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Postprint version. Published in *Systems and Control Letters*, Volume 52, Issue 1, May 2004, pages 49-58.
Publisher URL: http://dx.doi.org/10.1016/j.sysconle.2003.09.013

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Comments
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Bisimilar Control Affine Systems [1]

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Abstract

The notion of bisimulation plays a very important role in theoretical computer science where it provides several notions of equivalence between models of computation. These equivalences are in turn used to simplify verification and synthesis for these models as well as to enable compositional reasoning. In systems theory, a similar notion is also of interest in order to develop modular verification and design tools for purely continuous or hybrid control systems. In this paper, we introduce two notions of bisimulation for nonlinear systems. We present differential-algebraic characterizations of these notions and show that bisimilar systems of different dimensions are obtained by factoring out certain invariant distributions. Furthermore, we also show that all bisimilar systems of different dimension are of this form.

Key words: Bisimulation relations, bisimilar control systems, controlled invariance, symmetries.

1 Introduction

In theoretical computer science the notion of bisimulation inspired the definition of various notions of equivalence between models of computation. Each of these equivalences identifies classes of systems with similar properties, so that proving a property for a certain system can be done on a smaller equivalent system, thereby simplifying the proof process.

Similar notions are also important in the context of hybrid systems, where the inherent complexity of the hybrid model render its analysis or design very difficult. Motivated by this, we were naturally led to understand the continuous counterpart of this notion. Previous steps towards this objective have been given in [16] where linear control systems are embedded in the class of transition systems for which the notion of bisimulation was originally introduced by [20] and also [12]. It is shown in [16] that different embeddings give rise to semantically different notions of bisimulation being characterized by different conditions. For nonlinear systems no such attempt has appeared in the literature so far, except for [6] where the notion of bisimulation is presented in a sufficiently abstract categorical context to unify discrete and continuous interpretations. Compared to the work in [6], this paper seeks not to unify, but to characterize the notion by easily checkable (algebraic) conditions.

A characterization of bisimulation for nonlinear systems is important for several reasons that go beyond its application in hybrid systems. In the series of papers [18,19,21], a methodology has been introduced to compute abstractions of linear and nonlinear control systems. These abstractions are clearly important for verification problems, but also for hierarchical synthesis. For example, in [17] hierarchical stabilization of linear systems is discussed in the framework of abstractions. The ability to perform hierarchical synthesis depends on finding low-level trajectories that implement or refine trajectories of the abstracted model. A sufficient condition is given by bisimilarity, and this fact constitutes another reason to provide algebraic tests for its characterization.

The notion of bisimulation is also very interesting from a system theoretic point of view as

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[1] This research is partially supported by NSF Information Technology Research Grant CCR01-21431 and NSF CAREER CCR-01-32716.

it provides an equivalence relation on the class of control systems. This can be regarded as another tool in the quest of classifying nonlinear control systems. Furthermore, this equivalence relation has the important property of rendering as equivalent, control systems of possibly different dimensions. This contrasts with other known equivalences such as diffeomorphisms [11], or feedback transformations [3,8,10]. Furthermore, the notion of bisimulation also has interesting connections with other well known notions in systems theory such as controlled invariance [7,9,13] and symmetries for nonlinear control systems [5,14].

In this paper we introduce two notions of bisimulation for nonlinear control system systems based on the original definition in [12]. We then focus on control affine systems and relations between them defined by submersions, and provide algebraic characterizations for these notions. These characterizations turn out to be related with the notion of Φ-related control systems introduced in [18]. We then show that by factoring out certain invariant distributions one obtains bisimilar systems and that all bisimilar systems are obtained in this way. The distinguishing power of the two introduced notions is also discussed by showing that, locally, they are equivalent up to a feedback transformation. This is achieved by relating the introduced notions of bisimulation with controlled invariance.

2 Geometrical Preliminaries

Let M be a differentiable manifold and T_xM its tangent space at $x \in M$. In this paper, we will consider that all the manifolds are smooth, that is C^∞, and that all related mathematical objects are also smooth. The tangent bundle of M is denoted by $TM = \cup_{x\in M} T_xM$ and π_M is the canonical projection map $\pi_M : TM \longrightarrow M$ taking a tangent vector $X(x) \in T_xM \subset TM$ to the base point $x \in M$. Given manifolds M and N and given a map $\phi : M \longrightarrow N$, we denote by $T_x\phi : T_xM \longrightarrow T_{\phi(x)}N$ the induced tangent map which maps tangent vectors X at T_xM to tangent vectors $T_x\phi \cdot X$ at $T_{\phi(x)}N$. If ϕ is such that $T_x\phi$ is surjective at $x \in M$ then we say that ϕ is a submersion at x. When ϕ is a submersion at every $x \in M$ we simply say that it is a submersion. If furthermore, the submanifolds $\phi^{-1}(y) = \{x \in M \ : \ \phi(x) = y\} \subset M$ are connected, we say that ϕ has connected fibers. When ϕ has an inverse which is also smooth we call ϕ a diffeomorphism.

