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SIMULATION RELATIONS FOR DISCRETE-TIME LINEAR SYSTEMS

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Abstract: Simulation relations of labeled transition systems are used in theoretical computer science in order to formally establish notions of modeling abstraction and refinement in hierarchical systems. In this paper, we establish and characterize simulation relations for arbitrary discrete-time, linear control systems. More precisely, given two discrete-time systems, we consider various embeddings into labeled transition systems, that differ in the amount of timing information that is maintained in the transition relation. For each embedding, we obtain necessary and sufficient conditions for one discrete-time system simulating the transitions of the other. Naturally, the simulation characterizations become weaker as more information is abstracted away in the embedding.

Keywords: Transition systems, discrete-time systems, simulation relations, reachability, control invariant subspaces.

1. INTRODUCTION

Theoretical computer science, and, in particular, the areas of concurrency theory (Milner, 1989), and computer aided verification (Manna and Pnueli, 1995) have established formal notions of abstraction and model refinement which are used to tackle the exponential explosion arising in the formal analysis and design of purely discrete systems. In the control community, similar notions have been considered in the hierarchical, supervisory control of discrete event systems (Caines and Wei, 1995; Wong and Wonham, 1995), and hybrid systems (see surveys by Alur *et al.* (2000) and Koutsoukos *et al.* (2000)).

Simulation relations of labeled transition systems is one such formal notion of abstraction (Milner, 1989). Roughly, transition system T_2 simulates transition system T_1, if every transition taken by T_1 can be matched by a similar transition taken by T_2. System T_2 may, however, contain transitions infeasible in T_1 and thus overapproximates the transitions of T_1. Simulation relations are used in order to establish modeling consistency between various levels of hierarchical systems, as transitions of the coarser (higher level) system T_1 can be matched by the more complicated (lower level) system T_2. In the case where T_2 is smaller in size than T_1, simulation relations allow reducing the complexity of T_1, as it is sufficent to analyze T_2.

As mentioned in van der Schaft and Schumacher (2001), simulation relations have escaped the world of purely continuous systems. More recently, a notion of simulation was introduced for continuous-time systems in Pappas *et al.* (2000). Given a continuous system and quotient map, a formal construction was provided for extracting quotient systems that simulated the trajectories of the original system. Furthermore, we characterized linear quotient maps that preserve control theoretic properties such as controllability (Pappas *et al.*, 2000), and stabilizability (Pappas and Lafferriere, 2001). Similar results have also been established for nonlinear systems in Pappas and Simic (2001).

In this paper, we consider for the first time simulation relations for *any* two discrete-time linear systems in the exact sense that the notion is used in theoretical computer science. More precisely, given any two discrete-time linear control systems, we consider a variety of labeled transition systems that are generated by the linear systems. In particular, we consider *timed* and *time-abstract* transition systems generated by linear systems, depending on whether we wish to maintain or ignore timing information as labels on the transitions. In all cases, however, we abstract away control information, in contrast to model reduction results which reduce systems while preserving control input information (Aoki, 1968).

Given two discrete-time systems and the associated embeddings into transition systems, we obtain necessary and sufficient conditions under which one system simulates the transitions of the other. The more information that is abstracted away in the embedding, the weaker the conditions become. In the special case of surjective simulation relations, we obtain as a corollary the discrete-time analogues of the results in Pappas *et al.* (2000).

In Section 2 we briefly review simulation relations for labeled transition systems. In Section 3, we embedd discrete-time linear systems into various transition systems, and in Section 4 we characterize simulation relations for transition systems generated by linear systems. Section 5 contains various special cases of the main results.

2. SIMULATIONS OF LABELED TRANSITION SYSTEMS

In this section we review the standard definitions of simulation relations for transition systems (Milner, 1989).

Definition 2.1. A labeled transition system is a tuple $T = (\mathrm{Q}, \Sigma, \longrightarrow)$ that consists of:

- A (possibly infinite) set Q of states,
- A (possibly infinite) set Σ of labels,
- A transition relation $\longrightarrow \subseteq \mathrm{Q} \times \Sigma \times \mathrm{Q}$,

The transition $(q_1, \sigma, q_2) \in \longrightarrow$ is commonly denoted as $q_1 \xrightarrow{\sigma} q_2$. The transition system is called *finite* is Q and Σ are finite, and *infinite* otherwise. A *region* is a subset $P \subseteq Q$ of the states. The σ-*successor* of a region P is defined as the set that can be reached from P with one σ-transition. More precisely,

$$Post_\sigma(P) = \{q \in Q \mid \exists p \in P \text{ with } p \xrightarrow{\sigma} q\} \quad (1)$$

Simulation relations between transition systems are used to formally define when one transition system implements another.

