilde{h}_{kl}(y)} = -\frac{1}{3}\tilde{h}^{ij}\tilde{\pi}^{kl}\delta^3(x-y), \tag{53}$$

and

$$\frac{\delta\tilde{h}_{ij}(x)}{\delta\tilde{\pi}^{kl}(y)} = 0, \qquad \frac{\delta\tilde{\pi}^{ij}(x)}{\delta\tilde{\pi}^{kl}(y)} = \tilde{\delta}^{ij}_{kl}\delta^3(x-y), \tag{54}$$

from which follow the operator identities

$$\frac{\delta}{\delta\tilde{h}_{ij}} = \tilde{\delta}^{ij}_{ab}\frac{\delta}{\delta\tilde{h}_{ab}}, \qquad \frac{\delta}{\delta\tilde{\pi}^{ij}} = \tilde{\delta}^{ab}_{ij}\frac{\delta}{\delta\tilde{\pi}^{ab}}. \tag{55}$$

Defining the Poisson bracket appropriate to the reduced phase space,

$$\{A, B\} \equiv \int d^3x\left(\frac{\delta A}{\delta\tilde{h}_{ij}(x)}\frac{\delta B}{\delta\tilde{\pi}^{ij}(x)} - \frac{\delta A}{\delta\tilde{\pi}^{ij}(x)}\frac{\delta B}{\delta\tilde{h}_{ij}(x)}\right), \tag{56}$$

any quantity $A(\tilde{h}_{ij}, \tilde{\pi}^{ij}, t)$ obeys the equation of motion

$$\dot{A} = \frac{\partial A}{\partial t} + \{A, H''\}. \tag{57}$$

Variation of the action with respect to N^i yields the three constraints

$$\tilde{\mathcal{H}}_i \sim 0. \tag{58}$$

As before, the time evolution of N^i is unconstrained by the action; in the absence of a gauge-fixing procedure, the three functions N^i are arbitrary.

D. Constraint properties and degrees of freedom

By lengthy direct calculation, it is possible to prove that the constraints $\tilde{\mathcal{H}}_i$ are first class, i.e., $\{\tilde{\mathcal{H}}_i(x), \tilde{\mathcal{H}}_j(y)\} \sim 0$. Furthermore, by applying the equations of motion to $\tilde{\mathcal{H}}_i$, it is possible to show that $\dot{\tilde{\mathcal{H}}}_i \sim 0$, so the constraints are preserved by the equations of motion. We defer demonstrations of these two facts to Sec. V, where we will examine general relativity in the context of a class of realistic theories. This is an important consistency check, because *a priori* it is not clear that our procedure for solving the Hamiltonian constraint will yield a consistent action on the reduced phase space.

As a final check, we revisit the counting of degrees of freedom in spatially covariant general relativity. After imposing cosmological gauge and solving the Hamiltonian constraint $\mathcal{H}_0 \sim 0$, general relativity is a theory of a unit-determinant spatial metric $\tilde{h}_{ij}$ and its traceless conjugate momentum $\tilde{\pi}^{ij}$, so the theory contains ten canonical (or 5 real) variables. This reduction in the size of the phase space is compensated by a corresponding reduction in the number of constraints and arbitrary functions: the theory contains three first class constraints $\tilde{\mathcal{H}}_i \sim 0$, and its equations of motion involve three arbitrary functions N^i.

$$5\cdot\tilde{h}_{ij}{}'s + 5\cdot\tilde{\pi}^{ij\prime}s - 3\cdot\tilde{\mathcal{H}}_i{}'s - 3\cdot N^{i\prime}s$$
$$= 4\,\text{canonical DoF}. \tag{59}$$

Spatially covariant general relativity thus contains four canonical (or two real) degrees of freedom, the same number as fully covariant general relativity.

E. Modifying spatially covariant general relativity

We have two criteria in mind for our modified theories of gravity: two graviton degrees of freedom, and manifest spatial covariance. Our starting point is the action (49) of spatially covariant general relativity, which has both of these properties. To modify general relativity, we will change the functional form of the scalar quantity π_ω, which in general relativity obeys $\pi_\omega = \pi_{\rm GR}$. This yields the action

[7] The distinction between $\mathcal{H}_i$ and $\tilde{\mathcal{H}}_i$ in Eq. (33) vanishes identically after solving $\mathcal{H}_0$.

$$S = \int dt d^3x (\tilde{\pi}^{ij} \dot{\tilde{h}}_{ij} + \dot{\omega} \pi_\omega - N^i \tilde{\mathcal{H}}_i) \tag{60}$$

with constraints

$$\tilde{\mathcal{H}}_i = -2\tilde{h}_{ij} \tilde{\nabla}_k \tilde{\pi}^{jk} - \omega \tilde{\nabla}_i \pi_\omega, \tag{61}$$

where π_ω is an unspecified scalar function of t, the phase space variables $\tilde{h}_{ij}$ and $\tilde{\pi}^{ij}$, and spatial derivatives:

$$\pi_\omega = \pi_\omega(t, \tilde{h}_{ij}, \tilde{\pi}^{ij}, \partial_i). \tag{62}$$

This action leads to the equation of motion

$$\dot{A} = \frac{\partial A}{\partial t} + \{A, H\}, \tag{63}$$

where the Hamiltonian H is

$$H = \int d^3x (-\dot{\omega} \pi_\omega + N^i \tilde{\mathcal{H}}_i), \tag{64}$$

and the Poisson bracket is

$$\{A, B\} \equiv \int d^3x \left(\frac{\delta A}{\delta \tilde{h}_{ij}(x)} \frac{\delta B}{\delta \tilde{\pi}^{ij}(x)} - \frac{\delta A}{\delta \tilde{\pi}^{ij}(x)} \frac{\delta B}{\delta \tilde{h}_{ij}(x)} \right). \tag{65}$$

Retaining the manifest spatial covariance of the theory amounts to demanding (1) that the modified $\tilde{\mathcal{H}}_i$ remain first class, i.e.,

$$\{\tilde{\mathcal{H}}_i(x), \tilde{\mathcal{H}}_j(y)\} \sim 0, \tag{66}$$

and (2) that the modified constraints be preserved by the modified equations of motion, i.e.,

$$\dot{\tilde{\mathcal{H}}}_i \sim 0. \tag{67}$$

Any theory satisfying these two points will be manifestly covariant under spatial diffeomorphisms, with the constraints $\tilde{\mathcal{H}}_i$ acting as the generators of the gauge symmetry. Moreover, the presence of three first class constraints $\tilde{\mathcal{H}}_i$ on the phase space $(\tilde{h}_{ij}, \tilde{\pi}^{ij})$ guarantees that such a theory contains two local degrees of freedom, exactly as desired.

In the remainder of the paper, we examine two classes of theories. First, for pedagogical purposes, we assume that π_ω does not contain spatial derivatives; this is the *ultralocal* case. Second, to make contact with general relativity, we allow π_ω to depend on spatial derivatives through $\tilde{R}$, the Ricci scalar of $\tilde{h}_{ij}$; this is the *local* case. Forthcoming work will examine more general classes of scalar momenta [20]. In Sec. IV, we use the ultralocal case to introduce the formalism needed to determine when the constraints $\tilde{\mathcal{H}}_i$ remain first class and when the constraints are preserved by the equations of motion. In Sec. V, we apply the formalism to the local case. In both the ultralocal and the local case, the consistency of the constraints with the equations of motion requires π_ω to satisfy an analogue of the renormalization group equation; scalar momenta satisfying this equation are manifestly invariant under spatial conformal rescaling of the volume factor ω. In the ultralocal case, this is the only consistency condition that arises. In the local case, demanding that the constraints $\tilde{\mathcal{H}}_i$ satisfy a first class algebra is equivalent to demanding that π_ω obey a rather complicated differential equation.

IV. ULTRALOCAL MODIFIED GRAVITY

The ultralocal limit of a theory is achieved by neglecting all terms in the action which are second order or higher in spatial derivatives. Conceptually, this is the limit in which each point in space evolves independently of the points around it. The ultralocal truncation of a theory is a good approximation to the full theory whenever spatial gradients of fields are small compared to the fields themselves and their time derivatives. This makes it a natural limit to take in cosmology.

In general relativity, the ultralocal limit simplifies the form of the Hamiltonian constraint while (1) preserving the momentum constraints and (2) maintaining a consistent constraint algebra [56]. This approximation has proven fruitful for analyzing both long wavelength cosmological perturbations [57] and for studying physics near cosmological singularities [58,59]. The idea of using cosmological gauge in the ultralocal limit, sometimes referred to as the separate universe picture, is treated in [56].

In our approach, the ultralocal limit simplifies the form of the physical Hamiltonian density π_ω while preserving the form of the momentum constraints $\tilde{\mathcal{H}}_i$. There are two terms in the action (60) which contain π_ω, namely $\dot{\omega}\pi_\omega$ and $\omega N^i \partial_i \pi_\omega$. Though π_ω appears in the action without a spatial gradient acting on it, vector indices in our theory only arise from spatial gradients, so π_ω cannot contain terms linear in spatial gradients. In the ultralocal limit, π_ω is thus a scalar function of t, $\tilde{h}_{ij}$, and $\tilde{\pi}^{ij}$ that does not contain spatial derivatives, i.e.,

$$\pi_\omega = \pi_\omega(t, \tilde{h}_{ij}, \tilde{\pi}^{ij}). \tag{68}$$

When π_ω is of this form, we will say that π_ω is an *ultralocal function* of the phase space variables. We will now show that this form for π_ω leads to a first class constraint algebra.

A. Constraint algebra

In this section, we will compute the Poisson bracket $\{\tilde{\mathcal{H}}_i(x), \tilde{\mathcal{H}}_a(y)\}$ assuming that π_ω is an ultralocal function, and use the result to determine when the constraints $\tilde{\mathcal{H}}_i$ remain first class. To simplify the calculation of $\{\tilde{\mathcal{H}}_i(x), \tilde{\mathcal{H}}_a(y)\}$, we split $\tilde{\mathcal{H}}_i$ into a tensor part $\mathcal{J}_i$ and a scalar part $\mathcal{K}_i$. Concretely, we define the vector densities

$$\mathcal{J}_i \equiv -2\tilde{h}_{ij}\tilde{\nabla}_k\tilde{\pi}^{jk}, \qquad \mathcal{K}_i \equiv -\omega\tilde{\nabla}_i\pi_\omega, \tag{69}$$

in terms of which $\tilde{\mathcal{H}}_i$ becomes simply

$$\tilde{\mathcal{H}}_i = \mathcal{J}_i + \mathcal{K}_i. \tag{70}$$

The Poisson bracket $\{\tilde{\mathcal{H}}_i(x), \tilde{\mathcal{H}}_a(y)\}$ can then be written as the sum of more manageable brackets,

$$\{\tilde{\mathcal{H}}_i(x), \tilde{\mathcal{H}}_a(y)\} = \{\mathcal{J}_i(x), \mathcal{J}_a(y)\} + \{\mathcal{K}_i(x), \mathcal{K}_a(y)\} + \{\mathcal{J}_i(x), \mathcal{K}_a(y)\} + \{\mathcal{K}_i(x), \mathcal{J}_a(y)\}. \tag{71}$$

To simplify the evaluation of these component Poisson brackets, we will first compute the Poisson brackets of the smoothing functionals

$$F_J \equiv \int d^3x f^i \mathcal{J}_i, \qquad F_K \equiv \int d^3x f^i \mathcal{K}_i,$$
$$G_J \equiv \int d^3y g^a \mathcal{J}_a, \qquad G_K \equiv \int d^3y g^a \mathcal{K}_a, \tag{72}$$

where the functions f^i and g^i are time-independent smoothing functions. We make the key assumption that the smoothing functions decay so rapidly at infinity that when we integrate by parts inside the smoothing functionals, the boundary term vanishes identically; the smoothing functions are otherwise arbitrary. With the freedom to integrate by parts at will, it is straightforward to compute variational derivatives of the smoothing functionals, and thereby to obtain explicit expressions for their Poisson brackets. To obtain the brackets of the vector densities from the brackets of the smoothing functionals, we will use the relations

$$\{F_J, G_J\} = \int d^3x d^3y f^i(x) g^a(y) \{\mathcal{J}_i(x), \mathcal{J}_a(y)\},$$
$$\{F_J, G_K\} + \{F_K, G_J\} = \int d^3x d^3y f^i(x) g^a(y) \times (\{\mathcal{J}_i(x), \mathcal{K}_a(y)\} + \{\mathcal{K}_i(x), \mathcal{J}_a(y)\}),$$
$$\{F_K, G_K\} = \int d^3x d^3y f^i(x) g^a(y) \{\mathcal{K}_i(x), \mathcal{K}_a(y)\}. \tag{73}$$

The fact that Eq. (73) must hold for all sufficiently well-behaved functions f and g will allow us to derive explicit expressions for the Poisson brackets involving $\mathcal{J}_i$ and $\mathcal{K}_i$.