A fiber bundle is a tuple $(B, M, \pi_B, \mathcal{F}, \{O_i\}_{i\in I})$, where B, M and $\mathcal{F}$ are manifolds called the *total space*, the *base space* and *standard fiber* respectively. The map $\pi_B : B \longrightarrow M$ is a

surjective submersion and $\{O_i\}_{i\in I}$ is an open cover of M such that for every $i \in I$ there exists a diffeomorphism $\Psi_i : \pi_B^{-1}(O_i) \longrightarrow O_i \times \mathcal{F}$ satisfying $\pi_{o_i} \circ \Psi_i = \pi_B$, where π_{o_i} is the projection from $O_i \times \mathcal{F}$ to O_i. The submanifold $\pi_B^{-1}(x)$ is called the fiber at $x \in M$ and is diffeomorphic to $\mathcal{F}$. Since a fiber bundle is locally a product, we can always find local coordinates, which we shall call trivializing coordinates, of the form (x, b), where x are coordinates for the base space and b are coordinates for the local representative of the standard fiber.

Definition 2.1 (Control System) *A control system $\Sigma_M = (M \times V, F_M)$ consists of smooth manifolds M called the state space, V called the input space and a smooth map $F_M : M \times V \longrightarrow TM$ that assigns a vector $X \in T_xM$ to each pair $(x, v) \in M \times V$.*

Although the previous definition captures the usual notion of control systems, in certain situations it is more natural to model available inputs as being dependent on the state. This dependence can be captured by replacing the product $M \times V$ by a fiber bundle. In this situation, we define a control system as $\Sigma_M = (U_M, F_M)$ consisting of a fiber bundle $\pi_{U_M} : U_M \longrightarrow M$ called the control bundle and a map $F_M : U_M \longrightarrow TM$ making the following diagram commutative:

$$\begin{tikzcd} U_M \arrow[r, "F_M"] \arrow[d, "\pi_{U_M}"'] & TM \arrow[dl, "\pi_M"] \\ M & \end{tikzcd} \tag{2.1}$$

that is, $\pi_M \circ F_M = \pi_{U_M}$, where $\pi_M : TM \longrightarrow M$ is the tangent bundle projection.

In trivializing coordinates (x, v), the map $F_M : U_M \longrightarrow TM$ reduces to the familiar expression $\dot{x} = f(x, v)$ with $v \in \pi_{U_M}^{-1}(x)$. In the special case where the control bundle is trivial, that is, $U_M = M \times V$ we recover Definition 2.1.

Having defined control systems the concept of trajectories or solutions of a control system is naturally expressed as follows:

Definition 2.2 (Trajectories of Control Systems) *A smooth curve $c : I \longrightarrow M$, $I = [0, \tau[\subseteq \mathbb{R}_0^+$ is called a trajectory of control system $\Sigma_M = (U_M, F_M)$, if there exists a (not necessarily smooth) curve $c^V : I \longrightarrow V$ such that:*

$$\frac{d}{dt}c(t) = F(c(t), c^V(t)) \qquad \forall\, t \in I \tag{2.2}$$

We shall also say that a control system is forward complete [2] if solutions exist for all positive times, that is, if we can take τ to be $+\infty$.

When we need to consider a fiber bundle U_M instead of the product $M \times V$, we replace c^V by $c^U : I \longrightarrow U_M$ and require commutativity of the following diagrams:

$$\begin{tikzcd} & U_M \arrow[d, "\pi_{U_M}"] \\ I \arrow[ur, "c^U"] \arrow[r, "c"'] & M \end{tikzcd} \qquad \begin{tikzcd} & U_M \arrow[d, "F_M"] \\ I \arrow[ur, "c^U"] \arrow[r, "Tc"'] & TM \end{tikzcd} \tag{2.3}$$

where we have identified I with TI. These commutative diagrams are equivalent to the equalities $\pi_{U_M} \circ c^U = c$ and $Tc = F_M(c^U)$ which express in the language of fiber bundles equality (2.2).

A (left) action of a Lie group G on a manifold M is a map $\theta : G \times M \longrightarrow M$ such that $\theta(e, x) = x$ and $\theta(g_1 g_2, x) = \theta(g_1, \theta(g_2, x))$, where e is the group identity and $g_1, g_2 \in G$ (see [2]). Given a point $x \in M$ we can define the orbit of θ thru x to be the following subset of M:

$$\{x' \in M \ : \ x' = \theta(x, g) \text{ for some } g \in G\}$$

An action is said to be free when $\theta(g, x) = x \Rightarrow g = e$ and proper when the map $\overline{\theta}(g, x) = (x, \theta(g, x))$ is proper. When $\theta : G \times M \longrightarrow M$ is a free and proper action, then M/G, the space of orbits of θ is a smooth manifold and the projection $\pi : M \longrightarrow M/G$ taking each point in M to its orbit is a smooth surjective submersion [2]. Furthermore by fixing any $g \in G$ we obtain $\theta(g, -) = \theta_g : M \longrightarrow M$ a diffeomorphism of M.