Definition 2.2. Let $T_1 = (\mathrm{Q}_1, \Sigma, \longrightarrow_1)$ and $T_2 = (\mathrm{Q}_2, \Sigma, \longrightarrow_2)$ be two transition systems over the same label set Σ . The relation $S \subseteq \mathrm{Q}_1 \times \mathrm{Q}_2$ is called a simulation relation if for all $(q_1, q_2) \in S$, the following property holds:

- if $q_1 \xrightarrow{\sigma} q_1'$, then there exists $q_2' \in Q_2$ with $q_2 \xrightarrow{\sigma} q_2'$ and $(q_1', q_2') \in S$

If such a simulation relation exists, then T_2 simulates (or implements) T_1, since every transition taken by T_1 can be matched by T_2. Note that the label set Σ is common to both transition systems. In general T_2 may have many more transitions, and may be a much more complicated system. Transition system T_1 can also serve as a more abstract description of transition system T_2.

In terms of σ-successors, Definition 2.2 can be rephrased as.

Definition 2.3. Transition system T_2 simulates T_1 iff for all $(q_1, q_2) \in S$ and for all $\sigma \in \Sigma$ we have

$$\forall q_1' \in \mathrm{Post}_\sigma(q_1), \exists q_2' \in \mathrm{Post}_\sigma(q_2) : (q_1', q_2') \in S.$$

The goal of this paper is to establish simulation relations for linear systems. In order to achieve this, we must relate the world of linear systems with the world of transition systems.

3. EMBEDDING LINEAR SYSTEMS INTO TRANSITION SYSTEMS

Every discrete-time linear system can be embedded into various labeled transition systems. We consider various embeddings that differ in the amount of timing information retained or abstracted away on the transitions. For each embedding, the semantics (and eventually the characterization) of the simulation relation will be different.

Consider discrete-time, linear control systems:

$$\Delta : \qquad x_{k+1} = Ax_k + Bu_k \quad (2)$$

with time $k \in \mathbb{N}_+$, state $x_k \in \mathbb{R}^n$, control $u_k \in \mathbb{R}^m$, and matrices A, B of appropriate dimension. From linear systems theory (Wonham, 1985), we know that given an initial condition x_0 at time zero, and an input sequence $\{u_i\}_{i=0}^{k-1} = \{u_0, u_1, \ldots, u_{k-1}\}$, then the state x_k at time k is

$$x_k = A^k x_0 + \sum_{i=0}^{k-1} A^{k-i-1} B u_i \quad (3)$$

The first embedding of discrete-time systems into control systems maintains only transitions that happen in one time step, abstracting away the control that was used for the transition.

Definition 3.1. The one-step transition system $T_\Delta^1 = (Q, \Sigma, \longrightarrow)$ generated by Δ consists of:

- State space $Q = \mathbb{R}^n$,
- Unique label $\Sigma = \{1\}$,
- Transition relation $\longrightarrow \subseteq Q \times \{1\} \times Q$ with

$$x \xrightarrow{1} x' \Leftrightarrow \exists u \text{ such that } x' = Ax + Bu$$

The transitions of the one-step transition system naturally correspond to evolution of the discrete-time system in one time step (hence the unique label 1 on the transitions). Furthermore, the transitions of Definition 3.1 are *control abstract* in the sense that the transition system does not care which particular control u is responsible for the transition of the discrete-time system.

There are two natural variations of Definition 3.1. The first variation maintains not only 1-step transitions, but also k-step transitions for any $k \in \mathbb{N}_+$, whereas the second variation does not care how many time steps are needed for a transition.

Definition 3.2. The timed transition system $T_\Delta^{\mathbb{N}_+} = (Q, \Sigma, \longrightarrow)$ generated by Δ consists of:

- State space $Q = \mathbb{R}^n$,
- Label set $\Sigma = \mathbb{N}_+$,
- Transition relation $\longrightarrow \subseteq Q \times \mathbb{N}_+ \times Q$ with

$$x \xrightarrow{k} x' \Longleftrightarrow \exists \{u_i\}_{i=0}^{k-1} \text{ such that } x' = A^k x + \sum_{i=0}^{k-1} A^{k-i-1} B u_i$$

More intuitively, there exists a transition $x \xrightarrow{k} x'$ if there is an appropriate sequence of control inputs $\{u_i\}_{i=0}^{k-1}$ that in *exactly* k time steps will result in the discrete-time system Δ reaching state x' from state x.

Contrary to $T_\Delta^{\mathbb{N}_+}$ which maintains all timing information, the following transition system does not care about the exact number of time steps needed to reach a state. Since it abstracts away timing information, it is called a time-abstract transition system.

