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Essays in Information Economics

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Abstract
This dissertation is composed of three essays considering the role of private information in economic environments. The first essay considers efficient investments into technologies such as auditing and enforcement systems that are designed to mitigate information and enforcement frictions that impede the provision of first best insurance against income risk. In the model, the principal can choose a level of enforceability that inhibits an agent's ability to renege on the contract and a level of auditing that inhibits his ability to conceal income. The dynamics of the optimal contract imply an endogenous lower bound on the lifetime utility of an agent, strictly positive auditing at all points in the contract and positive enforcement only when the agent's utility is sufficiently low. Furthermore, the two technologies operate as complements and substitutes at alternative points in the state space. The second essay considers a planning problem with hidden actions and hidden states where the component of utility affected by the unobservable state is separable from component governed by the hidden action. I show how this problem can be written recursively with a one dimensional state variable representing a modified version of the continuation utility promise. I apply the framework to a model in which an agent takes an unobservable decision to invest in human capital using resources allocated to him by the planner. Unlike similar environments without physical investment, it is shown numerically the immiserising does not necessarily hold. In the third essay, with Kyungmin Kim, I examine the effects of commitment on information transmission. We study and compare the behavioral consequences of honesty and white lie in communication. An honest agent is committed to telling the truth, while a white liar may manipulate information but only for the sake of the principal. We identify the effects of honesty and white lie on communication and show that the principal is often better off with a possibly honest agent than with a potential white liar. This result provides a fundamental rationale on why honesty is thought to be an important virtue in many contexts.

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ABSTRACT

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Jonathan T Pogach

Harold Cole

This dissertation is composed of three essays considering the role of private information in economic environments. The first essay considers efficient investments into technologies such as auditing and enforcement systems that are designed to mitigate information and enforcement frictions that impede the provision of first best insurance against income risk. In the model, the principal can choose a level of enforceability that inhibits an agent's ability to renege on the contract and a level of auditing that inhibits his ability to conceal income. The dynamics of the optimal contract imply an endogenous lower bound on the lifetime utility of an agent, strictly positive auditing at all points in the contract and positive enforcement only when the agent's utility is sufficiently low. Furthermore, the two technologies operate as complements and substitutes at alternative points in the state space. The second essay considers a planning problem with hidden actions and hidden states where the component of utility affected by the unobservable state is separable from component governed by the hidden action. I show how this problem can be written recursively with a one dimensional state variable representing a modified version of the continuation utility promise. I apply the framework to a model in which an agent takes an unobservable decision to invest in human capital using resources allocated to him by the planner. Unlike similar environments without physical investment, it is shown numerically the immiserising does not necessarily hold. In the third essay, with Kyungmin Kim, I examine the effects of commitment on information transmission. We study and compare the behavioral consequences of honesty and white lie in communication. An honest agent is committed to telling the truth, while a white liar may manipulate information but only for the sake of the principal. We identify the effects of honesty and white lie on communication and show that the

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Efficient Auditing and Enforcement in Dynamic Contracts

1.1 Introduction

Limited commitment and private information are two sources commonly identified as impediments to an agent's ability to insure himself against idiosyncratic income risk. However, models of efficient insurance with limited commitment generally ignore all considerations of private information and implicitly assume both that enforcement is prohibitively costly and that information is completely costless. Similarly, models of efficient insurance under private information often abstract away from the issue of enforcement, assuming that it is costless to compel agents to remain in the contract.

In this chapter, I consider the provision of efficient insurance under both limited commitment and private information when the insurer has access to enforcement and auditing technologies that, respectively, reduce the agent's returns from walking away from the contract and concealing income. I depart from the literature in that I consider both information and enforcement frictions and also allow for an insurer's access to technologies designed to mitigate these frictions in a dynamic environment.[1] I show that the implications on an agent's consumption, along with use of these technologies, critically depend on the presence of both frictions and the relative costs of auditing and enforcement. The analysis of this chapter therefore provides an understanding of efficient insurance that is absent in models that ignore one dimension of the problem.

In environments with limited commitment, an agent is assumed to have access to an "outside option," typically taken to be his value of consuming his contemporaneous income and autarky consumption thereafter. Should the contract ever be sufficiently undesirable, the agent may then choose to break the contract and receive the value of this option. A first best contract that requires the transfer of income from an agent with high income to one with low income may then be impossible to implement as an agent receiving a large income may prefer a high consumption today and the outside option to paying the transfer.

Meanwhile, in environments with private information, low income agents are typically indistinguishable from high income agents. Thus, first best insurance is impossible to implement as all agents would choose to mimic the agent who is due the largest net transfer; any insurance plan that transfers resources from high income to low income agents must also provide high income agents the appropriate incentives to reveal their information.

Most applications of insurance problems not only are likely to entail some level of both these frictions, but also endogenous choices on the part of insurers that affect the extent to which one would expect these frictions to impede first best insurance. The analysis in this chapter is applicable to a wide range of standard contracting and insurance questions including optimal social insurance policy, managerial compensation, and informal risk sharing. Consider the social insurance program of unemployment insurance. Implementation of efficient unemployment insurance entails

[1] One paper that considers both frictions and an endogenous enforcement choice is Krasa and Villamil (2000) who study a three period model with private information and enforcement.

a government's decisions on the monitoring of agents' income claims and job offers to mitigate these sources of information frictions. Moreover, efficient unemployment insurance also relies on a choice of investment in a court system, law enforcement, and even prisons to compel employed agents with high income to pay taxes.

In the managerial compensation literature, Jensen (1986) hypothesizes that firm structure is the result of the optimal balancing of agency and information frictions between institutional investors and management. However, these frictions can typically be reduced by institutional investors via internal audits and the ability to hire lawyers to sue opportunistic managers. Such choices will then have implications on managerial compensation and corporate financial structure. Finally, this chapter may also extend into the informal risk sharing literature[2] to the extent that small villages suffer from these frictions and have the capacity to mitigate them.

I study three dynamic principal agent problems in which the principal is risk neutral and the agent is risk averse. The first is one of limited commitment, full information, and an enforcement technology and is referred to as the *Enforcement Model*, the second is one of private information, full commitment, and an auditing technology and is referred to as the *Auditing Model*, and the third features both sets of frictions and technologies and is referred to as the *General Model*. From the first two models, I show how the principal optimally uses enforcement and auditing, respectively, to insure the agent. In the *General Model*, I show how the implications of a joint model differ dramatically from the joint implications of the separate models.

Models of limited commitment, such as Thomas and Worrall (1988), often feature an increasing level of consumption and long run consumption constant and equal to an amount at which the highest income agent would not choose to renege. Efficient insurance that requires the transfer of income from high to low income states must also provide an agent who realizes high income with an incentive to not renege. Such a transfer can be implemented by increasing future consumption across future shocks

[2]See Townsend (1994) and Ligon, et al (2002).

for an agent with high income. Higher consumption across future income states then diminishes the agent's future incentives to renege and, eventually, first best insurance is possible.

The *Enforcement Model* is a version of the limited commitment and endogenous enforcement model studied in Koeppl (2007) and the results from this model are used to contrast and provide insight into the role of enforcement in the *General Model*. With the addition of endogenous enforcement to a limited commitment environment, enforcement is decreasing in the agent's promised utility. Enforcement alters the dynamics of the contract by slowing the growth of consumption to allow for smoother inter-temporal consumption. However, in the long run insurance is first best and the principal uses no enforcement. Notice that this long run result relies heavily on the costless information assumption, as long run consumption patterns are identical for high and low income agents. Such a model should then be viewed skeptically if it is believe that there is any level of information friction.

Models of efficient insurance with private information[3] deliver a conflicting prediction on consumption. In these models, providing an agent with high income realization the incentives to reveal his information is "cheap" at lower levels of consumption and utility. The reason for this is that when marginal utility of consumption is high, agents with high income and agents with low income evaluate quite differently small differences in transfers. This eases the ability of the principal to separate income realizations and yields the well known immiseration result in the long run in which agents are driven down to their lower bound of utility (or toward $-\infty$ if utility is unbounded below) almost surely. This extreme prediction of models with private information is a direct consequence of the implicit costless enforcement assumption; given any capacity to renege an agent would surely undertake such an option to avoid the consequences of the implied contract.

[3] See, for example, Green (1987), Atkeson and Lucas (1992), and Thomas and Worrall (1990)

In the *Auditing Model* I show that while the auditing technology slows this process, it does not prevent immiseration and thus, also relies heavily on costless enforcement. Furthermore, I show that as the agent is immiserised, the principal's use of auditing also vanishes. A novel feature of the *Auditing Model* is its specification of the auditing technology, which allows for more complete analytic results than have been obtained in other dynamic contracting environments with private information and some form of auditing. For example, Wang (2005) studies a dynamic version of costly state verification á la Townsend (1979). However, the complex nature of his problem hampers his ability to make statements on, for example, the qualitative response of the agent's future utility to income shocks. The auditing technology I consider is such that the level of investment into the technology reduces the agent's return to misreporting directly, but does not itself reveal information. This greatly simplifies the information structure of the problem and allows for more analytic results.

The dependence of the *Enforcement* and *Auditing* models on costless information and enforcement, respectively, reveals the tension that exists in a model where both technologies are assumed to be costly. In the *General Model*, I study such a model. Past models that consider environments with both sources of frictions include Atkeson and Lucas (1995), who study unemployment insurance when job offers are unobservable and there is an exogenous lower bound on promised utility. Phelan (1995) similarly studies an environment of limited commitment and private information in which competitive insurance contracts determine the lower bound on an agent's utility. However, neither chapter allows a principal any recourse against these frictions, thus assuming that both information and enforcement are infinitely costly.

The key difference between the *General Model* studied in this chapter and other models in the literature is that in other models it is the high income agent who must be kept from reneging while in this model, at low levels of promised utility, it is an individual with a *low* income realization for whom the enforcement constraint binds. The intuition behind this result is that agent's utility from misreporting income depends on the level of consumption promised in the contract while the agent's utility from reneging does not. Therefore, as the consumption from the contract diminishes and the marginal utility increases, a high income agent finds misreporting income relatively more attractive than reneging. Consequently, to deliver on the promised utility, it is the low income agent who must be compelled to remain in the contract via enforcement.

One implication of this is that the opposing forces on promised utility of limited commitment and private information give rise to an endogenous lower bound. To contrast this result to those aforementioned papers with both frictions, their lower bounds are achieved via the (sometimes implicit) assumption of infinitely costly enforcement.[4] That is, the principal can take no action to reduce the agent's utility below some specified level. I do not rely on infinitely costly enforcement to establish this lower bound and show that such a bound exists and immiseration is prevented whenever enforcement costs are convex, independent of its costs relative to auditing and consumption.

Furthermore, I show that auditing and enforcement technologies act alternatively as both complements and substitutes at alternative points in the state space. Consider the case when promised utility is at a level such that the high income agent is indifferent between remaining in the contract and reneging. Additional auditing allows the principal to better smooth the agent's consumption between high and low states, reducing the principal's consumption cost. However, this smoothing can be done only when accompanied by additional enforcement cost, as the high income agent must be kept from reneging. Thus, the two technologies are *complements*. On the other hand, when promised utility is sufficiently low, it is the agent with a low income realization who is indifferent between reneging and remaining in the

[4]The lower bound in Phelan paper is still endogenous via competitive insurance markets.

contract. In this case, an increase in auditing that increases the low income agent's utility and smooths utility across states is accompanied by a decrease in the necessary enforcement. Thus, the two technologies are *substitutes* in this region.

The remainder of the chapter is organized as follows. Section 1.2 lays out the general problem and shows how the problem can be written recursively, which is used as the basis for the rest of the chapter. Sections 1.3 and 1.4 examine the *Enforcement* (costless information) and *Auditing* (costless enforcement) models, respectively, while Section 1.5 analyzes the *General Model*. Section 1.6 provides s numerical example of the Enforcement, Auditing, and General models. Section 1.7 concludes.

1.2 Model

A risk neutral principal looks to provide insurance to a risk averse agent who is subject to a risky income stream. Each period, an agent's income shock $\theta_t \in \Theta = \{\theta_1, .., \theta_S\}$ is realized with θ_t i.i.d. over time with $\theta_s < \theta_{s+1}$. Generic elements in Θ are denoted by θ_s and occur with probability p_s where $\sum_s p_s = 1$. I assume that the principal and the agent share a common discount factor $\beta \in (0, 1)$. In the *General Model*, there is an information friction and an enforcement friction. The enforcement friction implies that the agent can renege on the contract at any point in time with the consequence of being restricted from future insurance. In the absence of enforcement, I take the future value of the walk away option, U_0, to be the autarky value: $U_0 = \sum_s p_s u(\theta_s)$.

Before each period's realization of income the principal can invest in two technologies to ameliorate these frictions: an auditing technology $\alpha \in A = [0, 1]$ and an enforcement technology $\gamma \in \Gamma = \mathbf{R}_+$ at costs of $f(\alpha)$ and $g(\gamma)$, respectively. The choice of investment is observed by the agent and the investment is effective only for one period.[5] After observing his private income, θ_t, and the principal's choice of

[5]The fact that the auditing choice is made and observed before the agent's income is realized implies that the principal would never have an ex-post incentive to reverse his auditing choice.

auditing and enforcement, the agent reports income to the principal. Based on the public history, defined below, the principal provides transfers τ_t to the agent each period, so that his consumption is $\tau_t + \theta_t$ and his contemporaneous utility given this consumption is $(1-\beta)u(\tau_t + \theta_t)$. It is assumed that the agent is an expected utility maximizer and that utility is separable across periods.

The role of the enforcement technology is as follows. In any period after observing his income, the agent could walk away from the contract, consume that period's income and his autarky consumption at any point thereafter.[6] However, with an investment of $\gamma \in \Gamma = \mathbf{R}_+$ into enforcement, his utility from leaving the contract is shifted down by γ units of utility.[7] That is, given an enforcement level γ and an income θ_s, an agent's valuation of his outside option is $(1-\beta)u(\theta_s) + \beta U_0 - \gamma$. We could think of this cost as being a disutility of court proceedings to break a contract or even more extreme measures such as debtor prisons.

The nature of the auditing technology makes hiding income (or consumption) costly to the agent. Namely, for each unit of output that he tries to hide, α will be lost in an effort to avoid detection, where $\alpha \in A = [0, 1]$. For simplicity, I assume that should the agent choose to lie about his output (which he will, of course, not do in the optimal allocation) he will also choose to hide his income. The cost to hiding income can be interpreted as an agent having to consume less desirable goods. For example, when a high level of auditing is chosen by the principal the agent may have

In the costly state verification literature, the auditing technology allows the principal to decide, conditional on the agent's announced income, to observe the true state at some cost. However, as the optimal contract is written as one in which the agent never misreports, the principal has no ex-post incentive to monitor the agent given the income announcement. See Khalil (1997).

[6] I could consider the agent as having an option to recontract with another principal should he choose to renege on the contract. Under the enforcement environment in this chapter, this set up would be identical to one in which the agent's outside option is the value of this alternative contract rather than autarky. See Krueger and Uhlig (2006).

[7] The fact that γ is a utility cost to the agent is not critical for many of the results in the chapter. What drives many of the results is that the effect of γ on the outside option is independent of the contract, while the auditing choice affects the agent's incentives in a way that depends on the rest of the contract. The idea that enforcement has a level effect in the contract and the effect of auditing is not level reflects the nature of the commitment and information frictions they are, respectively, designed to mitigate.

to buy a smaller house or less fancy car to remain inconspicuous. Thus, the purpose of auditing solely serves the function of reducing the incentives of the agent to lie. This model is one of costless enforcement and private information and will be referred to as the *Auditing Model*. It is worthwhile to note that the auditing technology in this chapter differs from the common "costly state verification" auditing model. In the model here, auditing does not change the principal's information on the agent's true income, but rather, decreases the agent's return to misreporting. The third model combines the first two models, so that there are both enforcement and information frictions and the principal has access to auditing and enforcement to mitigate each friction.

Thus, the agent's t-period consumption when his true income is θ_s under a falsely reported income, $\theta_{\tilde{s}}$, is $\tau_{\tilde{s}} + \theta_s - \alpha(\theta_s - \theta_{\tilde{s}})$. For ease of notation, define $x_s \equiv \tau_s + \theta_s$, so that the agent's consumption under truth telling is x_s and his consumption under a false report, $\theta_{\tilde{s}}$, is $x_{\tilde{s}} + (1-\alpha)(\theta_s - \theta_{\tilde{s}})$. Let the period t public information of reports, truthful consumption level (which has a one-to-one mapping with transfers), auditing levels and enforcement levels be denoted by $h_t = (\tilde{\theta}_t, x_t, \alpha_t, \gamma_t)$ with private information $\tilde{h}_t = (\theta_t, h_t)$. Denote the t-period public history $h^t = (h_0, .., h_t) \in H^t$, where H^t is the set of t-period public histories. Denote the set of private histories by $\tilde{H}^t$ with generic element $\tilde{h}^t$.

Define a *plan*, $\nu = (\nu_0, \nu_1, ..,)$, where $\nu_t = (\nu_t^1, \nu_t^2)$ such that $\nu_t^1 : H^t \to A \times \Gamma$ and $\nu_t^2 = H^t \times A \times \Gamma \times \Theta \to \mathbf{R}$, as a sequence of duple mappings of the principal's enforcement and auditing choices given public histories and the principal's choice of transfers given the public history, this period income report, and this period's auditing and enforcement choices. The set of plans is denoted by N. An agent's *strategy* is the mapping of private histories and this period's auditing, enforcement, and income realizations into income reports.[8] Let $\sigma_t : \tilde{H}^{t-1} \times A \times \Gamma \times \Theta \to \Theta$ be agent's the t-period announcement. Denote $\Sigma = \{(\sigma_0, \sigma_1, ..)\}_{t=0}^{\infty}$. I make one

[8]Note that I ignore the agent's decision to walk away. See the discussion in the following paragraph.

additional and non-standard restriction on allowable strategies; namely, given an income θ_t, the agent cannot report any income $\theta > \theta_t$. The thought behind this assumption is that it costless to validate that the agent does not have a high income by virtue of his inability to show resources when asked. This assumption is also useful in simplifying the constraints in the recursive formulation of the *General Model* as well as proving concavity of the value function.

Note that if at any point in the contract it is optimal to let the agent walk away then there also must exist an optimal contract in which the agent chooses to not walk away and the principal replicates the autarky consumption profile by setting transfers equal to 0 every period. Furthermore, a version of the Revelation Principle applies. Thus, the principal's problem is to choose the plan that maximizes resources such that the agent never reneges and that the agent has no incentive to lie. Denote the probability of a history given a plan ν and a strategy σ by $\pi(\tilde{h}^t|\sigma,\nu)$, where the conditioning variables are dropped when there is no confusion.

Given a plan ν and a strategy σ, an agent's utility at time t' given history $\tilde{h}^{t'}$ is

$$\begin{aligned} U_{t'}(\sigma|\nu,\tilde{h}^{t'-1}) &= \sum_{t=t'}^{\infty}(1-\beta)\beta^{t-t'} \sum_{\tilde{h}^t\in\tilde{H}^t|\tilde{h}^{t'}} u\Big(x(h^{t-1},\alpha_t,\gamma_t,\tilde{\theta}_t) \\ &+ (1-\alpha_t(h^{t-1}))(\theta_t(\tilde{h}^t)-\sigma(\tilde{h}^{t-1},\alpha_t,\gamma_t,\theta_t))\Big)\pi(\tilde{h}^t|\tilde{h}^{t'-1}) \end{aligned}$$

where x,α denote the consumption component and auditing components of plan ν, $\tilde{\theta}_t$ is the agent's t period announcement given the private history, and the notation $\tilde{h}^t \in \tilde{H}^t|\tilde{h}^{t'}$ denotes the set of histories that are consistent with the t'-period private history $h^{t'}$ and h^t denotes the public history consistent with private history $\tilde{h}^t$. Denote the truth-telling strategy σ^* as the strategy such that for all histories and periods $\sigma^*_t(\tilde{h}^t)=\theta_t$.

