Conformal Smectics and their Many Metrics

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Abstract
We establish that equally spaced smectic configurations enjoy an infinite-dimensional conformal symmetry and show that there is a natural map between them and null hypersurfaces in maximally symmetric spacetimes. By choosing the appropriate conformal factor it is possible to restore additional symmetries of focal structures only found before for smectics on flat substrates.

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Conformal smectics and their many metrics

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We establish that equally spaced smectic configurations enjoy an infinite-dimensional conformal symmetry and show that there is a natural map between them and null hypersurfaces in maximally symmetric spacetimes. By choosing the appropriate conformal factor it is possible to restore additional symmetries of focal structures only found before for smectics on flat substrates.

Not only do symmetries characterize and constrain the structure of physical theories, they also allow us to choose convenient frames, coordinates, and variables to analyze and formulate our questions. Symmetries of the ground state manifold are especially interesting, being the deep origin of Nambu-Goldstone modes and the consequential topological defects. Ground states in smectics have broken rotational and translational symmetries which lead to disclinations and defects. Ground states in smectics have broken rotational and translational symmetries which lead to disclinations and dislocations as topological excited states. However, smectics are easily identified in the laboratory through the formation of focal conic domains—which defects take the shape of conic sections—which are a hallmark of layer order [1]. In prior work [2] it was found that there exists a hidden symmetry of these focal conic domains, namely, they admit a natural action of the Poincaré group on a Minkowski spacetime that extends the space on which the smectic lives. In this Rapid Communication we show how this formalism extends to describe smectics on curved substrates while retaining the hidden symmetry among textures through an infinite dimensional conformal freedom in the choice of spacetime metric.

Geometry and topology play a prominent role in determining the order and properties of soft materials [3–5]. Textures that would be suppressed by a large energetic cost in flat space can become energetically preferred, or may even be an unavoidable requirement of topology. For instance, smectics on bumpy surfaces show an accumulation of dislocations in regions of positive Gaussian curvature, and the flat space can become energetically preferred, or may even be an unavoidable requirement of topology.

Curved geometries and the formation of cusps [6,7].

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Geometry and topology play a prominent role in determining the order and properties of soft materials [3–5]. Textures that would be suppressed by a large energetic cost in flat space can become energetically preferred, or may even be an unavoidable requirement of topology. For example (Fig. 1, upper panel), the surface consisting of the pair of light cones with vertices at $P_1 = (x_1, \phi_1)$ and $P_2 = (x_2, \phi_2)$ has a “focal” curve in $\mathbb{M}$ (and a corresponding, experimentally visible, projection onto $\mathbb{R}^2$) where they intersect, which, directly from the definition of conic sections, is a conic. We use the term “focal curve”
null surfaces in $\mathbb{S}^2$. The general apparatus that was constructed for relating null surfaces in $\mathbb{M}$ to focal conics in flat space [2], carries over in its entirety to furnish a description of equally spaced smectic textures on the sphere (in any dimension, in fact), simply by switching between the usual Cartesian coordinates $(t,x,y)$ of $\mathbb{M}$ and the Carter-Penrose coordinates $(\phi,\alpha,\beta)$ of the conformal version of the metric (2). Since the spacetime $\mathbb{M}$, we recover the natural action of the Poincaré group that mixes space and time coordinates, revealing once again a hidden symmetry between the smectic textures; see Fig. 1.

Moreover, the conformal freedom reveals a much greater structure, for not only are seemingly distinct smectic textures on $\mathbb{S}^2$ related via a spacetime symmetry, they are also related to textures (and symmetries) of smectics in flat space. They are, after all, the same null surface, in the same spacetime, just being viewed by different observers. Similarly, by choosing coordinates $(\phi,\alpha,\beta)$ defined by $t = e^\theta \cosh \alpha, x + iy = e^\theta \sinh \alpha e^{i\phi}$, we cover the interior of the future light cone through the origin in $\mathbb{M}$ with a metric conformal to the standard one on $\mathbb{H}^2 \times \mathbb{R}$. Thus with this choice of coordinates the same null surfaces in $\mathbb{M}$ can be used to describe equally spaced smectic textures on the hyperbolic plane.

