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Hierarchical Trajectory Refinement for a Class of Nonlinear Systems ⋆

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Abstract

Trajectory generation for nonlinear control systems is an important and difficult problem. In this paper, we provide a constructive method for hierarchical trajectory refinement. The approach is based on the recent notion of ϕ-related control systems. Given a control affine system satisfying certain assumptions, we construct a ϕ-related control system of smaller dimension. Trajectories designed for the smaller, abstracted system are guaranteed, by construction, to be feasible for the original system. Constructive procedures are provided for refining trajectories from the coarser to the more detailed system.

Key words: Trajectory refinement, ϕ-related control systems.

1 Introduction

Research in trajectory generation for classes of nonlinear control systems has resulted in various approaches for nonholonomic systems [MS93] as well as real-time trajectory generation methods [vNM98] for differentially flat systems [FLMR95]. The rapidly growing interest in unmanned aerial vehicles (UAVs) has also emphasized the need to generate aggressive trajectories for individual UAVs ([FDF01,HJ00]) as well as large numbers of autonomous UAVs ([BK04]).

One approach to handle the complexity of trajectory generation for nonlinear systems is the adoption of hierarchical design principles. In this paper we present the fundaments of such hierarchical approach to trajectory generation. The proposed methodology builds upon the notion of ϕ-related systems, which has been introduced in [PLS00]. Given a control system Σ_M with state space M, and a map $\phi : M \to N$, a ϕ-related system is an abstracted control system Σ_N on the smaller state space N, that captures the ϕ-image of all Σ_M trajectories. A construction is provided in [PS02] which given nonlinear model Σ_M and map ϕ, generates the abstracted model Σ_N. Furthermore, given control theoretic properties such as controllability and stabilizability, we can obtain natural conditions on the map ϕ in order for Σ_M and Σ_N to have equivalent properties. These include controllability for linear [PLS00], nonlinear [PS02], and Hamiltonian systems [TP03] and stabilizability of linear systems [PL01].

In this paper we present a constructive solution to following problem: *Given a trajectory of the abstracted model Σ_N, refine this trajectory to a trajectory of the original model Σ_M*. A solution to the above problem provides a hierarchical approach to trajectory generation, since we can transfer trajectory generation problems from Σ_M to Σ_N, solve the trajectory generation problem on the simpler model Σ_N using any existing method, and then refine the trajectory back to Σ_M. The explicit construction of refined trajectories along with conditions guaranteeing its feasibility are the main contributions of this paper.

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The idea of reducing the synthesis of control systems to simpler, lower dimensional systems has appeared in various forms in the literature. For mechanical systems, one such approach is based on the existence of symmetries, which enable the reduction of a given control system to a simpler quotient system [dA89,KM97]. Recently, a different approach has been reported in [BL01,BL04], where kinematic models of mechanical systems (kinematic reductions) generating trajectories refinable to trajectories of the full dynamical model are introduced. In the same spirit, the so-called inclusion principle [SS02] allows us to carry analysis and design of systems to simpler models. Trajectory morphing [HM98] is a homotopy based approach that is, in spirit, hierarchical. The related problem of characterizing regularity of the original system input trajectories from regularity of the map ϕ and the abstracted system input trajectories is discussed in [Gra03].

Backstepping has been a very successful approach for the recursive (or hierarchical) design of stabilizing controllers for nonlinear systems [SJK97] and was a source of inspiration for the results presented in this paper. However, the focus of this paper is trajectory refinement and not controller design. Our results systematically lead to a formal methodology that can be thought of as open-loop backstepping.

A different approach which bears some connections with the proposed approach is flatness [FLMR95]. Flatness can also be used for hierarchical trajectory generation, since curves on the flat output space uniquely define state/input trajectories for the original system. Our approach differs from flatness based approaches in that not every trajectory of the abstraction can be concretized in the original system. In addition, it is also not the case that trajectories of the abstraction *uniquely* define state/input trajectories of the original system as is the case for flat systems. On the other hand, these relaxations enable the refinement of curves in spaces that do not necessarily correspond to a flat output space. Another important difference lies in the constructive nature of the proposed methodology, providing checkable conditions for its use.

The structure of this paper is the following. In Section 2 we introduce some notation, review the notion of ϕ-related control systems and present a construction of such control systems. Section 3 contains constructive solutions for trajectory refinement which constitute the main contribution of the paper. The presented results are then discussed in Section 4, which finalizes the paper.