Definition 3.3. The time-abstract transition system $T_\Delta^\tau = (Q, \Sigma, \longrightarrow)$ generated by Δ consists of:

- State space $Q = \mathbb{R}^n$,
- Unique label $\Sigma = \{\tau\}$,
- Transition relation $\longrightarrow \subseteq Q \times \{\tau\} \times Q$ with

$$x \xrightarrow{\tau} x' \Longleftrightarrow \exists k \in \mathbb{N}_+ \; \exists \{u_i\}_{i=0}^{k-1} \text{ such that } x' = A^k x + \sum_{i=0}^{k-1} A^{k-i-1} B u_i$$

In other words, a transition $x \xrightarrow{\tau} x'$ occurs if x' is reachable from x in any number of steps by an appropriate sequence of control inputs. Therefore, T_Δ^τ is both time-abstract and control-abstract.

4. SIMULATION CHARACTERIZATIONS

Consider two discrete-time linear systems, Δ_1 and Δ_2 of the form,

$$\Delta_1 \quad x_{k+1} = \mathsf{A}x_k + \mathsf{B}u_k, \quad x \in \mathbb{R}^n, u \in \mathbb{R}^m \tag{4}$$

$$\Delta_2 \quad z_{k+1} = \mathsf{F}z_k + \mathsf{G}v_k, \quad z \in \mathbb{R}^r, v \in \mathbb{R}^s \tag{5}$$

where A,B,F, and G are of appropriate dimension. Both linear systems, Δ_1 and Δ_2 generate the transition systems described in Section 3. We are now interested in determining necessary and sufficient conditions on the discrete time systems under which $T_{\Delta_2}^1$ simulates $T_{\Delta_1}^1$, $T_{\Delta_2}^{\mathbb{N}_+}$ simulates $T_{\Delta_1}^{\mathbb{N}_+}$, and $T_{\Delta_2}^\tau$ simulates $T_{\Delta_1}^\tau$. Clearly, different embeddings will result in different characterizations.

The simulation relations that we shall consider in this paper are of the form $S \subseteq \mathbb{R}^n \times \mathbb{R}^r$ where

$$(x, \mathsf{H}x + y) \in S \subseteq \mathbb{R}^n \times \mathbb{R}^r, \quad y \in \mathrm{Y} \tag{6}$$

where $\mathsf{H} \in \mathbb{R}^{r \times n}$ is an arbitrary linear map, and $\mathrm{Y} \subseteq \mathbb{R}^r$ is a subspace. The relation S can be regarded as a set valued map assigning to each $x \in \mathbb{R}^n$ an affine set $\mathsf{H}x + \mathrm{Y} \subseteq \mathbb{R}^r$.

The structure of the relations (6) considered in this paper captures at least two important cases. In the first case, where $Y = 0$ and the map Hx is surjective, we are interested in simulating the transitions of Δ_1 by a system Δ_2, which should be smaller in size, thus performing complexity reduction. In the second case, where the map Hx is injective and $Y = R(H)^\perp$, we are interested in the more complicated system Δ_2 simulating the transitions of the simpler system Δ_1, thus refining the transitions from the simpler to the more complicated model.

Theorem 4.1. (One-Step Simulation). Consider discrete time systems Δ_1 and Δ_2 given in equations (4, 5), and a relation S of the form (6). Then $T_{\Delta_2}^1$ simulates $T_{\Delta_1}^1$ if and only if

$$\mathscr{R}(\mathsf{HA} - \mathsf{FH}) + \mathscr{R}(\mathsf{HB}) \subseteq \mathscr{R}(\mathsf{G}) + \mathrm{Y} \tag{7}$$

$$F\mathrm{Y} \subseteq \mathrm{Y} + \mathscr{R}(\mathsf{G}) \tag{8}$$

Proof: By definition, Δ_2 one-step simulates Δ_1 with respect to the relation S iff $\forall (x,z) \in S$:

$$\forall x' \in \mathrm{Post}_1(x), \exists z' \in \mathrm{Post}_1(z) : (x', z') \in S.$$

Given the structure of S we have $\forall (x,z) \in S$:

$$\forall x' \in \mathrm{Post}_1(x), \exists y_1 \in \mathrm{Y} : z' = \mathsf{H}x + y_1 \in \mathrm{Post}_1(z).$$

Substituting the Post_1 operators for the one-step transition systems generated by Δ_1 and Δ_2, the above becomes: $\forall (x,z) \in S, \forall u \in \mathbb{R}^m, \exists y_1 \in \mathrm{Y}, \exists v \in \mathbb{R}^s$:

$$\mathsf{F}z + \mathsf{G}v = \mathsf{H}(\mathsf{A}x + \mathsf{B}u) + y_1$$

and since $(x,z) \in S \Leftrightarrow z = \mathsf{H}x + \mathrm{Y}$, then $\forall x \in \mathbb{R}^n$:

$$\mathsf{F}(\mathsf{H} + y_2) + \mathsf{G}v = \mathsf{H}(\mathsf{A}x + \mathsf{B}u) + y_1$$