Given an initial reservation value of autarky, the principal's problem, $(P1)$, can be written as the maximum net present value of resources such that truth-telling is incentive compatible and the agent never chooses to renege:

$$
\begin{aligned}
(P1) \quad \tilde{V} \quad &= \quad \max_{\nu \in N} \sum_t \beta^t \sum_{\tilde{h}^t} \Big[(1-\beta)\left(\theta_t(\tilde{h}^t) - x(h^{t-1}, \alpha_t, \gamma_t, \tilde{\theta}_t)\right) - \\
& \qquad f\left(\alpha(h^t)\right) - g\left(\gamma(h^t)\right)\Big] \pi(\tilde{h}^t|\sigma, \nu) \\
s.t.(PK) \qquad & U(\sigma^*|\nu) = U_0 \\
\forall\, \tilde{\sigma} \in \Sigma; t',\ h^{t'}(IC) \qquad & U_{t'+1}(\sigma^*|\nu, \tilde{h}^{t'}) \geq U_{t'+1}(\tilde{\sigma}|\nu, \tilde{h}^{t'}) \\
\forall\, s,\ t',\ h^{t'-1}(E) \qquad & (1-\beta)u(x(h^{t-1}, \nu_{t',1}(h^{t'-1}), \theta_s)) + \beta U_{t'+1}(\sigma^*|\nu, \tilde{h}^{t'}) \geq \\
& \quad (1-\beta)u(\theta_s) + \beta U_0 - \gamma(h^{t'-1})
\end{aligned}
$$

A dynamic model of costless information and limited commitment is one where $f(\alpha) = 0$ for all α. Similarly, a dynamic model of asymmetric information and costless enforcement is one in which $g(\gamma) = 0$ for all γ. These models are considered separately, respectively, in Sections 1.3 and 1.4. The *General Model* is analyzed in Section 1.5.

Before analyzing the separate models, I first state a proposition following Spear and Srivastava (1987) that allows us to write the problem recursively using the agent's expected utility of the continuation contract, ω as a state variable. The interpretation of the state variable is that this represents the agent's wealth (or indebtedness). I use the recursive model as the starting point for the *Enforcement*, *Auditing*, and *General* models in subsequent sections.

Proposition 1 *The problem* $(P1)$ *can be written recursively as:*

$$
\begin{aligned}
V(\omega) \quad &= \quad \max_{\alpha, \gamma, \{x_s, \omega_s'\}_{s \in S}} \sum_s [(1-\beta)(\theta_s - x_s) - \beta V(\omega_s')]p_s - f(\alpha) - g(\gamma) \\
s.t.\ (IC) \qquad & (1-\beta)u(x_s) + \beta\omega_s' \geq (1-\beta)u(x_{\tilde{s}} + \alpha(\theta_s - \theta_{\tilde{s}})) + \beta\omega_{\tilde{s}}',\ \forall\, s > \tilde{s} \in S \\
(E) \qquad & (1-\beta)u(x_s) + \beta\omega_s' \geq (1-\beta)u(\theta_s) + \beta U_0 - \gamma,\ \forall\, s \in S \\
(PK) \qquad & \sum_s [(1-\beta)u(x_s) + \beta\omega_s']p_s = \omega
\end{aligned}
$$

All omitted proofs appear in Section 1.8.

Notice that the expected transfer to the agent each period is $\mathbf{E}[\theta_s - x_s]$. As $\mathbf{E}[\theta_s]$ is constant, I drop it hereafter when writing the principal's maximization problem.

The following assumptions that will be used throughout the remainder of the chapter:

Assumption 1 *i)* $u : [\underline{c}, \infty) \to (-\infty, \bar{u}]$ *is unbounded below with* $u', -u'' > 0$

ii) u''/u' *is non-decreasing.*

iii) $\lim_{x \to 0} u(x) = -\infty$ *and* $\lim_{x \to +\infty} = \bar{u}$

iv) $f', f'', g', g'' > 0$.

v) $\lim_{\alpha \to 0} f'(\alpha) = 0$, $\lim_{\alpha \to 1} f'(\alpha) = +\infty$, $\lim_{\gamma \to 0} g'(\gamma) = 0$, $\lim_{\gamma \to +\infty} g'(\gamma) = +\infty$

Assumptions i, iii, iv, and v are fairly common assumptions on the regularity properties of the cost functions and the utility function. Note that utility is unbounded below and bounded above and consumption is unbounded above and may or may not be bounded from below. Assumption ii is common in dynamic contracts with private information and states that the agent's absolute risk aversion is non-increasing and is a sufficient condition for the concavity of the value function in the dynamic programming problem.

1.3 Enforcement Model

In this section I consider a special case of the *General Model* in which information is free ($f(\alpha) = 0$ for all $\alpha \in [0, 1]$). Namely, the principal's problem is:

$$V(U_0) \quad = \quad \max_{\gamma, \{x_s, \omega'_s\}_{s \in S}} \sum_s [-(1-\beta)x_s + \beta V(\omega'_s)] - g(\gamma)$$

subject to (E) for all $s \in S$ and (PK). Concavity and differentiability of the value function are established for the *General Model* in Section 1.5. The proof here follows accordingly and is omitted.

Letting λ be the multiplier on the promise keeping constraints and $\xi_s p_s$ be the multipliers on enforcement constraints, the first order conditions and envelope condition for this problem are:

$$\begin{aligned} u'(x_s)(\lambda + \xi_s) - 1 &= 0 \\ V'(\omega'_s) + \lambda + \xi_s &= 0 \\ g'(\gamma) - \sum_s \xi_s p_s &= 0 \\ V'(\omega'_s) &= -\lambda'_s \end{aligned}$$

As in other environments of limited commitment, one implication from this model is that the highest income agent determines the level of enforcement. Namely, if (E) binds for some state s, then it also binds for all $\tilde{s} > s$. It is worthwhile to note that while this is a standard result with limited commitment, it does not hold in the *General Model* considered in Section 1.5.

Lemma 1 *If $\xi_s = 0$ then $\xi_{\tilde{s}} = 0$ for all $\tilde{s} < s$. That is, if the enforcement constraint binds for some income realization $\tilde{s}$ then it also binds for all higher income realizations.*

Proof. Suppose not. Then $(1-\beta)u(x_s) + \beta\omega'_s > (1-\beta)u(\theta_s) + \beta U_0 - \gamma$ and $(1-\beta)u(x_{\tilde{s}}) + \beta\omega'_{\tilde{s}} = (1-\beta)u(\theta_{\tilde{s}}) + \beta U_0 - \gamma$ where $\theta_{\tilde{s}} < \theta_s$ so that the RHS of the former is greater than that of the latter. This implies that either (i) $x_s > x_{\tilde{s}}$ or (ii) $\omega'_s > \omega'_{\tilde{s}}$. For whichever of (i) or (ii) holds, we can increase the variable (say x) for $\tilde{s}$ and decrease it for s in such a way that γ is unchanged and PK holds. The concavity of u and the value function imply that this yields strictly more resources to the principal. ■

The previous lemma is what would be expected from results in the limited enforcement literature. Namely, for any level of promised utility, an agent with the highest income realization has the largest incentive to walk away from the contract.

For low income realizations, an agent does not have an incentive to renege and the principal can perfectly smooth consumption across such realizations. Thus, for each level of incoming promise there is some threshold level of income below which consumption and future promises are smoothed across states and above which the principal receives a net payment from the agent, $\theta_s - x_s > 0$. An agent with a high income realization is then deterred from reneging via increased consumption in future periods.

The dynamics of the agent's promised utility are simple in this environment and can easily be seen from the first order conditions. If the agent receives an income θ_s for which the enforcement constraint does not bind, then $\xi_s = 0$ and $\lambda'_s = \lambda$, implying that the agent's promised utility remains constant. Likewise, consumption is equalized over states in which the enforcement constraint does not bind and is increased when income shocks are sufficiently high. At the moment when the constraint binds, promised utility increases and the process repeats itself. Thus, the continuation promise is weakly increasing over time, eventually reaching a maximum value of $(1-\beta)u(\theta_S) + \beta U_0$ when the highest income state is realized.

In models of efficient insurance and limited commitment, the set of continuation promises that can be reached with positive probability from any state ω (with the exception of that amount) are a subset of $\mathcal{U} \equiv \{(1-\beta)u(\theta_s) + \beta U_0\}_{s \in S}$. The reason for this is that the outside option of an agent who receives an income realization of θ_s is equal to $(1-\beta)u(\theta_s) + \beta U_0$ independent of his promised utility upon entering the period. As the optimal mix of consumption and future promises to delivering this level of utility does not depend on history, the set of possible values for ω is always a subset of $\mathcal{U}$. However, in a model with endogenous enforcement, the enforcement variable and, consequently, the value of the agent's outside option, depend on the contractual state variable ω. Thus, in a model with endogenous enforcement, the continuation promise of an agent receiving income θ_s is history dependent.

In particular, as the continuation promise increases and the autarky option is less attractive to the agent, then enforcement also decreases. This is established in the following proposition and is similar to the result achieved in Koeppl (2007). He studies a case in which two risk averse agents split a fixed amount of resources and finds that enforcement increasing in inequality. Alternatively phrased, his result shows that enforcement decreases as the promised utility of the poorest agent increases. A parallel result is achieved here.

Proposition 2 γ *is strictly decreasing in* ω *when* $\gamma > 0$*.*

Given the dynamics of the continuation promise, the dynamics of enforcement are easily determined. Namely, the use of the enforcement technology is constant over time so long as the enforcement constraint does not bind and decreases when sufficiently high income shocks cause the enforcement constraint to bind and increase future ω. One consequence of this is that in the long run, enforcement goes to zero almost surely. Therefore, the enforcement technology is eventually obsolete.

Proposition 3 $\gamma_t \to 0$ *almost surely.*

Note also that the enforcement technology implies that the principal could implement *any* level of ω, in particular those below U_0. In the *Enforcement Model* where there are no information frictions, the agent's continuation promise is increasing and implementing $\omega < U_0$ is off the equilibrium path. However, this will not be the case in later sections. Furthermore, costly enforcement implies that V is increasing for ω sufficiently small. This can easily be seen as $\gamma \geq U_0 - \omega$ from the enforcement constraints.

1.4 Auditing Model

In this section I consider a model in which the agent has private information on his income shocks and a planner has access to an auditing technology; enforcement is costless ($g(\gamma) = 0$ for all γ).

The efficient contracting problem with only auditing is as follows:

$$\begin{aligned} V(\omega) &= \max_{\alpha,x,\omega'} \sum_s [-(1-\beta)x_s - \beta V(\omega'_s)]p_s - f(\alpha) \\ s.t.\ (IC) & \quad u(x_s) + \beta\omega'_s \geq u(x_{\tilde{s}} + \alpha(\theta_s - \theta_{\tilde{s}})) + \omega'_{\tilde{s}} \qquad \forall \tilde{s} < s \\ (PK) & \quad \sum_s [u(x_s) + \beta\omega'_s]p_s = \omega \end{aligned}$$

Concavity and differentiability of the value function follow a similar argument to the one presented in Proposition 9. The proof is therefore skipped.[9]

The problem can be simplified further by showing that it is enough to consider only local incentive compatibility constraints and that these constraints bind. That is, if for each state s the agent does not have the incentive to lie downward to $s-1$, then all of the incentive compatibility constraints are satisfied. To do this, I first establish that higher income realizations receive higher consumption and lower transfers than lower income realizations in the optimal allocation. This is a straightforward result, as one would expect insurance schemes to generally give a larger transfer payment (or demand a smaller fee) to an agent who realizes low income than to one who realizes high income. The second lemma establishes that it is enough to only consider "local" lies. Define $\Delta_s \equiv \theta_s - \theta_{s-1}$.

Lemma 2 *For all s, $x_{s-1} \leq x_s \leq x_{s-1} + \Delta_s(1-\alpha)$.*

Throughout the remainder of the chapter let $\tilde{u}(x, \alpha, \theta_s - \theta_{\tilde{s}}) = u(x + (1-\alpha)(\theta_s - \theta_{\tilde{s}}))$ denote an agent's contemporaneous utility from reporting an income $\theta_{\tilde{s}}$ when his true income is θ_s.

[9] The one major difference is the domain of V, which is unbounded below here. To adapt the argument for this environment it is sufficient to note that the value function defined here lies between the functions F_1 and F_2 defined in the proof. Furthermore, absent enforcement, the difference between these functions is bounded for all ω, established in the proof of the proposition and the fact that the slope of the value function tends toward 0 as $\omega \to -\infty$. See Thomas and Worrall (1990).

Lemma 3 *Local constraints are enough. Let* $C_{\tilde{s},s} = (1-\beta)[u(x_{\tilde{s}}) - \tilde{u}(x_s, \alpha, \theta_{\tilde{s}} - \theta_s)] + \beta[\omega'_{\tilde{s}} - \omega'_s]$*. Then,* $C_{s+1,s} \geq 0$ *and* $C_{s,s-1} \geq 0$ *for all* s *imply that* $C_{k,s} \geq 0$*.*

Furthermore, it is also the case that all of the local constraints bind. Fixing any level of auditing, if the local constraint were to ever not bind, then the principal could increase consumption to the agent when a low state is realized and decrease consumption when a high state is realized without affecting any of the other IC constraints. From Lemma 2, the marginal consumption under low realizations is greater than under high realizations so that such a scheme would increase the principal's utility.

Lemma 4 *Local constraints bind so that auditing is strictly positive* ($\alpha > 0$) *for all* ω*.*

Auditing is useful to the principal in that it allows for better smoothing of agent's consumption across states. In a world with costless information, if marginal utility is higher in low states than in high states, then a risk neutral planner can increase his utility while holding the agent's utility constant by decreasing consumption in the high state and increasing consumption in the low state. With costly auditing, this requires an increase in auditing costs to keep an agent with high income from lying. The following proposition derives an expression for the optimal auditing when $S = 2$, with $\Delta = \theta_H - \theta_L > 0$ so that the marginal cost of auditing is equal to the principal's benefit from consumption smoothing. A similar, but more cumbersome, expression is easily derived for the case where $S > 2$. The left hand side of the expression can be interpreted as the marginal auditing cost of smoothing consumption, while the right hand side is the marginal benefit to the principal of smoothing the agent's consumption.

Proposition 4 *The optimal allocation satisfies*

$$\frac{f'(\alpha)}{\Delta}\left[1 + \frac{u'(x_L)p_L}{\tilde{u}'(x_L, \alpha, \Delta)p_H}\right] = p_L\left[\frac{u'(x_L)}{u'(x_H)} - 1\right]$$

Proof. Consider the following scheme. Increase α by ϵ/Δ, increase x_L by η_L and decrease x_H by η_H. The cost of this scheme is $-\frac{f'(\alpha)}{\Delta} - \eta_L p_L + \eta_H p_H$. Now, we must show that we can choose values such that IC and PK are satisfied, while decreasing resource costs. Choose $\eta_H = \eta_L \frac{u'(x_L)p_L}{u'(x_H)p_H}$ so that the PK constraint holds. Under this scheme, the LHS of the IC constraint is decreased by $\eta_L \frac{u'(x_L)p_L}{p_H}$ while the RHS is decreased by $(\epsilon - \eta_L)u'(x_L, \alpha, \Delta)$. Choosing $\epsilon = \eta_L \left[1 + \frac{u'(x_L)p_L}{u'(x_L,\alpha,\Delta)p_H}\right]$, the total cost of the scheme is then

$$\eta_L \left[-\frac{f'(\alpha)}{\Delta} \left(1 + \frac{u'(x_L)p_L}{u'(x_L, \alpha, \Delta)p_H} \right) - p_L + \frac{u'(x_L)p_L}{u'(x_H)} \right]$$

which must be less than or equal to zero in an optimal allocation. The opposite perturbation implies that the expression must be greater than or equal to zero. ■

Although auditing is positive for all ω, it vanishes as ω tends toward $-\infty$ and, under an additional assumption on the utility function (which is satisfied by CRRA utility functions), auditing also vanishes as ω approaches its upper bound $\bar{u}$. Auditing tends toward zero in the extremes for different reasons. As ω becomes small, providing incentives for truth-telling to the agent is cheap. The reason for this is that as the marginal utility from consumption increases, small consumption differences across states are enough to induce truth telling. Meanwhile, the agent's high marginal utility implies that the value function is flat as ω diverges to $-\infty$. As the incentive compatibility constraint is inexpensive to satisfy via variations in current and future consumption, the auditing technology becomes unnecessary. In the case of ω large, the agent is already being promised high consumption. The fixed differences between income levels combined with the concavity of the utility function guarantees that the marginal value to the agent from lying is small. Thus, less auditing is necessary to deter him from doing so.

Proposition 5 *Auditing vanishes in the lower utility limit.* $\alpha \to 0$ *as* $\omega \to -\infty$

Proposition 6 *Auditing vanishes in the upper utility limit. If* $\lim_{x \to +\infty} u'(x)/u'(x+\Delta) = 1$ *for* $\Delta = \theta_S - \theta_1$ *(which is true under CRRA with a coefficient greater than 1), then* $\alpha \to 0$ *as* $\omega \to \bar{u}$*.*

Consequently, α reaches a maximum in the interior of $(-\infty, \bar{u}]$ and vanishes in the limits. This is in contrast to the *Enforcement Model* in which enforcement costs are high when the promises are low and are decreasing in ω. One could extend these insights by thinking of this model as the "component planner" problem[10] in a model of efficient insurance with a continuum of agents and a period by period resource constraint equal to the expected value of income. The implications in a more general model with a resource constraint would be that inequality requires large enforcement costs to overcome the friction arising from the agent's limited commitment. However, the *Auditing Model* implies that costs of auditing are *lower* when there are large inequalities and agents are toward the utility extremes and higher when there is more equality.

Perhaps surprisingly, the long run dynamics of the *Auditing Model* resemble that of a model in which information is prohibitively costly. That is, despite the presence of an auditing technology designed to mitigate the information friction, the principal's incentive to front-load consumption and push ω toward where incentives are "cheap" dominates and in the long run the auditing technology becomes obsolete.

Proposition 7 $\omega \to -\infty$ *almost surely.*

Proof. First, it is straightforward to show that λ_t is a martingale by summing over states the first order conditions on ω'_s. Furthermore, as V is decreasing it must be the case that λ is bounded below by 0. Then, by the Martingale Convergence Theorem, λ_t converges almost surely. Suppose λ_t converges to any interior value. This is a contradiction as for every $\omega \in \Omega$ it is the case that $\omega'_L < \omega < \omega'_H$. This implies that $\omega \to -\infty$. ∎

[10]See Atkeson and Lucas (1992)

The robustness of the immiseration result to auditing can be understood in relation to the relative costs of auditing when consumption is front-loaded and back-loaded. An agent with low income or high income realizations values continuation contracts identically, but values current transfers differently. Therefore, the more front-loaded the contract facing an agent, the easier the principal can induce truth-telling by manipulating this difference. Consider an optimal insurance plan and imagine reallocating consumption for the low income realization from the future period into the current period. That is, increase x_L and decrease ω'_L such that the agent's overall utility (and his utility from the low state in particular) is unchanged. What effect does this have on auditing? Denote by ϵ_x and ϵ_ω the increase and decrease, respectively of current consumption and future utility. Then $(1-\beta)\epsilon_x u'(x_L) - \beta\epsilon_\omega = 0$ by assumption. It is clear that such a reallocation of future consumption to current consumption relaxes the IC constraint as the change in the right hand side of the constraint is $(1-\beta)\epsilon_x \tilde{u}'(x_L, \alpha, \Delta) - \beta\epsilon_\omega < 0$ while the left hand side of the IC constraint is unchanged. It follows that front-loading consumption to agent reduces auditing costs. Thus, as in standard models of insurance with private information, the principal continues to have an incentive to front-load payments to the agent.

1.5 General Model

From the *Enforcement* and *Auditing* models one is led to ask, to what extent do the conclusions of these models rely on the assumptions of costless information and costless enforcement, respectively? Looking at the *Auditing Model*, the principal relies heavily on cheap truth-telling incentives for the agent at low levels of ω. However, we know from the *Enforcement Model* that this is precisely when the principal chooses the largest levels of enforcement. Conversely, in the *Enforcement Model* the principal back-loads the agent's consumption, with long run utility equal to the walk-away option when income is high and there is no enforcement. From the *Auditing Model* we know that it is in this interior region where the information friction binds the most tightly.