To compute variational derivatives of the smoothing functional F_J, first integrate by parts to obtain

$$F_J = 2\int d^3x \tilde{h}_{ij}\tilde{\pi}^{jk}\tilde{\nabla}_k f^i, \tag{74}$$

from which it follows that

$$\delta F_J = \int d^3x \{2\tilde{\pi}^{jk}(\tilde{\nabla}_k f^i)\delta\tilde{h}_{ij} + 2\tilde{h}_{ij}(\tilde{\nabla}_k f^i)\delta\tilde{\pi}^{jk} + 2\tilde{h}_{ij}\tilde{\pi}^{jk}\delta\tilde{\nabla}_k f^i\}. \tag{75}$$

The first two terms in this integral are in a convenient form for taking variational derivatives with respect to $\tilde{h}_{ij}$ and $\tilde{\pi}^{jk}$, but the third term requires finessing. To evaluate $\delta\tilde{\nabla}_k f^i$, expand the covariant derivative as $\tilde{\nabla}_k f^i = \partial_k f^i + \tilde{\Gamma}^i_{kr} f^r$, where $\tilde{\Gamma}^i_{jk}$ is the connection of the metric $\tilde{h}_{ij}$. It follows immediately that $\delta\tilde{\nabla}_k f^i = f^r \delta\tilde{\Gamma}^i_{kr}$. The identity

$$\delta\tilde{\Gamma}^i_{kr} = \tfrac{1}{2}\tilde{h}^{im}(\nabla_r\delta\tilde{h}_{km} + \nabla_k\delta\tilde{h}_{rm} - \nabla_m\delta\tilde{h}_{rk}) \tag{76}$$

thus implies that $2\tilde{\pi}^{jk}\tilde{h}_{ij}\delta\tilde{\nabla}_k f^i = f^i\tilde{\pi}^{jk}\tilde{\nabla}_i\delta\tilde{h}_{jk}$, so Eq. (75) becomes

$$\delta F_J = \int d^3x \{2\tilde{\pi}^{jk}(\tilde{\nabla}_k f^i)\delta\tilde{h}_{ij} + 2\tilde{h}_{ij}(\tilde{\nabla}_k f^i)\delta\tilde{\pi}^{jk} + f^i\tilde{\pi}^{jk}\tilde{\nabla}_i\delta\tilde{h}_{jk}\}. \tag{77}$$

Integrating by parts, this reduces to

$$\delta F_J = \int d^3x \{2\tilde{\pi}^{jk}(\tilde{\nabla}_k f^i)\delta\tilde{h}_{ij} - \tilde{\nabla}_i(f^i\tilde{\pi}^{jk})\delta\tilde{h}_{jk} + 2\tilde{h}_{ij}(\tilde{\nabla}_k f^i)\delta\tilde{\pi}^{jk}\}. \tag{78}$$

From this expression, it is straightforward to compute variational derivatives of F_J,

$$\frac{\delta F_J}{\delta\tilde{h}_{mn}} = 2\tilde{\delta}^{mn}_{ij}\tilde{\pi}^{jk}\tilde{\nabla}_k f^i - \tilde{\nabla}_i(f^i\tilde{\pi}^{mn}) - \frac{2}{3}\tilde{\pi}^{mn}\tilde{\nabla}_i f^i,$$
$$\frac{\delta F_J}{\delta\tilde{\pi}^{mn}} = 2\tilde{\delta}^{jk}_{mn}\tilde{h}_{ij}\tilde{\nabla}_k f^i. \tag{79}$$

The corresponding results for G_J are

$$\frac{\delta G_J}{\delta\tilde{h}_{mn}} = 2\tilde{\delta}^{mn}_{ab}\tilde{\pi}^{bc}\tilde{\nabla}_c g^a - \tilde{\nabla}_a(g^a\tilde{\pi}^{mn}) - \frac{2}{3}\tilde{\pi}^{mn}\tilde{\nabla}_a g^a,$$
$$\frac{\delta G_J}{\delta\tilde{\pi}^{mn}} = 2\tilde{\delta}^{bc}_{mn}\tilde{h}_{ab}\tilde{\nabla}_c g^a. \tag{80}$$

The variational calculation for the smoothing functional F_K is less straightforward. After integrating by parts, F_K becomes

$$F_K = \omega\int d^3x(\partial_i f^i)\pi_\omega, \tag{81}$$

from which it follows that

$$\delta F_K = \omega\int d^3x(\partial_i f^i)\delta\pi_\omega. \tag{82}$$

To evaluate $\delta\pi_\omega$ in full generality would be very difficult, so we will make some simplifying assumptions about the form of π_ω. In this section, we will assume that π_ω is an ultralocal function of t, $\tilde{h}_{ij}$, and $\tilde{\pi}^{ij}$.

To facilitate calculations, we will enumerate all the scalars that can be built by contracting factors of $\tilde{h}_{ij}$ against

factors of $\tilde{\pi}^{ij}$. We begin by recursively defining $\Pi^{ij}(n)$, the linked chain of n factors of $\tilde{\pi}^{ij}$. The chain of zero factors of $\tilde{\pi}^{ij}$ is simply

$$\Pi^{ij}(0) \equiv \tilde{h}^{ij}. \tag{83}$$

The process of adding a link to the chain is defined by

$$\Pi^{ij}(n+1) \equiv \tilde{\pi}^{i}{}_{k}\Pi^{kj}(n). \tag{84}$$

By closing the chain, one obtains scalars,

$$\phi(n) \equiv \Pi^{i}{}_{i}(n). \tag{85}$$

The $\phi(n)$ are the only scalars that can be built out of connected contractions of $\tilde{h}_{ij}$ and $\tilde{\pi}^{ij}$. Since $\phi(0) = 3$ and $\phi(1) = 0$, $\delta\phi(0) = \delta\phi(1) = 0$. For an arbitrary ultralocal function π_ω, it follows that

$$\delta\pi_\omega = \sum_{n=2}^{\infty} \frac{\partial \pi_\omega}{\partial \phi(n)} \delta\phi(n). \tag{86}$$

For $n \geq 2$, the variational derivatives of the $\phi(n)$ are

$$\frac{\delta\phi(n)(x)}{\delta\tilde{h}_{mn}(y)} = \left(\tilde{\delta}^{mn}_{ab} n\Pi(n)^{ab} - \frac{1}{3}\tilde{\pi}^{mn} n\phi(n-1)\right)\delta^3(x-y),$$
$$\frac{\delta\phi(n)(x)}{\delta\tilde{\pi}^{mn}(y)} = \tilde{\delta}^{ab}_{mn} n\Pi(n-1)_{ab}\delta^3(x-y). \tag{87}$$

The variational derivatives of F_K are thus

$$\frac{\delta F_K}{\delta\tilde{h}_{mn}} = \omega(\partial_i f^i)\sum_{n=2}^{\infty} n\frac{\partial\pi_\omega}{\partial\phi(n)}\left(\tilde{\delta}^{mn}_{jk}\Pi(n)^{jk} - \frac{1}{3}\tilde{\pi}^{mn}\phi(n-1)\right),$$
$$\frac{\delta F_K}{\delta\tilde{\pi}^{mn}} = \omega(\partial_i f^i)\sum_{n=2}^{\infty} n\frac{\partial\pi_\omega}{\partial\phi(n)}\tilde{\delta}^{jk}_{mn}\Pi(n-1)_{jk}. \tag{88}$$

Similarly, the variational derivatives of G_K are

$$\frac{\delta G_K}{\delta\tilde{h}_{mn}} = \omega(\partial_a g^a)\sum_{m=2}^{\infty} m\frac{\partial\pi_\omega}{\partial\phi(m)} \times\left(\tilde{\delta}^{mn}_{bc}\Pi(m)^{bc} - \frac{1}{3}\tilde{\pi}^{mn}\phi(m-1)\right),$$
$$\frac{\delta G_K}{\delta\tilde{\pi}^{mn}} = \omega(\partial_a g^a)\sum_{m=2}^{\infty} m\frac{\partial\pi_\omega}{\partial\phi(m)}\tilde{\delta}^{bc}_{mn}\Pi(m-1)_{bc}. \tag{89}$$

We emphasize that these results for F_K and G_K rely on the ultralocality assumption, and will be modified in Sec. V.

We are now in a position to compute the Poisson brackets of the smoothing functionals, from which we will extract the Poisson brackets of the vector densities $\mathcal{J}_i$ and $\mathcal{K}_i$.

(i) $\{\mathcal{J}_i(x), \mathcal{J}_a(y)\}$

To obtain the bracket $\{\mathcal{J}_i(x), \mathcal{J}_a(y)\}$, we first compute $\{F_J, G_J\}$. Combining the F_J and G_J variations into the bracket $\{F_J, G_J\}$ yields

$$\{F_J, G_J\} = 2\int d^3z\{(\tilde{\nabla}_c f^i)(\tilde{\nabla}_i g^a)\tilde{h}_{ab}\tilde{\pi}^{bc} - (\tilde{\nabla}_k g^a)(\tilde{\nabla}_a f^i)\tilde{h}_{ij}\tilde{\pi}^{jk} + (\tilde{\nabla}_k f^i)\tilde{\nabla}_a(g^a\tilde{h}_{ij}\tilde{\pi}^{jk}) - (\tilde{\nabla}_c g^a)\tilde{\nabla}_i(f^i\tilde{h}_{ab}\tilde{\pi}^{bc})\}. \tag{90}$$

After integrating by parts, using the definition $\mathcal{J}_i = -2\tilde{h}_{ij}\tilde{\nabla}_k\tilde{\pi}^{jk}$, and using the identity $(\tilde{\nabla}_i\tilde{\nabla}_j - \tilde{\nabla}_j\tilde{\nabla}_i)V^a = \tilde{R}^a{}_{bij}V^b$, this reduces to

$$\{F_J, G_J\} = \int d^3z\{f^i\mathcal{J}_a\tilde{\nabla}_i g^a - g^a\mathcal{J}_i\tilde{\nabla}_a f^i + 2f^i g^a\tilde{\pi}^{jk}(\tilde{R}_{jika} + \tilde{R}_{jaik})\}. \tag{91}$$

From the symmetries of the Riemann tensor[8] and the traceless momentum tensor,[9] it follows that $\tilde{\pi}^{jk}(\tilde{R}_{jika} + \tilde{R}_{jaik}) = 0$, so the last term in the integrand vanishes. The connection terms inside the remaining covariant derivatives cancel to yield

$$\{F_J, G_J\} = \int d^3z(f^i\mathcal{J}_a\partial_i g^a - g^a\mathcal{J}_i\partial_a f^i). \tag{92}$$

To extract the bracket $\{\mathcal{J}_i(x), \mathcal{J}_a(y)\}$ from this result, first relabel dummy indices

$$\{F_J, G_J\} = \int d^3x f^i\mathcal{J}_a\partial_i g^a - \int d^3y g^a\mathcal{J}_i\partial_a f^i. \tag{93}$$

Under the spatial derivatives in this equation, insert the identities

$$g^a(x) = \int d^3y\delta^3(x-y)g^a(y),$$
$$f^i(y) = \int d^3x\delta^3(x-y)f^i(x), \tag{94}$$

to obtain

$$\{F_J, G_J\} = \int d^3x d^3y f^i(x)g^a(y)(\mathcal{J}_a(x)\partial_{x^i}\delta^3(x-y) - \mathcal{J}_i(y)\partial_{y^a}\delta^3(x-y)). \tag{95}$$

Comparing this expression to Eq. (73) yields the identity

$$\{\mathcal{J}_i(x), \mathcal{J}_a(y)\} = \mathcal{J}_a(x)\partial_{x^i}\delta^3(x-y) - \mathcal{J}_i(y)\partial_{y^a}\delta^3(x-y). \tag{96}$$

This is the same algebra obeyed by the $\mathcal{H}_i$ in Eq. (17). This result is completely independent of our choice of π_ω, and will carry over unchanged into Sec. V.

[8] $\tilde{R}_{abcd} = \tilde{R}_{cdab}$, $\tilde{R}_{abcd} = -\tilde{R}_{bacd} = -\tilde{R}_{abdc}$.
[9] $\tilde{\pi}^{ij} = \tilde{\pi}^{ji}$.

(ii) $\{\mathcal{J}_i(x), \mathcal{K}_a(y)\} + \{\mathcal{K}_i(x), \mathcal{J}_a(y)\}$
To obtain $\{\mathcal{J}_i(x), \mathcal{K}_a(y)\} + \{\mathcal{K}_i(x), \mathcal{J}_a(y)\}$, we first compute $\{F_J, G_K\} + \{F_K, G_J\}$. Assembling the F_J and G_K variations into the Poisson bracket $\{F_J, G_K\}$ yields

$$\{F_J, G_K\} = -\omega \int d^3z(\partial_a g^a) \times \sum_{m=2}^{\infty} \frac{\partial \pi_\omega}{\partial \phi(m)} m\Pi(m-1)_{bc}\tilde{\nabla}_i(f^i \tilde{\pi}^{bc}). \tag{97}$$

By expanding the covariant derivative, simplifying the ensuing total derivative of π_ω, and recalling that $\mathcal{K}_i = -\omega\tilde{\nabla}_i\pi_\omega$, this expression reduces to

$$\{F_J, G_K\} = \int d^3z f^i \mathcal{K}_i \partial_a g^a - \omega \int d^3z(\partial_i f^i)(\partial_a g^a) \times \sum_{m=2}^{\infty} \frac{\partial \pi_\omega}{\partial \phi(m)} m\phi(m). \tag{98}$$

Similarly,

$$\{F_K, G_J\} = -\int d^3z g^a \mathcal{K}_a \partial_i f^i + \omega \int d^3z(\partial_i f^i)(\partial_a g^a) \times \sum_{m=2}^{\infty} \frac{\partial \pi_\omega}{\partial \phi(m)} m\phi(m), \tag{99}$$

so the sum of the two brackets simplifies considerably,

$$\{F_J, G_K\} + \{F_K, G_J\} = \int d^3z(f^i \mathcal{K}_i \partial_a g^a - g^a \mathcal{K}_a \partial_i f^i). \tag{100}$$

Integrating by parts and invoking the identity $\partial_i \mathcal{K}_a = \partial_a \mathcal{K}_i$ yields

$$\{F_J, G_K\} + \{F_K, G_J\} = \int d^3z(f^i \mathcal{K}_a \partial_i g^a - g^a \mathcal{K}_i \partial_a f^i). \tag{101}$$

To extract the quantity $\{\mathcal{J}_i(x), \mathcal{K}_a(y)\} + \{\mathcal{K}_i(x), \mathcal{J}_a(y)\}$, relabel dummy indices and insert the identities in Eq. (94) to obtain

$$\{F_J, G_K\} + \{F_K, G_J\} = \int d^3x d^3y f^i(x) g^a(y)(\mathcal{K}_a(x)\partial_{x^i}\delta^3(x-y) - \mathcal{K}_i(y)\partial_{y^a}\delta^3(x-y)). \tag{102}$$

Combined with Eq. (73), this result implies that

$$\{\mathcal{J}_i(x), \mathcal{K}_a(y)\} + \{\mathcal{K}_i(x), \mathcal{J}_a(y)\} = \mathcal{K}_a(x)\partial_{x^i}\delta^3(x-y) - \mathcal{K}_i(y)\partial_{y^a}\delta^3(x-y). \tag{103}$$

This expression depends strongly on the assumed form for π_ω. This result is modified heavily in Sec. VA, where π_ω is allowed to depend on $\tilde{R}$.