2 ϕ-Related Control Systems

We will assume familiarity with basic differential geometric objects used in geometric control theory [NvdS95,Isi96]. In particular, we will say that a given object is smooth when it is infinitely differentiable. In this paper all the objects will be assumed smooth unless explicitly stated. Given a map $\phi : M \to N$ between manifolds M and N, we say that ϕ is a submersion when its associated tangent map $T_x\phi$ is surjective for every $x \in M$. We will denote by $[X, Y]$ the Lie bracket between vector fields X and Y and consider both distributions and affine distributions. While a distribution Δ_M on manifold M is a smooth assignment to each $x \in M$ of a vector subspace of T_xM, an affine distribution $\mathcal{A}_M$ is a smooth assignment of an affine subspace of T_xM at each $x \in M$. In this paper all distributions will be assumed to locally have constant rank. This assumption guarantees the existence of a local basis of vector fields $X_M^0, X_M^1, \ldots, X_M^l$ for each $x \in M$ spanning $\mathcal{A}_M(x)$ and $\Delta_M(x)$, that is, $\mathcal{A}_M(x) = X_M^0(x) + \mathrm{span}\{X_M^1(x), \ldots, X_M^l(x)\}$ and $\Delta(x) = \mathrm{span}\{X_M^1(x), \ldots, X_M^l(x)\}$. Furthermore, given two distributions Δ_M^1 and Δ_M^2, we denote by $\Delta_M^1 + \Delta_M^2$ the distribution pointwise defined by the subspace of T_xM formed all the vectors $X = X_1 + X_2$ with $X_1 \in \Delta_M^1(x)$ and $X_2 \in \Delta_M^2(x)$. In the same spirit we will denote by $[X_M, \Delta_M]$ the distribution pointwise defined by the subspace of T_xM formed by all vector fields X such that $X(x) = [X_M, Y](x)$ for some $Y \in \Delta_M$. This notation is extended to $[\Delta_M^1, \Delta_M^2]$ by considering the sum $\sum_{X \in \Delta_M^1} [X, \Delta_M^2]$. A submersion $\phi : M \to N$ defines a distribution on M, denoted by $\ker(T\phi)$ and defined by $\ker(T\phi)(x) = \{X \in T_xM \mid T_x\phi \cdot X = 0\}$. We will also use the notation $\phi^{-1}(y)$ to denote the set of points $\{x \in M \mid \phi(x) = y\}$.

In this paper, we shall consider control systems which are affine in the control inputs.

Definition 2.1 *A control affine system $\Sigma_M = (M, \mathbb{R}^r, F_M)$ consists of manifold M as state space, $\mathbb{R}^r$ as input space, and system map $F_M : M \times \mathbb{R}^r \to TM$ of the form:*

$$F_M(x, \eta) = X_M^0(x) + \sum_{i=1}^{r} X_M^i(x)\eta_i \qquad (2.1)$$

where $X_M^0, X_M^1, \ldots, X_M^r$ are smooth vector fields on M.

A control affine system $\Sigma_M = (M, \mathbb{R}^r, F_M)$ defines an affine distribution on M by:

$$\mathcal{A}_M(x) = X_M^0(x) + \mathrm{span}\{X_M^1(x), \ldots, X_M^r(x)\}$$

We will usually denote by $\Delta_M^1(x)$ the distribution $\mathrm{span}\{X_M^1(x), \ldots, X_M^r(x)\}$ which allows us to write the affine distribution $\mathcal{A}_M$ in the compact form $\mathcal{A}_M = X_M^0 + \Delta_M^1$. Affine distributions are important since many properties of control systems are completely characterized by the induced affine distributions. When working with an affine distribution $\mathcal{A}_M$ defined by the vector fields $X_M^0, X_M^1, \ldots, X_M^r$ we will be implicitly considering control system $(M, \mathbb{R}^r, F_M)$ with system map (2.1).

Trajectories of affine control systems are defined as follows:

Definition 2.2 *Let $\Sigma_M = (M, \mathbb{R}^r, F_M)$ be a control affine system and $I \subseteq \mathbb{R}$ an open interval containing the origin. A smooth curve $\mathbf{x} : I \to M$ is said to be a state trajectory if there exists a (not necessarily smooth) input curve $\eta : I \to \mathbb{R}^r$ satisfying the differential equation*

$$\dot{\mathbf{x}}(t) = F_M(\mathbf{x}(t), \eta(t))$$

for almost all $t \in I$.

With respect to the affine distribution $\mathcal{A}_M$, a trajectory can be defined as a smooth map $\mathbf{x} : I \to M$ satisfying $\dot{\mathbf{x}}(t) \in \mathcal{A}_M(\mathbf{x}(t))$. Trajectories of different models are related by the notion of ϕ-related control systems:

Definition 2.3 (ϕ-related control systems [PLS00]) *Let $\Sigma_M = (M, \mathbb{R}^r, F_M)$ and $\Sigma_N = (N, \mathbb{R}^l, F_N)$ be control affine systems defining affine distributions $\mathcal{A}_M$ and $\mathcal{A}_N$, respectively, and let $\phi : M \to N$ be a smooth map. Control system Σ_N is said to be ϕ-related to control system Σ_M if for every $x \in M$:*

$$T_x\phi\big(\mathcal{A}_M(x)\big) \subseteq \mathcal{A}_N \circ \phi(x) \tag{2.2}$$

In the context of hierarchical trajectory generation we are interested in ϕ-related control systems where Σ_N is lower dimensional than Σ_M, therefore $\dim(M) \geq \dim(N)$. The notion of ϕ-related control systems allows us to relate the trajectories of the two control systems.