To answer this question we turn to the *General Model* of costly auditing and enforcement. The conclusions from this section demonstrate that these technologies interact in a non-uniform manner. Thus, in environments with both limited commitment and private information where technologies ameliorating each friction are costly, examining the problems separately may provide misleading conclusions.

In this section I take $\Theta = \{\theta_L, \theta_H\}$ with $\Delta = \theta_H - \theta_L > 0$. The principal solves:

$$\begin{aligned}
V(\omega) \quad &= \quad \max_{\alpha,x,\omega'} \sum_{s=L,H} [-(1-\beta)x_s - \beta V(\omega'_s)]p_s - f(\alpha) \\
s.t.\ (IC) \quad & \qquad (1-\beta)u(x_H) + \beta\omega'_H \geq (1-\beta)u(x_L + (1-\alpha)\Delta) + \beta\omega'_L \\
(E) \quad & \qquad (1-\beta)u(x_s) + \beta\omega'_s \geq (1-\beta)u(\theta_s) + \beta U_0 - \gamma \\
(PK) \quad & \qquad \sum_{s=L,H} [(1-\beta)u(x_s) + \beta\omega'_s]p_s = \omega
\end{aligned}$$

First I establish the regularity properties of the value function. The assumption of non-increasing absolute risk aversion plays an important role in establishing concavity.

Proposition 8 *V is concave.*

The following proposition establishes the differentiability of the value function over a bounded interval of ω. Setting the lower bound sufficiently low, I use differentiability to establish the properties of the value function. It is later shown in Proposition 10 that bounding utility from below is without loss of generality, as low enough values of ω are never reached.

Proposition 9 *For any $\underline{\omega} < \bar{u}$, V is differentiably continuous on $[\underline{\omega}, \bar{u})$.*

Denote the Lagrange multipliers for the IC, E, and PK constraints as η_s, ξ_s, and λ, respectively. Then, the necessary first order conditions and the envelope condition are[11]:

$$
\begin{array}{ll}
x_H: & p_H = (p_H\lambda + \eta + p_H\xi_H)u'(x_H) \\
x_L: & p_L = (p_L\lambda - \eta + p_L\xi_H)u'(x_H) \\
\omega'_H: & p_H V'(\omega'_H) = p_H\lambda + \eta + p_H\xi_H \\
\omega'_L: & p_L V'(\omega'_L) = p_L\lambda - \eta + p_L\xi_L \\
\alpha: & f' = \Delta\tilde{u}'(x_L, \alpha, \Delta)\eta \\
\gamma: & g' = \sum_s p_s\xi_s \\
EC: & V'(\omega) = \lambda
\end{array}
$$

Summing up the first order conditions on ω'_s yields $\lambda + g' = \mathbf{E}[\lambda']$. Therefore λ is a sub-martingale with $\mathbf{E}[\lambda'] > \lambda$ when at least one of the enforcement constraints binds and $\mathbf{E}[\lambda'] = \lambda$ otherwise. There are two opposing forces at work. The first force is from the enforcement constraint. Because keeping agents at a suppressed level of utility is costly, the optimal allocation features an upward push on utility when ω is sufficiently low and is seen in the g' term. When continuation promises are higher and enforcement for both income realizations is unnecessary, the contract resembles that in the *Auditing Model*. In this region, the marginal cost of delivering utility to the planner today is equal to the expected marginal cost of delivering utility to the agent tomorrow. The concavity of the value function then suggests that the agent's expected promise is less than his current promise.

As in the case of the *Auditing Model*, the high income agent never receives a larger transfer $(x_s - \theta_s)$ than the low income agent. This is naturally the case absent the enforcement constraints given that smoothing consumption across states and holding the agent's utility constant reduces the principal's consumption cost. The following lemma establishes that this continues to hold in the *General Model*.

[11] As in the case of Thomas and Worrall (1990) it is not the case that these equations are sufficient. However, optimality requires that they are necessary

Lemma 5 $x_H \leq x_L + (1-\alpha)\Delta$.

In the *Auditing Model*, the value function is downward sloping because the principal could always decrease the agent's utility by decreasing the agent's consumption for the lowest income realization without affecting any of the incentive compatibility constraints. In the *Enforcement Model*, the value function is downward sloping over the set of utilities reached with positive probability. The reason for this is the same as in the *Auditing Model*, noting that starting at U_0, the lowest enforcement constraint never binds. In the model with both auditing and enforcement, this is not necessarily the case. The information friction forces utility below U_0 when the low state is realized (proven in Corollary 2), thereby forcing an increase in future enforcement costs. Therefore, one cannot exclude the possibility that the agent reaches some level of ω for which the value function is upward sloping. Such a level of utility would not be Pareto efficient ex-post, but it may be optimal ex-ante in providing the appropriate truth-telling incentives. The following establishes that the value function is upward sloping for some ω, following from the costly enforcement necessary to induce such a level of utility.[12]

Lemma 6 *There is some ω^* such that $V'(\omega) > 0$ for all $\omega < \omega^*$.*

Notice also that the first order conditions on x_H and ω'_H immediately imply that $V'(\omega'_H) < 0$. This implies that the principal will choose to increase the continuation promise to the decreasing portion of the value function whenever the high state is realized.

Binding Patterns

I now turn to the binding patterns of the enforcement constraints and the incentive compatibility constraints. Studying these binding patterns is useful in determining

[12] I am not currently able to conclude whether such an ω is necessarily reached with positive probability.

the nature of the frictions at alternative points in the state space and, consequently, the principal's choices of enforcement and auditing.

Lemma 7 *For ω sufficient small, EL binds.*

It is shown further, in Lemma 11 in the omitted proofs, that if EL binds for some ω then it also binds for all $\hat{\omega} < \omega$. This is useful in establishing the relationship between enforcement and auditing later in the chapter. The fact that the low enforcement constraint binds when ω is sufficiently low represents an important distinction from similar environments studied in the literature in which incentives to renege are always highest when income is high. I later show that such a value of ω is reached with positive probability.

The following lemma establishes that the auditing technology is used at all nodes in the optimal dynamic contract. This is in contrast to the enforcement technology which is used only when the agent's promised utility is sufficiently low. Unlike the case with only auditing, the fact that the marginal cost of auditing at zero is zero does not guarantee that positive auditing is optimal. The reason for this is that interstate smoothing of consumption may still be costly via an increased need for enforcement. That is, decreasing high income consumption and increasing low income consumption may require an increase in enforcement costs when EH is already binding. The following lemma shows that some positive auditing is always optimal.

The idea behind the proof of Lemma 8 is as follows. An agent with high income has both the option to renege and the option to misreport income that period. If IC were ever to not bind, the agent could hide his entire misreported income and the gap between a high and low income agent's utility under the contract must be large enough to deter him from misreporting. As this gap would be zero under first best insurance, it must be implied from the enforcement constraints. When IC does not bind, the *General Model* resembles the *Enforcement Model* and so, the high income agent's enforcement constraint binds first. The difference in outside option under a high and low income agent is then bounded above by $(1-\beta)[u(\theta_H)-u(\theta_L)]$. As the agent's gain from hiding Δ income must be smaller than this, x_L must be sufficiently large. The agent's promised utility must then be sufficiently high so that no enforcement is necessary, yielding the contraction.

Lemma 8 *The incentive compatibility constraint binds for all ω.*

Binding Low Enforcement Constraint

Having established that auditing is positive for all ω, I now turn to the interaction of the low enforcement constraint with the incentive compatibility constraint. This interaction provides the following lemma, which is unique to this environment and as the primary basis for the difference of this model with others in the literature. In other environments of limited commitment, the common result (as in Lemma 1) is that it is always an agent with the highest income who has the greatest incentive to renege. However, in the *General Model* it is the case that as continuation promises decrease, eventually only the low state enforcement constraint and the incentive compatibility constraint bind. The reason for this is that satisfying the IC constraint when promises are low enough guarantees that the high state enforcement constraint is also satisfied. To understand this, imagine that x_L is small. The IC constraint ensures that the utility in the high state is at least $u(x_L + \Delta(1-\alpha)) - u(x_L) \approx u'(x_L)(1-\alpha)\Delta$ greater than it is if the low state is realized. Meanwhile, if both EL and EH bind, then the utility differential is exactly $u(\theta_H) - u(\theta_L)$. In other words, the gap between the agent's valuation of the outside option is constant across x_L, whereas the implied gap between the high and low income agents' utilities from IC is increases as x_L decreases. As x_L decreases, the IC constraint will then imply that EH holds.

Lemma 9 *Only EL and IC bind as ω sufficiently small.*

As the cost of enforcement increases with lower promised utility, the principal has higher incentives to back-load consumption to the agent and forego future enforcement costs. Consequently, for sufficiently low levels of promised utility the principal increases the agent's promised utility independent of the income shock. This fact implies that ω'_L crosses ω from above and establishes an endogenous lower bound on entitlements.

The lower bound on the agent's utility is determine by the relative costs of consumption, auditing, and enforcement to the principal. However, such a bound necessarily exists under the assumptions of the *General Model*. That is, so long as costs of enforcement are convex, the immiseration result is not obtained. Other papers in the literature establish a similar result by assuming that a principal cannot affect the agent's valuation of his outside option, implicitly assuming that enforcement is infinitely costly. I show that a lower bound on an agent's utility and consumption is necessary even when enforcement is not prohibitively costly.

Proposition 10 *$\omega'_L \geq \omega$ for some ω. There exists some $\underline{\omega} < U_0$ such that for all periods and histories, $\omega \geq \underline{\omega}$.*

Having established the binding patterns when ω is small, I show that at the initial utility level, U_0, only the enforcement constraint for the high income agent binds with the incentive compatibility constraint. This is useful in showing that U_0 falls in the interior of set $[\underline{\omega}, \bar{u})$ and also allows for the demonstration of other properties of the value function and optimal contract at this point.

Proposition 11 *At $\omega = U_0$, only the high enforcement constraint and IC bind.*

Corollary 1 *V is downward sloping at U_0.*

The following corollary follows directly from the previous proposition and first order conditions on continuation promises. The corollary implies that at the autarky

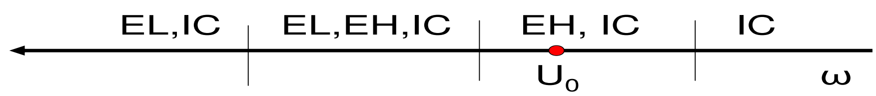

Figure 1.1: Binding Constraints for Values of ω

level of utility, the principal finds it optimal to decrease the agent's promised utility under low income realizations.

Corollary 2 *At U_0 continuation utilities are such that $\omega'_L < U_0 < \omega'_H$.*

Figure 1.1 summarizes the binding patterns of the enforcement and incentive compatibility constraints over the state space. When ω is high, the agent does not have incentives to renege on the contract and the principal uses the auditing technology as a means of mitigating the information friction. For levels of ω around the autarky level, smoothing the agent's consumption across the high and low income states increases the agent's incentives to renege in the high income state and, consequently, high income enforcement constraint binds. For lower levels of ω the benefit to smoothing the agent's consumption across states diminishes and the tension between the incentive compatibility constraint and the enforcement constraints increase. Thus, as ω decreases the low enforcement constraint begins to bind as well. Finally, for ω sufficient low, the tension between the constraints is such that the high enforcement constraint ceases to bind.

The theory of the maximum guarantees that the multipliers and decision variables are continuous in ω. For the region below U_0 it is not possible that only IC binds, as the non-binding of both enforcement constraints (combined with the implied zero enforcement) would contradict the PK constraint. However, we know that as ω

decreases only the EL and IC constraints bind. Therefore, there must be a region in which EL, EH, and IC each bind. Within this region, α and γ are both decreasing in ω. Additionally, it must be the case that $\underline{\omega}$ is contained in a region where EL binds. This follows from the fact that $p_L \xi_L - \eta = 0$ at $\underline{\omega}$ so that EL must bind at the point where the low continuation promise is equal to the current continuation promise.

An interesting implication of the model is that there is a positive probability that the agent will reach a node at which he consumes less than θ_L, the low level of income. That is, after some histories an agent receiving low income is a payor rather than recipient of transfers as part of the efficient insurance plan. While this appears as contrary to the notion of insurance, it is necessary to guarantee the appropriate incentives arising from the information friction.[13] More specifically, low continuation promises are used to induce truth-telling, but are also costly to the principal in terms of enforcement. The latter effect pushes the principal to compensate the agent with more consumption in future periods, reducing future enforcement costs. When ω is sufficiently small and enforcement costs sufficiently high, this effect dominates.

Lemma 10 *For some $\omega > \underline{\omega}$, reached with positive probability, $x_L < \theta_L$.*

Proof. It is enough to show that $x_L < \theta_L$ at $\underline{\omega}$ given continuity of consumption in the state variable. $\underline{\omega}$, by definition, is such that $\omega'_L = \underline{\omega}$ so that $\xi_L = \eta > 0$ and EL binds. The enforcement constraints and promise keeping imply that $U_0 - \gamma \leq \underline{\omega}$. Meanwhile, EL implies that $(1-\beta)u(x_L) + \beta\underline{\omega} = (1-\beta)u(\theta_L) + \beta U_0 - \gamma$. Together, these imply $(1-\beta)(u(x_L) - u(\theta_L)) = \beta(U_0 - \underline{\omega}) - \gamma < 0$. ■

Unfortunately, there is no immediate proof for the existence of an ergodic distribution. While it must be the case that $\omega_t \in [\underline{\omega}, \bar{u})$, there is no uniform bound on the probability that ω reaches any particular value. In fact, as the set of histories is countable, there are only a countable number of values of ω that can be reached with positive probability. As such, for there to exist an ergodic distribution it would

[13] A similar result is obtained in Atkeson (1991)

have to be the case that from some node ω^* there is a non-zero probability that the contract returns to exactly this value. However, the forces of the model suggest that the process ω_t is mean reverting. This follows from the downward pressure on ω' when promises are high and the upward pressure on ω' when promises are low.

1.5.1 Optimal use of Enforcement and Auditing Technologies

This section examines the use of the enforcement and auditing technologies in the optimal contract. While the qualitative use of enforcement in the optimal contract is similar to that when there was no information friction, the presence of the enforcement friction qualitatively changes the use of auditing. The next proposition is akin to that in the *Enforcement Model* and states that enforcement is decreasing in promised utility.

Proposition 12 *Enforcement (γ) is decreasing in ω.*

However, efficient auditing differs dramatically in the presence of the enforcement constraint. The reason for this is that the principal uses auditing not only to smooth the agent's consumption across states, but also to diminish the costs of enforcement that arise at lower levels of promised utility. Unlike standard models of enforcement, enforcement is determined in for some states of ω by the low income realization. Therefore, by increasing auditing and smoothing the agent's consumption across states, the principal can relax the level of enforcement. Because this incentive for auditing increases with lower levels of promised utility, auditing is decreasing for levels of promised utility in which EL binds.

Proposition 13 *Auditing (α) is decreasing in ω whenever EL binds.*

When EL does not bind, auditing is qualitatively similar to the *Auditing Model*. As in Proposition 6, auditing vanishes in the upper limit when the assumption of the proposition is satisfied. When EL does not bind, but enforcement is still positive, the comparative statics of α are unclear for the same reason they are unclear in the *Auditing Model*. Namely, as consumption increases marginal utility from lying is decreased. However, manipulating future promises to the agent to induce truth-telling is also more costly to the principal. The balance of these two forces determines the direction of auditing as ω increases.

Propositions 12 and 13 together imply that an agent near the lower utility bound $\underline{\omega}$ is costly to the principal in terms of auditing and enforcement. While the consumption cost to the principal of an agent with a low promised utility is low, the investment required to suppress the agent at these levels of utility are high. Thus, as the agent's promised utility decreases in this region, an increasingly large percentage of the principal's costs stem from the need to satisfy the enforcement and incentive compatibility constraints rather than the direct costs of providing the agent with consumption. This is examined further in the numerical example.

Auditing and Enforcement as Complements and Substitutes

In this section I address the nature of the relationship between auditing and enforcement. When promised utility is high and EH binds, auditing and enforcement serve as complements as a means of smoothing consumption across the agent's income states. However, when promised utility is low and only EL binds, auditing substitutes for enforcement in that it enables the principal to reduce enforcement investment needed to keep the low income agent suppressed at a low level of utility.

To address this relationship, I suppose that there is an exogenous increase in the level of enforcement and evaluate how this exogenous increase of enforcement affects the principal's choice of auditing. If an exogenous increase in enforcement causes the principal to increase auditing as well I will call auditing and enforcement *complements* and if it causes the principal to decrease auditing then I will call them *substitutes*.

The intuition for this is as follows. Suppose that EL binds and there is an increase in enforcement. The effect of the increase is that the enforcement constraint is loosened. This enables the principal to reduce auditing investment at the cost of de-smoothing consumption across high and low states. When consumption is low, as it is when EL binds, the principal may use small changes in x to de-smooth income and will find it optimal to do so. This is evidenced by the expression at the optimal allocation[14]

$$\begin{aligned} f'\left[1+\frac{p_L u'(x_L)}{p_H u'(x_L+(1-\alpha)\Delta)}\right] &= p_L\left[\frac{u'(x_L)}{u'(x_H)}-1+g'\right] \\ &> p_L\left[\frac{u'(x_L)}{u'(x_H)}-1\right] \end{aligned}$$

The LHS of the equation is the cost of auditing, while the RHS of the inequality is the benefit to the principal of smoothing the agent's consumption across states. Supposing that enforcement were increased so that the EL constraint was slack, the inequality implies that the principal would find it optimal to decrease auditing and un-smooth consumption across states.

On the other hand, as EH binds and EL is slack, increased enforcement allows the principal to better smooth consumption, but only at the cost of additional auditing. In this region, the auditing and enforcement technologies are *complements.* Suppose enforcement was increased in the region in which EH binds and EL is slack. This increase allows the principal to better smooth income across high and low states so long as auditing is also increased. At levels of consumption where EH binds, the increased benefit to the principal from smoothing the agents consumption is greater than the cost of additional auditing. If we look at the region in which EH and IC

[14] To derive this expression, one can use first order conditions. Alternatively, imagine an allocation in which x_L is raised and x_H is decreased. So that PK still holds. When only EL and IC bound initially, this scheme requires an increase of α and decrease of γ to maintain that IC and EL continue to bind. This scheme cannot yield any more resources to the principal than the optimal scheme. We can perform the opposite perturbation as well which then implies the expression.

bind (but not EL), the optimal allocation satisfies:

$$
\begin{aligned}
f'\left[1+\frac{p_L u'(x_L)}{p_H u'(x_L+(1-\alpha)\Delta)}\right] &= p_L\left[\frac{u'(x_L)}{u'(x_H)}-1-\frac{g' u'(x_L)}{p_H u'(x_H)}\right] \\
&< p_L\left[\frac{u'(x_L)}{u'(x_H)}-1\right]
\end{aligned}
$$

Supposing now that enforcement were increased so that the EH constraint were slack, the inequality implies that the cost of auditing in this case is less than the benefit to the principal from smoothing the agent's consumption across states. Therefore, an increase in enforcement in this case is accompanied with an increase in auditing as well.

The argument above, together with a threshold value of ω for which EL binds (see Lemma 11 in the omitted proofs) imply that auditing and enforcement are substitutes below this threshold and complements above it.

1.6 Numerical Example

In this section I explore the properties of auditing and enforcement through a numerical example.

Let $\Theta = \{\theta_L = 1, \theta_H = 5\}$ with $p_L = p_H = 0.5$ and $\beta = 0.9$. I take $u = \frac{c^{1-\sigma}}{1-\sigma}$ with $\sigma = 2$. These imply that $U_0 = -0.6$. For the cost functions let $g(\gamma) = \gamma^2$ and $f(\alpha) = \frac{\alpha^2}{250(1-\alpha)^4}$, $\alpha \in [0,1]$. Note that $f'(0) = g'(0) = 0$, $g', g'', f', f'' > 0$ and $\lim_{\alpha\to 1} f'(\alpha) = \lim_{\gamma\to+\infty} g(\gamma) = +\infty$.

First, consider the *Enforcement Model*. Figure 1.2 shows the decreasing relationship of enforcement, γ, in continuation promises, ω, as is consistent with Proposition 2. As ω_t is weakly increasing over time, ω is shown only for those values greater than the initial promise U_0. Note that some of those ω plotted will not be reached with positive probability, see the left panel of Figure 1.3.