(iii) $\{\mathcal{K}_i(x), \mathcal{K}_a(y)\}$
To obtain $\{\mathcal{K}_i(x), \mathcal{K}_a(y)\}$, we first compute the bracket $\{F_K, G_K\}$. Substituting the F_K and G_K variations into the Poisson bracket $\{F_K, G_K\}$ yields

$$\{F_K, G_K\} = \omega^2(\partial_i f^i)(\partial_a g^a) \sum_{m=2}^{\infty}\sum_{n=2}^{\infty} mn \frac{\partial \pi_\omega}{\partial \phi(m)} \times \frac{\partial \pi_\omega}{\partial \phi(n)}(\Pi(n)^{bc}\Pi(m-1)_{bc} - \Pi(m)^{jk}\Pi(n-1)_{jk}). \tag{104}$$

From the definition of the momentum chain $\Pi(n)^{ij}$, it follows that $\Pi(n)^{bc}\Pi(m-1)_{bc} = \Pi(m)^{jk}\Pi(n-1)_{jk} = \phi(n+m-1)$. The terms of the sum thus vanish order by order, so the bracket reduces to

$$\{F_K, G_K\} = 0. \tag{105}$$

By comparing this result to Eq. (73), it is apparent that

$$\{\mathcal{K}_i(x), \mathcal{K}_a(y)\} = 0. \tag{106}$$

When π_ω is an ultralocal function of the phase space variables, the Poisson bracket $\{\mathcal{K}_i(x), \mathcal{K}_a(y)\}$ vanishes identically. This will not be the case when π_ω depends nontrivially on $\tilde{R}$, as in Sec. VA.

By substituting Eqs. (96), (103), and (106) into Eq. (71), and recalling that $\tilde{\mathcal{H}}_i = \mathcal{J}_i + \mathcal{K}_i$, we obtain

$$\{\tilde{\mathcal{H}}_i(x), \tilde{\mathcal{H}}_j(y)\} = \tilde{\mathcal{H}}_j(x)\partial_{x^i}\delta^3(x-y) - \tilde{\mathcal{H}}_i(y)\partial_{y^j}\delta^3(x-y). \tag{107}$$

This is the same algebra obeyed by the $\mathcal{H}_i$ in Eq. (17), and by the $\mathcal{J}_i$ in Eq. (96). Since $\tilde{\mathcal{H}}_i \sim 0$, this result implies that $\{\tilde{\mathcal{H}}_i(x), \tilde{\mathcal{H}}_j(y)\} \sim 0$, so the constraints $\tilde{\mathcal{H}}_i$ are first class. To establish this result, we assumed only that π_ω was an arbitrary ultralocal function of t, $\tilde{h}_{ij}$, and $\tilde{\pi}^{ij}$; we showed that this was equivalent to making π_ω a function of t and the scalars $\phi(n)$ defined in Eq. (85). Evidently, π_ω can be made any ultralocal function of the phase space variables and the momentum constraints will remain first class.

B. Consistency of constraints with equations of motion

In this section, we will compute the time derivative $\dot{\tilde{\mathcal{H}}}_i$ assuming that π_ω is an ultralocal function, and use the result to determine when the constraints $\tilde{\mathcal{H}}_i$ are preserved by the equations of motion. The time evolution of $\tilde{\mathcal{H}}_i$ is determined by the equation of motion

$$\dot{\tilde{\mathcal{H}}}_i = \frac{\partial \tilde{\mathcal{H}}_i}{\partial t} + \{\tilde{\mathcal{H}}_i, H\}, \tag{108}$$

where

$$H = \int d^3x(-\dot{\omega}\pi_\omega + N^i \tilde{\mathcal{H}}_i). \tag{109}$$

Since $\tilde{\mathcal{H}}_i = \mathcal{J}_i + \mathcal{K}_i$ and $\partial \mathcal{J}_i/\partial t = 0$, it follows that $\partial \tilde{\mathcal{H}}_i/\partial t = \partial \mathcal{K}_i/\partial t$. Recalling that $\mathcal{K}_i = -\omega \partial_i \pi_\omega$, the first term in Eq. (108) becomes

$$\frac{\partial \tilde{\mathcal{H}}_i}{\partial t} = -\partial_i\left(\dot{\omega}\pi_\omega + \omega \frac{\partial \pi_\omega}{\partial t}\right). \tag{110}$$

To simplify the bracket $\{\tilde{\mathcal{H}}_i, H\}$, we define

$$\Pi_\omega \equiv \int d^3x \pi_\omega, \tag{111}$$

so that H can be written as

$$H = -\dot{\omega}\Pi_\omega + \int d^3x N^i \tilde{\mathcal{H}}_i. \tag{112}$$

From the first class character of the constraints $\tilde{\mathcal{H}}_i$, it follows that $\{\tilde{\mathcal{H}}_i, H\} \sim -\dot{\omega}\{\tilde{\mathcal{H}}_i, \Pi_\omega\}$. Since $\tilde{\mathcal{H}}_i = \mathcal{J}_i + \mathcal{K}_i$, the second term in Eq. (108) becomes

$$\{\tilde{\mathcal{H}}_i, H\} \sim -\dot{\omega}\{\mathcal{J}_i, \Pi_\omega\} - \dot{\omega}\{\mathcal{K}_i, \Pi_\omega\}. \tag{113}$$

To compute the brackets $\{\mathcal{J}_i, \Pi_\omega\}$ and $\{\mathcal{K}_i, \Pi_\omega\}$, we first compute the smoothing functional brackets

$$\{F_J, \Pi_\omega\} = \int d^3x f^i(x)\{\mathcal{J}_i(x), \Pi_\omega\}$$
$$\{F_K, \Pi_\omega\} = \int d^3x f^i(x)\{\mathcal{K}_i(x), \Pi_\omega\}. \tag{114}$$

We have already done all the work needed to evaluate these two brackets: since Π_ω can be obtained from G_K by the substitution $\partial_a g^a \to \omega^{-1}$, brackets involving Π_ω can be obtained by applying this substitution to brackets involving G_K.

(i) $\{\mathcal{J}_i, \Pi_\omega\}$

To compute the bracket $\{\mathcal{J}_i, \Pi_\omega\}$, we first compute the bracket $\{F_J, \Pi_\omega\}$. Applying $\partial_a g^a \to \omega^{-1}$ to Eq. (98) and integrating by parts yields

$$\{F_J, \Pi_\omega\} = \int d^3x f^i \partial_i\left(-\pi_\omega + \sum_{m=2}^{\infty} m\phi(m)\frac{\partial \pi_\omega}{\partial \phi(m)}\right). \tag{115}$$

It follows by comparing this result with Eq. (114) that

$$\{\mathcal{J}_i, \Pi_\omega\} = \partial_i\left(-\pi_\omega + \sum_{m=2}^{\infty} m\phi(m)\frac{\partial \pi_\omega}{\partial \phi(m)}\right). \tag{116}$$

(ii) $\{\mathcal{K}_i, \Pi_\omega\}$

To compute the bracket $\{\mathcal{K}_i, \Pi_\omega\}$, we first compute the bracket $\{F_K, \Pi_\omega\}$. By applying the transformation $\partial_a g^a \to \omega^{-1}$, Eq. (105) becomes

$$\{F_K, \Pi_\omega\} = 0. \tag{117}$$

Along with Eq. (114), this implies that

$$\{\mathcal{K}_i, \Pi_\omega\} = 0. \tag{118}$$

By substituting Eqs. (116) and (118) into Eq. (113), we obtain

$$\{\tilde{\mathcal{H}}_i, H\} \sim \dot{\omega}\partial_i\left(\pi_\omega - \sum_{m=2}^{\infty} m\phi(m)\frac{\partial \pi_\omega}{\partial \phi(m)}\right). \tag{119}$$

Upon inserting Eqs. (119) and (110) into the equation of motion (108), the $\dot{\omega}\partial_i\pi_\omega$ terms cancel to yield

$$\dot{\tilde{\mathcal{H}}}_i \sim -\partial_i\left(\omega\frac{\partial \pi_\omega}{\partial t} + \dot{\omega}\sum_{m=2}^{\infty} m\phi(m)\frac{\partial \pi_\omega}{\partial \phi(m)}\right). \tag{120}$$

Demanding $\dot{\tilde{\mathcal{H}}}_i \sim 0$ implies the consistency condition

$$\omega\frac{\partial \pi_\omega}{\partial t} + \dot{\omega}\sum_{m=2}^{\infty} m\phi(m)\frac{\partial \pi_\omega}{\partial \phi(m)} \sim f(t), \tag{121}$$

where $f(t)$ is an arbitrary function of time. We observe that the equation of motion (63) is invariant under $\pi_\omega \to \pi_\omega + g(t)$, where $g(t)$ is an arbitrary function of time, so we are free to apply this transformation to simplify our consistency condition. If we choose $g(t)$ so that $\omega g'(t) = f(t)$, the consistency condition becomes

$$\omega\frac{\partial \pi_\omega}{\partial t} + \dot{\omega}\sum_{m=2}^{\infty} m\phi(m)\frac{\partial \pi_\omega}{\partial \phi(m)} \sim 0. \tag{122}$$

By assumption, $\omega(t)$ is an invertible function of time, so $\partial/\partial t = \dot{\omega}\partial/\partial\omega$. Our consistency condition can thus be written as

$$\Delta\pi_\omega \sim 0, \tag{123}$$

where we have defined the operator

$$\Delta \equiv \omega\frac{\partial}{\partial\omega} + \sum_{m=2}^{\infty} m\phi(m)\frac{\partial}{\partial\phi(m)}. \tag{124}$$

To rule out the possibility of a π_ω which satisfies $\Delta\pi_\omega \sim 0$ while $\Delta\pi_\omega \neq 0$, we note that the constraints $\tilde{\mathcal{H}}_i$ contain one power of spatial derivatives, while by assumption the scalar momentum π_ω is ultralocal. To satisfy $\Delta\pi_\omega \sim 0$, the quantity $\Delta\pi_\omega$ would need to depend on the constraints $\tilde{\mathcal{H}}_i$, and would thus need to contain at least one power of spatial derivatives. However, applying Δ to π_ω does not increase the number of spatial derivatives. It follows that $\Delta\pi_\omega$ cannot contain any spatial derivatives, and thus

cannot depend on $\tilde{\mathcal{H}}_i$. The consistency condition can therefore be promoted to

$$\Delta \pi_\omega = 0. \tag{125}$$

To obtain the most general solution to this equation, we first note that $\Delta(\omega^{-n}\phi(n)) = 0$, which motivates us to define

$$\bar{\phi}(n) \equiv \frac{\phi(n)}{\omega^n(t)}. \tag{126}$$

The most general solution to the condition $\Delta\pi_\omega = 0$ is an arbitrary function of the $\bar{\phi}(n)$. The explicit time dependence of π_ω is thus determined by its dependence on the phase space variables.

To understand this result, we return briefly to the phase space (h_{ij}, π^{ij}). To construct three-scalars out of the tensor h_{ij} and the traceless tensor $\tilde{\pi}^{ij}_T$, we begin by recursively defining $\Pi^{ij}_T(n)$, a chain of n factors of $\tilde{\pi}^{ij}_T$ linked together by factors of h_{ij}. In analogy with our construction of the $\phi(n)$ of Eq. (85), we define

$$\Pi^{ij}_T(0) \equiv h^{ij} = \Omega^{-1}\Pi^{ij}(0), \tag{127}$$

and

$$\Pi^{ij}_T(n+1) \equiv \tilde{\pi}^{ia}_T h_{ab}\Pi^{bj}(n) = \omega^{-1}\tilde{\pi}^{ia} f_{ab}\Pi^{bj}(n), \tag{128}$$

from which it follows that $\Pi^{ij}_T(n) = \Omega^{-1}\omega^{-n}\Pi^{ij}(n)$. The contraction $h_{ij}\Pi^{ij}_T(n)$ yields the desired scalars,

$$\phi_T(n) \equiv h_{ij}\Pi^{ij}_T(n) = \frac{\phi(n)}{\omega^n}. \tag{129}$$

The $\phi_T(n)$ are the only scalars that can be built out of fully connected contractions of h_{ij} and $\tilde{\pi}^{ij}_T$. In the presence of the constraint $\omega \sim \omega(t)$, it follows that

$$\phi_T(n) \sim \bar{\phi}(n). \tag{130}$$

In other words, the $\bar{\phi}(n)$ are the scalars on the phase space $(\tilde{h}_{ij}, \tilde{\pi}^{ij})$ which have the correct conformal weight to have been derived from three-scalars on the phase space (h_{ij}, π^{ij}). It follows that the $\bar{\phi}(n)$ are invariant under spatial conformal transformations which rescale the volume factor ω, and the condition $\Delta\pi_\omega = 0$ is thus analogous to a renormalization group equation.

C. Summary

In this section, we developed a formalism for testing when our modified theories of gravity lead to a consistent first class constraint algebra, and hence contain two degrees of freedom. To develop the formalism, we made the simplifying assumption that the scalar momentum π_ω is an ultralocal function of time t and the phase space variables $\tilde{h}_{ij}$ and $\tilde{\pi}^{ij}$. This assumption is sufficient to guarantee that the constraints $\tilde{\mathcal{H}}_i$ remain first class. However, for the constraints to be consistent with the equations of motion, π_ω must be invariant under renormalization of the volume factor ω. Concretely, π_ω must obey the renormalization group equation

$$\Delta\pi_\omega = 0, \tag{131}$$

where

$$\Delta \equiv \omega\frac{\partial}{\partial\omega} + \sum_{m=2}^{\infty} m\phi(m)\frac{\partial}{\partial\phi(m)}. \tag{132}$$

Satisfying this equation completely fixes the dependence of π_ω on $\omega(t)$. In the next section, we will generalize our results to a realistic class of scalar momenta.

V. LOCAL MODIFIED GRAVITY

The ultralocal ansatz has the virtue of simplifying calculations, but the laws of nature are local, not ultralocal. In this section, we will apply the formalism developed in the last section to theories in which π_ω depends on spatial derivatives of the metric $\tilde{h}_{ij}$ through a dependence on the Ricci scalar $\tilde{R}$. Since the $\pi_{\rm GR}$ of spatially covariant general relativity belongs to this class [see Eq. (48)], this is a realistic class of theories. As we will demonstrate, such π_ω must obey stringent consistency conditions in order for the $\tilde{\mathcal{H}}_i$ to generate a consistent first class constraint algebra.

A. Constraint algebra

In this section, we will compute $\{\mathcal{H}_i(x), \mathcal{H}_a(y)\}$ assuming that π_ω is a function of t, the phase space variables $\tilde{h}_{ij}$ and $\tilde{\pi}^{ij}$, and the Ricci scalar $\tilde{R}$. We will then use the result to determine when the constraints $\tilde{\mathcal{H}}_i$ remain first class.