Collecting terms and eliminating the quantifiers:

$$\mathscr{R}(\mathsf{HA}-\mathsf{FH})+\mathscr{R}(\mathsf{HB})+\mathsf{FY}=\mathscr{R}(\mathsf{G})+\mathrm{Y} \quad (9)$$

which is a necessary and sufficient condition for simulation, derived from the definition. We proceed to show that (7)-(8) are equivalent to (10)

($\Rightarrow$): from condition (8) we get that $\forall y \in \mathrm{Y}, \exists v_1 \in \mathbb{R}^s, \exists y_1 \in \mathrm{Y}: \mathsf{F}y = \mathsf{G}v_1 + y_1$. Similarly, from (7), $\forall x \in \mathbb{R}^n, \forall u \in \mathbb{R}^m, \exists v_2 \in \mathbb{R}^s, \exists y_2 \in \mathrm{Y}: (\mathsf{HA}-\mathsf{FH})x + \mathsf{HB}u = \mathsf{G}v_2 + y_2$. Adding these equalities yields $\forall x \in \mathbb{R}^n, \forall u \in \mathbb{R}^m, \forall y \in \mathrm{Y}, \exists v_1, v_2 \in \mathbb{R}^s, \exists y_1, y_2 \in \mathrm{Y}$:

$$(\mathsf{HA}-\mathsf{FH})x+\mathsf{HB}u+\mathsf{F}y=\mathsf{G}(v_1+v_2)+(y_1+y_2)$$

and since Y is a subspace we can write that $\forall x \in \mathbb{R}^n, \forall u \in \mathbb{R}^m, \forall y_1 \in \mathrm{Y}, \exists v \in \mathbb{R}^s, \exists y_2 \in \mathrm{Y}$:

$$(\mathsf{HA}-\mathsf{FH})x+\mathsf{HB}u+\mathsf{F}y_1=\mathsf{G}v+y_2$$

which by eliminating the quantifiers becomes:

$$\mathscr{R}(\mathsf{HA}-\mathsf{FH})+\mathscr{R}(\mathsf{HB})+\mathsf{FY}\subseteq\mathscr{R}(\mathsf{G})+\mathrm{Y}$$

($\Leftarrow$): From (10), since x, u and y_2 belong to subspaces we can have:

(1) for $y_1 = 0 \Rightarrow \forall x, \forall u, \exists v, \exists y_2 : (\mathsf{HA}-\mathsf{FH})x + \mathsf{HB}u = \mathsf{G}v + y_2 \Rightarrow \mathscr{R}(\mathsf{HA}-\mathsf{FH}) + \mathscr{R}(\mathsf{HB}) \subseteq \mathscr{R}(\mathsf{G}) + \mathrm{Y}$,
(2) for $x = 0, u = 0 \Rightarrow \forall y_1, \exists v, \exists y_2 : \mathsf{F}y_1 = \mathsf{G}v + y_2 \Rightarrow \mathsf{FY} \subseteq \mathscr{R}(\mathsf{G}) + \mathrm{Y}$.

□

Notice that condition (7) relates the system dynamics and control of the two systems in question, whereas condition (8) is a condition only on system Δ_2 and the simulation relation S. This is because the dynamics on the Y subspace have meaning only in the the Δ_2 system which may contain more states that Δ_1.

Since the transitions of $T^1_{\Delta_1}$ generate the transitions of $T^{\mathbb{N}_+}_{\Delta_1}$, it should come as no suprise that simulating 1-step transitions is necessary and sufficient to simulate k-step transitions.

Theorem 4.2. (Timed Simulation). Consider the discrete time systems Δ_1 and Δ_2 given in equations (4, 5), and a relation S of the form (6). Then $T^{\mathbb{N}_+}_{\Delta_2}$ simulates $T^{\mathbb{N}_+}_{\Delta_1}$ iff conditions (7)-(8) are satisfied:

$$\mathscr{R}(\mathsf{HA}-\mathsf{FH})+\mathscr{R}(\mathsf{HB})\subseteq\mathscr{R}(\mathsf{G})+\mathrm{Y}$$
$$F\mathrm{Y}\subseteq\mathrm{Y}+\mathscr{R}(\mathsf{G})$$