The left panel of Figure 1.3 shows the continuation promises ω'_H and ω'_L as a

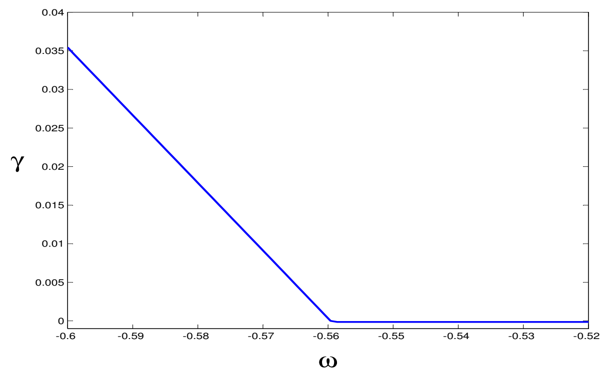

Figure 1.2: Enforcement as a Function of ω

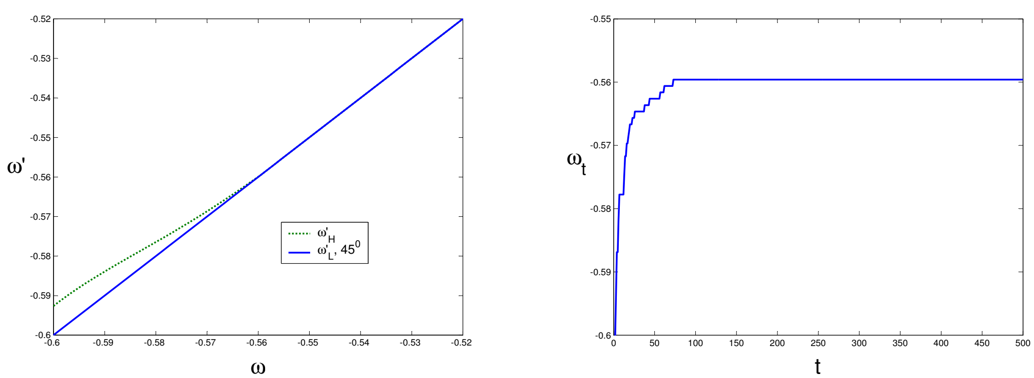

Figure 1.3: Left Panel: Continuation Promises, Right Panel: Sample Path ω

function of state ω as well as the 45^0 line. As there are only two shocks, the enforcement constraint binds only if $\theta_t = \theta_H$, so that $\omega'_L = \omega$ and $\omega'_H \geq \omega$ with strict equality if and only if $\omega < (1-\beta)u(\theta_H)+\beta U_0 = -0.56$. The right panel of Figure 1.3 shows a sample path of ω_t for a draw of income shocks $\{\theta_t\}_{t=1}^{500}$. For the remainder of the section, all sample paths will use this same draw. From the left panel, we see that promised utility will increase whenever a high income draw is realized and remains unchanged when a low draw is realized. As more high draws are realized, the agent's utility approaches -0.56. Contrast this to the standard case in which there is no endogenous enforcement case. In that case, the first draw of θ_H would push ω immediately to -0.56. When enforcement is endogenous, the growth of ω is slowed by the fact that enforcement keeps the high income agent's outside option below -0.56. Thus, promised utility approaches -0.56 more slowly than in the model with no enforcement.

Next, consider the *Auditing Model*. Figure 1.4 shows the relationship of auditing, α, to continuation promises, ω. Note that auditing is vanishing in the limits consistent with Propositions 5 and 6. In this and other numerical examples, auditing is single peaked in promised utility, but I am unable to generally prove that this is the case.

The left panel of Figure 1.5 shows the continuation promises ω'_H and ω'_L as a function of state ω as well as the 45^0 line. Auditing is positive for all interior ω and the incentive compatibility constraint, combined with Lemma 5 implies that $\omega'_L < \omega < \omega'_H$. The right panel of Figure 1.5 shows a sample path of ω_t for the same sample draw of $\{\theta_t\}_{t=1}^{500}$ when auditing costs are specified as above. Note the downward drift of ω_t corresponding to the immiseration result.

Finally, I turn to the *General Model*. As determined in Proposition 12, enforcement is decreasing in ω and is show in Figure 1.8. Note that it behaves qualitatively similarly to that in the *Enforcement Model* depicted in Figure 1.2. Auditing in the *General Model*, on the other hand, differs qualitatively from that in the *Auditing*

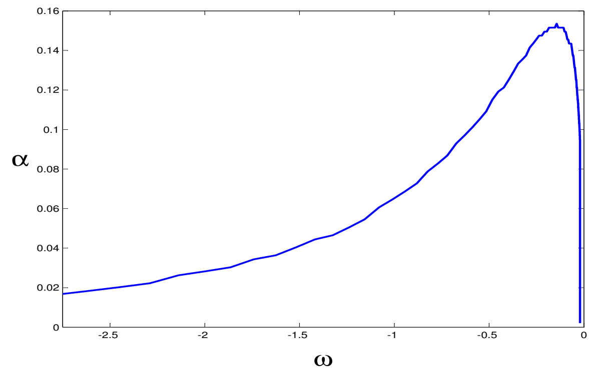

Figure 1.4: Auditing as a Function of ω

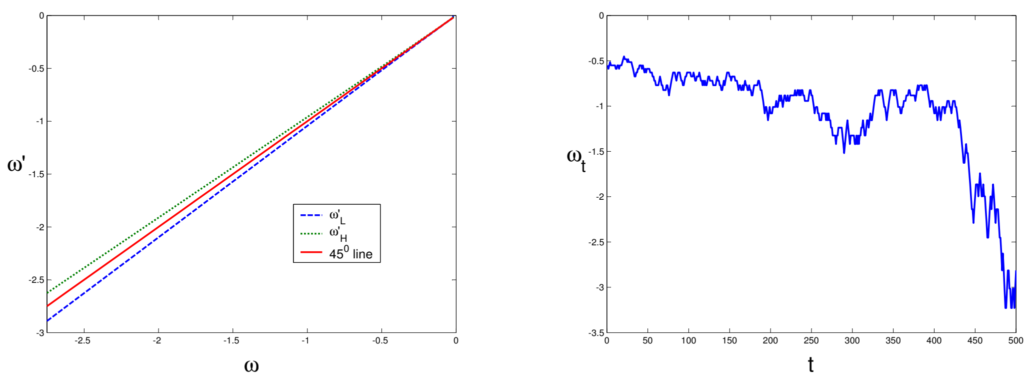

Figure 1.5: Left Panel: Continuation Promises, Right Panel: Sample Path ω

Model. At lower values of ω, auditing is sharply decreasing in ω. In this region, the IC constraint combined with the PK constraint interact so that the low income agent's enforcement constraint binds. The principal then uses auditing as a substitute for enforcement, as it allows for greater inter-state income smoothing. At the point where EL ceases to bind, auditing behaves more similarly to that in the *Auditing Model*. Namely, it is single peaked in this region and vanishes in the upper limit.

As discussed in Section 1.5, there exists an endogenous lower bound $\underline{\omega}$ on the

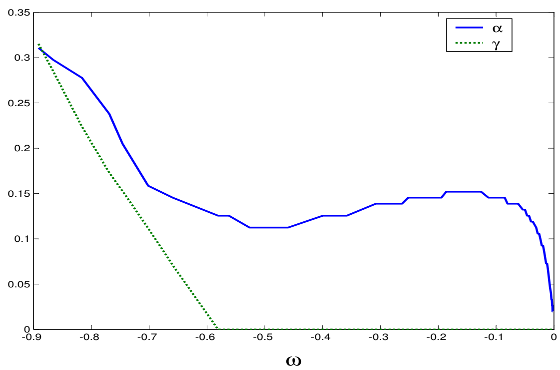

Figure 1.6: Auditing and Enforcement as a Function of ω

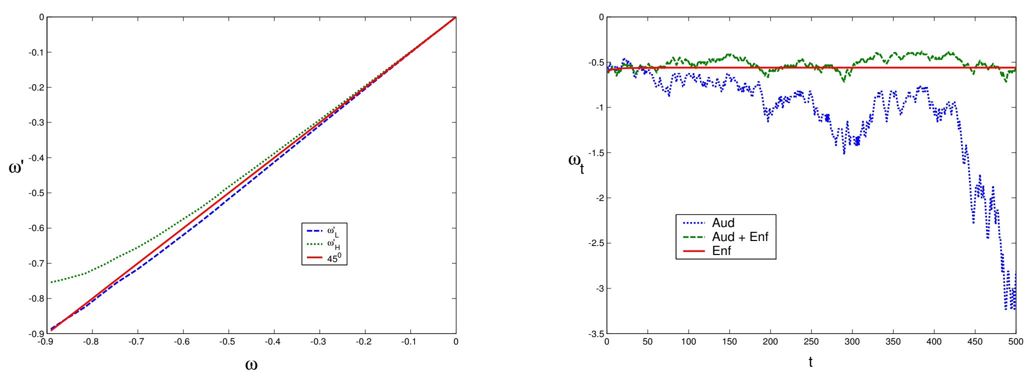

Figure 1.7: Left Panel: Continuation Promises, Right Panel: Sample Path ω

continuation promises of the agent. This is consistent with the left panel of Figure 1.7, which shows the continuation promises from high and low income shocks. When $\omega = \underline{\omega}$, continuation promises are qualitatively like that in the *Enforcement Model*, namely: promises respond positively to high income shocks, but are unchanged by low income shocks. The implication of this is that consumption is heavily back-loaded in this region. As promised utility increases and enforcement decreases, the dynamics more closely resemble that of the *Auditing Model*; high (low) income shocks propel increases (decreases) in the utility promise and, as enforcement vanishes, the utility promise begins to experience a downward drift. In this region, the consumption is predominantly front-loaded.

Given the draw of income shocks as above, the right panel of Figure 1.7 depicts the sample path ω_t. Unlike the other models, ω exhibits mean reversion as a consequence of the forces described in the previous paragraph. The relative costs of auditing and enforcement then play a role in establishing both the mean and volatility of this process.

Finally, consider the components of the principal's costs in the *General Model*. First, there is the principal's cost of delivering consumption to the agent $\mathbf{E}[x_s]$; second, there are the *institutional* costs of auditing and enforcement $f(\alpha)+g(\gamma)$ that are used in providing the appropriate incentives for the agent, but do not directly provide the agent with utility. At the point where EL binds, Propositions 12 and 13 establish that the *institutional* costs increase as ω decreases. Meanwhile, as ω decreases the principal also delivers fewer consumption resources to the agent. Thus, the fraction of that period's costs devoted to consumption, $\mathbf{E}[x_s]/\left(\mathbf{E}[x_s] + f(\alpha) + g(\gamma)\right)$ shrinks as ω declines.

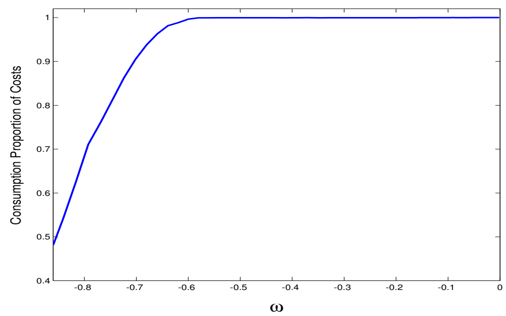

Figure 1.8: Proportion of Costs Devoted to Consumption

1.7 Remarks

In this chapter, I evaluated a problem of efficient insurance under limited commitment and private information frictions when the principal has access to auditing and enforcement technologies. I showed that long run use of auditing depends on the presence of the limited commitment friction and, likewise, long run usage of enforcement depends on the information friction. Furthermore, the combination of frictions implies a lower bound on the promised utility of the agent with the lower bound determined endogenously by the interaction of frictions and technologies. At the lower bound the contract resembles that of the *Enforcement Model* and the incentives to back-load the agent's consumption dominate. At higher levels of promised utility, the contract resembles the *Auditing Model* and the contract is front-loaded. Finally, I show alternatively how the auditing technology acts as a complement to enforcement to smooth consumption at higher levels of promised utility, but is used instead to substitute enforcement at low levels of promised utility.

By examining the extent to which agents are able to insure themselves against idiosyncratic risk, future work could use the analysis of this chapter to determine the relative costs of mitigating information and enforcement frictions. In particular, the cheaper is enforcement relative to auditing, the more front-loaded one should expect consumption. On the other hand, as enforcement is more expensive relative to auditing, the more one should expect consumption to resemble that of the *Enforcement Model* and contracts should be increasingly back-loaded. The framework provided in this chapter may then allow future work to tease out these costs.

Additionally, future work may address the ramifications on institutional choices of inequality. In the single principal and single agent model, I show that an agent with low promised utility requires large investments in both auditing and enforcement in providing incentives. Therefore, one might expect that societies in which a large number of agents near their lower consumption bound would devote a smaller percentage of resources toward consumption.

1.8 Omitted Proofs

Proposition 15: **Proof.** Consider an optimal allocation under $(P1)$. Now, suppose that the principal could reoptimize after some history $h^{t'}$. Clearly, the principal can be no worse off at that node given the option to change plans. Then, it is sufficient to show that the original plan does as well as when given the chance to reoptimize after the history $h^{t'}$. Consider some alternative plan, ν' that is equal to ν along every path, but switches to the reoptimized plan after $h^{t'}$.

By the assumption of reoptimization, the profile ν' satisfies enforcement constraints for all period after t' and incentive compatibility constraints after t'. Furthermore, after history $h^{t'}$ it must guarantee the agent a promised utility of $U_{t'}(\sigma|\nu, h^{t'})$.

Because all future nodes of ν' other than $h^{t'}$ are identical to ν and ν' satisfies the constraints for all periods after $h^{t'}$, it is only possible that ν' violates some constraint for $k < t'$ along the nodes consistent with history $h^{t'}$. Rewrite the agent's utility at some time $k < t'$ and some realized income θ_s as:

$$(1-\beta)u(x_s) + \sum_{t=k+1}^{t'} \sum_{\tilde{h}^t|\tilde{h}^k} [\beta^{t-k}(1-\beta)u(x_t(\cdot)) + \beta^{t'-k}U_{t'}(\sigma|\nu', h^t)]\pi(\tilde{h}^t)$$

It immediately follows that ν' necessarily satisfies promise keeping and enforcement as the LHS of each of those conditions is unchanged moving to plan ν' from ν.

Suppose that ν' violated the IC constraint. Then there is some σ, some $k < t'$ such that $U_k(\sigma|\nu', h^k) > U_k(\sigma^*|\nu', h^k)$. In turn, this implies that:

$$\sum_{t=0}^{k}\sum_{h^t}(1-\beta)\beta^t u(x(\cdot)) \quad + \quad \beta^k U_k(\sigma|\nu', h^{t'}) > \\ \sum_{t=0}^{k}\sum_{h^t}(1-\beta)\beta^t u(x(\cdot)) + \beta^k U_k(\sigma^*|\nu', h^{t'})$$

Because the plans are identical up until time t', this can be rewritten as:

$$\sum_{t=0}^{t'}\sum_{h^t}(1-\beta)\beta^t u(x(h^t)) \quad + \quad \beta^{t'} U_{t'}(\sigma|\nu', h^{t'}) > \\ \sum_{t=0}^{t'}\sum_{h^t}(1-\beta)\beta^t u(x(h^t)) + \beta^{t'} U_{t'}(\sigma^*|\nu', h^{t'}) \tag{1.1}$$

However, by assumption we know that the continuation utilities at t' are identical between ν and ν' so the conditioning on ν' in Inequality (1.1) can be replaced with ν. This implies that the original plan was not incentive compatible yielding a contradiction.

■

Proposition 2: Proof. First note that the concavity of u and convexity of g imply that V is strictly concave (only weak concavity is proven for the general model in Proposition 8). Suppose that the statement did not hold so that for some $\hat{\hat{\omega}} < \tilde{\omega}$ it was the case that $\hat{\hat{\gamma}} \leq \tilde{\gamma}$. As γ is decreasing for ω sufficiently large (it must be 0 in the region for which first best insurance is possible), there is some $\hat{\omega} > \tilde{\omega}$ for which $\hat{\gamma} = \hat{\hat{\gamma}}$. Then, note that for any state s at which the enforcement constraint binds it is the case that $(1-\beta)u(\hat{x}_s)+(1-\beta)\hat{\omega}'_s = (1-\beta)u(\hat{\hat{x}}_s)+(1-\beta)\hat{\hat{\omega}}'_s$ and by the concavity of V and u it must be that for any such state $\hat{x}_s = \hat{\hat{x}}_s$ and $\hat{\omega}'_s = \hat{\hat{\omega}}'_s$. Let $\underline{s}$ be the lowest state for which both $\hat{\xi}_s > 0$ and $\hat{\hat{\xi}}_s > 0$. Consequently, $\sum_{s=\underline{s}}^{S} p_s[\hat{\lambda} + \hat{\xi}_s] = \sum_{s=\underline{s}}^{S} p_s[\hat{\hat{\lambda}} + \hat{\hat{\xi}}_s]$. As $\hat{\lambda} > \hat{\hat{\lambda}}$ it must be the case that $\sum_{s=\underline{s}}^{S} p_s\hat{\xi}_s < \sum_{s=\underline{s}}^{S} p_s\hat{\hat{\xi}}_s = g'(\hat{\gamma})$ so that there is some state $s' < \underline{s}$ for which

$$\begin{aligned}(1-\beta)u(\theta_{s'}) + \beta U_0 - \hat{\gamma} &= (1-\beta)u(\hat{x}_{s'}) + \beta\hat{\omega}'_{s'} \\ &< (1-\beta)u(\hat{\hat{x}}_{s'}) + \beta\hat{\hat{\omega}}'_{s'}\end{aligned}$$

Additionally, by the fact that $\hat{\omega} > \hat{\hat{\omega}}$ there must also exist some state $s'' < s'$ where: $(1-\beta)u(\hat{x}_{s''}) + \beta\hat{\omega}'_{s''} > (1-\beta)u(\hat{\hat{x}}_{s''}) + \beta\hat{\hat{\omega}}'_{s''}$. As utility is constant across states of the double-hat profile for which enforcement constraint does not bind this implies that $(1-\beta)u(\hat{x}_{s''}) + \beta\hat{\omega}'_{s''} > (1-\beta)u(\hat{x}_{s'}) + \beta\hat{\omega}'_{s'}$ yielding a contradiction. ■

Proposition 3: Proof. Note that first order conditions imply that $\mathbf{E}[\lambda'] = \lambda + g'$ so that λ_t is a sub-martingale. Furthermore, when $\omega \geq u(\theta_S) + \beta U_0 = \bar{\omega}$, first best insurance is possible without any enforcement costs so that no enforcement constraints bind and $\omega'_s = \omega$ for all s. For $\omega > \bar{\omega}$ it is the case that the cost of providing the agent additional utility is simply $\lambda = u^{-1}(\omega)$ and as λ'_s is continuous in ω from the Theory of the Maximum, $\max_{\omega,s} \lambda'_s(\omega)$ is bounded on $\omega \in [U_0, \bar{\omega}]$. Therefore, λ_t is a submartingale that is bounded from above and converges almost surely by the Martingale Convergence Theorem. If ever an enforcement constraint binds, it must be that $\lambda'_s > \lambda$, so it must be that $g' = 0$ almost surely. ■

Lemma 2: Proof. Suppose that there are also upward constraints (specified by the C function next defined). It will then be shown that the upward constraints do not bind. Let $C_{s,s-1} = u(x_s) + \beta\omega'_s - u(x_{\tilde{s}} + (1-\alpha)(\theta_s - \theta_{\tilde{s}}))$. Then, consider $C_{s,s-1} + C_{s-1,s} = u(x_s) - u(x_s - \Delta_s(1-\alpha)) + u(x_{s-1}) - u(x_{s-1} + \Delta_s(1-\alpha)) \geq 0$. The concavity of u guarantees that this expression is true only if $x_s \geq x_{s-1}$.