As before, we decompose $\tilde{\mathcal{H}}_i$ into a tensor part $\mathcal{J}_i \equiv -2\tilde{h}_{ij}\tilde{\nabla}_k\tilde{\pi}^{jk}$ and a scalar part $\mathcal{K}_i \equiv -\omega\tilde{\nabla}_i\pi_\omega$. Computing $\{\mathcal{H}_i(x), \mathcal{H}_a(y)\}$ is then a matter of computing the four brackets in Eq. (71). The result for $\{\mathcal{J}_i(x), \mathcal{J}_a(y)\}$ carries over unchanged from Eq. (96), but we will have to revisit the brackets involving $\mathcal{K}_i$. To do so, we will first evaluate the smoothing functional brackets $\{F_J, G_K\} + \{F_K, G_J\}$ and $\{F_K, G_K\}$. By comparing the ensuing explicit expressions to the formal expressions in Eq. (73), we will derive explicit expressions for the Poisson brackets involving $\mathcal{K}_i$.

Our analysis of the variational derivatives of the smoothing functional F_K defined in Eq. (72) proceeds exactly as in the ultralocal case up to Eq. (82), where the quantity $\delta\pi_\omega$ arises. In this section, we assume that π_ω is a function of t, $\tilde{h}_{ij}$, $\tilde{\pi}^{ij}$, and $\tilde{R}$. To simplify calculations, note that this is equivalent to making π_ω a function of t, $\tilde{R}$, and the $\phi(n)$ defined in Eq. (85). It follows from this assumption that

$$\delta\pi_\omega = \sum_{n=2}^{\infty}\frac{\partial\pi_\omega}{\partial\phi(n)}\delta\phi(n) + \frac{\partial\pi_\omega}{\partial\tilde{R}}\delta\tilde{R}. \tag{133}$$

Substituting this result into Eq. (82), using the identity $\delta\tilde{R} = -\tilde{R}^{jk}\delta\tilde{h}_{jk} + \tilde{\nabla}^k\tilde{\nabla}^j\delta\tilde{h}_{jk}$, and integrating by parts yields

$$\delta F_K = \omega\int d^3x(\partial_i f^i)\left(\sum_{n=2}^{\infty}\frac{\partial\pi_\omega}{\partial\phi(n)}\delta\phi(n) - \frac{\partial\pi_\omega}{\partial\tilde{R}}\tilde{R}^{jk}\delta\tilde{h}_{jk}\right) + \omega\int d^3x\tilde{\nabla}^j\tilde{\nabla}^k\left((\partial_i f^i)\frac{\partial\pi_\omega}{\partial\tilde{R}}\right)\delta\tilde{h}_{jk}. \tag{134}$$

Using Eq. (87), it is now straightforward to compute the variational derivatives of F_K,

$$\frac{\delta F_K}{\delta\tilde{h}_{mn}} = \omega(\partial_i f^i)\sum_{n=2}^{\infty} n\frac{\partial\pi_\omega}{\partial\phi(n)} \times\left(\tilde{\delta}^{mn}_{jk}\Pi(n)^{jk} - \frac{1}{3}\tilde{\pi}^{mn}\phi(n-1)\right) - \omega(\partial_i f^i)\frac{\partial\pi_\omega}{\partial\tilde{R}}\tilde{\delta}^{mn}_{jk}\tilde{R}^{jk} + \omega\tilde{\delta}^{mn}_{jk}\tilde{\nabla}^j\tilde{\nabla}^k\left((\partial_i f^i)\frac{\partial\pi_\omega}{\partial\tilde{R}}\right),$$

$$\frac{\delta F_K}{\delta\tilde{\pi}^{mn}} = \omega(\partial_i f^i)\sum_{n=2}^{\infty} n\frac{\partial\pi_\omega}{\partial\phi(n)}\tilde{\delta}^{jk}_{mn}\Pi(n-1)_{jk}. \tag{135}$$

The corresponding results for G_K are

$$\frac{\delta G_K}{\delta\tilde{h}_{mn}} = \omega(\partial_a g^a)\sum_{m=2}^{\infty} m\frac{\partial\pi_\omega}{\partial\phi(m)} \times\left(\tilde{\delta}^{mn}_{bc}\Pi(m)^{bc} - \frac{1}{3}\tilde{\pi}^{mn}\phi(m-1)\right) - \omega(\partial_a g^a)\frac{\partial\pi_\omega}{\partial\tilde{R}}\tilde{\delta}^{mn}_{bc}\tilde{R}^{bc} + \omega\tilde{\delta}^{mn}_{bc}\tilde{\nabla}^b\tilde{\nabla}^c\left((\partial_a g^a)\frac{\partial\pi_\omega}{\partial\tilde{R}}\right),$$

$$\frac{\delta G_K}{\delta\tilde{\pi}^{mn}} = \omega(\partial_a g^a)\sum_{m=2}^{\infty} m\frac{\partial\pi_\omega}{\partial\phi(m)}\tilde{\delta}^{bc}_{mn}\Pi(m-1)_{bc}. \tag{136}$$

We are now in a position to compute the brackets involving $\mathcal{K}_i$.

(i) $\{\mathcal{J}_i(x), \mathcal{K}_a(y)\} + \{\mathcal{K}_i(x), \mathcal{J}_a(y)\}$
To compute $\{\mathcal{J}_i(x), \mathcal{K}_a(y)\} + \{\mathcal{K}_i(x), \mathcal{J}_a(y)\}$, we first compute $\{F_J, G_K\} + \{F_K, G_J\}$. We begin by substituting
Eqs. (79) and (136) into the bracket $\{F_J, G_K\}$. After expanding and simplifying a total derivative of $\phi(n)$, $\{F_J, G_K\}$ turns into

$$\{F_J, G_K\} = -\omega\int d^3z f^i(\partial_a g^a)\sum_{m=2}^{\infty}\frac{\partial\pi_\omega}{\partial\phi(m)}\tilde{\nabla}_i\phi(m) + 2\omega\int d^3z(\partial_a g^a)\frac{\partial\pi_\omega}{\partial\tilde{R}}\tilde{R}_i{}^k\tilde{\nabla}_k f^i - 2\omega\int d^3z(\tilde{\nabla}_k f^i)\tilde{\nabla}_i\tilde{\nabla}^k\left((\partial_a g^a)\frac{\partial\pi_\omega}{\partial\tilde{R}}\right) + \frac{2}{3}\omega\int d^3z(\partial_i f^i)\tilde{\nabla}_c\tilde{\nabla}^c\left((\partial_a g^a)\frac{\partial\pi_\omega}{\partial\tilde{R}}\right) - \omega\int d^3z(\partial_i f^i)(\partial_a g^a)\left(\frac{2}{3}\tilde{R}\frac{\partial\pi_\omega}{\partial\tilde{R}} + \sum_{m=2}^{\infty} m\phi(m)\frac{\partial\pi_\omega}{\partial\phi(m)}\right). \tag{137}$$

To finesse this expression, integrate by parts, use the identities $\tilde{\nabla}_i\tilde{\nabla}_j V^i = \tilde{\nabla}_j\tilde{\nabla}_i V^i + \tilde{R}_{ij}V^i$ and $2\tilde{\nabla}_j\tilde{R}_i{}^j = \tilde{\nabla}_i\tilde{R}$, simplify a total derivative of π_ω, use the identity $\mathcal{K}_i = -\omega\tilde{\nabla}_i\pi_\omega$, and expand to obtain

$$\{F_J, G_K\} = \int d^3z f^i\mathcal{K}_i\partial_a g^a + \frac{4}{3}\omega\int d^3z\tilde{\nabla}_k(\partial_i f^i)\tilde{\nabla}^k\left((\partial_a g^a)\frac{\partial\pi_\omega}{\partial\tilde{R}}\right) - \omega\int d^3z(\partial_i f^i)(\partial_a g^a)\left(\frac{2}{3}\tilde{R}\frac{\partial\pi_\omega}{\partial\tilde{R}} + \sum_{m=2}^{\infty} m\phi(m)\frac{\partial\pi_\omega}{\partial\phi(m)}\right). \tag{138}$$

Similarly,

$$\{F_K, G_J\} = -\int d^3z g^a\mathcal{K}_a\partial_i f^i - \frac{4}{3}\omega\int d^3z\tilde{\nabla}_k(\partial_a g^a)\tilde{\nabla}^k\left((\partial_i f^i)\frac{\partial\pi_\omega}{\partial\tilde{R}}\right) + \omega\int d^3z(\partial_i f^i)(\partial_a g^a)\left(\frac{2}{3}\tilde{R}\frac{\partial\pi_\omega}{\partial\tilde{R}} + \sum_{m=2}^{\infty} m\phi(m)\frac{\partial\pi_\omega}{\partial\phi(m)}\right), \tag{139}$$

so the sum of the two brackets reduces to

$$\{F_J, G_K\} + \{F_K, G_J\} = \int d^3z f^i\mathcal{K}_i\partial_a g^a - \int d^3z g^a\mathcal{K}_a\partial_i f^i + \frac{4}{3}\omega\int d^3z\tilde{\nabla}_k(\partial_i f^i)\tilde{\nabla}^k\left((\partial_a g^a)\frac{\partial\pi_\omega}{\partial\tilde{R}}\right) - \frac{4}{3}\omega\int d^3z\tilde{\nabla}_k(\partial_a g^a)\tilde{\nabla}^k\left((\partial_i f^i)\frac{\partial\pi_\omega}{\partial\tilde{R}}\right). \tag{140}$$

After integrating by parts, expanding, and using the identity $\partial_i\mathcal{K}_a = \partial_a\mathcal{K}_i$, this becomes

$$\{F_J, G_K\} + \{F_K, G_J\} = \int d^3z f^i\mathcal{K}_a\partial_i g^a - \int d^3z g^a\mathcal{K}_i\partial_a f^i + \int d^3z(\partial_i f^i)(\partial_k\partial_a g^a)\mathcal{M}^k - \int d^3z(\partial_a g^a)(\partial_k\partial_i f^i)\mathcal{M}^k, \tag{141}$$

where

$$\mathcal{M}_k \equiv -\frac{4}{3}\omega\tilde{\nabla}_k\frac{\partial\pi_\omega}{\partial\tilde{R}}. \tag{142}$$

To extract the bracket $\{\mathcal{J}_i(x), \mathcal{K}_a(y)\} + \{\mathcal{K}_i(x), \mathcal{J}_a(y)\}$, integrate by parts, relabel dummy indices, and insert the identities in Eq. (94) to yield

$$\{F_J, G_K\} + \{F_K, G_J\} = \int d^3x d^3y f^i(x) g^a(y)(\mathcal{K}_a(x)\partial_{x^i}\delta^3(x-y) - \mathcal{K}_i(y)\partial_{y^a}\delta^3(x-y)) + \int d^3x d^3y f^i(x) g^a(y)\partial_{x^i}(-\mathcal{M}^k(x)\partial_{x^k}\partial_{x^a}\delta^3(x-y)) - \int d^3x d^3y f^i(x) g^a(y)\partial_{y^a}(-\mathcal{M}^k(y)\partial_{y^k}\partial_{y^i}\delta^3(x-y)). \tag{143}$$

By comparing this expression to Eq. (73), it is clear that

$$\{\mathcal{J}_i(x), \mathcal{K}_a(y)\} + \{\mathcal{K}_i(x), \mathcal{J}_a(y)\} = \mathcal{K}_a(x)\partial_{x^i}\delta^3(x-y) - \mathcal{K}_i(y)\partial_{y^a}\delta^3(x-y) + \partial_{x^i}(-\mathcal{M}^k(x)\partial_{x^k}\partial_{x^a}\delta^3(x-y)) - \partial_{y^a}(-\mathcal{M}^k(y)\partial_{y^k}\partial_{y^i}\delta^3(x-y)). \tag{144}$$

(ii) $\{\mathcal{K}_i(x), \mathcal{K}_a(y)\}$

To compute $\{\mathcal{K}_i(x), \mathcal{K}_a(y)\}$, we first compute the bracket $\{F_K, G_K\}$. Substituting Eqs. (135) and (136) into the bracket $\{F_K, G_K\}$ yields

$$\{F_K, G_K\} = \omega^2 \int d^3z (\partial_a g^a) \frac{\partial \pi_\omega}{\partial \tilde{\pi}^{jk}} \tilde{\nabla}^j \tilde{\nabla}^k \left((\partial_i f^i) \frac{\partial \pi_\omega}{\partial \tilde{R}} \right) - \omega^2 \int d^3z (\partial_i f^i) \frac{\partial \pi_\omega}{\partial \tilde{\pi}^{jk}} \tilde{\nabla}^j \tilde{\nabla}^k \left((\partial_a g^a) \frac{\partial \pi_\omega}{\partial \tilde{R}} \right), \tag{145}$$

where

$$\frac{\partial \pi_\omega}{\partial \tilde{\pi}^{jk}} = \tilde{\delta}^{bc}_{jk} \sum_{n=2}^{\infty} n \frac{\partial \pi_\omega}{\partial \phi(n)} \Pi(n-1)_{bc}. \tag{146}$$

After integrating by parts and expanding, the bracket becomes

$$\{F_K, G_K\} = \int d^3z (\partial_i f^i)(\partial_k \partial_a g^a) \mathcal{N}^k - \int d^3z (\partial_a g^a)(\partial_k \partial_i f^i) \mathcal{N}^k, \tag{147}$$

where

$$\mathcal{N}_k \equiv \omega^2 \frac{\partial \pi_\omega}{\partial \tilde{R}} \tilde{\nabla}^j \frac{\partial \pi_\omega}{\partial \tilde{\pi}^{jk}} - \omega^2 \frac{\partial \pi_\omega}{\partial \tilde{\pi}^{jk}} \tilde{\nabla}^j \frac{\partial \pi_\omega}{\partial \tilde{R}}. \tag{148}$$

To extract the bracket $\{\mathcal{K}_i(x), \mathcal{K}_a(y)\}$, integrate by parts, relabel dummy indices, and insert the identities in Eq. (94) to obtain

$$\{F_K, G_K\} = \int d^3x d^3y f^i(x) g^a(y) \partial_{x^i} \times (-\mathcal{N}^k(x)\partial_{x^k}\partial_{x^a}\delta^3(x-y)) - \int d^3x d^3y f^i(x) g^a(y) \partial_{y^a} \times (-\mathcal{N}^k(y)\partial_{y^k}\partial_{y^i}\delta^3(x-y)). \tag{149}$$

Comparing this expression to Eq. (73), it follows that

$$\{\mathcal{K}_i(x), \mathcal{K}_a(y)\} = \partial_{x^i}(-\mathcal{N}^k(x)\partial_{x^k}\partial_{x^a}\delta^3(x-y)) - \partial_{y^a}(-\mathcal{N}^k(y)\partial_{y^k}\partial_{y^i}\delta^3(x-y)). \tag{150}$$