Theorem 2.4 ([PLS00]) *Control system Σ_N is ϕ-related to control system Σ_M if and only if for every trajectory $\mathbf{x}$ of Σ_M, $\phi \circ \mathbf{x}$ is a trajectory of Σ_N.*

Even though Σ_N captures the ϕ-image of every trajectory of Σ_M, it may also generate trajectories that are not feasible for the Σ_M model. The goal of this paper is to reverse the direction of the above theorem, and hence refine trajectories of the coarser model Σ_N to trajectories of the more detailed model Σ_M. This frequently occurs when, for example, trajectories of kinematic models must be refined to trajectories of dynamic models. In particular, in this paper, we shall address the following two problems.

Problem 2.5 (Trajectory Refinement I) *Let Σ_N be a control system that is ϕ-related to a control system Σ_M. Given a state trajectory $\mathbf{y}$ of Σ_N corresponding to smooth input trajectory ζ, construct an input trajectory η for Σ_M such that the resulting state trajectory $\mathbf{x}$ satisfies the relation $\phi \circ \mathbf{x} = \mathbf{y}$.*

Problem 2.6 (Trajectory Refinement II) *Let Σ_N be a control system that is ϕ-related to a control system Σ_M. Consider desired initial and final states $x_0, x_F \in M$ for system Σ_M. Given a state trajectory $\mathbf{y}$ of Σ_N satisfying $\mathbf{y}(0) = \phi(x_0)$ and $\mathbf{y}(T) = \phi(x_F)$ for given time $T \in \mathbb{R}^+$, construct an input trajectory η for Σ_M such that the resulting state trajectory $\mathbf{x}$ satisfies $\phi \circ \mathbf{x} = \mathbf{y}$, $\mathbf{x}(0) = x_0$ and $\mathbf{x}(T) = x_F$.*

Even if Σ_N is ϕ-related to Σ_M, Σ_N may generate trajectories that not feasible for Σ_M. Hence, in addition to ϕ-relatedness, additional conditions will be required to solve Problems 2.5 and 2.6. In [PS02] a construction is introduced to obtain ϕ-related affine control systems Σ_N from arbitrary affine control systems Σ_M and submersions $\phi : M \to N$. In this paper we restrict attention to a special class of control systems characterized by the following assumptions which will hold throughout the paper:

A.I The manifold M is diffeomorphic to $N \times \mathbb{R}^k$ via diffeomorphism $\psi = (\phi, \phi^\perp)$ with $\phi : M \to N$, $\phi^\perp : M \to \mathbb{R}^k$ and $k = \dim \ker(T\phi)$;

A.II $[\ker(T\phi), [\ker(T\phi), \mathcal{A}_M]] \subseteq \Delta_M^1 + \ker(T\phi) + [\ker(T\phi), \mathcal{A}_M]$.

The refinement results proposed in this paper rely on identifying some inputs of Σ_N with states of Σ_M. This identification immediately imposes restrictions on manifold M since we are modeling the input space as $\mathbb{R}^r$. Assumption **A.I** captures precisely these restrictions on the state space structure and is always locally satisfied. Globally, topological properties of M may prevent the existence of a map ϕ such that **A.I** holds. Given the identification of M with $N \times \mathbb{R}^k$ we will denote a point in M as x or (y, z) where $y \in N$ and $z \in \mathbb{R}^k$. We will also make frequent use of the standard basis for $\ker(T\phi) \cong \mathbb{R}^k$ defined by the vector fields $\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \ldots, \frac{\partial}{\partial z_k}$. Assumption **A.II** greatly simplifies the relation between state/inputs of Σ_M and state/inputs of Σ_N. In particular, it reduces the construction of ϕ-related control systems given in [PS02] to the sequence of seven steps described in the following construction:

Construction 2.7
Input: *Affine distribution $\mathcal{A}_M$ satisfying Assumptions* **A.I** *and* **A.II** *with respect to surjective submersion $\phi : M \to N$.*

Step 1: $\Delta_M^2(x) := [\ker(T\phi), X_M^0](x)$

Step 2: $\Delta_M^3(x) := [\ker(T\phi), \Delta_M^1](x)$

Step 3: $X_N^0(y) := T_{(y,0)}\phi \cdot X_M^0(y, 0)$

Step 4: $\Delta_N^1(y) := T_{(y,0)}\phi(\Delta_M^1(y, 0))$

Step 5: $\Delta^2_N(y) := T_{(y,0)}\phi(\Delta^2_M(y,0))$

Step 6: $\Delta^3_N(y) := T_{(y,0)}\phi(\Delta^3_M(y,0))$

Step 7: $\mathcal{A}_N := X^0_N + \Delta^1_N + \Delta^2_N + \Delta^3_N$

Output : *Affine distribution* $\mathcal{A}_N$.