3 Bisimulation Relations

The notion of bisimulation is originally credited to [20] and [12], and since then many authors have made important contributions to its development. In the context of continuous control systems, bisimulations have been discussed for the first time in [16] for linear control systems. We start by recalling the concept of transition system and bisimulation as presented in [12].

[2] Determining if a control system is forward complete can be a nontrivial task. Standard sufficient conditions include compactness of M or compact support of F_M.

Definition 3.1 (Transition Systems) *A transition system is a tuple $T = (S, L, \longrightarrow)$ consisting of:*

- *A set of states S;*
- *A set of labels L;*
- *A transition relation $\longrightarrow \subseteq S \times L \times S$.*

We use the graphical representation $q_1 \xrightarrow{l} q_2$ to denote $(q_1, l, q_2) \in \longrightarrow$. Intuitively, one can regard a transition system as a nondeterministic control system. Given a state $s \in S$, one interprets the set of labels $l \in L$ such that $s \xrightarrow{l} s'$ for some $s' \in S$, as the set of control inputs available at state s. Choosing one of those inputs will make the transition system evolve to the new state or states s' satisfying $(s, l, s') \in \longrightarrow$. The nondeterminism is captured by the fact that different triples (s, l, s') and (s, l, s'') may belong to $\longrightarrow$. This is the analogy that we shall make use to provide a continuous counterpart of the notion of bisimulation that we now recall.

Definition 3.2 (Bisimulation relation) *Let $T_1 = (S_1, L, \longrightarrow_1)$ and $T_2 = (S_2, L, \longrightarrow_2)$ be transition systems. A relation $F \subseteq S_1 \times S_2$ is said to be a bisimulation relation between T_1 and T_2 if $(s_1, s_2) \in F$ implies for all $l \in L$:*

- *if $s_1 \xrightarrow{l}_1 s_1'$ then there exists a $s_2' \in S_2$ such that $s_2 \xrightarrow{l}_2 s_2'$ and $(s_1', s_2') \in F$.*
- *if $s_2 \xrightarrow{l}_1 s_2'$ then there exists a $s_1' \in S_1$ such that $s_1 \xrightarrow{l}_2 s_1'$ and $(s_1', s_2') \in F$.*

To import this notion into the continuous context we face the difficulty of not being able to express the continuous dynamics in terms of the "atomic" jumps $s_1 \xrightarrow{l} s_1'$. We shall, therefore, replace the atomic jumps for *any* evolution, that is, we will ask a control system to match the evolution of another control system for *every* instant of time. Furthermore, as trajectories must be obtained by using the same input trajectory, the input space cannot depend on the state space. We shall, therefore assume, that the control bundle is a product $U_M = M \times V$, being V the input space.

Naturally, this leads to the following notion of bisimulation for control systems:

Definition 3.3 (Bisimulation of Control Systems) *Let $\Sigma_M = (U_M, F_M)$ and $\Sigma_N = (U_N, F_N)$ be control systems such that $U_M = M \times V$ and $U_N = N \times V$. A relation $F \subseteq M \times N$ is said to be a bisimulation relation between Σ_M and Σ_N if $(x, y) \in F$ implies:*

(1) for any state trajectory $c_M : I \longrightarrow M$ of Σ_M with $c_M(0) = x$ determined by input trajectory

$c^V : I \longrightarrow V$ *there exists a state trajectory* $c_N : I \longrightarrow N$ *of* Σ_N *with* $c_N(0) = y$ *determined by input trajectory* $c^V : I \longrightarrow V$ *such that* $(c_M(t), c_N(t)) \in F$ *for every* $t \in I$.

(2) for any state trajectory $c_N : I \longrightarrow N$ *of* Σ_N *with* $c_N(0) = y$ *determined by input trajectory* $c^V : I \longrightarrow V$ *there exists a state trajectory* $c_M : I \longrightarrow M$ *of* Σ_M *with* $c_M(0) = x$ *determined by input trajectory* $c^V : I \longrightarrow V$ *such that* $(c_M(t), c_N(t)) \in F$ *for every* $t \in I$.

As we shall see soon, this notion of bisimulation will be quite restrictive. This will motivate more relaxed notions of bisimulation, and in particular, we shall consider an input abstract version. This new notion relaxes the requirement that both systems have the same input trajectories and furthermore can be easily expressed without the assumption of trivial control bundles, being therefore, better suited for global analysis of control systems.

Definition 3.4 (Input Abstract Bisimulation of Control Systems) *Let* $\Sigma_M = (U_M, F_M)$ *and* $\Sigma_N = (U_N, F_N)$ *be control systems. A relation* $F \subseteq M \times N$ *is said to be an input abstract bisimulation relation between* Σ_M *and* Σ_N *if* $(x, y) \in F$ *implies:*

(1) for any state trajectory $c_M : I \longrightarrow M$ *of* Σ_M *with* $c_M(0) = x$ *there exists a state trajectory* $c_N : I \longrightarrow N$ *of* Σ_N *with* $c_N(0) = y$ *such that* $(c_M(t), c_N(t)) \in F$ *for every* $t \in I$.