By substituting Eqs. (96), (144), and (150) into Eq. (71), and recalling that $\tilde{\mathcal{H}}_i = \mathcal{J}_i + \mathcal{K}_i$, we obtain the identity

$$\{\tilde{\mathcal{H}}_i(x), \tilde{\mathcal{H}}_j(y)\} = \tilde{\mathcal{H}}_j(x)\partial_{x^i}\delta^3(x-y) - \tilde{\mathcal{H}}_i(y)\partial_{y^j}\delta^3(x-y) + \partial_{x^i}(-\mathcal{I}^k(x)\partial_{x^k}\partial_{x^j}\delta^3(x-y)) - \partial_{y^j}(-\mathcal{I}^k(y)\partial_{y^k}\partial_{y^i}\delta^3(x-y)), \tag{151}$$

where $\mathcal{I}_k \equiv \mathcal{M}_k + \mathcal{N}_k$, or

$$\mathcal{I}_k = \omega^2 \frac{\partial \pi_\omega}{\partial \tilde{R}} \tilde{\nabla}^j \frac{\partial \pi_\omega}{\partial \tilde{\pi}^{jk}} - \omega^2 \frac{\partial \pi_\omega}{\partial \tilde{\pi}^{jk}} \tilde{\nabla}^j \frac{\partial \pi_\omega}{\partial \tilde{R}} - \frac{4}{3}\omega \tilde{\nabla}_k \frac{\partial \pi_\omega}{\partial \tilde{R}}. \tag{152}$$

Expanding the derivatives in this expression and using the fact that $\tilde{\mathcal{H}}_i \sim 0$ yields

$$\{\tilde{\mathcal{H}}_i(x), \tilde{\mathcal{H}}_j(y)\} \sim -(\mathcal{I}^k(x) + \mathcal{I}^k(y))\partial_{x^i}\partial_{x^j}\partial_{x^k}\delta^3(x-y) - (\partial_{x^i}\mathcal{I}^k(x))\partial_{x^j}\partial_{x^k}\delta^3(x-y) + (\partial_{y^j}\mathcal{I}^k(y))\partial_{y^i}\partial_{y^k}\delta^3(x-y). \tag{153}$$

The three terms of this equation are algebraically independent, so the necessary and sufficient condition for the Poisson bracket $\{\tilde{\mathcal{H}}_i(x), \tilde{\mathcal{H}}_j(y)\}$ to vanish is

$$\mathcal{I}_k \sim 0. \tag{154}$$

In the ultralocal case the constraints were automatically first class, but to generate a first class constraint algebra in

the local case, the scalar momentum π_ω must obey the fearsome looking differential equation $\mathcal{I}_k \sim 0$.

As a check, we will now compute the $\mathcal{I}_k$ arising from the $\pi_{\rm GR}$ of spatially covariant general relativity. Recall from Eq. (48) that

$$\pi_{\rm GR} = -\sqrt{\frac{8}{3}}\sqrt{\omega^{-2}\phi(2) - \omega^{-2/3}\tilde{R} + 2\Lambda}. \tag{155}$$

Since $\pi_{\rm GR}$ is a function only of $\phi(2)$ and $\tilde{R}$, its partial derivative with respect to $\tilde{\pi}^{ij}$ simplifies,

$$\frac{\partial \pi_{\rm GR}}{\partial \tilde{\pi}^{jk}} = 2\frac{\partial \pi_{\rm GR}}{\partial \phi(2)}\tilde{\pi}_{jk}. \tag{156}$$

After substituting this relation into the definition of $\mathcal{I}_k$ and recalling that $\mathcal{J}_i = -2\tilde{h}_{ij}\tilde{\nabla}_k\tilde{\pi}^{jk}$, $\mathcal{I}_k$ becomes

$$\mathcal{I}_k(\pi_{\rm GR}) = -\frac{4}{3}\omega\tilde{\nabla}_k\frac{\partial \pi_{\rm GR}}{\partial \tilde{R}} - \omega^2\frac{\partial \pi_{\rm GR}}{\partial \tilde{R}}\frac{\partial \pi_{\rm GR}}{\partial \phi(2)}\mathcal{J}_k + 2\omega^2\tilde{\pi}_{jk}\left(\frac{\partial \pi_{\rm GR}}{\partial \tilde{R}}\tilde{\nabla}^j\frac{\partial \pi_{\rm GR}}{\partial \phi(2)} - \frac{\partial \pi_{\rm GR}}{\partial \phi(2)}\tilde{\nabla}^j\frac{\partial \pi_{\rm GR}}{\partial \tilde{R}}\right). \tag{157}$$

Upon substituting the derivatives

$$\frac{\partial \pi_{\rm GR}}{\partial \phi(2)} = \frac{4}{3\omega^2}\frac{1}{\pi_{\rm GR}}, \qquad \frac{\partial \pi_{\rm GR}}{\partial \tilde{R}} = -\frac{4}{3\omega^{2/3}}\frac{1}{\pi_{\rm GR}}, \tag{158}$$

into $\mathcal{I}_k(\pi_{\rm GR})$, the term in parentheses vanishes. By using the relations $\mathcal{K}_i = -\omega\tilde{\nabla}_i\pi_\omega$ and $\mathcal{H}_i = \mathcal{J}_i + \mathcal{K}_i$, we obtain

$$\mathcal{I}_k(\pi_{\rm GR}) = \frac{16}{9\omega^{2/3}\pi_{\rm GR}^2}\mathcal{H}_k. \tag{159}$$

Since $\tilde{\mathcal{H}}_i \sim 0$, the scalar momentum $\pi_{\rm GR}$ satisfies $\mathcal{I}_k \sim 0$. The constraints $\tilde{\mathcal{H}}_i$ of spatially covariant general relativity thus generate a first class constraint algebra.

B. Consistency of constraints with equations of motion

In this section, we will compute the time derivative $\dot{\tilde{\mathcal{H}}}_i$ for local π_ω assuming that the constraints $\tilde{\mathcal{H}}_i$ are first class, and use the result to determine when the constraints $\tilde{\mathcal{H}}_i$ are also preserved by the equations of motion. The analysis of $\dot{\tilde{\mathcal{H}}}_i$ proceeds exactly as in the ultralocal case until we arrive at the expression,

$$\dot{\tilde{\mathcal{H}}}_i = -\partial_i\left(\dot{\omega}\pi_\omega + \omega\frac{\partial \pi_\omega}{\partial t}\right) - \dot{\omega}\{\mathcal{J}_i, \Pi_\omega\} - \dot{\omega}\{\mathcal{K}_i, \Pi_\omega\}, \tag{160}$$

where, as before,

$$\Pi_\omega \equiv \int d^3x\pi_\omega. \tag{161}$$

The point of departure from the ultralocal case is the evaluation of the two Poisson brackets $\{\mathcal{J}_i, \Pi_\omega\}$ and $\{\mathcal{K}_i, \Pi_\omega\}$. To compute them, we first compute the smoothing functional brackets $\{F_J, \Pi_\omega\}$ and $\{F_K, \Pi_\omega\}$. As in the ultralocal case, we will obtain brackets involving Π_ω by applying the substitution $\partial_a g^a \to \omega^{-1}$ to brackets involving G_K.

(i) $\{\mathcal{J}_i, \Pi_\omega\}$

To obtain the bracket $\{\mathcal{J}_i, \Pi_\omega\}$, we first compute the bracket $\{F_J, \Pi_\omega\}$. Applying $\partial_a g^a \to \omega^{-1}$ to Eq. (138) and integrating by parts yields

$$\{F_J, \Pi_\omega\} = \int d^3x f^i\partial_i\left(-\pi_\omega + \frac{2}{3}\tilde{R}\frac{\partial \pi_\omega}{\partial \tilde{R}} + \sum_{m=2}^{\infty} m\phi(m)\frac{\partial \pi_\omega}{\partial \phi(m)} - \omega^{-1}\tilde{\nabla}_k\mathcal{M}^k\right), \tag{162}$$

where, as before,

$$\mathcal{M}_k = -\frac{4}{3}\omega\tilde{\nabla}_k\frac{\partial \pi_\omega}{\partial \tilde{R}}. \tag{163}$$

It follows from an application of Eq. (114) that

$$\{\mathcal{J}_i, \Pi_\omega\} = \partial_i\left(-\pi_\omega + \frac{2}{3}\tilde{R}\frac{\partial \pi_\omega}{\partial \tilde{R}} + \sum_{m=2}^{\infty} m\phi(m)\frac{\partial \pi_\omega}{\partial \phi(m)} - \omega^{-1}\tilde{\nabla}_k\mathcal{M}^k\right). \tag{164}$$

(ii) $\{\mathcal{K}_i, \Pi_\omega\}$

To compute the bracket $\{\mathcal{K}_i, \Pi_\omega\}$, we first compute the bracket $\{F_K, \Pi_\omega\}$. After substituting $\partial_a g^a \to \omega^{-1}$ and integrating by parts, Eq. (147) becomes

$$\{F_K, \Pi_\omega\} = \int d^3x f^i\partial_i(-\omega^{-1}\nabla_k\mathcal{N}^k), \tag{165}$$

where, as before,

$$\mathcal{N}_k = \omega^2\frac{\partial \pi_\omega}{\partial \tilde{R}}\tilde{\nabla}^j\frac{\partial \pi_\omega}{\partial \tilde{\pi}^{jk}} - \omega^2\frac{\partial \pi_\omega}{\partial \tilde{\pi}^{jk}}\tilde{\nabla}^j\frac{\partial \pi_\omega}{\partial \tilde{R}}. \tag{166}$$

Comparing with Eq. (114) yields

$$\{\mathcal{K}_i, \Pi_\omega\} = \partial_i(-\omega^{-1}\nabla_k\mathcal{N}^k). \tag{167}$$

After substituting Eqs. (164) and (167) into the equation of motion (160) and recalling that $\mathcal{I}_k = \mathcal{M}_k + \mathcal{N}_k$, we obtain

$$\dot{\tilde{\mathcal{H}}}_i = -\partial_i\left(\omega\frac{\partial \pi_\omega}{\partial t} + \frac{2}{3}\dot{\omega}\tilde{R}\frac{\partial \pi_\omega}{\partial \tilde{R}} + \dot{\omega}\sum_{m=2}^{\infty} m\phi(m)\frac{\partial \pi_\omega}{\partial \phi(m)} - \dot{\omega}\omega^{-1}\tilde{\nabla}_k\mathcal{I}^k\right). \tag{168}$$

Since the $\tilde{\mathcal{H}}_i$ are assumed to be first class, it follows necessarily that $\mathcal{I}^k \sim 0$. Demanding $\dot{\tilde{\mathcal{H}}}_i \sim 0$ thus implies the consistency condition

$$\omega\frac{\partial\pi_\omega}{\partial t}+\frac{2}{3}\dot{\omega}\tilde{R}\frac{\partial\pi_\omega}{\partial\tilde{R}}+\dot{\omega}\sum_{m=2}^{\infty}m\phi(m)\frac{\partial\pi_\omega}{\partial\phi(m)}\sim f(t), \quad (169)$$

where $f(t)$ is an arbitrary function of time. Recall once again that the equation of motion (63) is invariant under $\pi_\omega \to \pi_\omega + g(t)$, where $g(t)$ is an arbitrary function of time. By choosing a function $g(t)$ such that $\omega g'(t) = f(t)$, the consistency condition becomes

$$\omega\frac{\partial\pi_\omega}{\partial t}+\frac{2}{3}\dot{\omega}\tilde{R}\frac{\partial\pi_\omega}{\partial\tilde{R}}+\dot{\omega}\sum_{m=2}^{\infty}m\phi(m)\frac{\partial\pi_\omega}{\partial\phi(m)}\sim 0. \quad (170)$$

Since $\omega(t)$ is assumed to be an invertible function of time, $\partial/\partial t = \dot{\omega}\partial/\partial\omega$. In analogy with our approach in the ultralocal case, we rewrite the consistency condition as

$$\Delta\pi_\omega \sim 0, \quad (171)$$

where we have redefined the operator Δ as

$$\Delta \equiv \omega\frac{\partial}{\partial\omega}+\frac{2}{3}\tilde{R}\frac{\partial}{\partial\tilde{R}}+\sum_{m=2}^{\infty}m\phi(m)\frac{\partial}{\partial\phi(m)}. \quad (172)$$

To rule out the possibility of a π_ω which satisfies $\Delta\pi_\omega \sim 0$ while $\Delta\pi_\omega \neq 0$, we note that the constraints $\tilde{\mathcal{H}}_i$ contain a term $\tilde{\nabla}_i\pi_\omega$, making the constraints higher order in spatial derivatives than π_ω itself. However, by examining a series expansion of π_ω in the parameter $\tilde{R}$, one can verify that applying Δ to π_ω does not alter its order in spatial derivatives.[10] It follows that $\Delta\pi_\omega$ cannot depend on $\tilde{\mathcal{H}}_i$. The condition $\Delta\pi_\omega \sim 0$ is therefore equivalent to the apparently stronger condition

$$\Delta\pi_\omega = 0. \quad (173)$$

Since $\Delta(\omega^{-n}\phi(n)) = 0$ and $\Delta(\omega^{-2/3}\tilde{R}) = 0$, we are led to define the quantities

$$\bar{\phi}(n) \equiv \frac{\phi(n)}{\omega^n(t)}, \qquad \bar{R} \equiv \frac{\tilde{R}}{\omega^{2/3}}. \quad (174)$$

The most general solution to the condition $\Delta\pi_\omega = 0$ is an arbitrary function of $\bar{R}$ and the $\bar{\phi}(n)$. In this manner, the dependence of π_ω on $\omega(t)$ is determined by its dependence on the phase space variables.

As before, to understand this result, we return briefly to the phase space (h_{ij}, π^{ij}). As shown in Sec. IV B, the only scalars that can be built out of the tensor h_{ij} and the traceless tensor $\tilde{\pi}^{ij}_T$ are the $\phi_T(n) = \omega^{-n}\phi(n)$. If we impose the gauge-fixing constraint $\omega \sim \omega(t)$, then $\phi_T(n) \sim \bar{\phi}(n)$; likewise, the Ricci scalar R of the metric h_{ij} obeys $R \sim \bar{R}$.[11] This means that $\bar{R}$ and the $\bar{\phi}(n)$ have the correct conformal weight to have been derived from three-scalars on the phase space (h_{ij}, π^{ij}). The scalars $\bar{R}$ and the $\bar{\phi}(n)$ are thus invariant under spatial conformal transformations which rescale the volume factor ω, so once again $\Delta\pi_\omega = 0$ is revealed to be analogous to a renormalization group equation.