The affine distribution $\mathcal{A}_N$ defines control system Σ_N which is ϕ-related to Σ_M. The system map F_N of Σ_N takes the form:

$$F_N(y,(\alpha,\beta,\gamma)) = X^0_N(y) + \sum_{i=1}^{a} X^i_N(y)\alpha_i + \sum_{j=1}^{b} Y^j_N(y)\beta_j + \sum_{i=1,j=1}^{a,b} Z_N(y)^{ij}\gamma_{ij} \qquad (2.3)$$

with vector fields X^i_N, Y^j_N and Z^{ij}_N defined by:

$$X^i_N(y) = T_{(y,0)}\phi \cdot X^i_M(y,0)$$
$$Y^j_N(y) = T_{(y,0)}\phi \cdot [\frac{\partial}{\partial z_j}, X^0_M](y,0)$$
$$Z^{ij}_N(y) = T_{(y,0)}\phi \cdot [\frac{\partial}{\partial z_j}, X^i_M](y,0)$$

Note that vector fields X^i_N, Y^j_N and Z^{ij}_N are not necessarily linearly independent, however the above expression will be very convenient from a notational point of view. We now illustrate the above construction through a simple example. Consider the following control system:

$$\begin{aligned} \dot{x}_1 &= x_1 + x_2^2 x_3 + x_1 u_2 \\ \dot{x}_2 &= x_1 x_2 + x_1^2 + x_3 u_2 \\ \dot{x}_3 &= x_3 x_4 + (x_2^2 + x_1^4) u_1 \\ \dot{x}_4 &= x_1 x_4 x_2^2 + x_2 u_3 \end{aligned} \qquad (2.4)$$

and the surjective submersion:

$$(y_1, y_2) = \phi(x_1, x_2, x_3, x_4) = (x_1, x_2) \qquad (2.5)$$

Control system (2.4) defines the following vector fields:

$$X^0_M = (x_1 + x_2^2 x_3)\frac{\partial}{\partial x_1} + (x_1 x_2 + x_1^2)\frac{\partial}{\partial x_2} + (x_3 x_4)\frac{\partial}{\partial x_3} + (x_1 x_4 x_2^2)\frac{\partial}{\partial x_4}$$
$$X^1_M = (x_2^2 + x_1^3)\frac{\partial}{\partial x_3} \quad X^2_M = x_1\frac{\partial}{\partial x_1} + x_3\frac{\partial}{\partial x_2}$$
$$X^3_M = x_2\frac{\partial}{\partial x_4}$$

and map ϕ induces distribution $\ker(T\phi) = \text{span}\{\frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}\}$. It is not difficult to see that system (2.4) and map (2.5) satisfy Assumptions **A.I** and **A.II** for every $x \in \mathbb{R}^4$ such that $x_2 \neq 0$. We can thus use Construction 2.7 and compute:

$$\Delta^2_M(x) := [\ker(T\phi), X^0_M](x) = \text{span}\{x_2^2\frac{\partial}{\partial x_1} + x_4\frac{\partial}{\partial x_3}, x_3\frac{\partial}{\partial x_3} + x_1 x_2^2\frac{\partial}{\partial x_4}\}$$
$$\Delta^3_M(x) := [\ker(T\phi), \Delta^1_M](x) = \text{span}\{\frac{\partial}{\partial x_2}\}$$
$$X^0_N(y) = T_{(x_1,x_2,0)}\phi \cdot X^0_M(x) = y_1\frac{\partial}{\partial y_1} + (y_1 y_2 + y_1^2)\frac{\partial}{\partial y_2}$$
$$\Delta^1_N(y) = T_{(x_1,x_2,0)}\phi(\Delta^1_M(x)) = \text{span}\{y_1\frac{\partial}{\partial y_1}\}$$
$$\Delta^2_N(y) = T_{(x_1,x_2,0)}\phi(\Delta^2_M(x)) = \text{span}\{y_2^2\frac{\partial}{\partial y_1}\}$$
$$\Delta^3_N(y) = T_{(x_1,x_2,0)}\phi(\Delta^3_M(x)) = \text{span}\{\frac{\partial}{\partial y_2}\}$$

The resulting control system is then given by:

$$\begin{aligned} \dot{y}_1 &= y_1 + y_1\alpha_1 + y_2^2\beta_1 \\ \dot{y}_2 &= y_1 y_2 + y_1^2 + \gamma_{11} \end{aligned} \qquad (2.6)$$

Comparing the first equation in (2.6) with the first equation in (2.4) we see that we can identify α_1 with u_2 and β_1 with x_3. This example illustrates that while some inputs of (2.6) correspond to inputs of (2.4), other inputs can be identified with states of (2.4). However, γ_{11} cannot be identified neither with an input nor with a state of (2.4). The correct interpretation of term γ_{11} is as the product $\beta_1\alpha_1$. This decomposition of inputs as a product of other inputs is in fact critical to enable trajectory refinement as discussed in the next section.