Can we go further? Are there other choices of coordinates such that $\mathbb{M}$ is conformal to $\mathbb{U} \times \mathbb{R}$ for an arbitrary spatial surface $\mathbb{U}$? The answer is no: Maximally symmetric spacetimes can only accommodate metrics of the form (1) with maximally symmetric spatial sections [17], i.e., $\mathbb{U}$ is either (i) the plane $\mathbb{R}^d$, (ii) the sphere $\mathbb{S}^d$, or (iii) the hyperbolic plane $\mathbb{H}^d$. Other spatial sections require a less symmetric spacetime. However, it is also interesting to observe that we need not restrict ourselves to Minkowski spacetime as the underlying spacetime: The conformal freedom in (1) allows us both to view smectics on different spatial sections with the same spacetime and to view smectics on the same spatial section with different spacetimes. For instance, smectics on each of the three maximally symmetric spaces ($\mathbb{R}^d, \mathbb{S}^d, \mathbb{H}^d$) can be described using an optical metric (1) corresponding to any of the three maximally symmetric spacetimes: Minkowski
(dS), de Sitter (dS), and anti-de Sitter (AdS). This then raises
the question: Why use Minkowski? And, is there a reason to
choose one representation over the others?

Certainly for smectic textures in flat space, Minkowski
spacetime may seem to be the natural choice. But is it still
the natural choice for smectics on the sphere? We suggest that
here the use of anti-de Sitter has some potential advantages.
First, AdS is distinguished among the three by having a
periodic time direction and since we wish to associate this
to the smectic phase $\phi$—itself a periodic quantity—the use
of AdS may bring certain benefits, especially if we consider
smectic textures with dislocations. A second distinguishing
feature of AdS is that it is the only homogeneous spacetime
in which we can view space as a sphere and where the time
coordinate $t$ is associated to a proper Killing vector field,
rather than just a conformal one; for the same reason we might
choose $\mathbb{M}$ and dS for smectics on $\mathbb{R}^2$ and $\mathbb{H}^2$, respectively.
Although this is not crucial for the equally spaced textures
we consider here, it may prove useful when the compression
is non-zero. For these reasons we summarize briefly the use
of AdS in describing smectic textures on $S^2$. Although a
complete classification of all possible textures is challenging,
for the same reasons as described in Ref. [2] a very large class
of textures, covering all experimentally observed focal conic
textures, may be pieced together from the intersections of null
planes and light cones. Thus it suffices to describe the generic
features of these simple null surfaces, from which any desired
texture can be constructed. Note that this construction works
in any dimension, but we will focus on the relevant case of
two-dimensional smectics.

Recall that AdS$_3$ can be isometrically embedded in $\mathbb{R}^{d,2}$,
with its natural metric, as the hyperboloid

$$ t^2 + x^2 - x^2 = 1. \tag{3} $$

The isometries of AdS are then generated by the action of
SO(d,2) on this hyperboloid and they descend to a natural
action of SO(d,2) on the set of smectic textures on $S^2$.
The choice of coordinates $t_1 = t_2$ sec $\alpha$ $e^{i\phi}$, $x_1 + i x_2 =
tan \alpha$ $e^{i\theta}$ in $\mathbb{R}^{d,2}$ provides a parametrization of AdS$_3$ such
that the induced metric is of the form (1) with $\Omega = sec \alpha$ and
where the spatial metric $d^2$ is the usual round metric
on $S^2$. The coordinates $\phi, \alpha, \beta$ cover the entire AdS$_3$ with
$\phi \in [0,2\pi)$, $\alpha \in [0,\pi/2)$, $\beta \in [0,2\pi)$, and thus only cover one
hemisphere of the $S^2$, with the conformal factor $sec \alpha$ taking
the equator to infinity in AdS$_3$.