As a check, we will now apply the renormalization group equation to the scalar momentum $\pi_{\rm GR}$ of spatially covariant general relativity. Since the constraints of spatially covariant general relativity are first class (see Sec. V A), the necessary and sufficient condition for the constraints to be preserved by the equation of motion is for $\pi_{\rm GR}$ to satisfy Eq. (173). From Eq. (48), it follows that

$$\pi_{\rm GR} = -\sqrt{\frac{8}{3}}\sqrt{\bar{\phi}(2) - \bar{R} + 2\Lambda}, \quad (175)$$

so the scalar momentum $\pi_{\rm GR}$ depends only on the invariant quantities $\bar{\phi}(2)$ and $\bar{R}$. It follows at once that $\pi_{\rm GR}$ satisfies the condition $\Delta\pi_{\rm GR} = 0$, which implies that the constraints of the theory are preserved by the equations of motion. It is now clear within the context of our formalism that the constraints $\tilde{\mathcal{H}}_i$ of spatially covariant general relativity generate a consistent first class algebra. This result justifies the assertions we made in the first paragraph of Sec. III D.

C. Summary

In this section, we applied the formalism developed in Sec. IV to determine when scalar momenta π_ω built out of $\tilde{h}_{ij}$, $\tilde{\pi}^{ij}$, and $\tilde{R}$ yield a consistent first class constraint algebra. To ensure the first class character of the constraints $\tilde{\mathcal{H}}_i$, it is necessary and sufficient for π_ω to obey the condition

$$\mathcal{I}_k \sim 0, \quad (176)$$

where

$$\mathcal{I}_k = \omega^2\frac{\partial\pi_\omega}{\partial\tilde{R}}\tilde{\nabla}^j\frac{\partial\pi_\omega}{\partial\tilde{\pi}^{jk}} - \omega^2\frac{\partial\pi_\omega}{\partial\tilde{\pi}^{jk}}\tilde{\nabla}^j\frac{\partial\pi_\omega}{\partial\tilde{R}} - \frac{4}{3}\omega\tilde{\nabla}_k\frac{\partial\pi_\omega}{\partial\tilde{R}}. \quad (177)$$

If $\partial\pi_\omega/\partial\tilde{R} = 0$, then $\mathcal{I}_k = 0$, so ultralocal scalar momenta satisfy this condition trivially. The scalar momentum $\pi_{\rm GR}$ of spatially covariant general relativity depends essentially on $\tilde{R}$, and thus satisfies this condition nontrivially.

To guarantee the preservation of the constraints $\tilde{\mathcal{H}}_i$ by the equations of motion, the scalar momentum π_ω must also be invariant under renormalization of the volume factor ω. This requires π_ω to obey the renormalization group equation

$$\Delta\pi_\omega = 0, \quad (178)$$

where

[10] Spatial derivatives enter π_ω solely through $\tilde{R}$, so the derivative expansion of π_ω can be written $\pi_\omega = \sum_{k=0}^{\infty} c_k\tilde{R}^k$, where the coefficients c_k depend on ω and the $\phi(n)$. Applying the Δ operator to π_ω changes the functional form of the c_k, but does not generate higher order powers of $\tilde{R}$.

[11] See Eq. (C20) in Appendix C.

$$\Delta \equiv \omega \frac{\partial}{\partial \omega} + \frac{2}{3}\tilde{R}\frac{\partial}{\partial \tilde{R}} + \sum_{m=2}^{\infty} m\phi(m)\frac{\partial}{\partial \phi(m)}. \tag{179}$$

This generalizes Eq. (131) to include a possible dependence of π_ω on $\tilde{R}$. The scalar momentum $\pi_{\rm GR}$ satisfies $\Delta\pi_{\rm GR} = 0$ in addition to $\mathcal{I}_k(\pi_{\rm GR}) \sim 0$, so the constraints of spatially covariant general relativity generate a consistent first class constraint algebra.

VI. CONCLUSIONS

In this paper, we developed a general formalism for verifying the consistency of spatially covariant modified theories of the transverse, traceless graviton degrees of freedom. It was a long road, so it is worth retracing our steps to see the logic of our path.

In Sec. II, we showed how to express general relativity as a theory of a spatial metric h_{ij} and its conjugate momentum π^{ij}. In this language, the general covariance and local Lorentz covariance of the theory are encoded by the Dirac algebra obeyed by the four constraints $\mathcal{H}_\mu$. In Sec. III, we showed how to obtain a spatially covariant version of general relativity. We began in Sec. III A by splitting the phase space (h_{ij}, π^{ij}) into the phase space (ω, π_ω) of the spatial volume factor and the phase space $(\tilde{h}_{ij}, \tilde{\pi}^{ij})$ of the unit-determinant metric. In the context of cosmology on an FRW background, it is natural to represent time diffeomorphism symmetry on the phase space (ω, π_ω) and to represent spatial diffeomorphisms on the phase space $(\tilde{h}_{ij}, \tilde{\pi}^{ij})$; in Sec. III B, we showed how to achieve this splitting using a cosmological gauge condition. On an expanding background, ω drops out of the dynamical phase space of the theory, and its conjugate momentum π_ω becomes the scalar part of the physical Hamiltonian density on the phase space $(\tilde{h}_{ij}, \tilde{\pi}^{ij})$; in Sec. III C, we showed how to reduce the phase space by solving the Hamiltonian constraint in cosmological gauge. By successfully projecting the degrees of freedom of general relativity onto the reduced phase space $(\tilde{h}_{ij}, \tilde{\pi}^{ij})$, we have shown how to represent the graviton dynamics of general relativity on the class of conformally equivalent spatial metrics.

To modify general relativity, we simply modified the functional form of the scalar momentum π_ω while retaining the explicit spatial diffeomorphism symmetry generated by the three constraints $\tilde{\mathcal{H}}_i$. In Sec. IV, we considered the case in which π_ω is an ultralocal function of the phase space quantities $\tilde{h}_{ij}$ and $\tilde{\pi}^{ij}$. In this case, the consistency of the constraints $\tilde{\mathcal{H}}_i$ imposes a single nontrivial condition on the form of π_ω, namely, that it must satisfy a renormalization group equation with flow parameter ω. The renormalization group equation encodes the fact that π_ω must be invariant under flow through the space of conformally equivalent spatial metrics. In Sec. V, we applied our formalism to the case in which π_ω is also allowed to depend on $\tilde{R}$, the Ricci scalar of the metric $\tilde{h}_{ij}$. In this case, π_ω must satisfy a corresponding renormalization group equation, but its form is further restricted by a differential equation that relates its dependence on $\tilde{R}$ to its dependence on the phase space variables $\tilde{h}_{ij}$ and $\tilde{\pi}^{ij}$.

As a proof of principle, this paper demonstrates the possibility of consistently modifying the graviton equations of motion, but more remains to be done. In forthcoming work [20], we will apply our formalism to search for viable alternatives to general relativity by attempting to modify the π_ω of general relativity parametrically in the infrared. If we discover nontrivial modifications of general relativity that contain only two degrees of freedom, it could open up new lines of theoretical and experimental research. A null result, on the other hand, would serve as further evidence of the uniqueness of general relativity. It will be interesting to see just how far we can push this program.

ACKNOWLEDGMENTS

We thank N. Afshordi, K. Hinterbichler and M. Trodden for helpful discussions. We also thank the Perimeter Institute for Theoretical Physics in Waterloo, Ontario, where portions of this research were completed. This work is supported in part by the U.S. Department of Energy under Grant No. DE-AC02-76-ER-03071 (J. K. and G. E. J. M.) and the Alfred P. Sloan Foundation (J. K.).

APPENDIX A: COVARIANT CONSTRAINT ALGEBRA OF GR

Recall the Poisson bracket of GR

$$\{A, B\} \equiv \int d^3z \left(\frac{\delta A}{\delta h_{mn}(z)}\frac{\delta B}{\delta \pi^{mn}(z)} - \frac{\delta A}{\delta \pi^{mn}(z)}\frac{\delta B}{\delta h_{mn}(z)}\right) \tag{A1}$$

and the constraints

$$\mathcal{H}_0 \equiv -\sqrt{h}(R - 2\Lambda) + \frac{1}{\sqrt{h}}\left(\pi^{ij}\pi_{ij} - \frac{1}{2}(\pi^i{}_i)^2\right),$$
$$\mathcal{H}_i \equiv -2h_{ij}\nabla_k \pi^{jk}. \tag{A2}$$

Our object in this section is to derive the constraint algebra

$$\{\mathcal{H}_0(x), \mathcal{H}_0(y)\} = \mathcal{H}^i(x)\partial_{x^i}\delta^3(x-y) - \mathcal{H}^i(y)\partial_{y^i}\delta^3(x-y),$$
$$\{\mathcal{H}_0(x), \mathcal{H}_i(y)\} = \mathcal{H}_0(y)\partial_{x^i}\delta^3(x-y),$$
$$\{\mathcal{H}_i(x), \mathcal{H}_j(y)\} = \mathcal{H}_j(x)\partial_{x^i}\delta^3(x-y) - \mathcal{H}_i(y)\partial_{y^j}\delta^3(x-y). \tag{A3}$$

To evaluate these Poisson brackets, we first define the smoothing functionals

$$F_H \equiv \int d^3x f^0(x)\mathcal{H}_0(x), \quad F \equiv \int d^3x f^i(x)\mathcal{H}_i(x),$$
$$G_H \equiv \int d^3y g^0(y)\mathcal{H}_0(y), \quad G \equiv \int d^3y g^a(y)\mathcal{H}_a(y), \tag{A4}$$

where the functions f^0, f^i, g^0, and g^i are time-independent smoothing functions. We then compute the brackets

$$\{F_H, G_H\} = \int d^3x d^3y f^0(x) g^0(y) \{\mathcal{H}_0(x), \mathcal{H}_0(y)\},$$
$$\{F_H, G\} = \int d^3x d^3y f^0(x) g^a(y) \{\mathcal{H}_0(x), \mathcal{H}_a(y)\},$$
$$\{F, G\} = \int d^3x d^3y f^i(x) g^a(y) \{\mathcal{H}_i(x), \mathcal{H}_a(y)\}. \quad \text{(A5)}$$

As in Sec. IVA, we assume that the smoothing functions decay so rapidly that they eliminate all boundary terms generated by integration by parts, but that they are otherwise arbitrary. This greatly simplifies the explicit evaluation of the brackets of the smoothing functionals. By comparing the explicit forms of the brackets to the implicit forms in Eq. (A5), we will derive explicit formulae for the brackets of the $\mathcal{H}_\mu$'s.

To simplify the calculation of the variational derivatives of F_H, we will split the Hamiltonian constraint $\mathcal{H}_0$ into a kinetic piece $\mathcal{H}_T$ and a potential piece $\mathcal{H}_V$. Explicitly, we have $\mathcal{H}_0 = \mathcal{H}_T + \mathcal{H}_V$, where

$$\mathcal{H}_T \equiv \frac{1}{\sqrt{h}}\left(h_{ik}h_{jl} - \frac{1}{2}h_{ij}h_{kl}\right)\pi^{ij}\pi^{kl},$$
$$\mathcal{H}_V \equiv -\sqrt{h}(R - 2\Lambda). \quad \text{(A6)}$$

Similarly, $F_H = F_T + F_V$, where

$$F_T \equiv \int d^3x f^0(x)\mathcal{H}_T(x), \quad F_V \equiv \int d^3x f^0(x)\mathcal{H}_V(x). \quad \text{(A7)}$$

Computing the variation δF_T is straightforward:

$$\delta F_T = \int d^3x f^0\left(\frac{1}{\sqrt{h}}(2\pi^i{}_k\pi^{kj} - \pi^k{}_k\pi^{ij}) - \frac{1}{2}\mathcal{H}_T h^{ij}\right)\delta h_{ij} + \int d^3x f^0 \frac{1}{\sqrt{h}}(2\pi_{ij} - h_{ij}\pi^k{}_k)\delta\pi^{ij}. \quad \text{(A8)}$$

It follows that

$$\frac{\delta F_T}{\delta h_{mn}} = f^0\left(\frac{1}{\sqrt{h}}(2\pi^m{}_k\pi^{kn} - \pi^k{}_k\pi^{mn}) - \frac{1}{2}\mathcal{H}_T h^{mn}\right)$$
$$\frac{\delta F_T}{\delta \pi^{mn}} = f^0\frac{1}{\sqrt{h}}(2\pi_{mn} - h_{mn}\pi^k{}_k). \quad \text{(A9)}$$

Likewise,

$$\frac{\delta G_T}{\delta h_{mn}} = g^0\left(\frac{1}{\sqrt{h}}(2\pi^m{}_k\pi^{kn} - \pi^k{}_k\pi^{mn}) - \frac{1}{2}\mathcal{H}_T h^{mn}\right)$$
$$\frac{\delta G_T}{\delta \pi^{mn}} = g^0\frac{1}{\sqrt{h}}(2\pi_{mn} - h_{mn}\pi^k{}_k). \quad \text{(A10)}$$

Keeping in mind that $\delta R = -\delta h_{ij}R^{ij} + \nabla^j\nabla^i\delta h_{ij} - \nabla^k\nabla_k h^{ij}\delta h_{ij}$, computing δF_V is just as straightforward:

$$\delta F_V = \int d^3x f^0\left(\frac{1}{2}\mathcal{H}_V h^{ij} + \sqrt{h}R^{ij}\right)\delta h_{ij} + \int d^3x\sqrt{h}f^0(\nabla^k\nabla_k h^{ij}\delta h_{ij} - \nabla^j\nabla^i\delta h_{ij}). \quad \text{(A11)}$$

Before taking variational derivatives, we exploit our freedom to integrate by parts to pull the covariant derivatives off the metric variation δh_{ij}:

$$\delta F_V = \int d^3x f^0\left(\frac{1}{2}\mathcal{H}_V h^{ij} + \sqrt{h}R^{ij}\right)\delta h_{ij} + \int d^3x\sqrt{h}(h^{ij}\nabla_k\nabla^k f^0 - \nabla^i\nabla^j f^0)\delta h_{ij}. \quad \text{(A12)}$$