(2) for any state trajectory $c_N : I \longrightarrow N$ *of* Σ_N *with* $c_N(0) = y$ *there exists a state trajectory* $c_M : I \longrightarrow M$ *of* Σ_M *with* $c_M(0) = x$ *such that* $(c_M(t), c_N(t)) \in F$ *for every* $t \in I$.

We shall say that two control systems are (input abstract) bisimilar when there exists a (input abstract) bisimulation between then.

The above introduced notions of bisimulation are also important from a systems perspective since they allow a new type of classification of control systems. Indeed, it is not difficult to show that the notion of bisimulation defines an equivalence relation in the class of control systems.

Proposition 3.5 *Bisimulation and input abstract bisimulation are equivalence relations on the class of control systems.*

These equivalence relations have the important characteristic of rendering equivalent, systems of possibly different dimension. It therefore makes sense to consider as representative of each equivalence class, the system of smallest dimension, leading to notions of minimality.

4 A Characterization of Bisimulation

For presentation purposes, all proofs of the main results in this section can be found at Appendix A. We start by making some assumptions that will allow to provide simple characterizations of bisimilar control systems:

(1) The control systems are assumed to be control affine, that is, there are local (trivializing) coordinates (x, v) where the system map F_M takes the form $F_M(x, v) = f_M(x) + \sum_{i=1}^{k} g_M^i(x)v_i$.

(2) The associated affine distribution $\mathcal{D}_M = f_M + \Delta_M = f_M + span\{g_M^1, g_M^2, \ldots, g_M^k\}$ is of constant rank.

(3) The relation $F \subseteq M \times N$ is induced by a smooth map $f : M \longrightarrow N$, that is $(x, y) \in F$ iff $f(x) = y$ where f is a submersion, that is, $T_x f$ is surjective at every $x \in M$.

The first two assumptions are not very restrictive since the results obtained for affine control systems can be lifted to fully nonlinear control systems by making use of the notion of extended control system [15]. The third assumption is more restrictive but its justified by the fact that in [19] an algorithm has been presented for the computation of quotients of control systems based on such a quotient map. It is therefore of extreme importance to be able to determine when such quotients are in fact bisimilar to the original one with respect to the quotient map.

Before characterizing bisimulation we recall that given a control system Σ_M and a subset S of M, we say that S is invariant for Σ_M iff every trajectory of Σ_M starting at a point $x \in S$ remains in S for all time. It follows easily that S must contain the all the points reachable by Σ_M from S. This notion of invariance allows the characterization of bisimulation given in the next theorem:

Theorem 4.1 (Bisimilar control affine systems) *Let $\Sigma_M = (U_M, F_M)$ and $\Sigma_N = (U_N, F_N)$ be two forward complete control affine systems such that $U_M = M \times V$ and $U_N = N \times V$, and $f : M \longrightarrow N$ a submersion. Then, Σ_M is bisimilar to Σ_N via f iff for every $x \in M$:*

- *$f(M)$ is invariant for Σ_N*
- *$T_x f(\mathcal{D}_M(x)) = \mathcal{D}_N \circ f(x)$*
- *for every $X \in \mathcal{D}_M$ there exists a $Y \in \mathcal{D}_N$ such that $T_x f \cdot X(x) = Y \circ f(x)$.*

The above characterization shows how restrictive the notion of bisimulation is, since every vector field X in $\mathcal{D}_M$ must be f-related to some vector field Y in $\mathcal{D}_N$. Relaxing this condition was

the motivating factor behind the notion of input abstract bisimulation whose characterization is now presented.

Theorem 4.2 (Input abstract bisimilar control affine systems) *Let Σ_M and Σ_N be two forward compete control affine systems and $f : M \longrightarrow N$ a submersion. Then, Σ_M is input abstract bisimilar to Σ_N via f iff for every $x \in M$:*

- *$f(M)$ is invariant for Σ_N*
- *$T_x f(\mathcal{D}_M(x)) = \mathcal{D}_N \circ f(x)$*

Both in Theorem 4.1 as in Theorem 4.2 we assume forward completeness. This precludes the existence of finite explosion times and allows working with arbitrary time intervals I as required by the definition of (input abstract) bisimulation. If forward completeness does not hold, a weaker form of the above theorems still holds, where I is now taken as the intersection of the time domains of trajectories c_M and c_N.

We also note that since the relation $F \subseteq M \times N$ is defined by a map $f : M \longrightarrow N$, the first condition on the definition of input abstract bisimulation can be refrased as: *for every trajectory c_M of Σ_M, $f(c_M)$ must be a trajectory of Σ_N*. This was the basic definition of abstraction introduced in [18], so that it is natural that the characterization of (input abstract) bisimulation is a stronger version of the concept of f-related control systems, which is the algebraic characterization of abstractions. It is also interesting to note that the characterization of input abstract bisimilarity, given in Theorem 4.2 distinguishes these systems from general abstractions at the level of the structure of the control bundle as discussed in [21]. In fact, when a control system is bisimilar to its abstraction, no new inputs will appear on the abstraction, a phenomenon that does not occur for general abstracted systems [18].