From the point of view of smectics, the advantage of
embedding in $\mathbb{R}^{d,2}$ is that null surfaces in AdS$_3$ are also null
in $\mathbb{R}^{d,2}$, which provides a convenient means of constructing
them. Starting with the action of smectic ground states, we
can take as a representative null plane in AdS$_3$ the intersection
of the hyperboloid (3) with the plane $x_1 = t_1$. In terms of our
coordinates $(\alpha, \beta, \phi)$ this is the relation $sin \alpha$ $cos \beta = cos \phi$ and
thus equally spaced values of $\phi$ correspond to equally spaced
layers $x = const$ on the standard embedding of the sphere into
$\mathbb{R}^3$, $x^2 + y^2 + z^2 = 1$. Now, the plane $x_1 = t_1$ intersects AdS$_3$
in two disjoint pieces, $t_2 \geq 1$ and $t_2 \leq -1$, so that we are
really describing two null planes in AdS$_3$. However, as the
coordinates $\alpha, \beta$ only cover one hemisphere, this duplicity is
useful in defining the smectic texture on the entire $S^2$: The
two pieces can be mapped to different hemispheres and glued
together along the equator, corresponding to infinity in AdS$_3$.
Although this process leads to +1 point defects on the equator of
the sphere, these defects are nowhere to be found in AdS$_3$ since
it does not include the equator. More generally, when gluing together the two hemispheres, there is no reason for us
to choose the same orientation of plane waves: For instance, we
may take the second null plane to be given by the intersection
of the plane $t_1 = x_2$ with AdS$_3$ ($t_2 \leq -1$), as shown in Fig. 2.
When they are different, there are four +1/2 point defects on the
-equator, but otherwise the layers can be made to join so
that the normal is continuous.

Now we consider light cones in AdS$_3$ and their related
disclinations in $S^2$. Since AdS$_3$ is maximally symmetric, its
geometry looks exactly the same everywhere. A light cone at
$P = (1,0,0,0)$ gives rise to a point defect at the North pole
of $S^2$, with lines of constant latitude as layers. Rotations in
$t_1 \cap t_2$ and $x_1 \cap x_2$ shift $\phi$ and $\beta$, respectively. The former
evolve time and the latter just reparametrize the layers. Boosts
in $t_2 \cap x_1$ and $t_2 \cap x_2$ fix $P$ and thus also just reparametrize
the layers. However, a boost in $t_1 \cap x_1$ with velocity $v = \sin \psi$
maps the condition $t_1 = 1$ to $t_1 \sec \psi - x_1 \tan \psi = 1$, which
is equivalent to $\cos \phi = s \cdot n$, where $n = (\sin \psi, 0, \cos \psi)$ and
$s = (\sin \alpha \cos \beta, \sin \alpha \sin \beta, \cos \alpha)$ is an arbitrary point on the
sphere. As a result, layers of constant latitude are rotated around the axis $\hat{f}$ by an angle $\psi$. Similarly, a boost in $t_1 \cap x_2$
yields a rotation around $\hat{f}$.

We finally come to the question of focal sets. Consider a point $(s, \phi)$
at the intersection between the future light cone emanating from $(f_1, \phi_1)$ and the past light cone emanating from $(f_2, \phi_2)$. Denoting the distance along the sphere by $d$, this
means that $d(s, f_1) = \phi - \phi_1$ and $d(s, f_2) = \phi_2 - \phi$ so that
$d(s, f_1) + d(s, f_2) = \phi_2 - \phi_1$, the equation for an ellipse.
Similarly, the intersection of two future or two past light
cones yields $|d(s, f_1) - d(s, f_2)| = |\phi_1 - \phi_2|$, the equation for a
hyperbola. Note, however, that the distinction between ellipses
and hyperbolas is artificial on the sphere, since a light cone
always refocuses after a time $\pi$; the equation for a hyperbola
with foci at $f_1$, $f_2$ goes into the equation for an ellipse with a foci
at $f_1$, $f_2$, with $f_1$ the antipodal point of $f_1$. In fact, even a parabola
on the sphere is an ellipse: The locus of points equidistant
from an arbitrary point and a great circle is identical to the
locus generated by either an ellipse or hyperbola between
the arbitrary point and the conjugate poles of the great circle.