It follows that

$$\frac{\delta F_V}{\delta h_{mn}} = f^0\left(\frac{1}{2}\mathcal{H}_V h^{mn} + \sqrt{h}R^{mn}\right) + \sqrt{h}(h^{mn}\nabla_k\nabla^k f^0 - \nabla^m\nabla^n f^0)$$
$$\frac{\delta F_V}{\delta \pi^{mn}} = 0. \quad \text{(A13)}$$

Similarly,

$$\frac{\delta G_V}{\delta h_{mn}} = g^0\left(\frac{1}{2}\mathcal{H}_V h^{mn} + \sqrt{h}R^{mn}\right) + \sqrt{h}(h^{mn}\nabla_k\nabla^k g^0 - \nabla^m\nabla^n g^0)$$
$$\frac{\delta G_V}{\delta \pi^{mn}} = 0. \quad \text{(A14)}$$

Before computing δF, we integrate by parts inside F:

$$F = 2\int d^3x h_{ij}\pi^{jk}\nabla_k f^i. \quad \text{(A15)}$$

This simplifies the variational calculation:

$$\delta F = 2\int d^3x(\nabla_k f^i)\pi^{jk}\delta h_{ij} + 2\int d^3x(\nabla_k f^i)h_{ij}\delta\pi^{jk} + 2\int d^3x\pi^{jk}h_{ij}\delta\nabla_k f^i. \quad \text{(A16)}$$

To evaluate $\delta\nabla_k f^i$, first expand the covariant derivative as $\nabla_k f^i = \partial_k f^i + \Gamma^i_{ka}f^a$. It follows that $\delta\nabla_k f^i = f^a\delta\Gamma^i_{ka}$. The identity

$$\delta\Gamma^l_{ki} = \tfrac{1}{2}h^{lm}(\nabla_i\delta h_{km} + \nabla_k\delta h_{im} - \nabla_m\delta h_{ik}) \quad \text{(A17)}$$

implies that $2\pi^{jk}h_{ij}\delta\nabla_k f^i = f^i\pi^{jk}\nabla_i\delta h_{jk}$. Substituting this result into the expression for δF and integrating by parts yields

$$\delta F = 2\int d^3x(\nabla_k f^i)\pi^{jk}\delta h_{ij} - \int d^3x\nabla_i(f^i\pi^{jk})\delta h_{jk} + 2\int d^3x(\nabla_k f^i)h_{ij}\delta\pi^{jk}. \quad \text{(A18)}$$

It follows that

$$\frac{\delta F}{\delta h_{mn}} = 2(\nabla_k f^i)\pi^{jk}\delta^{mn}_{ij} - \nabla_i(f^i\pi^{mn})$$
$$\frac{\delta F}{\delta \pi^{mn}} = 2(\nabla_k f^i)h_{ij}\delta^{jk}_{mn}. \tag{A19}$$

Likewise,

$$\frac{\delta G}{\delta h_{mn}} = 2(\nabla_c g^a)\pi^{bc}\delta^{mn}_{ab} - \nabla_a(g^a\pi^{mn})$$
$$\frac{\delta G}{\delta \pi^{mn}} = 2(\nabla_c g^a)h_{ab}\delta^{bc}_{mn}. \tag{A20}$$

We are now in a position to compute the Poisson brackets of interest.

(i) $\{\mathcal{H}_i(x), \mathcal{H}_j(y)\}$

To calculate $\{\mathcal{H}_i(x), \mathcal{H}_j(y)\}$, we will first calculate $\{F, G\}$. Substituting Eqs. (A19) and (A20) into the Poisson bracket yields

$$\{F, G\} = 2\int d^3z h_{ab}\pi^{bc}(\nabla_c f^i)(\nabla_i g^a) - 2\int d^3z h_{ij}\pi^{jk}(\nabla_k g^a)(\nabla_a f^i) - 2\int d^3z(\nabla_c g^a)\nabla_i(f^i h_{ab}\pi^{bc}) + 2\int d^3z(\nabla_k f^i)\nabla_a(g^a h_{ij}\pi^{jk}). \tag{A21}$$

After integrating by parts, applying the identity $(\nabla_i\nabla_j - \nabla_j\nabla_i)u^a = R^a{}_{bij}u^b$, and recalling that $\mathcal{H}_i = -2h_{ij}\nabla_k\pi^{jk}$, this bracket becomes

$$\{F, G\} = \int d^3z(f^i\mathcal{H}_a\nabla_i g^a - g^a\mathcal{H}_i\nabla_a f^i) + 2\int d^3z f^i g^a\pi^{jk}(R_{jika} + R_{jaik}). \tag{A22}$$

It follows from the symmetry ($R_{abcd} = R_{cdab}$) and antisymmetry ($R_{abcd} = -R_{bacd} = -R_{abdc}$) properties of the Riemann tensor that $R_{jaik} = -R_{kija}$. The symmetry property ($\pi^{ij} = \pi^{ji}$) of the momentum tensor then implies that $\pi^{jk}(R_{jika} + R_{jaik}) = 0$, so

$$\{F, G\} = \int d^3z(f^i\mathcal{H}_a\nabla_i g^a - g^a\mathcal{H}_i\nabla_a f^i). \tag{A23}$$

Upon expanding the covariant derivatives, the connection terms cancel, yielding

$$\{F, G\} = \int d^3z(f^i\mathcal{H}_a\partial_i g^a - g^a\mathcal{H}_i\partial_a f^i). \tag{A24}$$

To extract the Poisson brackets $\{\mathcal{H}_i(x), \mathcal{H}_j(y)\}$, first relabel integration variables,

$$\{F, G\} = \int d^3x f^i(x)\mathcal{H}_a(x)\partial_{x^i}g^a(x) - \int d^3y g^a(y)\mathcal{H}_i(y)\partial_{y^a}f^i(y), \tag{A25}$$

then use the identities

$$g^a(x) = \int d^3y\delta^3(x-y)g^a(y),$$
$$f^i(y) = \int d^3x\delta^3(x-y)f^i(x), \tag{A26}$$

to write

$$\{F, G\} = \int d^3x d^3y f^i(x)g^a(y)(\mathcal{H}_a(x)\partial_{x^i}\delta^3(x-y) - \mathcal{H}_i(y)\partial_{y^a}\delta^3(x-y)). \tag{A27}$$

By comparing this expression to (A5), we obtain the identity

$$\{\mathcal{H}_i(x), \mathcal{H}_j(y)\} = \mathcal{H}_j(x)\partial_{x^i}\delta^3(x-y) - \mathcal{H}_i(y)\partial_{y^j}\delta^3(x-y). \tag{A28}$$

(ii) $\{\mathcal{H}_0(x), \mathcal{H}_i(y)\}$

To calculate $\{\mathcal{H}_0(x), \mathcal{H}_i(y)\}$, we will first calculate $\{F_H, G\} = \{F_T, G\} + \{F_V, G\}$. Substituting Eqs. (A13) and (A20) into the bracket $\{F_T, G\}$ yields

$$\{F_T, G\} = \int d^3z f^0\nabla_a(g^a\mathcal{H}_T). \tag{A29}$$

Assembling Eqs. (A13) and (A20) into the bracket $\{F_V, G\}$ yields

$$\{F_V, G\} = \int d^3z f^0(\nabla_a g^a)\mathcal{H}_V + 2\int d^3z\sqrt{h}f^0(\nabla_c g^a)R_a{}^c + 2\int d^3z\sqrt{h}(\nabla_a g^a)\nabla_c\nabla^c f^0 - 2\int d^3z\sqrt{h}(\nabla_c g^a)\nabla_a\nabla^c f^0. \tag{A30}$$

After integrating the last two terms by parts, the identity $(\nabla_a\nabla_c - \nabla_c\nabla_a)g^a = R_{ac}g^a$ implies that

$$\{F_V, G\} = \int d^3z f^0(\nabla_a g^a)\mathcal{H}_V + 2\int d^3z\sqrt{h}R_a{}^c\nabla_c(f^0 g^a). \tag{A31}$$

By integrating the last term by parts, using the identity $2\nabla_c R_a{}^c = \nabla_a R = \nabla_a(R - 2\Lambda)$, and recalling that $\mathcal{H}_V = \sqrt{h}(2\Lambda - R)$, the bracket becomes

$$\{F_V, G\} = \int d^3z f^0 \nabla_a (g^a \mathcal{H}_V). \tag{A32}$$

Combining $\{F_V, G\}$ with $\{F_T, G\}$ and recalling that $\mathcal{H}_0 = \mathcal{H}_T + \mathcal{H}_V$ yields

$$\{F_H, G\} = \int d^3z f^0 \nabla_a (g^a \mathcal{H}_0). \tag{A33}$$

Since g^a is a three-vector and $\mathcal{H}_0/\sqrt{h}$ is a three-scalar,

$$\nabla_a (g^a \mathcal{H}_0) = \partial_a (g^a \mathcal{H}_0), \tag{A34}$$

from which it follows that

$$\{F_H, G\} = \int d^3z f^0 \partial_a (g^a \mathcal{H}_0). \tag{A35}$$

To extract the bracket $\{\mathcal{H}_0(x), \mathcal{H}_i(y)\}$, first relabel the variable of integration,

$$\{F_H, G\} = \int d^3x f^0(x) \partial_{x^a} (g^a(x) \mathcal{H}_0(x)), \tag{A36}$$

then use the identity

$$g^a(x)\mathcal{H}_0(x) = \int d^3y \delta^3(x-y) g^a(y) \mathcal{H}_0(y) \tag{A37}$$

to write

$$\{F_T, G\} = \int d^3x d^3y f^0(x) g^a(y) \mathcal{H}_0(y) \partial_{x^a} \delta^3(x-y). \tag{A38}$$

By comparing this expression to (A5), we obtain the identity

$$\{\mathcal{H}_0(x), \mathcal{H}_i(y)\} = \mathcal{H}_0(y) \partial_{x^i} \delta^3(x-y). \tag{A39}$$

(iii) $\{\mathcal{H}_0(x), \mathcal{H}_0(y)\}$

To calculate $\{\mathcal{H}_0(x), \mathcal{H}_0(y)\}$, we will first calculate

$$\{F_H, G_H\} = \{F_T, G_T\} + \{F_T, G_V\} + \{F_V, G_T\} + \{F_V, G_V\}. \tag{A40}$$

It is straightforward to verify that the brackets $\{F_T, G_T\}$ and $\{F_V, G_V\}$ vanish identically. To compute $\{F_T, G_V\}$, substitute Eqs. (A9) and (A14) into the Poisson bracket to obtain

$$\{F_T, G_V\} = 2\int d^3z f^0 \pi^{mn} \nabla_m \nabla_n g^0 - \int d^3z f^0 g^0 \frac{1}{\sqrt{h}} \left(\frac{1}{2} \mathcal{H}_V \pi^k{}_k + 2\sqrt{h} R_{mn} \pi^{mn} \right). \tag{A41}$$

Likewise,

$$\{F_V, G_T\} = -2\int d^3z g^0 \pi^{mn} \nabla_m \nabla_n f^0 + \int d^3z f^0 g^0 \frac{1}{\sqrt{h}} \left(\frac{1}{2} \mathcal{H}_V \pi^k{}_k + 2\sqrt{h} R_{mn} \pi^{mn} \right). \tag{A42}$$

The sum of the four brackets reduces to

$$\{F_H, G_H\} = 2\int d^3z (f^0 \pi^{mn} \nabla_m \nabla_n g^0 - g^0 \pi^{mn} \nabla_m \nabla_n f^0). \tag{A43}$$

After integrating by parts and recalling that $\mathcal{H}_i = -2h_{ij}\nabla_k \pi^{jk}$, the bracket becomes

$$\{F_H, G_H\} = \int d^3z (f^0 \mathcal{H}^i \nabla_i g^0 - g^0 \mathcal{H}^i \nabla_i f^0). \tag{A44}$$

Upon expanding the covariant derivatives in terms of partial derivatives and connection terms, the connection terms cancel to yield

$$\{F_H, G_H\} = \int d^3z (f^0 \mathcal{H}^i \partial_i g^0 - g^0 \mathcal{H}^i \partial_i f^0). \tag{A45}$$

To extract the bracket $\{\mathcal{H}_0(x), \mathcal{H}_0(y)\}$, relabel integration variables and use the identities

$$g^0(x) = \int d^3y \delta^3(x-y) g^0(y),$$
$$f^0(y) = \int d^3x \delta^3(x-y) f^0(x) \tag{A46}$$

to write

$$\{F_H, G_H\} = \int d^3x d^3y f^0(x) g^0(y) (\mathcal{H}^i(x) \times \partial_{x^i} \delta^3(x-y) - \mathcal{H}^i(y) \partial_{y^i} \delta^3(x-y)). \tag{A47}$$

By comparing this expression to (A5), we obtain the identity

$$\{\mathcal{H}_0(x), \mathcal{H}_0(y)\} = \mathcal{H}^i(x) \partial_{x^i} \delta^3(x-y) - \mathcal{H}^i(y) \partial_{y^i} \delta^3(x-y). \tag{A48}$$

APPENDIX B: CONSTRAINT BRACKETS AFTER IMPOSING $\chi \sim 0$

We begin with the four constraints $\mathcal{H}_\mu$. After introducing the gauge-fixing constraint

$$\chi \equiv \sqrt{h} - \omega(t), \tag{B1}$$

we need to compute the brackets of each of the five constraints (including χ) with χ. We introduce the smoothing functionals

$$F_\chi \equiv \int d^3x f_\chi(x)\chi(x), \qquad G_\chi \equiv \int d^3y g_\chi(y)\chi(y), \quad \text{(B2)}$$

where f_χ and g_χ are arbitrary rapidly-decaying smoothing functions. We then compute the brackets

$$\{F_\chi, G_\chi\} = \int d^3x d^3y f_\chi(x) g_\chi(y)\{\chi(x), \chi(y)\},$$
$$\{F_H, G_\chi\} = \int d^3x d^3y f^0(x) g_\chi(y)\{\mathcal{H}_0(x), \chi(y)\},$$
$$\{F, G_\chi\} = \int d^3x d^3y f^i(x) g_\chi(y)\{\mathcal{H}_i(x), \chi(y)\}. \quad \text{(B3)}$$

The variation δF_χ is

$$\delta F_\chi = \int d^3x f_\chi \frac{1}{2}\sqrt{h}h^{ij}\delta h_{ij}, \quad \text{(B4)}$$

so

$$\frac{\delta F_\chi}{\delta h_{mn}} = f_\chi \frac{1}{2}\sqrt{h}h^{mn}, \qquad \frac{\delta F_\chi}{\delta \pi^{mn}} = 0. \quad \text{(B5)}$$

Likewise,

$$\frac{\delta G_\chi}{\delta h_{mn}} = g_\chi \frac{1}{2}\sqrt{h}h^{mn}, \qquad \frac{\delta G_\chi}{\delta \pi^{mn}} = 0. \quad \text{(B6)}$$

It follows at once that

$$\{F_\chi, G_\chi\} = 0. \quad \text{(B7)}$$

Comparing with (B3), we obtain the identity

$$\{\chi(x), \chi(y)\} = 0. \quad \text{(B8)}$$

We now turn to the brackets of χ with the $\mathcal{H}_\mu$.