We now clarify how different can (input abstract) bisimilar control systems be if they have different dimensions. For this we will assume that $dim(M) > dim(N)$, and recall the notions of invariant and controlled invariant distributions:

Definition 4.3 (Invariant and Controlled Invariant Distributions) *Let Σ_M be a control affine system and let $\mathcal{E}$ be a regular and involutive distribution on M. Distribution $\mathcal{E}$ is said to be invariant for Σ_M when:*

$$[\mathcal{D}_M, \mathcal{E}] \subseteq \mathcal{E}$$

Distribution $\mathcal{E}$ is said locally controlled invariant if there exist a local feedback transformation around each $x \in M$, such that $\mathcal{E}$ is invariant for the feedback transformed system.

Locally controlled invariant distributions also admit the following characterization:

Theorem 4.4 (Adapted from [4]) *Let Σ_M be a control affine system and $\mathcal{E}$ a regular and involutive distribution on M. The distribution $\mathcal{E}$ is locally controlled invariant for Σ_M iff:*

$$[\mathcal{D}_M, \mathcal{E}] \subseteq \mathcal{E} + \Delta_M$$

where $\Delta_M = span\{g_M^1, g_M^2, \ldots, g_M^k\}$.

Equipped with the notions of invariant and controlled invariant distributions we can now understand the relationship between (input abstract) bisimilar systems of different dimensions.

Theorem 4.5 *Let $\Sigma_M = (U_M, F_M)$ and $\Sigma_N = (U_N, F_N)$ be forward complete control affine systems such that $U_M = M \times V$, $U_N = N \times V$, $dim(M) > dim(N)$ and let $f : M \longrightarrow N$ be a surjective submersion with connected fibers. Then Σ_M is bisimilar to Σ_N via f iff $\ker(Tf)$ is invariant for Σ_M and Σ_N is defined by the affine distribution:*

$$\mathcal{D}_N \circ f(x) = \bigcup_{x' \in f^{-1} \circ f(x)} T_{x'} f(\mathcal{D}_M(x'))$$

For input abstract bisimulation we recover local controlled invariance:

Theorem 4.6 *Let Σ_M and Σ_N be forward complete control affine systems such that $dim(M) > dim(N)$ and $f : M \longrightarrow N$ a surjective subimmersion with connected fibers. Then Σ_M is input abstract bisimilar to Σ_N via f iff $\ker(Tf)$ is locally controlled invariant for Σ_M and Σ_N is defined by the affine distribution:*

$$\mathcal{D}_N \circ f(x) = \bigcup_{x' \in f^{-1} \circ f(x)} T_{x'} f(\mathcal{D}_M(x'))$$

As in Theorems 4.1 and 4.2, in the absence of forward completeness a weaker form of Theorems 4.5 and 4.6 obtained by restricting I still holds.

The previous characterization of (input abstract) bisimulation shows that although dimension is not constant on the equivalence classes of this equivalence, two control systems Σ_M and Σ_N of different dimensions are in the same equivalence class if and only if it is possible to obtain one from the other by factoring out (controlled) invariant distributions. As an immediate corollary of the previous results we have that factoring out symmetries also produces bisimilar systems:

Corollary 4.7 *Let Σ_M be an affine control system and $\theta : G \times M \longrightarrow M$ be a free and proper*

action of a Lie group G such that for every $X \in \mathcal{D}_M$ we have $\theta_g^* X = X$ for every $g \in G$. Then Σ_M is bisimilar via π to Σ_M/G defined by:

$$\mathcal{D}_N \circ \pi(x) = \bigcup_{x' \in \pi^{-1} \circ \pi(x)} T_{x'}\pi(\mathcal{D}_M(x'))$$

For input abstract bisimilar systems it is still the case that factoring out symmetries implies input abstract bisimilarity but we allow a larger class of symmetries (partial symmetries in the context of [14]):

Corollary 4.8 *Let Σ_M be an affine control system and $\theta : G \times M \longrightarrow M$ be a free and proper action of a Lie group G such that for every $X \in \mathcal{D}_M$ we have $\theta_g^* X \in \mathcal{D}_M$ for every $g \in G$. Then Σ_M is input abstract bisimilar via π to Σ_M/G defined by:*

$$\mathcal{D}_N \circ \pi(x) = \bigcup_{x' \in \pi^{-1} \circ \pi(x)} T_{x'}\pi(\mathcal{D}_M(x'))$$

We have not explicitly discussed the quotient system Σ_M/G control bundle geometry. We defer the reader to the reference [21] where these issues are addressed for general quotients and to [5,14] where symmetries are modeled by group actions acting on the control bundle as well.