3 Hierarchical trajectory refinement

For general control systems the relationships between state/inputs of the original and abstracted system can be very complex [TP04b]. As these relations are crucial for hierarchical trajectory generation we will focus on a particular class of nonlinear systems more amenable to analysis. This class of systems is characterized by Assumptions **A.I** and **A.II**, that we have already introduced, and also by assumption **A.III**:

A.III: $\ker(T\phi) \subseteq \Delta^1_M$

Assumption **A.III** requires states projected out in the abstraction process to be directly controlled. This will ensure the existence of control inputs to generate the desired refinements. Construction 2.7 guarantees that $T\phi(\mathcal{A}_M) \subseteq \mathcal{A}_N \circ \phi$. However, there are vectors in $\mathcal{A}_N$ which are not the image under $T\phi$ of any vector in $\mathcal{A}_M$. The first step towards refining trajectories is to identify which vectors in $\mathcal{A}_N$ come from vectors in $\mathcal{A}_M$.

Lemma 3.1 *Let Σ_M be an affine control system on M satisfying Assumptions* **A.I**, **A.II** *and* **A.III** *with respect to surjective submersion $\phi : M \to N$ and let Σ_N be the ϕ-related control system obtained by Construction 2.7. Then, for any $x \in M$ the following equality holds:*

$$T_x\phi(\mathcal{A}_M(x)) = \bigcup_{\alpha\in\mathbb{R}^a} F_N\left(\phi(x), (\alpha, \phi^\perp(x), \alpha\phi^\perp(x))\right)$$

Proof: Since M is diffeomorphic to $N\times\mathbb{R}^k$ we shall work on $N \times \mathbb{R}^k$, where ϕ takes the form of a projection map $\pi : N \times \mathbb{R}^k \to N$. Denote by $\mathcal{A}_M^y(z)$ the distribution obtained from $\mathcal{A}_M$ by fixing y, that is $\mathcal{A}_M^y(z) = \mathcal{A}_M(y, z)$. Expanding $T_{(y,z)}\pi(\mathcal{A}_M^y(z))$ in Taylor series around $0 \in \mathbb{R}^k$ we obtain:

$$T_{(y,0)}\pi(\mathcal{A}_M^y(0)) + T_{(y,0)}\pi\Big(\sum_{i=1}^{k}[\frac{\partial}{\partial z_i}, \mathcal{A}_M^y](0)z_i\Big)$$
$$+T_{(y,0)}\pi\Big(\frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{k}[\frac{\partial}{\partial z_i}, [\frac{\partial}{\partial z_j}, \mathcal{A}_M^y(z)]](0)z_iz_j\Big) + \dots$$

We now use the assumption $[\ker(T\phi), [\ker(T\phi), \mathcal{A}_M]] \subseteq \Delta_M^1 + \ker(T\phi) + [\ker(T\phi), \mathcal{A}_M]$ to simplify the series expansion to:

$$T_{(y,z)}\pi(\mathcal{A}_M^y(z)) = T_{(y,0)}\pi(\mathcal{A}_M^y(0)) + T_{(y,0)}\pi\Big(\sum_{i=1}^{k}[\frac{\partial}{\partial z_i}, \mathcal{A}_M^y](0)z_i\Big) \tag{3.1}$$

Expression (3.1) shows that the Taylor series of $T_{(y,z)}\pi(\mathcal{A}_M^y(z))$ is finite which implies that (3.1) is in fact valid not only on a neighborhood of $0 \in \mathbb{R}^k$, but for all $z \in \mathbb{R}^k$. Consider now a vector $X_N = F_N(y, (\alpha, z, \alpha z))$ with $\alpha \in \mathbb{R}^a$. Then, by Construction 2.7, X_N can be written as:

$$X_N = T_{(y,0)}\pi \cdot X_M^0(y, 0) + T_{(y,0)}\pi \cdot \sum_{i=1}^{r} X_M^i(y, 0)\alpha_i$$
$$+T_{(y,0)}\pi \cdot \sum_{j=1}^{k}\Big[\frac{\partial}{\partial z_j}, X_M^0\Big](y, 0)z_j$$
$$+T_{(y,0)}\pi \cdot \sum_{i=1}^{r}\sum_{j=1}^{k}\Big[\frac{\partial}{\partial z_j}, X_M^i\Big](y, 0)\alpha_iz_j$$

$$= T_{(y,z)}\pi\Big(X_M^0(y, 0) + \sum_{i=1}^{r} X_M^i(y, 0)\alpha_i\Big)$$
$$+T_{(y,0)}\pi \cdot \sum_{j=1}^{k}\Big[\frac{\partial}{\partial z_j}, X_M^0 + \sum_{i=1}^{r} X_M^i\alpha_i\Big](y, 0)z_j$$