(i) $\{\mathcal{H}_0(x), \chi(y)\}$

We split $\{F_H, G_\chi\}$ into $\{F_H, G_\chi\} = \{F_T, G_\chi\} + \{F_V, G_\chi\}$. Assembling Eqs. (A9) and (B5) into the Poisson bracket $\{F_T, G_\chi\}$ yields

$$\{F_T, G_\chi\} = \int d^3z f^0 g_\chi \frac{1}{2}\pi^k{}_k. \quad \text{(B9)}$$

The bracket $\{F_V, G_\chi\}$ vanishes identically, so

$$\{F_H, G_\chi\} = \int d^3x f^0 g_\chi \frac{1}{2}\pi^k{}_k. \quad \text{(B10)}$$

To extract the bracket $\{\mathcal{H}_0(x), \chi(y)\}$, use the identity

$$g_\chi(x) = \int d^3y g_\chi(y)\delta^3(x-y), \quad \text{(B11)}$$

which yields

$$\{F_H, G_\chi\} = \int d^3x d^3y f^0(x) g_\chi(y)\frac{1}{2}\pi^k{}_k(x)\delta^3(x-y). \quad \text{(B12)}$$

Comparing to (B3), we obtain the identity

$$\{\mathcal{H}_0(x), \chi(y)\} = \tfrac{1}{2}\pi^k_k(x)\delta^3(x-y). \quad \text{(B13)}$$

(ii) $\{\mathcal{H}_i(x), \chi(y)\}$

From Eq. (A19), it follows that

$$\{F, G_\chi\} = -\int d^3z g_\chi \sqrt{h}\nabla_i f^i. \quad \text{(B14)}$$

Integrating by parts and using the fact that g_χ is a scalar, this bracket becomes

$$\{F, G_\chi\} = \int d^3x f^i \sqrt{h}\partial_i g_\chi. \quad \text{(B15)}$$

To extract the bracket $\{\mathcal{H}_i(x), \chi(y)\}$, use the identity

$$g_\chi(x) = \int d^3y g_\chi(y)\delta^3(x-y) \quad \text{(B16)}$$

to write

$$\{F, G_\chi\} = \int d^3x d^3y f^i(x) g_\chi(y)\sqrt{h(x)}\partial_{x^i}\delta^3(x-y). \quad \text{(B17)}$$

Comparing to (B3), we obtain the identity

$$\{\mathcal{H}_i(x), \chi(y)\} = \sqrt{h(x)}\partial_{x^i}\delta^3(x-y). \quad \text{(B18)}$$

APPENDIX C: CONFORMAL DECOMPOSITION

Consider a metric $g_{\mu\nu}$ in a number of dimensions d. Denote the determinant of $g_{\mu\nu}$ by g. Define the positive conformal factor

$$\Omega \equiv |g|^{1/d} > 0 \quad \text{(C1)}$$

and the metric

$$\tilde{g}_{\mu\nu} \equiv |g|^{-1/d} g_{\mu\nu} \quad \text{(C2)}$$

so that

$$g_{\mu\nu} = \Omega \tilde{g}_{\mu\nu}. \quad \text{(C3)}$$

By construction, the signature of $\tilde{g}_{\mu\nu}$ is the same as that of $g_{\mu\nu}$. Denote the determinant of $\tilde{g}_{\mu\nu}$ by $\tilde{g}$. From the definition of $\tilde{g}_{\mu\nu}$, it follows that $\tilde{g} = g/|g|$, so $\tilde{g} = \pm 1$, depending on the signature of $g_{\mu\nu}$. We therefore call $\tilde{g}_{\mu\nu}$ a unit-determinant metric.

The inverse metrics are related by $g^{\mu\nu} = \tilde{g}^{\mu\nu}\Omega^{-1}$. We denote the covariant derivative with respect to $g_{\mu\nu}$ by ∇_μ, and the covariant derivative with respect to $\tilde{g}_{\mu\nu}$ by $\tilde{\nabla}_\mu$. The connection $\Gamma^\lambda_{\mu\nu}$ defined by $g_{\mu\nu}$ is

$$\Gamma^\lambda_{\mu\nu} = \tfrac{1}{2}g^{\lambda\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}), \quad \text{(C4)}$$

while the connection $\tilde{\Gamma}^\lambda_{\mu\nu}$ defined by $\tilde{g}_{\mu\nu}$ is

$$\tilde{\Gamma}^{\lambda}_{\mu\nu} = \tfrac{1}{2}\tilde{g}^{\lambda\sigma}(\partial_\mu \tilde{g}_{\nu\sigma} + \partial_\nu \tilde{g}_{\mu\sigma} - \partial_\sigma \tilde{g}_{\mu\nu}). \tag{C5}$$

The connection $\tilde{\Gamma}^{\lambda}_{\mu\nu}$ obeys $\tilde{\Gamma}^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} - C^{\lambda}_{\mu\nu}$, where

$$C^{\lambda}_{\mu\nu} = (\delta^{\lambda\sigma}_{\mu\nu} - \tfrac{1}{2}\tilde{g}^{\lambda\sigma}\tilde{g}_{\mu\nu})\partial_\sigma \log\Omega. \tag{C6}$$

For convenience, we can write Ω in terms of a scalar field φ and a constant Ω_0 as

$$\Omega \equiv \Omega_0 e^{2\varphi}, \tag{C7}$$

in which case

$$C^{\lambda}_{\mu\nu} = \delta^{\lambda}_{\mu}\tilde{\nabla}_\nu\varphi + \delta^{\lambda}_{\nu}\tilde{\nabla}_\mu\varphi - \tilde{g}_{\mu\nu}\tilde{\nabla}^{\lambda}\varphi. \tag{C8}$$

The Riemann tensor of $g_{\mu\nu}$ is

$$R^{\lambda}{}_{\kappa\mu\nu} = \partial_\mu\Gamma^{\lambda}_{\kappa\nu} - \partial_\nu\Gamma^{\lambda}_{\kappa\mu} + \Gamma^{\lambda}_{\mu\sigma}\Gamma^{\sigma}_{\kappa\nu} - \Gamma^{\lambda}_{\nu\sigma}\Gamma^{\sigma}_{\kappa\mu}, \tag{C9}$$

while the Riemann tensor of $\tilde{g}_{\mu\nu}$ is

$$\tilde{R}^{\lambda}{}_{\kappa\mu\nu} = \partial_\mu\tilde{\Gamma}^{\lambda}_{\kappa\nu} - \partial_\nu\tilde{\Gamma}^{\lambda}_{\kappa\mu} + \tilde{\Gamma}^{\lambda}_{\mu\sigma}\tilde{\Gamma}^{\sigma}_{\kappa\nu} - \tilde{\Gamma}^{\lambda}_{\nu\sigma}\tilde{\Gamma}^{\sigma}_{\kappa\mu}. \tag{C10}$$

Using $\Gamma^{\lambda}_{\mu\nu} = \tilde{\Gamma}^{\lambda}_{\mu\nu} + C^{\lambda}_{\mu\nu}$, the Riemann tensor $R^{\lambda}{}_{\kappa\mu\nu}$ can be rewritten as

$$\begin{aligned} R^{\lambda}{}_{\kappa\mu\nu} = {} & \tilde{R}^{\lambda}{}_{\kappa\mu\nu} + C^{\lambda}_{\mu\sigma}C^{\sigma}_{\kappa\nu} - C^{\lambda}_{\nu\sigma}C^{\sigma}_{\kappa\mu} + \partial_\mu C^{\lambda}_{\kappa\nu} \\ & + \tilde{\Gamma}^{\lambda}_{\mu\sigma}C^{\sigma}_{\kappa\nu} - \tilde{\Gamma}^{\sigma}_{\mu\kappa}C^{\lambda}_{\sigma\nu} - \partial_\nu C^{\lambda}_{\kappa\mu} \\ & - \tilde{\Gamma}^{\lambda}_{\nu\sigma}C^{\sigma}_{\kappa\mu} + \tilde{\Gamma}^{\sigma}_{\nu\kappa}C^{\lambda}_{\mu\sigma}. \end{aligned} \tag{C11}$$

Using

$$\begin{aligned} & \tilde{\nabla}_\mu C^{\lambda}_{\kappa\nu} - \tilde{\nabla}_\nu C^{\lambda}_{\kappa\mu} \\ & = \partial_\mu C^{\lambda}_{\kappa\nu} + \tilde{\Gamma}^{\lambda}_{\mu\sigma}C^{\sigma}_{\kappa\nu} - \tilde{\Gamma}^{\sigma}_{\mu\kappa}C^{\lambda}_{\sigma\nu} \\ & \quad - \partial_\nu C^{\lambda}_{\kappa\mu} - \tilde{\Gamma}^{\lambda}_{\nu\sigma}C^{\sigma}_{\kappa\mu} + \tilde{\Gamma}^{\sigma}_{\nu\kappa}C^{\lambda}_{\sigma\mu}, \end{aligned} \tag{C12}$$

$R^{\lambda}{}_{\kappa\mu\nu}$ becomes

$$\begin{aligned} R^{\lambda}{}_{\kappa\mu\nu} = {} & \tilde{R}^{\lambda}{}_{\kappa\mu\nu} + C^{\lambda}_{\mu\sigma}C^{\sigma}_{\kappa\nu} - C^{\lambda}_{\nu\sigma}C^{\sigma}_{\kappa\mu} \\ & + \tilde{\nabla}_\mu C^{\lambda}_{\kappa\nu} - \tilde{\nabla}_\nu C^{\lambda}_{\kappa\mu}. \end{aligned} \tag{C13}$$

The Ricci tensor of $g_{\mu\nu}$ is $R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu}$; the Ricci tensor of $\tilde{g}_{\mu\nu}$ is $\tilde{R}_{\mu\nu} = \tilde{R}^{\lambda}{}_{\mu\lambda\nu}$. Tracing Eq. (C13) appropriately yields

$$R_{\mu\nu} = \tilde{R}_{\mu\nu} + C^{\lambda}_{\lambda\sigma}C^{\sigma}_{\mu\nu} - C^{\lambda}_{\nu\sigma}C^{\sigma}_{\mu\lambda} + \tilde{\nabla}_\lambda C^{\lambda}_{\mu\nu} - \tilde{\nabla}_\nu C^{\lambda}_{\mu\lambda}. \tag{C14}$$

We now express $R_{\mu\nu}$ in terms of $\tilde{R}_{\mu\nu}$ and derivatives of φ. Recalling that $\delta^{\mu}_{\mu} = d$, we find

$$\begin{aligned} & C^{\lambda}_{\lambda\sigma} = d\tilde{\nabla}_\sigma\varphi \\ & C^{\lambda}_{\mu\sigma}C^{\sigma}_{\nu\lambda} = (d+2)(\tilde{\nabla}_\mu\varphi)(\tilde{\nabla}_\nu\varphi) - 2\tilde{g}_{\mu\nu}(\tilde{\nabla}_\alpha\varphi)(\tilde{\nabla}^\alpha\varphi), \end{aligned} \tag{C15}$$

so

$$\begin{aligned} R_{\mu\nu} = {} & \tilde{R}_{\mu\nu} + (d-2)(\tilde{\nabla}_\mu\varphi)(\tilde{\nabla}_\nu\varphi) \\ & - (d-2)\tilde{g}_{\mu\nu}(\tilde{\nabla}_\sigma\varphi)(\tilde{\nabla}^\sigma\varphi) \\ & - (d-2)\tilde{\nabla}_\mu\tilde{\nabla}_\nu\varphi - \tilde{g}_{\mu\nu}\tilde{\nabla}_\sigma\tilde{\nabla}^\sigma\varphi. \end{aligned} \tag{C16}$$

The Ricci scalar for $g_{\mu\nu}$ is $R = g^{\mu\nu}R_{\mu\nu}$; the Ricci scalar for $\tilde{g}_{\mu\nu}$ is $\tilde{R} = \tilde{g}^{\mu\nu}\tilde{R}_{\mu\nu}$. In terms of covariant derivatives of φ, we have

$$\begin{aligned} \Omega R = {} & \tilde{R} - (d-1)(d-2)(\tilde{\nabla}_\alpha\varphi)(\tilde{\nabla}^\alpha\varphi) \\ & - 2(d-1)\tilde{\nabla}^\sigma\tilde{\nabla}_\sigma\varphi. \end{aligned} \tag{C17}$$

In three dimensions, the Weyl tensor vanishes, so the Riemann tensor is completely determined by the Ricci tensor and the metric via

$$\begin{aligned} R_{lkmn} = {} & \frac{1}{d-2}(g_{lm}R_{kn} - g_{ln}R_{km} - g_{km}R_{ln} + g_{kn}R_{lm}) \\ & - \frac{1}{(d-1)(d-2)}(g_{lm}g_{kn} - g_{ln}g_{km})R. \end{aligned} \tag{C18}$$

In this case, it suffices to compute the Ricci tensor. When $d = 3$, our previous formulas reduce to

$$\begin{aligned} R_{ij} = {} & \tilde{R}_{ij} + (\tilde{\nabla}_i\varphi)(\tilde{\nabla}_j\varphi) - \tilde{g}_{ij}(\tilde{\nabla}_k\varphi)(\tilde{\nabla}^k\varphi) \\ & - \tilde{\nabla}_i\tilde{\nabla}_j\varphi - \tilde{h}_{ij}\tilde{\nabla}_k\tilde{\nabla}^k\varphi, \\ \Omega R = {} & \tilde{R} - 2(\tilde{\nabla}_k\varphi)(\tilde{\nabla}^k\varphi) - 4\tilde{\nabla}^k\tilde{\nabla}_k\varphi. \end{aligned} \tag{C19}$$

The condition $\omega \sim \omega(t)$ amounts to $\varphi \sim \varphi(t)$, so in cosmological gauge we have

$$R_{ij} \sim \tilde{R}_{ij}, \qquad \Omega R \sim \tilde{R}. \tag{C20}$$

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