It is clear that the equivalence relation defined by bisimulation is strictly finer (in the sense that it distinguishes more control systems) then the equivalence relation defined by input abstract bisimulation. However, locally, every two input abstract bisimilar control systems are bisimilar up to a feedback transformation. This fact is a simple consequence of Theorem 4.4. This proves the following result:

Proposition 4.9 *Let Σ_M and Σ_N be affine control systems input abstract bisimilar via $f : M \longrightarrow N$. Then, locally, there exists a feedback transformation for Σ_M rendering it bisimilar to Σ_N via f.*

Note that the previous result does not assert that Σ_M is bisimilar to Σ_N since the feedback transformation is not a bisimulation relation.

5 Conclusions

Motivated by notions of equivalence in computer science and hybrid systems, we have introduced the notion of (input abstract) bisimulation for nonlinear control systems. A differential algebraic characterization was given for the introduced notions which can be seen as a strengthening of the notion of f-related control systems of [18]. Although this notion constitutes an equivalence relation on the class of control systems which does not require the dimension of the systems to be an invariant, it was shown that bisimilar systems of different dimensions must be related in a special way. In fact, one of the systems must be obtained from the other by factoring out (controlled) invariant distributions.

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Appendix A - Proofs

Proof of Theorem 4.2.

(*Necessity*)

Since the relation F is defined by a map $f : M \longrightarrow N$, the first condition of bisimilarity implies that for every trajectory $c_M(t)$ of Σ_M, $f(c_M(t))$ must be a trajectory of Σ_N. This, in turn, implies:

$$T_x f(\mathcal{D}_M(x)) \subseteq \mathcal{D}_N \circ f(x) \tag{5.1}$$

as was shown in [18]. The second condition for bisimilarity implies that for every trajectory c_N of Σ_N such that $y = c_N(0)$ belongs to the range of f, there exists a trajectory c_M of Σ_M for every $x \in f^{-1}(c_N(0))$ satisfying $c_M(0) = x$ and $f(c_M) = c_N$. Let $X = \frac{d}{dt}\Big|_{t=0} c_M(t)$ and $Y = \frac{d}{dt}\Big|_{t=0} c_N(t)$. The equality $f(c_M) = c_N$ implies $T_x f \cdot X = Y$, and as c_N is any trajectory originating at y, Y is any vector belonging to $\mathcal{D}_N(y)$. As such, we must have:

$$\mathcal{D}_N(y) = \mathcal{D}_N \circ f(x) \subseteq T_x f(\mathcal{D}_M(x)) \tag{5.2}$$

for every $x \in M$. Combining (5.1) with (5.2) gives:

$$T_x f(\mathcal{D}_M(x)) = \mathcal{D}_N \circ f(x) \tag{5.3}$$

for every $x \in M$. Furthermore, $f(M)$ must be invariant under Σ_N, otherwise for any trajectory c_N of Σ_N starting in $f(M)$ and leaving $f(M)$, there would be no trajectory c_M of Σ_M satisfying $f(c_M) = c_N$.

(*Sufficiency*)

Condition $T_x f(\mathcal{D}_M(x)) = \mathcal{D}_N \circ f(x)$ implies $T_x f(\mathcal{D}_M(x)) \subseteq \mathcal{D}_N \circ f(x)$ from which follows that, for every trajectory c_M of Σ_M, $f(c_M)$ is a trajectory of Σ_N, as proved in [18]. This shows that the first condition of input abstract bisimulation is satisfied.

The second condition requires more work. Let c_N be a trajectory of Σ_N such that $c_N(0)$ belongs to the range of f. Consider the graph $\Gamma \subset I \times N$ of c_N which is a submanifold of $\overline{N} = I \times N$ as c_N is a smooth map. On Γ we define the following vector field $Y : \Gamma \longrightarrow T\Gamma$, $Y(t, c_N(t)) = (\frac{\partial}{\partial t}, T_t c_N \cdot \frac{\partial}{\partial t})$. Note that the integral curve of this vector field is precisely the graph of c_N. Consider now the manifold $\overline{M} = I \times M$ with local coordinates (t, x), where x