Let $\Delta_s = \theta_s - \theta_{s-1}$. For the second part of the statement, suppose that $x_s > x_{s-1} + \Delta_s(1-\alpha)$. Rearrange terms in the expression above to get,

$$\{u(x_s) - u(x_{s-1} + \Delta_s(1-\alpha))\} - \{u(x_s - \Delta_s(1-\alpha)) - u(x_{s-1})\} \geq 0$$

The LHS of the expression is decreasing in x_s and is equal to zero when x_s is equal to the upper bound specified. ■

Lemma 3: Proof. Consider the case of $k > s$. Note that

$$\begin{aligned} C_{k,s} &= \sum_{j=s}^{k} C_{j+1,j} + \left[\sum_{i=s+1}^{k-1} u(x_i + (1-\alpha)\Delta_{i+1}) - u(x_i)\right] - \\ &\quad [u(x_s + (\theta_k - \theta_s)(1-\alpha))u(x_s + \Delta_{s+1}(1-\alpha))] \end{aligned}$$

Lemma 2 implies that for $i > s$,

$$u(x_i + (1-\alpha)\Delta_{i-1}) - u(x_i) \geq u(x_s + (\theta_{i-1} - \theta_s)) - u(x_s + (\theta_{i-2} - \theta_s))$$

and the first term in brackets is thus larger than the second term in brackets. ■

Lemma 4: Proof.

To show this, assume V is concave and that the problem features upward constraints in addition to downward constraints. It is then shown that (i) Local downward constraints always bind and (ii) the upward constraints are not binding.

(i) To show that $C_{s,s-1} = 0$. Suppose to the contrary that $C_{s,s-1} > 0$, for some s. Then $\omega'_s > \omega'_{s-1}$ by Lemma 2. Consider changing $(x_i, \omega'_i)_{i \in S}$ as follows: keep ω'_1 fixed and if the downward constraint is slack, reduce ω'_s so that the downward constraint binds. Do the same for $s = 2, 3..$ until all downward constraints bind. Add a constant to each ω'_s so as to satisfy promise keeping. Each $\omega'_s - \omega'_{s-1}$ has been reduced increasing the principal's objective. The new contract offers the borrower the same utility and satisfies upward constraints because $x_s \leq x_{s-1} + \Delta_s(1-\alpha)$ still holds and this combined with the downward constraints binding implies that upward constraints also hold.

(ii) Suppose we ignore the constraint $C_{s-1,s}$. If $x_s \leq x_{s-1} + \Delta_s(1-\alpha)$ then by (i) the upward incentive constraint is automatically satisfied. So suppose that the solution has $x_s \geq x_{s-1} + \Delta_s(1-\alpha)$. Then $\omega'_s < \omega'_{s-1}$ and $C_{s-1,s} < 0$. But then replacing $x_{s-1} - (1-\alpha)\theta_{s-1}$ by $x_s - (1-\alpha)\theta_s$ and ω'_{s-1} by ω'_s cannot decrease the principal's objective and cannot violate incentive compatibility. However, by the concavity of the agent's utility function, this increases his utility, yielding a contradiction.

■

Proposition 5: Proof. At the lower limit: First order conditions establish that $\lambda = E[\lambda'_s]$. We know that λ tends toward 0 and given that $\lambda'_L \geq 0$, it must be that λ'_H vanishes as well. Then, from the first order condition $u'(x_H)\lambda'_H = 1$ it must be that $1/u'(x_H)$ vanishes. Then, dividing numerator and denominator by $u'(x_H)$ in Equation 1.2 it must be that the numerator tends toward zero. Given INADA conditions on f, α must be bounded strictly below 1 and the denominator of Equation 1.2 is bounded away from 0. ■

Proposition 6: Proof. At the upper bound: From earlier propositions we have that

$$f' = p_L \frac{\frac{u'(x_L)}{u'(x_H)} - 1}{1 + \frac{p_L u'(x_L)}{p_H u'(x_L+(1-\alpha)\Delta)}} < p_L \frac{\frac{u'(x_L)}{u'(x_L+(1-\alpha)\Delta)} - 1}{1 + \frac{p_L u'(x_L)}{p_H u'(x_L+(1-\alpha)\Delta))}} \tag{1.2}$$

Notice that the denominator is bounded away from zero. As for the numerator, $\lim_{x\to\infty} \frac{u'(x)}{u'(x+(1-\alpha)\Delta)} \leq \lim_{x\to\infty} \frac{u'(x)}{u'(x+\Delta)} = 1$, proving that auditing vanishes at the upper limit. ■

Proposition 8: Proof.

Consider the operator:

$$T\hat{V}(\omega) = \max_{\alpha,\gamma,x,\omega'} \sum_s p_s \left[-(1-\beta)x_s + \beta\hat{V}(\omega'_s)\right] - f(\alpha) - g(\gamma) \tag{1.3}$$

subject to the IC, E, and PK constraints. We will show that V is its fixed point. Consider the space of functions

$$F = \{\hat{F} \in \mathcal{C} : [\underline{\omega}, \bar{u}) \to \mathbf{R} | F_1(\omega) \leq F(\omega) \leq F_2(\omega), \omega \in [\underline{\omega}, \bar{u})\}$$

where $\mathcal{C}$ is the set of continuous functions defined on the appropriate domain. First, we show that F is a complete metric space in the supremum metric and T is a contraction mapping on F.

Using the sup norm, we will construct F_2 and F_1 so that the distance between these two functions bounded, which implies that F is a complete metric space. The upper bound function F_2 is the first best allocation, in which $F_2(\omega) = u^{-1}(\omega)$. The lower bound function F_1 grants each agent his income plus/minus a constant y and pays the necessary enforcement cost associated with this. To guarantee that this satisfies PK it must be that $\sum_s [u(y + \theta_s)]p_s = \omega$ and that the enforcement cost is

$$\bar{\gamma}(\omega) = \max_s \{(1-\beta)u(\theta_s) + \beta U_0 - [(1-\beta)u(y+\theta_s) + \beta\omega]\}.$$

As ω is bounded from below, the difference between these two functions is bounded. To show this, note that $\sum_{s\in S} u(y+\theta_s) = u^{-1}$ for all ω so that $y + \theta_1 \leq u^{-1}$ and for all s, $y + \theta_s + \theta_1 \leq y + \theta_N + \theta_1 \leq u^{-1} + \theta_N$. Then, the difference in cost to the principal (ignoring enforcement) between the two plans is $\sum_s [y+\theta_s - u^{-1}] \leq \theta_N - \theta_1$. Given that γ is bounded above from the plan associated with F_1, difference between F_1 and F_2 must be bounded. By construction, $T(V)$ lies in F and, in addition, Bellman's sufficient conditions are satisfied. Therefore, T is a contraction mapping on a complete metric space, so that there exists a unique fixed point V.

Next, we show that T maps concave functions into concave functions. As V is the fixed point of the contraction mapping T, and the set of concave functions is closed, V is then also concave. Consider any $\hat{\omega} < \hat{\hat{\omega}}$ with corresponding contracts $\{\{\hat{x_s}, \hat{\omega'}_s\}_{s\in S}, \hat{\alpha}, \hat{\gamma}\}$ and $\{\{\hat{\hat{x}}_s, \hat{\hat{\omega}}'_s\}_{s\in S}, \hat{\hat{\alpha}}, \hat{\hat{\gamma}}\}$. Consider some $\omega = (1-\delta)\hat{\omega} + \delta\hat{\hat{\omega}}$, $\delta \in [0,1]$. Assume that V is concave and consider the contraction mapping in Equation 1.3. We want to show that $TV(\omega) \geq (1-\delta)TV(\hat{\omega}) + \delta TV(\hat{\hat{\omega}})$.

Construct the following contract $\{\{x_s^*, \omega_s'\}_{s\in S}, \alpha^*, \gamma^*\}$ such that:

$$
\begin{aligned}
f(\alpha^*) &= (1-\delta)f(\hat{\alpha}) + \delta f(\hat{\hat{\alpha}}) \\
u(x_s^*) &= (1-\delta)u(\hat{x}_s) + \delta u(\hat{\hat{x}}_s) \\
\omega_s^{*\prime} &= (1-\delta)\hat{\omega}' + \delta\hat{\hat{\omega}}_s' \\
\gamma^* &= (1-\delta)\hat{\gamma} + \delta\hat{\hat{\gamma}}
\end{aligned}
$$

and note that by convexity of f, concavity of u, V, it is the case that $\alpha^* \geq (1-\delta)\hat{\alpha}+\hat{\hat{\alpha}}$, $x_s^* \leq (1-\delta)\hat{x}_s + \hat{\hat{x}}_s$ and that $V(\omega_s^{*\prime}) \geq (1-\delta)V(\hat{\omega}_s') + \delta V(\hat{\hat{\omega}}_s')$ so that this yields at least as must resources to the principal as the convex combination. Clearly, this scheme satisfies promise keeping and enforcement given that the plans at $\hat{\omega}$ and $\hat{\hat{\omega}}$ satisfied those constraints. Then, we must show that the starred plan satisfies incentive compatibility. There are two cases, one in which the α and x move in the same direction, in which case concavity follows from standard arguments. In the case where the α and x move in opposite directions $u(x^* + (1-\alpha^*)\Delta)$ need not be less than the convex combination of $u(\hat{x} + (1-\hat{\alpha})\Delta)$ and $u(\hat{\hat{x}} + (1-\hat{\hat{\alpha}})\Delta)$, complicating the proof.

Case 1: Suppose that α and x move in the same direction: $(\hat{\alpha} - \hat{\hat{\alpha}})(\hat{x} - \hat{\hat{x}}) \geq 0$.

We must show that $C^* = (1-\beta)u(x_H^*)+\beta\omega_H^{*\prime} - [(1-\beta)u(x_L^* + (1-\alpha^*)\Delta) + \beta\omega_L^{*\prime}] \geq 0$. Consider $(1-\delta)\hat{C}_{H,L} + \delta\hat{\hat{C}}_{H,L} \geq 0$, where the C terms represent the incentive constraints for the appropriate contracts. Note that

$$
\begin{aligned}
C^* = (1-\delta)\hat{C}_{H,L} + \delta\hat{\hat{C}}_{H,L} \quad &+ \quad (1-\beta)[(1-\delta)u(\hat{x} + (1-\hat{\alpha})\Delta) \\
&+ \quad \delta u(\hat{\hat{x}} + (1-\hat{\hat{\alpha}})\Delta) - u(x_L^* + (1-\alpha^*)\Delta)]
\end{aligned}
$$

so given that the IC constraints hold for the hat contracts by assumption, it is enough to show that $(1-\delta)u(\hat{x} + (1-\hat{\alpha})\Delta) + \delta u(\hat{\hat{x}} + (1-\hat{\hat{\alpha}})\Delta) - u(x_L^* + (1-\alpha^*)\Delta) \geq 0$. Replace all of the α terms with $(1-\delta)\hat{\alpha} + (1-\delta)\hat{\hat{\alpha}}$. This will decrease the sum of the first two terms as more weight is put on the term with the lower marginal utility (by the assumption of Case 1) and the second term is more negative as α^* is greater than this term. Then, the case is proven by the assumption u''/u' increasing.

Case 2: Suppose that α and x move in opposite directions: $(\hat{\alpha} - \hat{\hat{\alpha}})(\hat{x} - \hat{\hat{x}}) < 0$.

By way of contradiction, suppose that TV is not concave and assume that $TV(\omega)$ lies below secant connecting $TV(\hat{\omega})$ and $TV(\hat{\hat{\omega}})$ for δ sufficiently close to 0.[15] Then, it must be the case that the starred profile violates incentive compatibility for such a δ (otherwise the starred profile would be better than the convex combination). The following is true (using the inverse function theorem) of the starred profile at $\delta = 0$:

$$\frac{\partial \alpha^*}{\partial \delta} = \frac{f(\hat{\hat{\alpha}}) - f(\hat{\alpha})}{f'(\hat{\alpha})}$$
$$\frac{\partial x^*}{\partial \delta} = \frac{u(\hat{\hat{x}}) - u(\hat{x})}{u'(\hat{x})}$$

So that at $\delta = 0$,

$$\begin{aligned}\frac{\partial C^*}{\partial \delta} &= -u'(\hat{x} + (1-\hat{\alpha})\Delta)) \left[\frac{u(\hat{\hat{x}}) - u(\hat{x})}{u'(\hat{x})} - \frac{f(\hat{\hat{\alpha}}) - f(\hat{\alpha})}{f'(\hat{\alpha})} \right] \\ &\quad + u(\hat{\hat{x}} + (1-\hat{\hat{\alpha}})\Delta) - u(\hat{x} + (1-\hat{\alpha})\Delta)\end{aligned}$$

Suppose that $\hat{x} < \hat{\hat{x}}$ so that $\hat{\alpha} > \hat{\hat{\alpha}}$, an identical argument will hold for the opposite case. Then,

$$\begin{aligned}\frac{\partial C^*}{\partial \delta} &\geq -u'(\hat{x} + (1-\hat{\alpha})\Delta)) \left[\frac{u(\hat{\hat{x}}) - u(\hat{x})}{u'(\hat{x})} \right] + u(\hat{\hat{x}} + (1-\hat{\alpha})\Delta) - u(\hat{x} + (1-\hat{\alpha})\Delta) \\ &\geq 0 \\ \Leftrightarrow &\quad -\frac{u(\hat{\hat{x}}) - u(\hat{x})}{u'(\hat{x})} + \frac{u(\hat{\hat{x}} + (1-\hat{\hat{\alpha}})\Delta) - u(\hat{x} + (1-\hat{\hat{\alpha}})\Delta)}{u'(\hat{x} + (1-\hat{\alpha})\Delta))} \geq 0\end{aligned}$$

The following claim then concludes the proof.

Claim: For any $\Delta > 0$, $z(x, \Delta) = \frac{u(x+\Delta) - u(x)}{u'(x)}$ is increasing in x by Assumption ii.

Proof: Denoting z_x as the partial with respect to x,

$$z_x(x, \Delta) = \frac{[u'(x+\Delta) - u'(x)]u'(x) - u''(x)[u(x+\Delta) - u(x)]}{(u'(x))^2} \geq 0$$

[15] If this is not true, then define $\hat{\omega}$ as the ω at which this is true.

if and only if $[u'(x+\Delta)-u'(x)]u'(x)-u''(x)[u(x+\Delta)-u(x)] > 0$, which is increasing in Δ by Assumption ii. The claim is proven by noting $\lim_{\Delta\to 0} z_x(x,\Delta) = 0$. ■

Proposition 9: Proof.

To show that V is differentiably continuous we adapt the argument of Thomas and Worrall (1990). Consider a neighborhood of ω around any $\tilde{\omega}$, and consider a contract that satisfies IC, E, and PK in which α and ω' are held constant at their values implied by $\tilde{\omega}$ so that only x and γ are varied to yield ω in which IC continues to bind. There is a unique way to do this construction and the cost of doing this is differentiably continuous and yields a utility to the principal less than or equal to to $V(\omega)$ with equality holding at $\tilde{\omega}$. That V is continuously differentiable then follows from Lemma 1 of Benveniste and Scheinkman (1979). ■

Lemma 5: Proof. Suppose not. First, show that negation of the statement implies that IC binds. Suppose not so that $\omega'_L, \omega'_H > \omega$. Consider the cases (i) $x_L \geq \theta_L$ and (ii) $x_L < \theta_L$. For case (i) note that PK and $\omega'_L, \omega'_H > \omega$ imply that $\omega > p_H u(x_H) + p_L u(x_L) > U_0$ so that no constraints bind. This implies first best insurance yielding a contradiction. Case (ii) note that $x_L < \theta_L$ implies that if the IC and EL constraints holds, then the EH constraint holds and is slack as the high income agent's utility is more than $u(x_L+\Delta) - u(x_L)$ greater than the low income agent's utility. Then, the principal could be better off by raising x_L and decreasing x_H, fixing the agent's utility, with no additional enforcement or auditing costs. Thus, IC must bind.

Given that IC binds, if $x_L + (1-\alpha)\Delta < x_H$ then it must be that $\omega'_L \geq \omega'_H$. First order conditions imply that $u'(x_H)\lambda'_H = 1$ and $u'(x_L)\lambda'_L + \eta(u'(x_L) - u'(x_L, \alpha, \Delta))/p_L = 1$. Since $u'(x_H) < u'(x_L)$ and $\lambda'_H \leq \lambda'_L$, this is a contradiction. ■

Lemma 6: Proof. By the concavity of V we need only establish that there exists some ω such that $V'(\omega) > 0$. Note that the enforcement constraints combined with the promise keeping constraint imply that $U_0 - \gamma < \omega$ so that γ increases indefinitely as ω decreases. Now, $V(U_0) \geq E[\theta]$ as setting $\gamma = \alpha = 0$ and $x_s = \theta_s$ satisfies all conditions. Therefore, $V'(\omega) > 0$ for some $\omega < U_0$. ∎

Lemma 7: Proof. First order conditions imply that $(1-\beta)u'(x_L)(p_L\lambda+p_L\xi_L)-\eta\tilde{u}(x_L, \alpha, \Delta) > 0$. As $\lambda < 0$ for ω sufficient small, it must be that $\xi_L > 0$ for all such ω. ∎

Lemma 8: Proof. We show for each possibility of enforcement constraints binding that IC must bind as well:

i) Neither EL nor EH bind. It is immediate that IC must bind, otherwise this would imply implementation of the efficient allocation.

ii) EH and EL bind. This implies that the difference between the high income agent's utility and low income agent's utility is $(1-\beta)(u(\theta_H) - u(\theta_L))$. If IC does not bind, this implies that the difference between the high income agent's utility and low income agent's utility is greater than $u(x_L + \Delta) - u(x_L)$ which implies that $x_L > \theta_L$. Both EL and EH binding, but IC not binding implies that $\eta = 0$ so that $\omega'_L \geq \omega$ by the concavity of V. The PK constraint then implies that $(1-\beta)[p_H u(x_L + \Delta) + p_L u(x_L)] + \beta\omega'_L < \omega$ so that $p_H u(x_L + \Delta) + p_L u(x_L) < \omega$. However, by $x_L > \theta_L$ this implies that $\omega > U_0$ which implies that $\gamma = 0$, contradicting the binding of EL and EH.

iii) EL only binding. This implies that $\omega'_L > \omega = \omega'_H$. The IC constraint then implies that $u(x_H) > u(x_L + \Delta)$ contradicting Lemma 2.

iv) EH only binding. This implies that $(1-\beta)u(\theta_H)+\beta U_0-\gamma > (1-\beta)u(x_L+\Delta)+\beta\omega'_L > (1-\beta)u(\theta_L)+\beta U_0-\gamma$. This in turn implies that $u(\theta_H)-u(\theta_L) > u(x_L+\Delta)-u(x_L)$ so that $x_L > \theta_L$. Furthermore, $x_L+\Delta$ is an upper bound on x_H and FOC (along with IC, EL not binding) imply that $\omega'_L = \omega$. Together, these imply that $\omega < U_0$. The PK constraint and IC imply further that $(1-\beta)[p_H u(x_L+\Delta)+p_L u(x_L)]+\beta\omega < \omega$ implying that $U_0 < p_H u(x_L + \Delta) + p_L u(x_L) < \omega$ yielding a contradiction, where the first inequality follows from $x_L > \theta_L$.

