Conformal Smectics and their Many Metrics

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Abstract
We establish that equally spaced smectic configurations enjoy an infinite-dimensional conformal symmetry and show that there is a natural map between them and null hypersurfaces in maximally symmetric spacetimes. By choosing the appropriate conformal factor it is possible to restore additional symmetries of focal structures only found before for smectics on flat substrates.

Disciplines
Physical Sciences and Mathematics | Physics

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Conformal smectics and their many metrics

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(Received 30 November 2011; revised manuscript received 8 February 2012; published 11 May 2012)

We establish that equally spaced smectic configurations enjoy an infinite-dimensional conformal symmetry and show that there is a natural map between them and null hypersurfaces in maximally symmetric spacetimes. By choosing the appropriate conformal factor it is possible to restore additional symmetries of focal structures only found before for smectics on flat substrates.

DOI: 10.1103/PhysRevE.85.050701 PACS number(s): 61.30.Jf, 11.10.Lm, 61.30.Dk

Not only do symmetries characterize and constrain the structure of physical theories, they also allow us to choose convenient frames, coordinates, and variables to analyze and formulate our questions. Symmetries of the ground state manifold are especially interesting, being the deep origin of Nambu-Goldstone modes and the consequential topological defects. Ground states in smectics have broken rotational and translational symmetries which lead to disclinations and dislocations as topological excited states. However, smectics and translational symmetries which lead to disclinations and Nambu-Goldstone modes and the consequential topological manifold are especially interesting, being the deep origin of convenient frames, coordinates, and variables to analyze and structure of physical theories, they also allow us to choose the order and properties of soft materials [3–5].

Textures through an infinite dimensional conformal freedom in curved substrates while retaining the hidden symmetry among focal conic domains, namely, they admit a natural action of the unit normal to the smectic layers. More generally, at long length scales, bending energy is much less costly than compression. This favors configurations where the compression vanishes away from a set of defects, which may be idealized as curves and points. Geometric insight into these textures can be gained by considering the surface S given by the graph of φ inside R^d+1 [11]. When the compression vanishes these are constant angle surfaces [12] (in the Euclidean sense) with the surface normal [13] N and one basis vector of the tangent plane T making an angle of π/4 with the φ direction. A deeper insight is furnished by a component-by-component bijection of (d + 1)-tuples between (Euclidean) R^d+1 and Minkowski spacetime M ≡ R^d,1. Then with respect to the Minkowski metric ds^2 = dφ^2 – dx^2, both N and T are null in R^d,1 and the surface S is a null hypersurface. Since S is null, through every regular point passes a unique light ray (null geodesic) of M that is contained entirely in S. These light rays making up S project to a family of geodesics in R^d whose tangents at every point coincide precisely with the director field n = ∇φ, the unit normal to the smectic layers.

Though this general structure establishes the connection with the more traditional geometric approaches [6–9,14], the spacetime perspective adds the additional insight that different configurations of layers are connected through the symmetries of Minkowski spacetime. Null surfaces in M remain null under Lorentz transformations but space and time, i.e., φ, are mixed. Equal time slices of the same null surface using the new time coordinate φ' yields an equally spaced smectic texture that is distinct from the original one, but related to it through the spacetime transformation. For example (Fig. 1, upper panel), the surface consisting of the pair of light cones with vertices at P_1 = (x_1, φ_1) and P_2 = (x_2, φ_2) has a “focal” curve in M (and a corresponding, experimentally visible, projection onto R^2) where they intersect, which, directly from the definition of conic sections, is a conic. We use the term “focal curve”
null surfaces in a Lorentzian spacetime. However, it is also interesting to observe that we can recover the natural action of the Poincaré group on mixtures of space and time coordinates, revealing once again a hidden symmetry between the smectic textures; see Fig. 1.

Moreover, the conformal freedom reveals a much greater structure, for not only are seemingly distinct smectic textures on $S^2$ related via a spacetime symmetry, they are also related to textures (and symmetries) of smectics in flat space. They are, after all, the same null surface, in the same spacetime, just being viewed by different observers. Similarly, by choosing coordinates $(\phi, \alpha, \beta)$ defined by $t = e^\phi \cosh \alpha, x + iy = e^\phi \sinh \alpha e^{i\beta}$, we cover the interior of the future light cone through the origin in $M$ with a metric conformal to the standard one on $H^2 \times \mathbb{R}$. Thus with this choice of coordinates the same null surfaces in $M$ can be used to describe equally spaced smectic textures on the hyperbolic plane.