are coordinates for M and t coordinates for $\mathbb{R}$. On this manifold we introduce a new control system defined by the affine distribution $\overline{\mathcal{D}_M} = \{\overline{X} \in T\overline{M} \ : \ \overline{X} = X\frac{\partial}{\partial x} + \frac{\partial}{\partial t} \quad \text{for } X \in \mathcal{D}_M\}$. We will now restrict this control system so as to project on Y. To do so, we consider the map $\overline{f} = id_I \times f : \overline{M} \longrightarrow \overline{N}$ which being a submersion, since f is a submersion, allows to define the submanifold $\Pi = \overline{f}^{-1}(\Gamma)$ of $\overline{M}$. The set Π is a submanifold since $\overline{f}$ is transversal to the submanifold Γ [1]. Denoting by $(T_x\overline{f})^{-1}$ the set valued inverse of $T_x\overline{f}$, we restrict $\overline{\mathcal{D}_M}$ to obtain a new affine distribution $\widehat{\mathcal{D}_M}(x) = (T_x\overline{f})^{-1}(Y \circ f(x))$. Distribution $\widehat{\mathcal{D}_M}(x)$ is affine since it can be locally written as $\overline{X} + \ker(T\overline{f})$ for some locally defined vector field $\overline{X}$ satisfying $T\overline{f} \cdot \overline{X} = Y \circ \overline{f}$. We note that such vector field $\overline{X}$ exists in virtue of the equality $Tf(\mathcal{D}_M) = \mathcal{D}_N \circ f$ which holds by assumption. Furthermore, $\widehat{\mathcal{D}_M}$ is regular since $\overline{f}$ is a submersion. It then follows from this construction that $\widehat{\mathcal{D}_M}$ satisfies $T\overline{f}(\widehat{\mathcal{D}_M}) = Y \circ \overline{f}$ but, as shown in [18], this condition implies that every trajectory of the control system defined by $\widehat{\mathcal{D}_M}$ is mapped by $\overline{f}$ to a trajectory of the trivial control system defined by Y. Furthermore, as $\widehat{\mathcal{D}_M} \subseteq \overline{\mathcal{D}_M}$ we also know that trajectories of $\widehat{\mathcal{D}_M}$ are also trajectories of $\overline{\mathcal{D}_M}$. Finally, by realizing that trajectories of the control system defined by $\overline{\mathcal{D}_M}$ are simply the graph of (some) trajectories of Σ_M we conclude that for every $x \in f^{-1}(y)$ there exist a trajectory c_M of Σ_M satisfying $c_M(0) = x$ and $f(c_M(t)) = c_N(t)$ for every for every $I \subseteq \mathbb{R}_0^+$ and any $t \in I$ since $f(M)$ is invariant for Σ_N and both Σ_M and Σ_N are forward invariant. $\square$

Proof of Theorem 4.1.

(*Necessity*)

As bisimilar control systems are input abstract bisimilar we only have to show that the vector fields in $\mathcal{D}_M$ are f-related to the vector fields in $\mathcal{D}_N$. Let $c^U(t) = (u_1(t) \quad \ldots \quad u_k(t)) = (1 \quad 0 \quad \ldots \quad 0)$ be an input trajectory defining trajectory c_N of Σ_N. Since for the same input there must exist trajectories c_M of Σ_M for every $x \in f^{-1}(c_N(0))$ such that $c_M(0) = x$ and $f(c_M) = c_N$, it follows by differentiation that:

$$T_x f(f_M(x) + g_M^1(x) \cdot 1 + g_M^2(x) \cdot 0 + \ldots + g_M^k(x) \cdot 0) = f_N \circ f(x) + g_N^1 \circ f(x) \tag{5.4}$$

This means that $f_M + g_M^1$ is f-related to $f_N + g_N^1$. By choosing other constant inputs one similarly shows that every vector field in $\mathcal{D}_M$ is f-related to some vector field in $\mathcal{D}_N$.

(*Sufficiency*)

By Theorem 4.2 control systems Σ_M and Σ_N are input abstract bisimilar via f. It remains to

show that for any trajectory c_N of Σ_N generated by input trajectory c^U and such that $c_N(0)$ belongs to the range of f there exists trajectories c_M of Σ_M generated by the same input trajectory c^U for every $x \in f^{-1}(c_N(0))$ such that $c_M(0) = x$ and $f(c_M) = c_N$. However this follows at once from the discussion in the necessity part since every vector field X in $\mathcal{D}_M$ is f-related to its projection $Tf \cdot X = Y \circ f$ and $Y \circ f \in \mathcal{D} \circ f$ in virtue of the equality $Tf(\mathcal{D}_M) = \mathcal{D} \circ f$. $\square$

The proof of Theorem 4.6 requires the following lemma:

Lemma 5.1 *Let $\phi : M \longrightarrow N$ be a surjective submersion with connected fibers and $\mathcal{D} = X + \Delta$ an affine distribution on M. Distribution $\mathcal{D}$ satisfies $T_x\phi(\mathcal{D}(x)) = T_{x'}(\mathcal{D}(x'))$ for any $x, x' \in M$ such that $\phi(x) = \phi(x')$ iff $[\mathcal{D}, \ker(T\phi)] \subseteq \Delta + \ker(T\phi)$.*

Proof of Lemma 5.1

We reduce the proof of this lemma to the equivalence $TK_t(\mathcal{E}) = \mathcal{E} \circ K_t$ iff $[\mathcal{E}, \ker(T\phi)] \subseteq \Delta_{\mathcal{E}}$ shown in [19] for an affine distribution $\mathcal{E} = Y + \Delta_{\mathcal{E}}$ on M and the flow K_t of any vector field $K \in \ker(T\phi)$.