$\blacksquare$

Lemma 9: Proof. Suppose not, so that EH also binds. Both enforcement constraints binding imply $\tilde{u}(x_L, \alpha, \Delta) - u(x_L)) = u(\theta_H) - u(\theta_L)$ so that $\alpha = \phi(x_L) \equiv \frac{1}{\Delta}\left[x_L + 1 - u^{-1}(u(x_L) + u(\theta_H) - u(\theta_L))\right]$. From first order conditions $\eta = \frac{f'}{\Delta \tilde{u}'(x_L, \alpha, \Delta)}$. Using the inverse function theorem and L'Hopital's rule:

$$\lim_{x_L \to 0} \eta = \lim_{x_L \to 0} \frac{f''(\phi)\left[1 - u'(x_L)\frac{1}{u'(x_L)}\right]}{\tilde{u}''(x_L, \alpha, \Delta)(1 - \phi')} = 0$$

From first order conditions $\lambda_L' + \frac{1}{u'(x_L)}\eta(u'(x_L) - \tilde{u}'(x_L, \alpha, \Delta)) = \frac{p_L}{u'(x_L)}$ and $\lambda_H' = \frac{p_H}{u'(x_L)}$. As ω decreases it is then then the case that λ_s' tends toward zero for $s = L, H$ and continuation promises are equal, as η vanishes. By the IC constraint binding this implies that it cannot be the case that $\tilde{u}(x_L, \alpha, \Delta) - u(x_L) = u(\theta_H) - u(\theta_L)$ as must be true when IC, EH, and EL bind, yielding a contradiction. $\blacksquare$

Lemma 11 *If EL binds for ω, then it also binds for all $\hat{\omega} < \omega$.*

Proof. First, note that at the greatest ω for which EL binds it is the case that $\omega < U_0$ and so it must also be the case that EH binds as well (this follows the proof from Proposition 11). Furthermore, if EL does not bind at $\hat{\hat{\omega}}$ then it must be that $V'(\hat{\hat{\omega}}) < 0$. Finally, there must also be some $\hat{\omega} \in [\hat{\hat{\omega}}, \omega)$ where both EL and EH hold with equality and $\xi_L = 0$. Then, $\omega - \hat{\omega} = u(\theta_H) - u(\theta_L)$. Furthermore $\alpha = \phi(x_L)$ where $\tilde{u}(x_L, \phi, \Delta) - u(x_L) = u(\theta_H) - u(\theta_L)$. Then, as x_L decreases it is straightforward to show that $\eta = \frac{f'}{\Delta \tilde{u}(x_L, \phi(x_L), \Delta)}$ increases. Furthermore it must be that $\hat{\gamma} - \gamma = \omega - \hat{\omega}$. As $\hat{\eta} > \eta$ and $p_H \hat{\xi} = g'(\hat{\gamma}) > g'(\gamma) = p_H \xi_H + p_L \xi_L$ it must be that $\hat{\lambda}_H' - \hat{\lambda}_L' > \lambda_H' - \lambda_L'$, yielding a contradiction. $\blacksquare$

Proposition 10: Proof.

For some ω only EL and IC bind. First order conditions on continuation promises yield

$$p_L \lambda_L' + p_H \lambda_H' = \lambda + g'. \tag{1.4}$$

EH not binding implies $\xi_L = g'$ so that $u'(x_L)p_L(\lambda + g') - \eta u'(x_L + (1-\alpha)\Delta) = p_L$ implying that $\lambda + g' > 0$. From the first order conditions on ω'_H and x_H we have that $u'(x_H)\lambda'_H = p_H$ so that $\lambda'_H > 0$ for all ω. Moreover, as ω decreases it is the case that λ'_H tends toward 0. Thus, from the Equation 1.4 it cannot be the case that λ'_L decreases indefinitely with λ as ω decreases.

The lower bound is then established by the fact Theory of the Maximum, which guarantees that the multipliers are continuous in ω. Finally, the fact that $\underline{\omega} < U_0$ follows from Proposition 11. ■

Proposition 11: Proof. To show this I show that other possible binding combinations are not possible. i) Suppose neither EL nor EH were to bind. This immediately implies that $\gamma = 0$ and $\sum_s [(1-\beta)u(x_s) + \beta\omega'_s] > U_0$ yielding a contradiction.

ii) Suppose only EL were to bind. Imagine that the planner did not have to pay $g(\gamma)$ for the first period only, so that the objective function was $\sum_s[-(1-\beta)x_s + \beta V(\omega'_s)]p_s - f(\alpha)$ subject to PK, E, IC. Call this problem (P'). The solution to this revised problem looks like a problem with private information in which income is shared across states $(\theta_L < x_L < x_H < \theta_H)$, utility is shared across states $u(\theta_L) + \beta U_0 < u(x_L) + \beta\omega'_L < u(x_H) + \beta\omega'_H < u(\theta_H) + \beta U_0$ and the following relation holds: $u'(x_H)\lambda'_H = p_H$. Call the original problem (P) and the solution to this problem $\mathcal{C}_1$ and the maximum V^*. Clearly, the planner could implement this allocation in the original problem (in which the principal pays $g(\gamma)$ by choosing $\mathcal{C}_1$ and setting $\gamma_1 = u(\theta_H) + \beta\omega'_H - u(x_H) - \beta\omega'_H$. Call the principal's utility from this plan V_1 and note that $V^* = V_1 + g(\gamma_1)$. Now, suppose that EL bound in the optimal allocation, $\mathcal{C}_2$ and the maximum principal utility was V_2. Note that the fact that $\mathcal{C}_2$ was not chosen in the revised problem implies that $V^* > V_2 + g(\gamma_2)$, where γ_2 is the first period enforcement technology under the optimal allocation. Abusing notation, note that $u'(x_H)\lambda'_H = p_H$ holds in this case as well. Since $V_2 > V_1$ by assumption, it must be that $\gamma_2 < \gamma_1$, which will lead to the contradiction. Note that $\gamma_1 =$

$(1-\beta)[u(\theta_H)-u(x_{H,1})]+\beta(U_0-\omega'_{H,1})$ and $\gamma_2 > (1-\beta)[u(\theta_H)-u(x_{H,2})]+\beta(U_0-\omega'_{H,2})$, where subscript H, i infers the first period variable implied by $\mathcal{C}_i$. $\gamma_2 < \gamma_1$ together with $u'(x_H)\lambda'_H = p_H$ and the concavity of the value function imply that $x_{H,1} < x_{H,2}$. Consider a perturbation of the optimal allocation of the (P') problem in which $x_{L,1}$ is increased, $x_{H,1}$ is decreased and α_1 is increased and vice versa. Optimality requires that:

$$\frac{p_L}{p_H u'(x_{H,1})} = \frac{p_L}{u'(x_{L,1})} + \frac{f'}{\Delta}\left[\frac{p_L u'(x_{L,1}) + u'(x_{L,1} + (1-\alpha)\Delta)}{p_H u'(x_{L,1})u'(x_{L,1} + (1-\alpha_1)\Delta)}\right] \tag{1.5}$$

Note that a similar perturbation implies the following for problem (P):

$$g' + \frac{p_L}{p_H u'(x_{H,2})} = \frac{p_L}{u'(x_{L,2})} + \frac{f'}{\Delta}\left[\frac{p_L u'(x_{L,2}) + u'(x_{L,2} + (1-\alpha)\Delta)}{p_H u'(x_{L,2})u'(x_{L,1} + (1-\alpha_2)\Delta)}\right] \tag{1.6}$$

If EL binds, then IC, PK imply that $U_0 = u(x_{L,2}) + \beta U_0 + p_H[u(x_{L,2} + (1-\alpha)\Delta) - u(x_{L,2})]$ where the second term must be greater than $u(\theta_H) - u(\theta_L)$, which is possible only if $x_{L,2} \leq \theta_L$. Finally, note that $\alpha_1 > \alpha_2$ follows from the IC constraint and the fact that $x_{H,1} < x_{H,2}$, $x_{L,1} > x_{L,2}$ and that $u(x_{H,1}) + \beta\omega'_{H,1} < u(x_{H,2}) + \beta\omega'_{H,2}$. Looking toward equations 1.5 and 1.6 this yields a contradiction, as the LHS of equation 1.6 is greater than that of 1.5 and the RHS of 1.6 is less than that of equation 1.5, yielding a contradiction. The latter point requires that the term with f' is decreasing in α, which is easily shown.

iii) Both EL and EH bind. IC must bind, otherwise we can perturb the allocation to smooth utility across states at no cost $f'(0) = g'(0) = 0$. All constraints binding imply that $u(x_L + (1-\alpha)\Delta) - u(x_L) = u(\theta_H) - u(\theta_L)$ implying that $x_L \leq \theta_L$ from the curvature of u. Because $\gamma = 0$, first order conditions imply that $\omega'_L < U_0$. Thus, $(1-\beta)u(x_L) + \beta\omega'_L < (1-\beta)u(\theta_L) + \beta U_0$ yielding a contradiction.

iv) If EH only binds, the IC must bind. If EH binds then $(1-\beta)u(x_L) + \beta\omega'_L > (1-\beta)u(\theta_L) + \beta U_0 - \gamma$. The IC constraint and the concavity of u imply that if $\alpha = 0$ so IC does not bind, the high income agent's utility is at least $u(\theta_H) - u(\theta_L)$ higher than the low income agent's, yielding a contradiction. ■

Corollary 1: Proof. If only EH binds then consider reducing the agent's utility by some small amount ϵ. The principal can guarantee $U_0 - \epsilon$ level of utility by replicating the allocation at U_0 and reducing x_L by $\epsilon/u'(x_L)$. Because EL is not binding, for ϵ sufficiently small, the EL constrain will still hold. Furthermore, reducing x_L loosens the IC constraint, while having no impact on the EH constraint. Thus, the principal is strictly better off at $U_0 - \epsilon$. ■

Lemma 12: Proof. Claim: if EH and EL both bind for any ω then for any $\hat{\omega} < \omega$ it is the case that the corresponding levels of enforcement satisfy $\hat{\gamma} > \omega$. Suppose not so $\hat{\gamma} \leq \gamma$. Then, at ω it is the case that

$$(1-\beta)u(x_s) + \beta\omega_s' = (1-\beta)u(\theta_s + \beta U_0 - \gamma$$
$$\leq (1-\beta)u(\theta_s) + \beta U_0 - \gamma \leq (1-\beta)u(\hat{x}_s) + \beta\hat{\omega}_s'$$

Summing over states this implies that $\omega \leq \hat{\omega}$ yielding a contradiction.

Given the claim, we need only prove that γ is strictly decreasing when only EH and IC bind. When EL does not bind, then $p_H\xi_H = g'$. Suppose that γ is not strictly decreasing over the region in which only EH and IC bind. Then, continuity implies that there must be some $\hat{\omega} < \hat{\hat{\omega}}$ at which $\hat{\gamma} = \hat{\hat{\gamma}}$. As only EH binds it must be the case that $(1-\beta)u'(\hat{x}_H) + \beta\hat{\omega}_H' = (1-\beta)u'(\hat{x}_H) + \beta\hat{\omega}_H'$ and by the concavity of the value function and u, this implies that $\hat{x}_H = \hat{\hat{x}}_H$ and $\hat{\omega}_H' = \hat{\hat{\omega}}_H'$. By IC binding it is also the case that $(1-\beta)\tilde{u}(\hat{\hat{x}}_L, \hat{\hat{\alpha}}, \Delta) + \beta\hat{\hat{\omega}}_L' = (1-\beta)\tilde{u}(\hat{x}_L, \hat{\alpha}, \Delta) + \beta\hat{\omega}_L'$ and by PK it must be that $p_L[(1-\beta)(u(\hat{x}_L) - u(\hat{\hat{x}}_L) + \beta(\hat{\omega}_L' - \hat{\hat{\omega}}_L')] = \hat{\hat{\omega}} - \hat{\omega}$. Together these imply that $\hat{x}_L - \hat{\hat{x}}_L < \hat{\hat{\alpha}} - \hat{\alpha} > 0$. Given optimality at $\hat{\omega}$, this implies the principal could be strictly better off by reducing $\hat{\hat{\alpha}}$ slightly and adjusting consumption and promised utility accordingly. ■

Proposition 13: Proof. Suppose that EL and EH bind. Then $u(x_L + (1 - \alpha)\Delta) - u(x_L) = u(\theta_H) - u(\theta_L)$. By the concavity of u, it is clear that α decreases as x_L increases.

Now, consider some $\hat{\omega}$ at which only EL binds and consider $\hat{\hat{\omega}} < \hat{\omega}$ sufficiently close to $\hat{\omega}$ such that only EL binds at $\hat{\hat{\omega}}$. Suppose, by way of contradiction, that $\hat{\hat{\alpha}} \leq \hat{\alpha}$. Using EL and IC, it is the case that $(1-\beta)u(\theta_L)+\beta U_0-\hat{\gamma}+(1-\beta)p_H[u(\hat{x_L}-(1-\hat{\alpha})\Delta)-u(\hat{x_L})] = \hat{\omega}$ and similarly for the double hat contract. Then:

$$\hat{\hat{\gamma}} - \hat{\gamma} + p_H(1 - \beta)\left[(u(\hat{x_L} - (1-\hat{\alpha})\Delta) - u(\hat{x_L})) - \left(u(\hat{\hat{x}}_L - (1-\hat{\hat{\alpha}})\Delta) - u(\hat{\hat{x}}_L)\right)\right] = \hat{\omega} - \hat{\hat{\omega}}$$

Notice that it must be the case that $\hat{\hat{\gamma}} - \hat{\gamma} - \hat{\gamma} \leq \hat{\omega} - \hat{\hat{\omega}}$ as the principal can do at least as well at $\hat{\hat{\omega}}$ by taking the allocation at $\hat{\omega}$, reducing $\hat{x}_L$ so that utility in the low state is reduced by $\hat{\omega} - \hat{\hat{\omega}}$ and increasing enforcement by $\hat{\omega} - \hat{\hat{\omega}}$. Therefore, $\left[(\tilde{u}(\hat{x}_L, \hat{\alpha}, \Delta) - u(\hat{x}_L)) - \left(\tilde{u}(\hat{\hat{x}}_L, \hat{\hat{\alpha}}, \Delta) - u(\hat{\hat{x}}_L)\right)\right] > 0$ which implies that $\hat{\alpha} \geq \hat{\hat{\alpha}}$ so long as $\hat{x}_L \geq \hat{\hat{x}}_L$, which is proven in Lemma 12. ■

Lemma 12 *x_L is increasing in ω.*

Lemma 12: Proof. Consider $\hat{\hat{\omega}} < \hat{\omega}$ with corresponding double hat and single hat allocations. Consider an allocation with the single hat contract and varying only x_L, γ, and α. Let $\hat{\hat{x}}^*_L, \hat{\hat{\gamma}}^*$ and $\hat{\hat{\alpha}}^*$ be so that the agent's utility is $\hat{\hat{\omega}}$. Let $\hat{x}^*_L, \hat{\gamma}^*$ and $\hat{\alpha}^*$ likewise denote the components of the allocation solving the principal's problem for $\hat{\omega}$ when the other variables are as specified in the double hat contract. Note that to satisfy PK, $\hat{\hat{x}}_L \leq \hat{x}^*_L$ and $\hat{\hat{x}}^*_L \leq \hat{x}_L$. Optimality of the hat and double hat contracts imply:

$$\begin{aligned}
-p_L\hat{\hat{x}}_L - p_H\hat{\hat{x}}_H &+ p_H V(\hat{\hat{\omega}}'_H) + p_L V(\hat{\hat{\omega}}'_L) - f(\hat{\hat{\alpha}}) - g(\hat{\hat{\gamma}}) \geq \\
&- p_L\hat{\hat{x}}^*_L - p_H\hat{x}_H + p_H V(\hat{\omega}'_H) + p_L V(\hat{\omega}'_L) - f(\hat{\hat{\alpha}}^*) - g(\hat{\hat{\gamma}}^*) \\
-p_L\hat{x}_L - p_H\hat{x}_H &+ p_H V(\hat{\omega}'_H) + p_L V(\hat{\omega}'_L) - f(\hat{\alpha}) - g(\hat{\gamma}) \geq \\
&- p_L\hat{x}^*_L - p_H\hat{\hat{x}}_H + p_H V(\hat{\hat{\omega}}'_H) + p_L V(\hat{\hat{\omega}}'_L) - f(\hat{\alpha}^*) - g(\hat{\gamma}^*) \\
\Rightarrow p_L(\hat{\hat{x}}^*_L - \hat{\hat{x}}_L) &+ f(\hat{\hat{\alpha}}^*) - f(\hat{\hat{\alpha}}) + g(\hat{\hat{\gamma}}^*) - g(\hat{\hat{\gamma}}) \geq \\
&p_L(\hat{x}_L - \hat{x}^*_L) + f(\hat{\alpha}) - f(\hat{\alpha}^*) + g(\hat{\gamma}) \geq -g(\hat{\gamma}^*)
\end{aligned}$$

As the starred and unstarred profiles differ only on x_L, α, and γ, then from the PK constraints

$$u(\hat{\hat{x}}_L^*) - u(\hat{\hat{x}}) = u(\hat{x}) - u(\hat{x}_L^*) \tag{1.7}$$

Suppose it were the case that $\hat{\hat{x}} > \hat{x}$, then it must be that

$$\hat{\hat{x}}_L^* - \hat{\hat{x}} > \hat{x}_L^* - \hat{x} \tag{1.8}$$

by the concavity of u. Similar calculations from the binding of EL constraints imply that $u(\hat{x}_L) - u(\hat{\hat{x}}_L^*) = \hat{\hat{\gamma}}^* - \hat{\gamma}$ and $u(\hat{\hat{x}}_L) - u(\hat{x}_L^*) = \hat{\gamma}^* - \hat{\hat{\gamma}}$. Summing together an combining with (1.7) implies that

$$\hat{\gamma} - \hat{\gamma}^* = \hat{\hat{\gamma}} - \hat{\hat{\gamma}}^* \tag{1.9}$$

Given that utility in the high state is equalized across the $\hat{\cdot}$ and $\hat{\hat{\cdot}}^*$ profiles, $\hat{\hat{x}} > \hat{x}$ implies that $\hat{\omega}'_L > \hat{\omega}'_L$. Furthermore, from the IC constraints it must be that $(1-\beta)\tilde{u}(\hat{x}_L, \hat{\alpha}, \Delta) + \beta\hat{\omega}'_L = (1-\beta)\tilde{u}(\hat{\hat{x}}_L^*, \hat{\hat{\alpha}}^*, \Delta) + \beta\hat{\hat{\omega}}'_L$ implying that $\hat{\alpha} >\geq \hat{\hat{\alpha}}^*$ from $\hat{x}_L \geq \hat{\hat{x}}_L^*$. A similar calculation shows $\hat{\hat{x}}_L \geq \hat{x}_L^*$. These two facts, combined with (1.9) and (1.8) contradict (1.7). ■

Chapter 2

On Efficient Allocations with Hidden Actions and States: An Application to Physical Investment in Human Capital

This chapter builds on a growing literature that analyzes efficient allocations in environments with unobservable states and actions. In environments with private information, the efficient allocations often depend in complicated ways on an agent's past history. The ability to summarize these histories with a small number of state variables is necessary for computation. Early work in the field shows that in simple environments of private information, the agent's valuation of the continuation contract is a sufficient state variable to write the problem recursively, allowing the use of dynamic programming techniques to (partially) characterize efficient allocations. For more complicated environments involving both hidden actions and hidden states, recent work shows how the problem can be written recursively, but requires either a state space with as many state variable as there are hidden states or an assumption on the validity of the first order approach. In this chapter I show that in environments in which hidden actions and hidden states affect utility separably, I can recover a one dimensional state variable. In this case, the state variable is a simple modification of the standard expected utility promise. I apply this result to a model with hidden physical investment and effort into human capital.

Recent work has shown how efficient allocations can be computed when there are both hidden states and hidden actions by expanding the state space, as the standard expected continuation utility promise[1] is not a sufficient state variable. The reason for this is that there is no point in time at which the planner and agent agree on the value of the continuation contract; an agent who deviates in his hidden action or report of the hidden state will evaluate the continuation contract differently than one who reports truthfully and follows the recommended action. One approach to these problems, the "first order approach," uses the standard continuation utility promise along with a marginal utility promise as state variables to allow for a recursive formulation. This approach assumes that the hidden action can be characterized with a first order condition. However, the first order condition is not generally sufficient and the optimality of the computed contract must be verified numerically ex-post. Alternatively, one can compute optimal allocations with hidden actions and hidden states by using a vector of *conditional* continuation utility promises as the state space. Because the agent and principal would agree upon the value of the continuation contract conditional on the unobserved state, they evaluate conditional promises similarly. The programming problem requires the addition of "threat keeping" constraints to deter an agent from jointly misreporting his hidden state and taking an incorrect action. This method can be used to compute the optimal contract without the need for ex-post verification. However, as the number of unobserved states increases, so too does the dimensionality of the state space, thereby hampering computation.