Can we go further? Are there other choices of coordinates such that $M$ is conformal to $U \times \mathbb{R}$ for an arbitrary spatial surface $U$? The answer is no: Maximally symmetric spacetimes can only accommodate metrics of the form (1) with maximally symmetric spatial sections [17], i.e., $U$ is either (i) the plane $\mathbb{R}^d$, (ii) the sphere $S^d$, or (iii) the hyperbolic plane $H^d$. Other spatial sections require a less symmetric spacetime. However, it is also interesting to observe that we need not restrict ourselves to Minkowski spacetime as the underlying spacetime: The conformal freedom in (1) allows us both to view smectics on different spatial sections with the same spacetime and to view smectics on the same spatial section with different spacetimes. For instance, smectics on each of the three maximally symmetric spaces $(\mathbb{R}^d, S^d, H^d)$ can be described using an optical metric (1) corresponding to any of the three maximally symmetric spacetimes: Minkowski...
(M), de Sitter (dS), and anti-de Sitter (AdS). This then raises the question: Why use Minkowski? And, is there a reason to choose one representation over the others?

Certainly for smectic textures in flat space, Minkowski spacetime may seem to be the natural choice. But is it still the natural choice for smectics on the sphere? We suggest that here the use of anti-de Sitter has some potential advantages. First, AdS is distinguished among the three by having a periodic time direction and since we wish to associate this to the smectic phase φ—itsself a periodic quantity—the use of AdS may bring certain benefits, especially if we consider smectic textures with dislocations. A second distinguishing feature of AdS is that it is the only homogeneous spacetime in which we can view space as a sphere and where the time coordinate t is associated to a proper Killing vector field, rather than just a conformal one; for the same reason we might choose M and dS for smectics on $\mathbb{R}^2$ and $\mathbb{H}^2$, respectively. Although this is not critical for the equally spaced textures we consider here, it may prove useful when the compression is nonzero. For these reasons we summarize briefly the use of AdS in describing smectic textures on $S^2$. Although a complete classification of all possible textures is challenging, for the same reasons as described in Ref. [2] a very large class of textures, covering all experimentally observed focal conic textures, may be pieced together from the intersections of null planes and light cones. Thus it suffices to describe the generic features of these simple null surfaces, from which any desired texture can be constructed. Note that this construction works in any dimension, but we will focus on the relevant case of two-dimensional smectics.

Recall that AdS$_{d+1}$ can be isometrically embedded in $\mathbb{R}^{d,2}$, with its natural metric, as the hyperboloid

$$t^2_i + t^2_j - x^2 = 1.$$  \hspace{1cm} (3)

The isometries of AdS are then generated by the action of SO(d,2) on this hyperboloid and they descend to a natural action of SO(d,2) on the set of smectic textures on $S^d$. The choice of coordinates $t_1 + i t_2 = \sec \alpha \, e^{i \theta}$, $x_1 + i x_2 = \tan \alpha \, e^{i \theta}$ in $\mathbb{R}^{2,2}$ provides a parametrization of AdS$_1$ such that the induced metric is of the form (1) with $\Omega = \sec \alpha$ and where the spatial metric $d l^2$ is the usual round metric on $S^2$. The coordinates $\phi, \alpha, \beta$ cover the entire AdS$_1$, with $\phi \in [0,2\pi)$, $\alpha \in [0,\pi/2)$, $\beta \in [0,2\pi)$, and thus only cover one hemisphere of the $S^2$, with the conformal factor $\sec \alpha$ taking the equator to infinity in AdS$_1$.