By noting that $T_{(y,0)}\pi\Big(X_M^0(y, 0)+\sum_{i=1}^{r} X_M^i(y, 0)\alpha_i\Big) \in T_{(y,0)}(\mathcal{A}_M^y(0))$ we immediately see from (3.1) that $X_N \in T_{(y,z)}(\mathcal{A}_M(y, z))$. Consider now a vector $X_M \in \mathcal{A}_M(y, z)$. Then $X_M = X_M^0 + \sum_{i=1}^{r} X_M^i\alpha_i$. From (3.1) we conclude that $T_{(y,z)}\pi \cdot X_M$ equals:

$$T_{(y,0)}\pi\Big(X_M^0(y, 0) + \sum_{i=1}^{r} X_M^i(y, 0)\alpha_i\Big)$$
$$+T_{(y,0)}\pi\Big(\sum_{i=1}^{k}\Big[\frac{\partial}{\partial z_i}, X_M^0 + \sum_{i=1}^{r} X_M^i\alpha_i\Big](y, 0)\Big)z_i$$

which is also given by $F_M(y, (\alpha, z, \alpha z))$. $\square$

The previous Lemma asserts that by imposing the restriction $\gamma = \alpha\beta$ we can lift vectors in $\mathcal{A}_N$ to vectors in $\mathcal{A}_M$. This restriction is in fact sufficient to lift not only vectors but also trajectories as described in the following result.

Theorem 3.2 (Hierarchical Trajectory Refinement) *Let Σ_M be a control affine system satisfying Assumptions* **A.I**, **A.II** *and* **A.III** *with respect to a surjective submersion $\phi : M \to N$ and let Σ_N be the ϕ-related control system obtained by Construction 2.7. Any smooth state trajectory $\mathbf{y}$ of Σ_N corresponding to a smooth input trajectory (α, β, γ) satisfying $\gamma_{ij} = \alpha_i\beta_j$ is refinable to a smooth trajectory $\mathbf{x}$ of Σ_M satisfying $\phi \circ \mathbf{x} = \mathbf{y}$. Furthermore, $\mathbf{x}$ is given by $\psi^{-1} \circ (\mathbf{y}, \beta)$.*

Proof: We will show that $\mathcal{A}_M$ is isomorphic to the dynamic extension of $\mathcal{A}_N^e$ defined on $N \times \mathbb{R}^k$ by the affine distribution $\mathcal{A}_N^e(y, z) = \{X \in T_{(y,z)}(N \times \mathbb{R}^k) \mid T_{(y,z)}\pi \cdot X = F_N(y, (\alpha, z, \alpha z))$ for some $\alpha \in \mathbb{R}^a\}$ where $\pi : N \times \mathbb{R}^k \to N$ is the natural projection on N. This will be done by proving that ψ is an isomorphism between $\mathcal{A}_M$ and $\mathcal{A}_N^e$, that is $T\psi(\mathcal{A}_M) = \mathcal{A}_N^e \circ \psi$. We start with the inclusion $T\psi(\mathcal{A}_M) \subseteq \mathcal{A}_N^e \circ \psi$. Let $X_M \in \mathcal{A}_M(x)$, then from Lemma 3.1 we conclude $T_x\phi \cdot X_M = F_N(\phi(x), (\alpha, \phi^\perp(x), \alpha\phi^\perp(x)))$. Since $\phi(x) = \pi \circ \psi(x)$ we also have $T_{\psi(x)}\pi(T_x\psi \cdot X_M) = F_N(\phi(x), (\alpha, \phi^\perp(x), \alpha\phi^\perp(x)))$. By definition of $\mathcal{A}_N^e$ now follows $T_x\psi \cdot X_M \in \mathcal{A}_N^e \circ \psi(x)$. We now prove the reverse inclusion $\mathcal{A}_N^e \circ \psi \subseteq T\psi(\mathcal{A}_M)$. We need to show that for any $X = (X_1, X_2) \in \mathcal{A}_N^e(y, z)$ there exists a $X_M \in \mathcal{A}_M(x)$ such that $T_x\psi \cdot X_M = X \circ \psi(x)$. By construction of $\mathcal{A}_M^e$, $X \in \mathcal{A}_M^e(y, z)$ implies $T_{(y,z)}\pi \cdot X = X_1 = F_N(y, (\alpha, z, \alpha z))$ for some $\alpha \in \mathbb{R}^a$. Furthermore, from Lemma 3.1 we know that there is a vector $X_M \in \mathcal{A}_M \circ \psi^{-1}(y, z)$ such that $T_x\phi \cdot X = X_1$. We now modify X_M to ensure $T_x\phi^\perp \cdot X_M = X_2$. Consider the vector $X_M + K$ with $K \in \ker(T\phi)(x)$. Since X_M belongs to $\mathcal{A}_M(x)$, then so does $X_M + K$ given the inclusion $\ker(T\phi)(x) \subseteq \Delta_M^1(x)$. Furthermore, $T_x\phi \cdot (X_M + K) = T_x\phi \cdot X_M$ for any $K \in \ker(T\phi)$. We thus conclude that K can always be chosen so as to satisfy $T_x\phi^\perp(X_M + K) = X_2$ since ψ being a diffeomorphism implies that $T_x\psi$ is a linear isomorphism. Hence, the inclusion $\mathcal{A}_N^e \circ \psi \subseteq T\psi(\mathcal{A}_M)$ follows and we conclude that ψ renders $\mathcal{A}_M$ isomorphic to $\mathcal{A}_N^e$.