This chapter studies environments of hidden actions and hidden states and shows

[1] as in Abreu Pearce Stacchetti (1990) or Spear and Srivastava (1987)

that under the condition of separability between the hidden action and hidden state, the problem can be written recursively in one dimension using a modification of the standard unconditional expected utility promise. The environment I study resembles that of Doepke and Townsend (2006) who consider a more general class of models. They show how the problem can be written recursively with a vector of conditional continuation utility promises and the addition of threat keeping constraints. While computation is feasible in their environment using a clever methodology to reduce in the number of threat keeping constraints, their method requires that the size of the unobserved state space is small. Furthermore, for problems with an infinite time horizon, their method enables one to compute only an inner approximation of the set of implementable utilities and it is not possible to characterize the entire set of such utilities. [2]

I consider a restricted environment and show how the problem can be written recursively in one dimension. The key to writing the problem recursively in one dimension is as follows. Suppose that the agent's per period utility can be written as $u(x_1, e) - v(x_2, \theta)$ where e denotes the hidden action, θ denotes the hidden state, x_1 and x_2 are quantities observable to the planner, and the future value of the hidden state θ depends on e. The separability of e and θ imply that, conditional on receiving x_1 and some continuation contract, the action that the agent takes is independent of the current value of θ and x_2. In particular, once the planner has given x_1 to the agent and set the continuation contract, the action that the agent will take is independent of whether the agent has lied this period or told the truth. The planner can then deduce the hidden action by virtue of setting the continuation contract and x_1. Therefore, the planner can assess the agent's valuation of a contract, less the component of utility that is state dependent. I use this valuation, which I call

[2]Their model is contrasted to that of Aragones and Palfrey (2002) who use a first order approach to solve such problems. The model presented here shares similar advantages and disadvantages to Doepke & Townsend relative to the first order approach. Namely, the first order approach works well with continuous choice variables, but the approach here can be used for any utility functions satisfying the separability assumption and allows for discrete actions.

a modified continuation utility promise, to write the contract recursively. I then show how standard techniques can be used to compute efficient allocations using the modified expected utility promise as a state variable. This technique allows allows for the analytic characterization of the set of implementable utilities.

I apply this framework to study the efficient allocation in an environment in which single period lived agents maximize dynastic utility and make unobservable physical and effort investments into the unobservable skills of their children. The planner can allocate physical resources to an agent and can require the agent to produce observable output, whose utility cost to the agent depends on his skill level. However, the planner cannot observe whether the agent uses the physical resources for consumption or investment into the skills of future generations. In this environment, I compute the efficient allocation under two alternative assumptions on the specification of investment. I show that these two assumptions provide different long run implications on dynastic insurance. When the hidden investment requires physical resources as described above the immiserizing result does not hold, while it does hold when investment is an effort cost separable from the utility consumption. The immiserizing result, common in these settings, implies that agents tend toward their utility bounds with probability one. The reason for this is that in order to provide incentives today for truthful revelation, the planner must raise (lower) future expected utility for high (low) types. Because the utility bounds are absorbing states, agents will be stuck at a utility bound with probability one. However, under the physical investment assumption, there is an efficiency cost of pushing agents toward lower utility, namely: low utility requires that the planner gives the agent few resources, which implies that the agent cannot take physical investments toward the skills of future generations. Because it is more costly for the planner to deliver utility when the agent is low skilled, the efficient allocation may involve a lower bound that is strictly greater than the minimum implementable level of utility.

This chapter builds upon two strands of literature. The first is characterizing and computing efficient allocations in dynamic environments with private information and hidden actions. Early contributors to this field include Rogerson (1985) and Spear and Srivastava (1987) who study a dynamic moral hazard problem. The former shows that an efficient contract does not satisfy the usual Euler equation, but rather an "inverse" Euler equation. The latter shows that the dynamic moral hazard problem can be written recursively by using the expected continuation utility as a state variable. Green (1987), Thomas and Worrall (1990), and Atkeson and Lucas (1992), among others, have studied efficient contracts in both partial and general equilibrium environments with (exogenous) private information. Phelan and Townsend (1991) show how the optimal contract can be computationally constructed using lotteries in a dynamical environment with hidden information. Analytic results on the optimal contract in the dynamic moral hazard problem have recently been achieved in continuous time by Sannikov (2007). Modeling output as a Brownian motion whose drift depends on the effort by the agent, Sannikov shows that in the optimal contract the principal cannot provide the agent with incentives to exert effort at the extremes of utility and almost surely "retires" all agents at the lower utility bound or some endogenous upper bound of utility. Williams investigates a more general problem with hidden state variables in continuous time.

A more recent literature has examined efficient contracts with hidden state variables that inhibit the usage of the continuation promise as a sufficient means to write the contract recursively. Fernandes and Phelan (2000) and Doepke and Townsend (2006) show that dynamic contracting problems with hidden state variables can be written recursively by using a vector of conditional continuation promises and modifying the constraints to include a "threat keeping" constraint. Aragones and Palfrey (2002) write the problem recursively using the usual continuation value and the agent's marginal rate of substitution as state variables. While their approach is computationally more efficient, their method does not necessarily yield the solution to the contracting problem and must be verified ex-post.

Another, overlapping, strand of literature that this chapter relates to is that concerning the immiserizing result common in environments of dynamic insurance with private information and/or hidden actions. The immiserizing result states that in the long run, models of dynamic insurance provide agents just the opposite, namely: agents are pushed to a lower or upper utility bound with probability one. Green recognized this result and it has been shown to be robust to many settings, including in a general equilibrium setting with private information as in Atkeson and Lucas (1992), a partial equilibrium environment of moral hazard shown numerically in Phelan and Townsend (1991) and analytically in a continuous time moral hazard setting as in Sannikov (2007). The immiserizing result has shown to fail in the case that the agent has a constant walk away option (Atkeson and Lucas (1995)), if the planner and agent have conflicting intertemporal preferences (Farhi and Werning (2007)), or if the planner cannot commit to driving the agent to a lower utility bound (Sleet and Yeltekin (2006)). Yazici (2010) also considers an environment in which parents undertake costly physical investment into the skills of their children, but parents possess private information on their own altruism. In his model, information asymmetries are resolved upon the public realization of the skills of children, but the planner cannot punish parents who under-invest in their children, as it assumed that skills are realized after parents gain utility from consumption. In his model, as here, the immiserizing result does not hold. However, there is no discrepancy over parental types in the model in this chapter: given the same resources, parents would make identical investments into the skills of their children. Furthermore, the nature of the informational environment in this chapter is more consistent with that of dynamic insurance literature; there is no temporal resolution of the informational friction between the planner and agent.

2.1 Setup

A risk neutral planner looks to provide efficient insurance to infinitely lived dynasties,[3] where one-period lived agents with skill level (θ_t) can produce output (y_t), consume (c_t), and undertake physical investment into the skills of their children through two channels: a physical investment component (i_t) and an effort cost (e_t) component. Because the agents are one period lived, human capital depreciates completely and I ignore persistence of human capital via genetics.[4]

Each agent is one period-lived and has a single child who is the beneficiary of the altruistic parental investments. While alive the agents' investments determine the probability distribution over the skills of the next generation according to $\pi(\cdot|g(i_t, e_t))$. Assume that $\pi(\cdot|g(i,e)) > 0$ for all $(i,e) \in I \times E$ so that all skill levels are possible regardless of parental investment.[5]

Given some skill level θ_t, each generation has a per period utility function given by:

$$u(c_t) - v(y_t)/\theta_t - \psi(e_t)$$

Although I will consider only the functional form above, the analysis and methods can apply to any problem in which the per period utility function is given by $u(x_{1t}, e_t) - v(x_{2t}, \theta_t)$ where $x_t = (x_{1t}, x_{2t})$ is observable, e_t is an unobservable action and θ_t is an unobservable state distributed according to $F(\cdot|e_{t-1})$.

The planner can observe only the output that the agent produces and the *total* amount of physical resources (a_t) that the agent may secretly divide between present

[3]This could be modeled equivalently as an infinitely lived agent whose human capital depreciates entirely after one period.

[4]I could also include, with some additional computational complexity, an *ability* component that is exogenous and persistent across generations. In this sense, ability should be interpreted as a genetic endowment that along with investment, maps into a future generation's probability distribution over skill levels. If ability is observable, then the model would be identical to the one presented here with an additional conditioning on observed ability. Unobservable ability would require an additional level of separation across individuals and could be included in the numerical method with only additional computational cost.

[5]This model could also be interpreted as an infinitely lived agent with full depreciation of the human capital investment.

consumption (c_t) and physical investment (i_t) toward human capital. The skill level of the agent is also private information as is the effort investment e_t toward human capital. Let θ_t denote the realization of a particular skill at time t and use $\theta^t = \{\theta_0, .., \theta_t\}$ to denote a history of realizations. Abusing notation slightly, $\theta_j(\theta^t)$ will denote the j-th period skill realization according to history θ^t.

To summarize, the timing of the model is as follows. At the beginning of period t, an agent is born with some skill level θ_t observed only by him. The agent sends some message to the planner, which by the revelation principal will be an announcement of his skill. Based on the message sent, the planner demands some level of output y_t from the agent, gives him a_t level of physical resources, and sends some message, which corresponds to a recommended action. The agent then makes an investment decision, which includes a physical investment, that maps stochastically into the skills of his child θ_{t+1}. The period t agent is altruistic, dies after period t and discounts the utility of future generations at $\beta \in (0, 1)$.

Let M_1 and M_2 denote generic message spaces between the planner and agent and a_t, y_t denote the observable t-period physical resources and output. A version of the revelation principle shows that any allocation feasible in the general mechanism is also feasible in the truth-telling-and-obedience mechanism. Let t-period public observations be denoted by $s_t = \{m_{1t}, m_{2t}, a_t, y_t\}$ with a history of public observations $s^t = \{s_0, .., s_t\}$. Let S^t denote the set of public histories until time t. The agent's t-period private observations are $h_t = \{s_t, \theta_t, i_t, e_t\}$ with private histories written accordingly as h^t and the set of private histories denoted by H^t. Let the null set of histories by $h^{-1} = \emptyset$. The planner chooses a plan, $\mu(y_t, a_t, m_{2t}|m_{1t}, s^{t-1})$, which is a probability distribution over output requirements from the agent, physical resources to the agent and action recommendation, conditional on the public history until that time and the message sent that period by the agent. Meanwhile, the agent chooses a *strategy* $\sigma = (\sigma_1, \sigma_2)$, which consists of a reporting strategy $\sigma_1(m_{1t}|h^{t-1}, \theta_t)$ where $\sigma_1 : H^{t-1} \times \Theta \to \Delta M_1$ and an action strategy $\sigma_2(i_t, e_t|a_t, y_t, m_{1t}, m_{2t}, h^{t-1}, \theta_t)$ where

$\sigma_2 : H^{t-1} \times M_1 \times M_2 \times \Theta \times A \times Y \to \Delta I \times \Delta E$. Let $\phi(h^t|\sigma,\mu)$ denote the conditional probability of observing a history given a plan and strategy.

Define the utility of the agent at time t' from plan $\hat{\mathcal{C}}$ given strategy σ by:

$$U_{t'}(\hat{\mathcal{C}},\sigma,h^{t'}) = \sum_{t=t'}^{\infty} \beta^{t-t'} \sum_{h^t} \Big[u(a(\sigma_1(h^t)) - \sigma_{2,i}(a_t,y_t,m_{1t},m_{2t},h^{t-1}) - \frac{v(y(\sigma_1(h^t)))}{\theta_t(\theta^t)} \\ - \psi(\sigma_{2,e}(a_t,y_t,m_{1t},m_{2t},h^{t-1})) \Big] \phi(h^t|\hat{\mathcal{C}},\sigma,h^{t'})$$

The planner's profits given a plan and agent's strategy is given by

$$V(\hat{\mathcal{C}},\sigma) = \sum_{t=0}^{\infty} \beta^t \sum_{h^t} \left[y(\sigma_1(h^t)) - a(\sigma_1(h^t)) \right] \phi(h^t|\hat{\mathcal{C}},\sigma,h^{t'}) \tag{2.1}$$

The planner's objective is then to maximize V over the set of contracts $\mathcal{C}$ such that the agent receives utility $\tilde{U}_0$ and that σ is incentive compatible for the agent. Given a reservation utility $\tilde{U}_0$, an allocation is said to be *feasible* if it is incentive compatible and guarantees the reservation utility, namely, that it satisfies:

$$\begin{array}{ll} (IR) & U_0(\hat{\mathcal{C}},\sigma,h^{-1}) \geq \tilde{U}_0 \\ (IC) & U_t(\hat{\mathcal{C}},\sigma,h^t) \geq U_t(\hat{\mathcal{C}},\hat{\sigma},h^t) \quad \forall\, t, h^{t-1}, \hat{\sigma} \end{array}$$

Definition 1 *An allocation is said to be an* optimal allocation *if it maximizes 2.1 subject to (IR) and (IC).*

A version of the revelation principal applies. The proof follows the usual form by considering a feasible allocation under arbitrary message spaces. By integrating over the message spaces, one can then show that there exists an equivalent contract in which, contingent on the announced state θ, the planner recommends an action, demands an output, and delivers an amount of physical resources with the cumulative probability that an agent of that type received that allocation and took those actions using an arbitrary message space. Feasibility demands that the original messages sent and actions taken by the agent were optimal. Because announcing θ and following the recommended action yields the same payoff to the agent as under the more general message space, there is no loss of generality is considering the reduced message space.

Proposition 14 *Any incentive compatible allocation with message space M_1 and M_2 can be attained as the outcome of a reporting game in which the message spaces are $M_1 = \Theta$, $M_2 = A \times E$, the agent reports his type truthfully and follows the agent's recommendations.*

Proof. See Doepke & Townsend (2002). ■

Denote by $\hat{\theta}$ the reported state and $\hat{i}, \hat{e}$ denote the recommended action by the planner. Then, as per the earlier definitions of public and private histories, $s^t = (\hat{\theta}^t, \hat{i}^t, \hat{e}^t, a^t, y^t)$ denotes the public history observed by the planner and agent and $h^t = (s^t, \theta^t, i^t, e^t)$ denotes the private history of the agent where i, e denote the actions taken by the agent. As before, the set of all public (private) histories up until time t are denoted by S^t (H^t). Write $\sigma = (\sigma^\theta, \sigma^i, \sigma^e)$ as the agent's strategy where $\sigma_t^x : H^t \rightarrow X$ for $x = \theta, i, e$, $X = \Theta, I, E$, respectively, and $\sigma^x = \{\sigma_t^x\}_{t=0}^\infty$. Let σ^* denote the truthtelling and obedient strategy where $\sigma_t^*(x^t) = x_t$ for all x^t. Let the set of all strategies be denoted by Σ.

Let $\hat{\mathcal{C}} = (a, y, \hat{i}, \hat{e})$ be a plan where $a = \{a_t\}_{t=0}^\infty$ and $a_t : S^t \rightarrow \Delta A$ (likewise for y) into this period's distribution of resource allocations and output requirements. Meanwhile, $\hat{i} : S^t \times A \times Y \rightarrow \Delta I$ (similarly for $\hat{e}$), is a mapping of reported histories and the outcome of lotteries on $A \times Y$ into a recommended action. Let C denote the set of all plans $\hat{\mathcal{C}}$.

Let $\phi(h^t | \hat{\mathcal{C}}, \sigma, h^{t'})$ be the conditional probability of an agent observing some history h^t at time $t' < t$ given some plan $\hat{\mathcal{C}}$, some strategy σ and some history $h^{t'}$. Assume that the initial distribution π_0 is known by the planner and that there is some initial level of utility $\tilde{U}_0$ that the planner must guarantee.

The planner's problem can then be written as follows.

$$\begin{aligned}
\max_{\hat{\mathcal{C}} \in C} \quad & \sum_{t=0}^{\infty} \beta^t \sum_{\theta^t} [y_t(s^t) - a_t(s^t)] \phi(h^t | \hat{\mathcal{C}}, \sigma^*) \\
s.t.\ \forall \sigma \in \Sigma, t, \theta^t \quad & U_t(\hat{\mathcal{C}}, \sigma^*, h^t) \geq U_t(\hat{\mathcal{C}}, \sigma, h^t) \\
& U_0(\hat{\mathcal{C}}, \sigma^*, h_0) \geq \tilde{U}_0
\end{aligned}$$

Notice first that y^t, a^t are S^t-measurable. Also the planner chooses probability distributions over output and resources.

2.2 Recursive Formulation

In general, contracts can depend in a complicated fashion on the history. Therefore, it is useful to rewrite the contract recursively allowing for the use of tools from dynamic programming. In many environments previously studied, like those in which private information has no persistence or those in which hidden actions map into observable states one can summarize the history of the contract by using the agent's expected utility upon entering a given period. The reason for this is that in cases such as these, there is a moment at which the planner and agent value the continuation contract identically. Because the manner in which a particular node is reached is irrelevant for delivering utility in the forward looking contract, the planner's problem each period is simply to deliver a particular level of utility at the minimum cost. In the case of private information independently distributed across periods, the planner and agent value the continuation contract of period $t + 1$ onward identically at period t, as the probability distribution does not depend on the realizations in period t and beforehand. That is, the planner can determine how the agent evaluates any forward looking contract independently of past realizations of private information and the agent's strategy in previous period. In environments with hidden actions and observable states, contracts are evaluated identically by the planner and agent after the realization of the publicly observable states independently of past actions under the assumption that past actions do not impact future states.

However, in the environment described in Section 2.1, there is no point in time at which agents and the planner identically value the continuation contracts. Entering a period (before the realization of skill level) the planner and the agent do not necessarily attach the same value to the future contract. This is because the probability distribution over skill level is known only by the agent, owing to the unobservable action in the previous period. Meanwhile, after skill is observed, the planner and agent again value the contract differently because the agent now knows his type with certainty.

To deal with cases of hidden states and hidden actions, past literature resorts to expanding the state space beyond the usual expected utility promise. One approach to solving such problems is the first order approach, which augments the state space with a marginal utility promise. Another approach is to use *conditional* continuation promises to the agent as the relevant state space, based on the observation that conditional on knowing a particular realization, the planner and agent would value the future contract identically. In this approach, the constraints are augmented to include threat keeping.

In this environment the hidden action by the agent implies that the planner and agent do not value a continuation contract the same. However, conditional on some report in period t, the planner and agent do agree on the value of the continuation contract ($t + 1$ onward) augmented by the period-t consumption utility and any effort cost. That is, the planner and agent value the contract identically other than the cost to the agent of producing output. This is because the separability of the hidden state and hidden action implies that this modified continuation value and the agent's optimal action are *independent* of whether the agent was truthful or not in his report.

Let $\mathcal{C}$ be an optimal contract. Define on-equilibrium-path modified continuation utilities ω at time τ given a particular distribution over allocation of resources to the agent by

$$
\begin{aligned}
\omega_\tau(\theta^\tau, \mathcal{C}, \sigma^*) \;=\;& u\left(a - i(h^\tau, a)\right) - \psi\left(e(h^\tau, a)\right) + \\
& \sum_{t=\tau+1}^{T} \beta^{t-\tau} \sum_{h^t} \left[u\left(a(h^t) - i(h^t, a)\right) - \frac{v\left(y(h^t)\right)}{\theta_t(h^t)} - \psi(e(h^t, a)) \right] \\
& \times \pi(h^t | h^\tau, \mathcal{C}, \sigma^*)
\end{aligned}
$$

Proposition 15 *The optimal contract $\mathcal{C}$ can be rewritten by reoptimizing at each node (s^τ, a) the modified continuation utilities.*

From Proposition 15 one can rewrite an auxiliary optimal contracting problem as follows:

$$
\begin{aligned}
V_t(\omega) = \quad & \max_{\mu} \sum_{\substack{a \in A, i \in I, e \in E \\ y \in Y^S, \omega' \in \Omega^S}} \left\{ -a + \beta \sum_{s \in S} [y_s + V_{t+1}(\omega'_s)] \right. \\
& \left. \times \pi(\theta_s | e, i) \right\} \mu(a, i, e, y, \omega') \\
s.t.\ (PK) \quad & \omega = \sum_{\substack{a \in A, i \in I, e \in E \\ \{y_{s'}, \omega'_{s'}\}_{s \in S} \in Y^S \times \Omega_F^S}} \left\{ u(a - i) - \psi(e) + \right. \\
& \left. \beta \sum_{s \in S} \left[-\frac{v(y_s)}{\theta_s} + \omega'_s \right] \pi(\theta_s | e, i) \right\} \mu(a, i, e, y, \omega') \\
(IC1) \quad & -\frac{v(y_s)}{\theta_s} + \omega'_s \geq -\frac{v(y_{\tilde{s}})}{\theta_s} + \omega'_{\tilde{s}} \ \forall s, \tilde{s} \in S \\
(IC2) \quad & (e, i) \in argmax_{\hat{e}, \hat{i}} u(a - \hat{i}) - \psi(\hat{e}) + \beta \sum_{s \in S} \left[-\frac{v(y_s)}{\theta_s} + \omega'_s \right] \pi(\theta_s | \hat{e}, \hat{i})
\end{aligned}
$$

The first constraint, PK, is the modified promise keeping condition that guarantees that the planner delivers the modified promised utility. The second constraint, $IC1$, guarantees that the agent finds it optimal to truthfully report his hidden state. The third constraint, $IC2$, guarantees that the agent takes the optimal action given physical resources and the continuation contracts. The separability of the components of utility affected by the hidden action and states is what allows for incentive compatibility to be solved simply through the final two constraints. In general, the problem of jointly deviating in report and action gives rise to the more cumbersome state space and constraints found, for example, in DT.