From the point of view of smectics, the advantage of embedding in $\mathbb{R}^{2,2}$ is that null surfaces in AdS$_3$ are also null in $\mathbb{R}^{2,2}$, which provides a convenient means of constructing them. Starting with the action of smectic group states, we can take as a representative null plane in AdS$_3$ the intersection of the hyperboloid (3) with the plane $x_1 = t_1$. In terms of our coordinates $(\alpha, \beta, \phi)$ this is the relation $\sin \alpha \cos \beta = \cos \phi$ and thus equally spaced values of $\phi$ correspond to equally spaced layers $x = \text{const}$ on the standard embedding of the sphere into $\mathbb{R}^3$, $x^2 + y^2 + z^2 = 1$. Now, the plane $x_1 = t_1$ intersects AdS$_3$ in two disjoint pieces, $t_2 \geq 1$ and $t_2 \leq -1$, so that we are really describing two null planes in AdS$_3$. However, as the coordinates $\alpha, \beta$ only cover one hemisphere, this duplication is useful in defining the smectic texture on the entire $S^2$. The two pieces can be mapped to different hemispheres and glued together along the equator, corresponding to infinity in AdS$_3$. Although this process leads to +1 point defects on the equator of the sphere, these defects are nowhere to be found in AdS$_3$ since it does not include the equator. More generally, when gluing together the two hemispheres, there is no reason for us to choose the same orientation of plane waves: For instance, we may take the second plane wave to be given by the intersection of the plane $t_1 = x_2$ with AdS$_3$ ($t_2 \leq -1$), as shown in Fig. 2. When they are different, there are four +1/2 point defects on the equator, but otherwise the layers can be made to join so that the normal is continuous.

Now we consider light cones in AdS$_3$ and their related disclinations in $S^2$. Since AdS$_3$ is maximally symmetric, its geometry looks exactly the same everywhere. A light cone at $\mathcal{P} = (1,0,0)$ gives rise to a point defect at the North pole of $S^2$, with lines of constant latitude as layers. Rotations in $t_1 \land t_2$ and $x_1 \land x_2$ shift $\phi$ and $\beta$, respectively. The former evolve time and the latter just reparametrize the layers. Boosts in $t_2 \land x_1$ and $t_2 \land x_2$ fix $\mathcal{P}$ and thus also just reparametrize the layers. However, a boost in $x_1 \land x_2$ with velocity $v \equiv \sin \psi$ maps the condition $t_1 = 1$ to $t_1 \sec \psi - x_1 \tan \psi = 1$, which is equivalent to $\cos \phi = s \cdot n$, where $n = (\sin \psi, 0, \cos \psi)$ and $s = (\sin \alpha \cos \beta, \sin \alpha \sin \beta, \cos \alpha)$ is an arbitrary point on the sphere. As a result, layers of constant latitude are rotated around the axis $\hat{x}$ by an angle $\psi$. Similarly, a boost in $t_1 \land x_2$ yields a rotation around $\hat{\phi}$. We finally come to the question of focal sets. Consider a point $(s, \phi)$ at the intersection between the future light cone emanating from $(\mathbf{f}_1, \phi_1)$ and the past light cone emanating from $(\mathbf{f}_2, \phi_2)$. Denoting the distance along the sphere by $d$, this means that $d(s, \mathbf{f}_1) = \phi - \phi_1$ and $d(s, \mathbf{f}_2) = \phi_2 - \phi$ so that $d(s, \mathbf{f}_1) + d(s, \mathbf{f}_2) = \phi_2 - \phi_1$, the equation for an ellipse. Similarly, the intersection of two future or two past light cones yields $|d(s, \mathbf{f}_1) - d(s, \mathbf{f}_2)| = |\phi_1 - \phi_2|$, the equation for a hyperbola. Note, however, that the distinction between ellipses and hyperbolas is artificial on the sphere, since a light cone always refocuses after a time $\pi$; the equation for a hyperbola with foci at $\mathbf{f}_1$, $\mathbf{f}_2$ goes into the equation for an ellipse with a foci at $\mathbf{f}_1$, $\mathbf{f}_2$, with $\mathbf{f}_1$ the antipodal point of $\mathbf{f}_1$. In fact, even a parabola on the sphere is an ellipse: The locus of points equidistant from an arbitrary point and a great circle is identical to the locus generated by either an ellipse or hyperbola between the arbitrary point and the conjugate poles of the great circle.