Proof: By definition, the necessary and sufficient condition for Δ_2 to k-step simulate Δ_1 with respect to the relation S is $\forall (x,z) \in S, \forall x' \in \mathrm{Post}_k(x), \exists z' \in \mathrm{Post}_k(z) : (x', z') \in S$. After some manipulation and substitution of the Post_k operators, this condition expands to:

$$\mathscr{R}(\mathsf{HA}^k-\mathsf{F}^k\mathsf{H})+\sum_{i=0}^{k-1}\mathscr{R}(\mathsf{HA}^i\mathsf{B})+\mathsf{F}^k\mathrm{Y}=\sum_{i=0}^{k-1}\mathscr{R}(\mathsf{F}^k\mathsf{G})+\mathrm{Y} \quad (10)$$

We show that (7)-(8) are equivalent to (10):

($\Rightarrow$) : The fact that (7)-(8) imply (10) can be shown by induction: if it holds for $p = k$ then

$$\mathsf{F}^{p+1}\mathsf{S}\subseteq\mathsf{F}(\sum_{i=0}^{p-1}\mathscr{R}(\mathsf{F}^i\mathsf{G})+\mathrm{Y})\subseteq\sum_{i=0}^{p}\mathscr{R}(\mathsf{F}^i\mathsf{G})+\mathrm{Y} \quad (11)$$

On the other hand, rewrite $\sum_{i=0}^{p}\mathscr{R}(\mathsf{HA}^i\mathsf{B})$ as $\sum_{i=0}^{p-1}\mathscr{R}(\mathsf{HA}^i\mathsf{B})+\mathscr{R}(\mathsf{HA}^p\mathsf{B}-\mathsf{F}^p\mathsf{HB}+\mathsf{F}^p\mathsf{HB})$ and $\mathscr{R}(\mathsf{HA}^{p+1}-\mathsf{F}^{p+1}\mathsf{H})$ as $\mathscr{R}(\mathsf{HA}^{p+1}-\mathsf{F}^{p+1}\mathsf{H}+\mathsf{F}^p\mathsf{HA}-\mathsf{F}^p\mathsf{HA})$. Then, by applying the conditions assumed true for $k = p$ and (11), one arrives at:

$$\mathscr{R}(\mathsf{HA}^{p+1}-\mathsf{F}^{p+1}\mathsf{H})+\sum_{i=0}^{p}\mathscr{R}(\mathsf{HA}^i\mathsf{B})\subseteq\sum_{i=0}^{p}\mathscr{R}(\mathsf{F}^i\mathsf{G})+\mathrm{Y} \quad (12)$$

By induction, (11)-(12) hold for every k and therefore imply (10).

($\Leftarrow$): From (10) if follows :$\forall x \in \mathbb{R}^n, \forall y_2 \in \mathrm{Y}, \forall \{u_i\}_{i=0}^{k-1} \in \mathbb{R}^m$, $\exists y_1 \in \mathrm{Y}, \exists \{v_i\}_{i=0}^{k-1} \in \mathbb{R}^s$: $(\mathsf{HA}^k - \mathsf{F}^k\mathsf{H})x + \sum_{i=0}^{k-1} \mathsf{HA}^i\mathsf{B}u_i + \mathsf{F}^k y_2 = \sum_{i=0}^{k-1} \mathsf{F}^i\mathsf{G}v_i + y_1$. Since x, u_i and y_2 belong to subspaces we can have:

(1) $y_2 = 0 \Rightarrow \forall x, \forall \{u_i\}_{i=0}^{k-1}$, $\exists \{v_i\}_{i=0}^{k-1}, \exists y_1$: $(\mathsf{HA}^k - \mathsf{F}^k\mathsf{H})x + \sum_{i=0}^{k-1} \mathsf{HA}^i\mathsf{B}u_i = \sum_{i=0}^{k-1} \mathsf{F}^i\mathsf{G}v_i + y_1 \Rightarrow \sum_{i=0}^{k-1} \mathscr{R}(\mathsf{HA}^k - \mathsf{F}^k\mathsf{H}) + \sum_{i=0}^{k-1} \mathscr{R}(\mathsf{HA}^i\mathsf{B}) \subseteq \sum_{i=0}^{k-1} \mathscr{R}(\mathsf{F}^i\mathsf{G}) + \mathrm{Y}$,
(2) $x = u_i = 0 \Rightarrow \forall y_2, \exists \{v_i\}_{i=0}^{k-1}, \exists y_1$: $\mathsf{F}^k y_2 = \sum_{i=0}^{k-1} \mathsf{F}^i\mathsf{G}v + y_1 \Rightarrow \mathsf{F}^k\mathrm{Y} \subseteq \sum_{i=0}^{k-1} \mathscr{R}(\mathsf{F}^i\mathsf{G}) + \mathrm{Y}$.

and since this is true for all k, it should also hold for $k = 1$ in which case we have:

$$\mathscr{R}(\mathsf{HA}-\mathsf{FH})+\mathscr{R}(\mathsf{HB})\subseteq\mathscr{R}(\mathsf{G})+\mathrm{Y}$$
$$\mathsf{FY}\subseteq\mathscr{R}(\mathsf{G})+\mathrm{Y}$$

□

In order to obtain the simulation characterizations for the time abstract transition systems, we shall need the following lemmas. By $R(A,B)$, we denote the reachable subspace from the origin, that is $R(A,B) = Im[\,B\,AB\,\ldots\,A^{n-1}B]$.