To finish the proof, it suffices to show that any trajectory of $\mathcal{A}_N$ can be lifted to a trajectory of $\mathcal{A}_N^e$ since $\mathcal{A}_N^e$ is isomorphic to $\mathcal{A}_M$. Diffeomorphism ψ^{-1} can then be used to transform a trajectory $\mathbf{y}^e$ of $\mathcal{A}_N^e$ into a trajectory $\psi^{-1} \circ \mathbf{y}^e$ of $\mathcal{A}_M$ since $\frac{d}{dt}\psi^{-1}(\mathbf{y}^e(t)) = T_{\mathbf{y}^e(t)}\psi^{-1} \cdot \dot{\mathbf{y}}^e(t) \subseteq T_{\mathbf{y}^e(t)}\psi^{-1}(\mathcal{A}_N^e \circ \mathbf{y}^e(t)) \subseteq \mathcal{A}_M \circ \psi^{-1}(\mathbf{y}^e(t))$. Let now $\mathbf{y}$ be a trajectory of $\mathcal{A}_N$ with corresponding smooth input trajectory $(\alpha, \beta, \alpha\beta)$. We claim that $(\mathbf{y}, \beta)$ is a trajectory of $\mathcal{A}_N^e$. To prove the claim we need to show that $(\dot{\mathbf{y}}(t), \dot{\beta}(t)) \in \mathcal{A}_N^e(\mathbf{y}(t), \beta(t))$. By definition of $\mathcal{A}_N^e$, $(\dot{\mathbf{y}}(t), \dot{\beta}(t)) \in \mathcal{A}_N^e(\mathbf{y}(t), \beta(t))$ holds iff $T_{(\mathbf{y}(t),\beta(t))}\pi \cdot (\dot{\mathbf{y}}(t), \dot{\beta}(t)) = F_N(\mathbf{y}(t), (\alpha(t), \beta(t), \alpha(t)\beta(t)))$ which is obviously satisfied. □

Theorem 3.2 can be used to provide a constructive solution to Problem 2.5 as we now illustrate with control system (2.4) and its abstraction (2.6). We first note that (2.4) satisfies Assumptions **A.I**, **A.II** and **A.III** with respect to the map (2.5). Assume now that we have designed a trajectory $\mathbf{y}$ of system (2.6) corresponding to a smooth input trajectory $(\alpha, \beta, \alpha\beta)$. Theorem 3.2 asserts that $(\mathbf{y}, \beta)$ is now the desired refinement of $\mathbf{y}$. However, while $(\mathbf{y}, \beta) \in (\mathbb{R}^3)^I$, trajectories of Σ_M live in $(\mathbb{R}^4)^I$ for some open interval $I \subseteq \mathbb{R}$ containing the origin. This apparent mismatch is resolved by rewriting the equations (2.6) so as to include all β terms as prescribed in (2.3):

$$\dot{y}_1 = y_1 + y_1\alpha_1 + y_2^2\beta_1 + 0\beta_2 \tag{3.2}$$

$$\dot{y}_2 = y_1y_2 + y_1^2 + 0\beta_1 + 0\beta_2 + \gamma_{11} \tag{3.3}$$

Equations (2.3) and (3.3) show that β_2 can be arbitrarily chosen as it appears multiplied by zero and this fact implies non-uniqueness of the refinement of $\mathbf{y}$. To obtain the input trajectory associated with the refinement $(\mathbf{y}, \beta)$, it suffices to solve (2.6) for the inputs upon substitution of $(\mathbf{y}, \beta)$. To make our discussion concrete, consider the following trajectory:

$$(\mathbf{y}_1(t), \mathbf{y}_2(t)) = (t, t), \qquad t \in [1, 2]$$

corresponding to the smooth input trajectory defined by:

$$\alpha_1(t) = \frac{1 - t - \sqrt{1 - 2t + t^3 - 4t^3 + 8t^5}}{2t}$$
$$\beta_1(t) = \frac{1 - t + \sqrt{1 - 2t + t^3 - 4t^3 + 8t^5}}{2t^2}$$
$$\gamma_{11}(t) = \alpha_1(t)\beta_1(t)$$