In closing, we note that whenever the focal curve is an ellipse on any space, then simple geometry allows us to see that

FIG. 2. (Color online) Smectic texture on a sphere derived from a pair of null “planes” in AdS$_3$. Left: The two planes have the same orientation. Right: The two planes have orthogonal orientations. See text for details.
the energy of a simple focal domain arises only from the focal curve. Consider the focal set depicted on the right in Fig. 1. The null surface only differs from that on the left along the cusp: Cutting along the rim and flipping the well over results in the null surface for a single disclination. Since no bending or stretching were necessary it follows that the only energy difference between the right and the left is concentrated on the focal curve. Thus, though the elastic energy for a smectic

$$F = \frac{1}{2} \int d^d x (B[(\nabla \phi)^2 - 1]^2 + K(\nabla^2 \phi)^2)$$

(4)

is not invariant under the hidden Poincaré, SO(2, 2), or SO(3, 1) symmetries of the underlying space, the energy of these focal domains transforms locally. The compression energy remains vanishing under such transformations and the bending energy is only different on the focal curve. This geometric transformation extends to all ellipsoidal focal sets in three dimensions, and so forth.

It does not appear, however, that any such simplification occurs for more complex domains, in particular the toric focal conic domains. Though the special conformal transformation of Minkowski spacetime can be used to map a circular focal set to any other conic [18], there exists an immediate obstruction to any sort of energetic comparison because the hyperbolic and parabolic focal sets are not compact. Either they run off to infinity or they end on point disclinations [2]. Once we admit focal curves we must also admit point defects; once we admit point defects we are compelled to set boundary conditions at infinity to conserve topology or, as is the usual case, we can add the point at infinity and study our problem on the sphere. Thus our ability to study smectics on compact surfaces such as $S^d$ becomes necessary to properly formulate these problems.

We note that the conformal freedom of the null hypersurfaces is a sort of pointwise realization of the projective geometry of light cones. Whether we can extend these ideas or exploit the full power of Lie sphere geometry [19] to understand the geometric symmetry of the full energy or the structure of higher genus spaces remains open.

In future work we will study these issues, generalizations to higher dimensions, and to more complex focal structures. In addition, we will explore the use of AdS to study the inclusion of dislocations and smectic textures with nonuniform spacing.

It is a pleasure to acknowledge discussions with D. Beller, B. G. Chen, and M. Trodden and, in particular, an August discussion with G. Jungman. R.A.M. acknowledges financial support from FAPESP. This work was supported in part by NSF Grant No. DMR05-47230.

[13] Bold symbols are always vectors in $\mathbb{R}^d$; $(d+1)$ vectors will be denoted as $\underline{v} = (v, v^{d+1})$. Inner products and lengths are calculated with respect to the ambient metric; when it is unclear by context, we will be explicit. Greek indices are used in $\mathbb{R}^{d-1}$, roman in $\mathbb{R}^d$.
[15] Let $f(x, x) = \frac{dt}{dt}$. Extremizing $\int \Omega ds = \int \Omega f dx$, gives the geodesic equation $0 = \Omega \left( -\frac{1}{2} \partial_x f + \partial_\alpha f \right) + \partial_\beta \Omega f$. Since $f = 0$ for null geodesics, null geodesics are the same in both metrics.
[18] It is amusing to note that in Minkowski space all the focal conic sections can be obtained via the special conformal transformation, $x^\mu \rightarrow (x^\mu - b^\mu x^\lambda)/(1 - 2b_\mu x^\mu + b^2)$, of the simplest circular focal set: Consider the two timelike separated light cones in $\mathbb{R}^{2,1}$ with vertices at $P_1 = (0, 0, 0)$ and $P_2 = (0, 0, 2r)$. These two cones intersect on a circle of radius $t$ at time $t$: $\Sigma = \{t \cos \theta, t \sin \theta, 0\} \in [0, 2\pi)$, which casts a focal circle on the $xy$ plane. Taking $b = (b, b, 0)$, $\Sigma$ becomes $\Sigma' = \{t \cos \theta, t \sin \theta, 1/(1 - 2b \cos \theta)^2 \in [0, 2\pi)\}$ which gives a planar curve in polar coordinates $r(\theta) = t/(1 - 2b \cos \theta)$; the general equation for a conic with eccentricity $2b$ and latus rectum $2t$. Were $b = (0, 0, \beta)$ timelike, the original circle of radius $t$ would transform into a circle of radius $t/(1 - \beta^2)$. 