Lemma 4.3. Let $\mathsf{A} \in \mathbb{R}^{n\times n}$, $\mathsf{H} \in \mathbb{R}^{s\times n}$, $\mathsf{F} \in \mathbb{R}^{r\times r}$, $\mathsf{G} \in \mathbb{R}^{r\times s}$ be matrices and G full rank with $s \le r$. If, $\mathscr{R}(\mathsf{HA}-\mathsf{FH}) \subseteq R(\mathsf{F},\mathsf{G})$, then for every k, it holds $\mathscr{R}(\mathsf{HA}^k - \mathsf{F}^k\mathsf{H}) \subseteq R(\mathsf{F},\mathsf{G})$

Proof: It is shown by induction: clearly it holds for $k = 1$. Assume it holds for $k = p$: $\exists p, \mathscr{R}(\mathsf{HA}^p - \mathsf{F}^p\mathsf{H}) \subseteq R(\mathsf{F},\mathsf{G})$. Then since $\mathsf{HA}^{p+1} - \mathsf{F}^{p+1}\mathsf{H} = \mathsf{HA}^{p+1} - \mathsf{F}^{p+1}\mathsf{H} + \mathsf{F}^p\mathsf{HA} - \mathsf{F}^p\mathsf{HA} = (\mathsf{HA}^p - \mathsf{F}^p\mathsf{H})\mathsf{A} + \mathsf{F}^p(\mathsf{HA} - \mathsf{FH})$, it will be that $\mathscr{R}(\mathsf{HA}^{p+1} - \mathsf{F}^{p+1}\mathsf{H}) \subseteq \mathscr{R}((\mathsf{HA}^p - \mathsf{F}^p\mathsf{H})\mathsf{A}) + \mathscr{R}(\mathsf{F}^p(\mathsf{HA} - \mathsf{FH})) \subseteq R(\mathsf{F},\mathsf{G}) + \mathsf{F}^p R(\mathsf{F},\mathsf{G})$.

However, $R(\mathsf{F},\mathsf{G})$ is $\mathsf{F}-$ invariant and therefore, $\mathsf{F}^p R(\mathsf{F},\mathsf{G}) \in R(\mathsf{F},\mathsf{G})$. Thus, $\mathscr{R}(\mathsf{HA}^{p+1} - \mathsf{F}^{p+1}\mathsf{H}) \subseteq R(\mathsf{F},\mathsf{G})$ and therefore by induction, $\forall k, \mathscr{R}(\mathsf{HA}^k - \mathsf{F}^k\mathsf{H}) \subseteq R(\mathsf{F},\mathsf{G})$

□

Lemma 4.4. Let Y be a subspace and $\mathsf{F} \in \mathbb{R}^{r\times r}$, $\mathsf{G} \in \mathbb{R}^{r\times s}$ be matrices and G full rank with $s \le r$. If $\mathsf{FY} \subseteq R(\mathsf{F},\mathsf{G})$, then for every k, it holds $\mathsf{F}^k\mathrm{Y} \subseteq R(\mathsf{F},\mathsf{G})$

Proof: It is shown by induction: obviously, it is true for $k = 1$. Suppose there exists a p for which: $\mathsf{F}^p\mathrm{Y} \subseteq R(\mathsf{F},\mathsf{G})$. Then for $k = p+1$, $\mathsf{F}^{p+1}\mathrm{Y} = \mathsf{F}\mathsf{F}^p\mathrm{Y} \subseteq \mathsf{F}R(\mathsf{F},\mathsf{G})$, and since $R(\mathsf{F},\mathsf{G})$ is $\mathsf{F}-$ invariant, $\mathsf{F}^{p+1}\mathrm{Y} \subseteq R(\mathsf{F},\mathsf{G})$. By induction we conclude that $\forall k \in \mathbb{N}_+, \mathsf{F}^k\mathrm{Y} \subseteq R(\mathsf{F},\mathsf{G})$.