Assume that $T_x\phi(\mathcal{D}(x)) = T_{x'}(\mathcal{D}(x'))$ holds for any $x, x' \in M$ such that $\phi(x) = \phi(x')$ and define $\mathcal{E}$ as $\mathcal{E} = \mathcal{D} + \ker(T\phi) = X + \Delta_{\mathcal{E}}$, $\Delta_{\mathcal{E}} = \Delta + \ker(T\phi)$. In particular we will assume that $x' \in M$ is such that there exists a flow K_t of a vector field $K \in \ker(T\phi)$ satisfying $K_t(x) = x'$. Note that $\phi(x) = \phi(x') = \phi \circ K_t(x)$ implies that:

$$T_x\phi(x) = T_{x'}\phi \circ T_xK_t \tag{5.5}$$

and this allows to rewrite $T_x\phi(\mathcal{D}(x)) = T_{x'}\phi(\mathcal{D}(x'))$ as:

$$\begin{aligned} T_x\phi(\mathcal{D}(x)) &= T_{x'}\phi(\mathcal{D}(x')) \\ \Leftrightarrow \quad T_x\phi\Big(\mathcal{D}(x) + \ker(T_x\phi)\Big) &= T_{x'}\phi(\mathcal{D} \circ K_t(x)) \\ \Leftrightarrow \quad T_{x'}\phi \circ T_xK_t\Big(\mathcal{D}(x) + \ker(T_x\phi)\Big) &= T_{x'}\phi(\mathcal{D} \circ K_t(x)) \\ \Leftrightarrow \quad T_xK_t(\mathcal{E}) &= \mathcal{D} \circ K_t(x) + \ker(T_{x'}\phi) = \mathcal{E} \circ K_t(x) \end{aligned}$$

From Proposition 5.1 in [19] now follows:

$$[\mathcal{E}, \ker(T\phi)] \subseteq \Delta_{\mathcal{E}}$$

This inclusion can be written as:

$$
\begin{aligned}
& [\mathcal{D} + \ker(T\phi), \ker(T\phi)] \subseteq \Delta + \ker(T\phi) \\
\Rightarrow \quad & [\mathcal{D}, \ker(T\phi)] + [\ker(T\phi), \ker(T\phi)] \subseteq \Delta + \ker(T\phi) \\
\Rightarrow \quad & [\mathcal{D}, \ker(T\phi)] \subseteq \Delta + \ker(T\phi) - [\ker(T\phi), \ker(T\phi)] \\
\Rightarrow \quad & [\mathcal{D}, \ker(T\phi)] \subseteq \Delta + \ker(T\phi) - \ker(T\phi) = \Delta + \ker(T\phi)
\end{aligned}
$$

which shows the desired inclusion. The converse is similarly proved since by connectedness of the fibers of ϕ any two points $x, x' \in M$ such that $\phi(x) = \phi(x')$ can be connected by local flows of vector fields in $\ker(T\phi)$. $\square$

Proof of Theorem 4.6.

Assume that Σ_M and Σ_N are input abstract bisimilar via f. Then by Theorem 4.2 we have:

$$T_x f(\mathcal{D}_M) = \mathcal{D}_N \circ f(x) \tag{5.6}$$

for every $x \in M$. This means that for every $x' \in f^{-1} \circ f(x)$ the equality:

$$T_x(\mathcal{D}_M(x)) = T_{x'} f(\mathcal{D}_M(x')) \tag{5.7}$$

holds and by Lemma 5.1 we have $[\mathcal{D}_M, \ker(Tf)] \subseteq \ker(Tf) + \Delta_M$. Furthermore, by Theorem 4.4 we have controlled invariance since f being a submersion ensures that $\ker(Tf)$ is regular and involutive. In addition, we also have:

$$
\begin{aligned}
\bigcup_{x' \in f^{-1} \circ f(x)} T_{x'} f(\mathcal{D}_M(x')) &= T_x f(\mathcal{D}_M(x)) \\
&= \mathcal{D}_N \circ f(x)
\end{aligned} \tag{5.8}
$$

where the first equality holds by (5.7) and the second equality holds by Theorem 4.2.

Assume now that $\ker(Tf)$ is controlled invariant for Σ_M, then the equality $T_x(\mathcal{D}_M(x)) = T_{x'} f(\mathcal{D}_M(x'))$ holds for every $x' \in f^{-1} \circ f(x)$ in virtue of Theorem 4.4 and Lemma 5.1. Therefore, since $\mathcal{D}_N$ is defined by:

$$\mathcal{D}_N \circ f(x) = \bigcup_{x' \in f^{-1} \circ f(x)} T_{x'} f(\mathcal{D}_M(x')) \tag{5.9}$$

it follows that $T_x f(\mathcal{D}_M(x)) = \mathcal{D}_N \circ f(x)$ for every $x \in M$. Recalling that f is surjective we also have that $N = F(M)$ is invariant for Σ_N, so the result now follows from Theorem 4.2. $\square$

Proof of Theorem 4.5.

The proof is similar to the proof of Theorem 4.6, except we now use the fact that $\ker(Tf)$ is invariant for Σ_M iff every vector field in $\mathcal{D}_M$ is f-related to its projection on N. $\square$