For simplicity we set $\beta_2 = 0$ and consider the refined trajectory $(\mathbf{y}_1, \mathbf{y}_2, \beta_1, 0)$. It is clear that $\phi(\mathbf{y}_1, \mathbf{y}_2, \beta_1, 0) = (y_1, y_2)$ and since $(\mathbf{y}_1, \mathbf{y}_2, \beta_1, 0)$ is guaranteed to be a trajectory of (2.4), we obtain the corresponding input by solving (2.4) for the inputs:

$$u_1(t) = \frac{\dot{x}_3(t) - x_3(t)x_4(t)}{x_2^2(t) + x_1^3(t)} = \frac{\dot{\beta}_1(t) - \beta_1(t)0}{t^2 + t^3} = \frac{\dot{\beta}_1(t)}{t^2 + t^3}$$
$$u_2(t) = \alpha_1(t)$$
$$u_3(t) = \frac{\dot{x}_4(t) - x_1(t)x_4(t)x_2^2(t)}{x_2(t)} = 0$$

Theorem 3.2 can be extended in two different directions. The first consists in eliminating the restriction $\gamma = \alpha\beta$ by further restricting the class of systems under consideration.

Corollary 3.3 *Let Σ_M be a control affine system satisfying Assumptions* **A.I**, **A.II** *and* **A.III** *with respect to a surjective submersion $\phi : M \to N$ and let Σ_N be the ϕ-related control system obtained by Construction 2.7. If the following inclusion holds:*

$$[\ker(T\phi), \Delta_M^1] \subseteq \ker(T\phi) + \Delta_M^1 + [\ker(T\phi), X_M^0] \tag{3.4}$$

then any *smooth state trajectory* $\mathbf{y}$ *of Σ_N corresponding to a smooth input trajectory is refinable to a smooth trajectory* $\mathbf{x}$ *of Σ_M satisfying $\phi \circ \mathbf{x} = \mathbf{y}$. Furthermore,* $\mathbf{x}$ *is given by $\psi^{-1} \circ (\mathbf{y}, \beta)$.*

Proof: From Construction 2.7 we see that when (3.4) is satisfied, then Δ_N^3 can be taken to be $\{0\}$, in which case the condition $\gamma = \alpha\beta$ is vacuously satisfied. □

The second direction consists in providing a constructive solution to Problem 2.6 by exploiting the equality $\mathbf{x} = \psi^{-1} \circ (\mathbf{y}, \beta)$ provided by Theorem 3.2:

Corollary 3.4 *Let Σ_M be a control affine system satisfying Assumptions* **A.I**, **A.II** *and* **A.III** *with respect to a surjective submersion $\phi : M \to N$ and let Σ_N be the ϕ-related control system obtained by Construction 2.7. Consider any two states x_0 and x_F in M and let* $\mathbf{y}$ *be any smooth state trajectory of Σ_N corresponding to a smooth input trajectory (α, β, γ) satisfying $\gamma_{ij} = \alpha_i\beta_j$, $\psi^{-1}(\mathbf{y}(0), \beta(0)) = x_0$ and $\psi^{-1}(\mathbf{y}(T), \beta(T)) = x_F$ for some $T \in \mathbb{R}^+$. Then, there exists a trajectory* $\mathbf{x}$ *of Σ_M satisfying $\phi \circ \mathbf{x} = \mathbf{y}$, $\mathbf{x}(0) = x_0$ and $\mathbf{x}(T) = x_F$.*

4 Discussion

In this paper we have presented a constructive hierarchical approach for trajectory refinement. The main contribution of this paper bridges a gap between the results reported in [PS02,TP04a,TP04b]. The results reported in [PS02] are restricted to control affine systems. However, projecting affine distribution $\mathcal{A}_M$ through $T\phi$ does not necessarily result in an affine distribution. This problem was addressed in [PS02] by constructing the smallest affine distribution on N containing $T\phi(\mathcal{A}_M)$. The resulting distribution adds new directions of motion to control system Σ_N allowing for trajectories that are not refinable. In a purely nonlinear context [TP04b] such problems do not appear and the relation between state/input trajectories of Σ_M and Σ_N can be clearly stated. The present paper thus provide the missing link between the two approaches by identifying within a control affine ϕ-related control system, which restrictions or which non-affine subsystem, describe refinable trajectories. The results presented in this paper can also be seen as complementary to [TP04a]. In this reference a very strong type of trajectory refinement is considered through the notion of bisimulation which requires a trajectory $\mathbf{y}$ of Σ_N be refinable not to one, but to a family of trajectories $\{\mathbf{x}_x\}_{x\in\phi^{-1}(\mathbf{y}(0))}$ each satisfying $\mathbf{x}_x(0) = x$. Clearly this strong requirement leads to a very special class of systems characterized by the existence of certain controlled invariant distributions. These results can now be obtained from Theorem 3.2 in the case where assumption **A.II** degenerates to $[\ker(T\phi), \mathcal{A}_M] \subseteq \Delta^1_M$.

The presented results also suggest interesting relations with other design approaches described in the literature such as backstepping [SJK97], flatness [FLMR95], and kinematic reductions [BL01]. Such relationships are the subject of current investigations.

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