□

Theorem 4.5. (Time-Abstract Simulation). Consider the discrete time systems Δ_1 and Δ_2 given in equations (4, 5), and consider a relation S of the form (6). Then $T^{\tau}_{\Delta_2}$ simulates $T^{\tau}_{\Delta_1}$ if and only if

$$\mathscr{R}(\mathsf{HA}-\mathsf{FH}) + \mathsf{H}R(\mathsf{A},\mathsf{B}) \subseteq R(\mathsf{F},\mathsf{G}) + \mathrm{Y} \quad (13)$$

$$\mathsf{FY} \subseteq \mathrm{Y} + R(\mathsf{F},\mathsf{G}). \quad (14)$$

Proof: By definition, for Δ_2 to simulate Δ_1 in abstract time, with respect to S it should hold that for every $(x,z) \in S \Rightarrow \forall x' \in \mathrm{Post}(x), \exists z' \in \mathrm{Post}(z) : (x',z') \in S$ Taking to account the structure of S and the expression for the Post operators: $\forall x \in \mathbb{R}^n$, $\mathsf{H}\bigcup_{k\ge 0}\mathsf{A}^k x + \mathsf{H}R(\mathsf{A},\mathsf{B}) + \bigcup_{k\ge 0}\mathsf{F}^k(\mathsf{H}x+\mathrm{Y}) \subseteq R(\mathsf{F},\mathsf{G}) + \mathrm{Y}$. Collecting terms and eliminating the quantifiers, $\bigcup_{k\ge 0}\mathscr{R}(\mathsf{HA}^k - \mathsf{F}^k\mathsf{H}) + \mathsf{H}R(\mathsf{A},\mathsf{B}) + \bigcup_{k\ge 0}\mathsf{F}^k\mathrm{Y} \subseteq R(\mathsf{F},\mathsf{G}) + \mathrm{Y}$. By lemmas 4.3 and 4.4 this reduces to:

$$\mathscr{R}(\mathsf{HA}-\mathsf{FH}) + \mathsf{H}R(\mathsf{A},\mathsf{B}) + \mathsf{FY} \subseteq R(\mathsf{F},\mathsf{G}) + \mathrm{Y} \quad (15)$$

Equation (15) is necessary and sufficient for Δ_2 to simulate Δ_1 in abstract time. It remains to establish the equivalence of (13)-(14) to (15):

($\Rightarrow$): from condition (14) we get that $\forall y \in \mathrm{Y}, \exists v_1 \in \mathbb{R}^s, \exists y_1 \in \mathrm{Y}$: $\mathsf{F}y = \mathsf{G}v_1 + y_1$. Similarly, from (13), $\forall x \in \mathbb{R}^n, \forall w \in R(\mathsf{A},\mathsf{B}), \exists v_2 \in \mathbb{R}^s, \exists y_2 \in \mathrm{Y}$: $(\mathsf{HA}-\mathsf{FH})x + \mathsf{HB}w = \mathsf{G}v_2 + y_2$. Adding these equalities yields $\forall x \in \mathbb{R}^n, \forall w \in R(\mathsf{A},\mathsf{B}), \forall y \in \mathrm{Y}, \exists v_1, v_2 \in \mathbb{R}^s, \exists y_1, y_2 \in \mathrm{Y}$:

$$(\mathsf{HA}-\mathsf{FH})x + \mathsf{H}w + \mathsf{F}y = \mathsf{G}(v_1+v_2) + (y_1+y_2)$$

and since Y is a subspace we have $\forall x \in \mathbb{R}^n, \forall w \in R(\mathsf{A},\mathsf{B}), \forall y_1 \in \mathrm{Y}, \exists v \in \mathbb{R}^s, \exists y_2 \in \mathrm{Y}$:

$$(\mathsf{HA}-\mathsf{FH})x + \mathsf{H}w + \mathsf{F}y_1 = \mathsf{G}v + y_2$$

which by eliminating the quantifiers becomes:

$$\mathscr{R}(\mathsf{HA}-\mathsf{FH}) + \mathsf{H}R(\mathsf{A},\mathsf{B}) + \mathsf{FY} \subseteq \mathscr{R}(\mathsf{G}) + \mathrm{Y}$$

($\Leftarrow$): From (15), since $x \in \mathbb{R}^n$, $w \in R(\mathsf{A},\mathsf{B})$ and $y_2 \in \mathrm{Y}$ belong to subspaces we have:

(1) for $y_1 = 0 \Rightarrow \forall x, \forall w \in R(\mathsf{A},\mathsf{B}), \exists v, \exists y_2 : (\mathsf{HA}-\mathsf{FH})x + \mathsf{H}w = \mathsf{G}v + y_2 \Rightarrow \mathscr{R}(\mathsf{HA}-\mathsf{FH}) + \mathsf{H}R(\mathsf{A},\mathsf{B}) \subseteq \mathscr{R}(\mathsf{G}) + \mathrm{Y}$,
(2) for $x = 0, w = 0 \Rightarrow \forall y_1, \exists v, \exists y_2 : \mathsf{F}y_1 = \mathsf{G}v + y_2 \Rightarrow \mathsf{FY} \subseteq \mathscr{R}(\mathsf{G}) + \mathrm{Y}$.