In closing, we note that whenever the focal curve is an
ellipse on any space, then simple geometry allows us to see that

![Figure 2](https://via.placeholder.com/150)
the energy of a simple focal domain arises only from the focal curve. Consider the focal set depicted on the right in Fig. 1. The null surface only differs from that on the left along the cusp: Cutting along the rim and flipping the well over results in the null surface for a single disclination. Since no bending or stretching were necessary it follows that the only energy difference between the right and the left is concentrated on the focal curve. Thus, though the elastic energy for a smectic or stretching were necessary it follows that the only energy is only different on the focal curve. This geometric transformation extends to all ellipsoidal focal sets in three dimensions, and so forth.

It does not appear, however, that any such simplification occurs for more complex domains, in particular the toric focal conic domains. Though the *special conformal transformation* of Minkowski spacetime can be used to map a circular focal set to any other conic [18], there exists an immediate obstruction to any sort of energetic comparison because the hyperbolic and parabolic focal sets are not compact. Either they run off to infinity or they end on point disclinations [2]. Once we admit focal curves we must also admit point defects; once we admit point defects we are compelled to set boundary conditions at infinity to conserve topology or, as is the usual case, we can add the point at infinity and study our problem on the sphere. Thus our ability to study smectics on compact surfaces such as $S^d$ becomes necessary to properly formulate these problems.

We note that the conformal freedom of the null hypersurfaces is a sort of pointwise realization of the projective geometry of light cones. Whether we can extend these ideas or exploit the full power of Lie sphere geometry [19] to understand the geometric symmetry of the full energy or the structure of higher genus spaces remains open.

In future work we will study these issues, generalizations to higher dimensions, and to more complex focal structures. In addition, we will explore the use of AdS to study the inclusion of dislocations and smectic textures with nonuniform spacing.

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[13] Bold symbols are always vectors in $\mathbb{R}^d$; $(d+1)$ vectors will be denoted as $\underline{v} = (v,v^{d+1})$. Inner products and lengths are calculated with respect to the ambient metric; when it is unclear by context, we will be explicit. Greek indices are used in $\mathbb{R}^{d-1}$, roman in $\mathbb{R}^d$.
[15] Let $f(x, t) = \frac{d}{dt}$. Extremizing $\int Ω dx = \int Ω t dt$, gives the geodesic equation $0 = Ω(-\frac{d}{dt}∂_t f + (∂_t f) + Ω t f$. Since $t = 0$ for null geodesics, null geodesics are the same in both metrics.
[18] It is amusing to note that in Minkowski space all the focal conic sections can be obtained via the *special conformal transformation*, $x^n \rightarrow (x^n - b^n x^0^2)/(1 - 2b_n x^n + b^n x^0^2)$, of the simplest circular focal set: Consider the two timelike separated light cones in $\mathbb{R}^{2,1}$ with vertices at $P_1 = (0,0,0)$ and $P_2 = (0,0,2)$. These two cones intersect on a circle of radius $\iota$ at time $\Sigma = \{(t \cos \theta, t \sin \theta, \iota)| \theta \in [0,2\pi]\}$, which casts a focal circle on the $xy$ plane. Taking $b = (0,0,0)$, $\Sigma$ becomes $\Sigma' = \{(t \cos \theta, t \sin \theta, 1/(1 - 2b \cos \theta))| \theta \in [0,2\pi]\}$ which gives a planar curve in polar coordinates $r(\theta) = \iota/(1 - 2b \cos \theta)$, the general equation for a conic with eccentricity $2b\iota$ and *latus rectum* $2\iota$. Were $b = (0,0,1)$ timelike, the original circle of radius $\iota$ would transform into a circle of radius $\iota/(1 - \beta_1)$. 