□

It is straightfoward that the simulation conditions (13,14) in the time-abstract case are clearly weaker that their counterparts (7, 8) in the timed-simulation case. Therefore, as timing information is abstracted away from the semantics of the transition systems, the simulation conditions become weaker.

5. SPECIAL CASES

Simulation relations induce partial order relations among transition systems. In the case where transition system T_2 simulates transition system T_1 with simulation relation S and, in addition, T_1 simulates T_2 with relation S^{-1}, then T_1 and T_2 are called bisimilar.

Consider now two discrete-time systems and a relation S of the form

$$(x, Hx) \in S \subseteq \mathbb{R}^n \times \mathbb{R}^r$$

where $\mathsf{H}x$ is a surjective map (that is $r < n$). The map $z = \mathsf{H}x$ is therefore grouping states in $\mathbb{R}^n$ to a single state in $\mathbb{R}^r$, thus inducing a partition of $\mathbb{R}^n$. This partition aims at complexity reduction from the original system Δ_1 to the quotient system Δ_2.

The results of the former section can be restated for this special case in the following corollaries.

Corollary 5.1. Transition system $T^1_{\Delta_2}$ ($T^{\mathbb{N}_+}_{\Delta_2}$) simulates $T^1_{\Delta_1}$ (respectively $T^{\mathbb{N}_+}_{\Delta_1}$) with simulation relation $S = \{(x,z) \in \mathbb{R}^n \times \mathbb{R}^r | z = \mathsf{H}x\}$ if and only if

$$\mathscr{R}(\mathsf{HA}-\mathsf{FH}) + \mathscr{R}(\mathsf{HB}) \subseteq \mathscr{R}(\mathsf{G}) \quad (16)$$

Condition (16) is the discrete-time analogue of the notion of H-related control systems, first introduced in Pappas *et al.* (2000). Given system Δ_1 with matrices A, B, and surjective map $z = Hx$, we can always construct system Δ_2 and matrices F, G, so that condition (16) is satisfied. Since H is full-row rank, one such choice is

$$F = HAH^+ \quad (17)$$

$$G = [HB \quad HAKer(H)] \quad (18)$$

With the above choice of matrices, transition system $T^1_{\Delta_2}$ simulates $T^1_{\Delta_1}$. In order for $T^1_{\Delta_2}$ to be bisimilar to $T^1_{\Delta_2}$, $T^1_{\Delta_2}$ must also simulate $T^1_{\Delta_1}$ with relation S^{-1}, which has the form

$$(z, H^+z + \mathrm{Ker}(H)) \in S^{-1} \subseteq \mathbb{R}^r \times \mathbb{R}^n$$

Therefore, in this case, if we apply conditions (7, 8) with $Y = Ker(H)$ (and the roles of A,B,H interchanged

with those of F,G,H^+) we obtain that in order for $T^1_{\Delta_2}$ to be bisimilar to $T^1_{\Delta_1}$ it is necessary that

$$\mathsf{A}\mathrm{Ker}(H) \subseteq Ker(H) + \mathscr{R}(\mathsf{B}) \tag{19}$$

In straightforward to show that for the choice of matrices (17,18) this condition is the only relevant necessary and sufficient condition, as (7) is automatically satisfied.

In the time abstract case the relation naturally loosens up since timing information is irrelavant and simulation actually involves aligning the reachable sets of the systems in question.

Corollary 5.2. Transition system $T^\tau_{\Delta_2}$ simulates $T^\tau_{\Delta_1}$ with simulation relation $S = \{(x,z) \in \mathbb{R}^n \times \mathbb{R}^r | z = \mathsf{H}x\}$ if and only if

$$\mathscr{R}(\mathsf{HA} - \mathsf{FH}) + \mathsf{H}R(\mathsf{A},\mathsf{B}) \subseteq R(\mathsf{F},\mathsf{G}).$$

6. CONCLUSIONS

In this paper we characterized simulation relations for various transition systems generated by discrete-time linear systems. The characterizations can be extended to continuous-time systems, as well as to nonlinear and hybrid systems for various transition system semantics.

Of particular interest is to consider such questions for discrete-time systems with state and input constraints. This will allow the not only establishing consistency between different models, but also between their respective constraint sets.

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