Given the initial utility promise $\tilde{U}_0$, the original planner's problem can be recovered by solving:

$$\begin{aligned} V_0(\tilde{U}_0) = \quad & \max_{\{\omega_s^0, y_s\}} \sum_{s\in S} [y_s + V_1(\omega_s^0)]\pi_0(\theta_s) \\ s.t.\ \forall s, \tilde{s} \quad & -\frac{v(y_s)}{\theta_s} + \omega_s^0 \geq -\frac{v(y_{\tilde{s}})}{\theta_s} + \omega_{\tilde{s}}^0 \\ & \sum_{s\in S} [-\frac{v(y_s)}{\theta_s} + \omega_s^0]\pi_0(\theta_s) \geq \tilde{U}_0 \end{aligned}$$

Notice that this is a linear programming problem, allowing us to use linear programming techniques. Furthermore, if the problem has an infinite time horizon, one can set up a contraction mapping and use value function iteration. Let $\mathbf{E}^\mu$ denote the expectation operator under the probability distribution μ.

Consider the following functional map.

$$\begin{aligned} T(V(\omega)) \quad = \quad & \max_{\mu(a,i,e,\{y_s,\omega_s'\}_{s\in S})} \mathbf{E}^\mu \left[-a + \beta \sum_{s\in S} [y_s + V(\omega_s')]\pi(\theta_s|e,i) \right] \\ s.t.\ (PK) \quad & \omega = \mathbf{E}^\mu \left[u(a-i) - \psi(e) + \beta \sum_{s\in S} \left[-\frac{v(y_s)}{\theta_s} + \omega_s' \right] \pi(\theta_s|e,i) \right] \\ (IC1) \quad & \forall s, \tilde{s}\ \mathbf{E}^\mu \left[-\frac{v(y_s)}{\theta_s} + \omega_s' \right] \geq \mathbf{E}^\mu \left[-\frac{v(y_{\tilde{s}})}{\theta_s} + \omega_{\tilde{s}}' \right] \\ (IC2) \quad & \forall a, y, \omega\ s.t.\ \omega(a,y,\omega,e,i) > 0 \\ (e,i) \quad & \in argmax_{\hat{e},\hat{i}} u(a-\hat{i}) - \psi(\hat{e}) + \mathbf{E}^\mu \left[\beta \sum_{s\in S} \left[-\frac{v(y_s)}{\theta_s} + \omega_s' \right] \pi(\theta_s|\hat{e},\hat{i}) \right] \end{aligned}$$

It is straightforward to show that this map satisfies Bellman's sufficient conditions of monotonicity and discounting, so that T is a contraction mapping. Consequently, there is a unique fixed point.

By allowing the planner to choose lotteries over the observables, it is guaranteed that the value function is (weakly) concave. Consequently, given a particular choice of a, the problem can be simplified slightly by showing that the planner need not to randomize over future promises and output to achieve an efficient allocation if the respective sets of Ω and Y are convex.

Lemma 13 *Suppose that under the optimal contract $\mathcal{C}$ at some promised modified utility ω and future state s it is the case that for some a there exists two distinct promises $(\omega'_{s,1}, y_{s,1})$ and $(\omega'_{s,2}, y_{s,2})$ such that $\mu(a, \omega'_{s,1}, y_{s,1}) > 0$ and $\mu(a, \omega'_{s,2}, y_{s,2}) > 0$. If Y is convex, then there exists another optimal contract such that for some $y^*_{s,a}$, $\omega^{*\prime}_{s,a}$, $\mu(a, y^*_{s,a}, \omega^{*\prime}_{s,a}) = 1$.*

Therefore, under the assumption of convex Y and Ω, one can focus on a recursive contract that maps current modified utility into a probability distribution over current physical resources and an associated unique vector of continuation promises and output demands for the following period.

Implementable Utility

One additional advantage to using a single dimensional state variable is that it allows for the precise determination of the set of implementable utilities when the time horizon is infinite. This is in contrast to the approach with a vector of conditional utility promises. In that approach, the vector of implementable utilities can be computed as part of an inner approximation, so that one can only be guaranteed if a vector of continuation utilities is implementable numerically, then it necessarily lies in the set of theoretically implementable utilities. In contrast, this environment allows for the following possibility.

Lemma 14 *All modified expected utility promises $\omega \in \mathbf{W} = [\underline{\omega}, \bar{\omega}]$ are implementable, where $\underline{\omega} = u(\underline{a} - i^*_1) - \psi(e^*_1) + \frac{\beta}{1-\beta}\mathbf{E}[\underline{\omega} - \frac{v(\bar{y})}{\theta} | i = i^*, e = e^*]$ and $\bar{\omega} = \frac{1}{1-\beta}\mathbf{E}[u(\bar{a} - i^*_2) - \frac{v(y)}{\theta} - \psi(e^*_2) | i = \underline{i}, e = e^*]$ where $(e^*_1, i^*_1) = argmax_{e,i} u(\underline{a} - i) - \psi(e) +$*

$\frac{\beta}{1-\beta}\mathbf{E}[\underline{\omega} - \frac{v(\bar{y})}{\theta}|i,e]$ *and* $(e_2^*, i_2^*) = argmax_{e,i} u(\bar{a} - i) - \psi(e) + \frac{\beta}{1-\beta}\mathbf{E}[\bar{\omega} - \frac{v(\bar{y})}{\theta}|i,e]$ *and the expectation operators* $\mathbf{E}$ *is that generated by* π *and* μ *such that mass is placed only on the extremum values of* a, y *and the respective optimal choices of* e, i.

Providing Incentives

To provide incentives in a particular period, contracting problems with private information typically require the principal to spread continuation promises to the agent, providing more future utility to high types relative to the entering level of utility and lower future utility to low types relative to the entering level. This is the driving force behind the immiserizing property; at any interior utility promise the optimal allocation promises higher future utility to one type than another type. In addition, the lower (upper) bound is an absorbing state that can be implemented only with continuation promises that are the lower (upper) bound as well as the minimum (maximum) resources and maximum (minimum) required output. Therefore, in the long run, all agents will end up absorbed at one or the other utility bounds.

The recursive structure provides insight here as to why that need not be the case. Future continuation payoffs in the recursive formulation include state by state both a continuation utility component and the output requirement from the agent conditional on his announced type. Then, providing incentives to the agent today via manipulation of future promises can be done on both the y_s as well as the ω_s. Consequently, the planner can provide incentives at interior modified continuation utilities in a way that cannot be done with the standard expected utility promises. In particular, when reducing ω has efficiency implications on the skill level in future periods, it may be optimal to not reduce ω below some threshold that is strictly greater than the lower utility bound.[6]

[6]Note that lower and upper utility bounds remain absorbing states.

2.2.1 Discretized Numerical Problem

In the theoretical formulation of the problem, the sets A, E, I, and Y were not specified. Furthermore, the modified continuation promises were allowed to take on a continuum of values. However, to solve the problem numerically, I rely on techniques in DT that convert the problem into a simple linear programming problem. To do so, first assume that $|A|, |Y|, |E| < \infty$ and $|I| \subseteq |A|$.

The linear programming problem will require us to solve for $\mu(a, y, \omega, i, e|\theta)$, where y and ω are vectors i, e will be restricted to lie in the set B described below. To compute this, one must discretize the modified continuation values to lie in some finite set $\mathbf{\Omega}_A$. While the finiteness of the other choice variables may be thought to be part of the physical environment, this restriction on the space of modified continuation values is for computational purposes only.

For the outcome $(\{a, y, \omega)$ of any lottery specified by μ, one can compute the optimal investment choices, i, e for the agent by solving:

$$\begin{aligned} B(a, y, \omega) &= \{(i, e)|(i, e) \in \\ &\quad argmax_{\hat{i} \in I, \hat{e} \in E} u(a - \hat{i}) - \psi(\hat{e}) + \beta \mathbf{E}^{\mu(a, \cdot)} \left[-v \left(\frac{y_{s'}}{\theta_{s'}} \right) + \omega'_{s'} | g(\hat{i}, \hat{e}) \right] \end{aligned}$$

The discretized version of the planners problem can then be written as follows:

$$
\begin{aligned}
V(\omega) = \quad & \max_{\mu \geq 0} \sum_{\substack{a \in A, i \in I, e \in E \\ \{y_{s'}, \omega'_{s'}\}_{s \in S} \in Y^S \times \Omega_F^S}} \left\{ -a + \beta \sum_s \left[-y_s + V(\omega'_s) \right] \pi(\theta_s | i, e) \right\} \\
& \qquad \times \mu(a, y, \omega', i, e) \\
s.t. \quad & (i(a, y, \omega'), e(a, y\omega')) \in B(a, y, \omega') \text{ if } \mu(a, y, \omega', i, e) > 0 \\
\forall s, \tilde{s} \quad & -v\left(\frac{y_s}{\theta_s}\right) + \omega'_s \geq -v\left(\frac{y_{\tilde{s}}}{\theta_s}\right) + \omega'_{\tilde{s}} \\
& \sum_{\substack{a \in A, i \in I, e \in E \\ \{y_{s'}, \omega'_{s'}\}_{s \in S} \in Y^S \times \Omega_F^S}} \left[u(a - i) - \psi(e) + \beta \sum_s \left[-v\left(\frac{y_s}{\theta_s}\right) + \omega'_s \right] \right. \\
& \qquad \left. \times \pi(\theta_s | i, e) \right] \mu(a, y, \omega') = \omega \\
& \sum_{\substack{a \in A, i \in I, e \in E \\ \{y_{s'}, \omega'_{s'}\}_{s \in S} \in Y^S \times \Omega_F^S}} \mu(a, y, \omega', i, e) = 1
\end{aligned}
$$

The advantage to working with a one dimensional state variable is that it does not suffer from the curse of dimensionality as do models with a vector of conditional utility promises. Doepke and Townsend introduce a method to overcome a curse of dimensionality on the number of "threat keeping" *constraints* required for such a recursive formulation, thereby allowing for a large number of actions, they are restricted to the number of hidden states that comprise the vector of conditional promises. As they state, "The main remaining limitation of our approach is that we are restricted to relatively small spaces for the [number of hidden states]." The approach here allows for a large number of hidden states unlike DT and is theoretically concise, without the need for numerical verification as in Aragones and Palfrey (2002). However, it comes at a loss of generality from those models, as it requires a separability condition that those models do not require.

2.3 Application to Model with Unobservable Investment

In this section, I consider two numerical environments addressed by the previous sections of the chapter. The first model includes no physical investment into human capital. The second model considers *only* physical investment into human capital and ignores the separable effort cost. While the first model yields the common immiserating result, the latter does not.

2.3.1 Economy with Hidden Effort

In this section I examine a moral hazard problem in which future skill level is determined only by an agent's effort whose cost in utility is separable from his utility from consumption and disutility of work. Letting $\pi(\theta_s|e)$ denote the probability of observing θ_s given levels of investment. This model is similar to that of Phelan and Townsend (1991) but with the additional unobserved state. The discretized auxiliary model therefore has per period utility given by $u(a) - \psi(e) - v(y)/\theta$ and there is no physical investment into human capital.

Assume the following functional forms and discretized state spaces. $u(c) = \sqrt{(}c)$, $\psi(e) = e^2$, $v(y) = 10y^2$, $\pi(\theta_L|e) = 1 - \exp(-2e - 0.001)$. $A = C = \{0, \frac{1}{6}, \frac{2}{6}, .., 1\}$, $Y = \{0, \frac{1}{3}, \frac{2}{3}, 1\}$, $E = \{0, 0.05, 0.1, .., 1\}$, $\Theta = \{\theta_L = 1, \theta_H = 4\}$. Finally, let $|\Omega_F| = 20$ with the lower and upper bounds as described in Lemma 14.

Given the parameterization above, the value function, output function ($\mathbf{E}^\mu[y]$), consumption function ($\mathbf{E}^\mu[c]$), and investment function ($\mathbf{E}^\mu[e]$) are given as a function as the modified continuation promise in corresponding Figures 2.1, 2.2, 2.3, respectively. Note the concavity of the value function and the increasing (decreasing) nature of consumption (output). Furthermore, note the shape of the investment function. At high levels of ω, the agent will not be forced to produce output and,

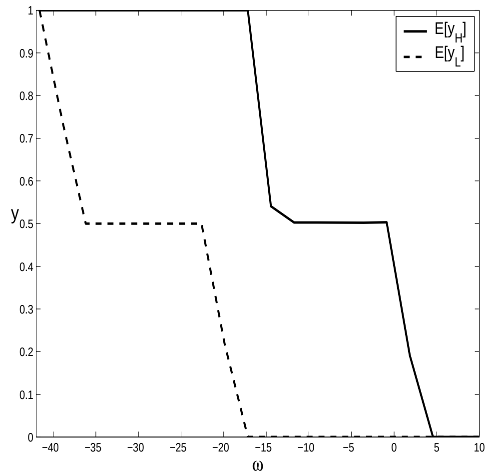

Figure 2.1: Future Output as a Function of ω

consequently, can not be provided the incentives to invest in future skills. Meanwhile, at low levels of ω, future output requirements from the agent will necessarily be large. As a result, the agent will exert the largest amount of effort into future skills at this level of ω.

The result that effort is largest at the lowest levels of ω is consistent with Sannikov (2007) who shows that equity and efficiency move in opposite directions at the low end of promised utility in a continuous time environment. The reason for the result here follows a similar logic to his chapter, namely: an agent cannot be provided incentives to work if his promised utility is high, as the planner is restricted in his ability to inflict punishment in the event of a bad outcome or report.

Furthermore, if the initial distribution of modified continuation promises is uniform across Ω_F, then Figure 2.4 tracks the evolution of continuation promises over various intervals. Notice that the probability density for all interior values of ω vanishes and all probability mass moves toward the minimum and maximum modified continuation promises.

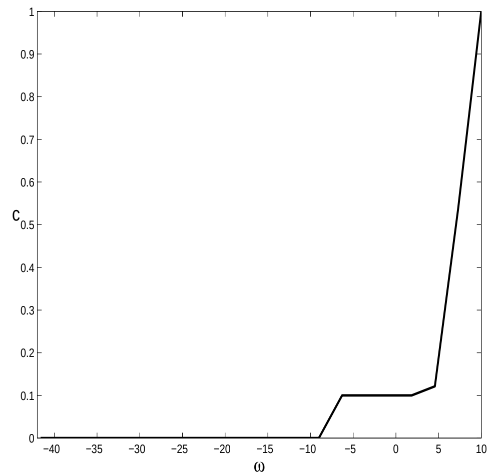

Figure 2.2: Consumption as a Function of ω

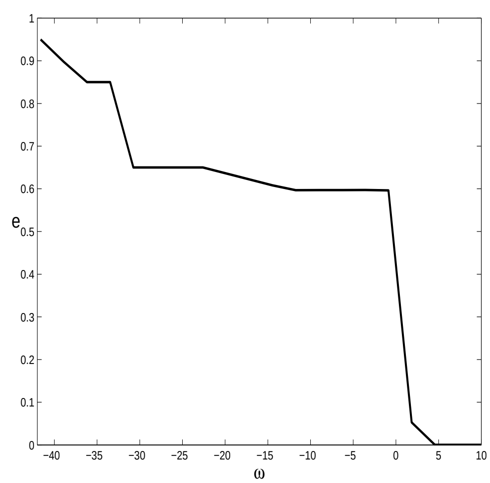

Figure 2.3: Effort Investment as a Function of ω

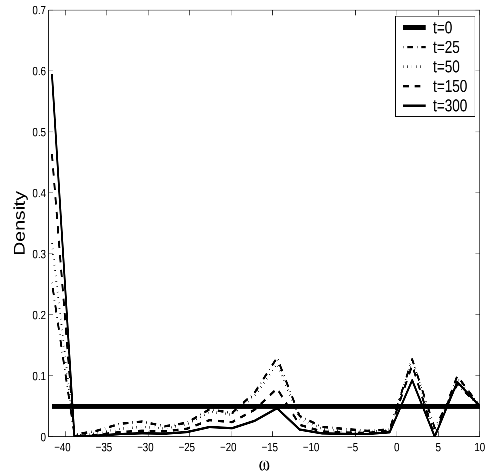

Figure 2.4: Density of ω Over Time

2.3.2 Economy with Hidden Physical Investment

In this subsection, consider an economy in which the investment into human capital is out of physical resources and that the planner cannot observe whether an agent uses those resources for consumption or investment. This differs from the previous example in important ways. In the previous example, the planner could simultaneously provide the agent incentive to exert effort and report truthfully by demanding high output independently of the reported level of θ. That is, immiserising the agent does not diminish the agent's ability to invest in future skills.

However, that logic fails in the case of hidden physical investment. In this case, the incentive to make a physical investment into human capital requires that the planner also deliver the agent resources with which to make the investment. Therefore, immiserising the agent in this scenario is necessarily accompanied by minimal investment into skill.

In this case, the discretized recursive version of the planner's problem is therefore is as above, with per period utility described by $u(a - i) - v(y)/\theta$ and all investment into human capital comes from diverting resources that are otherwise used for consumption.

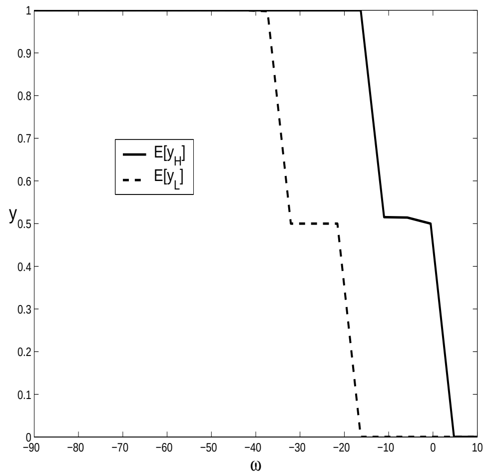

Figure 2.5: Future Output as a Function of ω

Assume the same functional forms as before with the following exceptions. Let $I = E$. $\pi(\theta_L|i) = 1 - \exp(-15i - 0.001)$ and $v(y) = 5y^2$. Also, let $u(c) = \sqrt{(}c)$ for $c \leq 1$ and $u(c) = 1$ for $c > 1$. Assume further that $A = \{0, \frac{1}{6}, \frac{2}{6}, .., 1, 1.5, 2\}$ so that any resources greater than 1 that are given to the agent will necessarily be invested into human capital.

Given the parameterization above, the value function, output function ($\mathbf{E}^{\mu}[y]$), consumption function ($\mathbf{E}^{\mu}[c]$), and investment function ($\mathbf{E}^{\mu}[i]$) are given as a function as the modified continuation promise in corresponding Figures 2.5, 2.6, 2.7, respectively. Note the concavity of the value function and the increasing (decreasing) nature of consumption (output). Furthermore, note the shape of the investment function. As in the previous subsection, at high levels of ω, the agent will not be forced to produce output and, consequently, can not be provided the incentives to invest in future skills. However, there is a drastic change on the agent's investment into future skills at at low levels of ω. In particular, a low level of ω implies *both* a high level of future output and a low level of physical resources. This latter effect of low ω implies that the agent cannot make investments into skill to reduce the utility cost of producing high output in future periods. This gives rise to the curve seen in

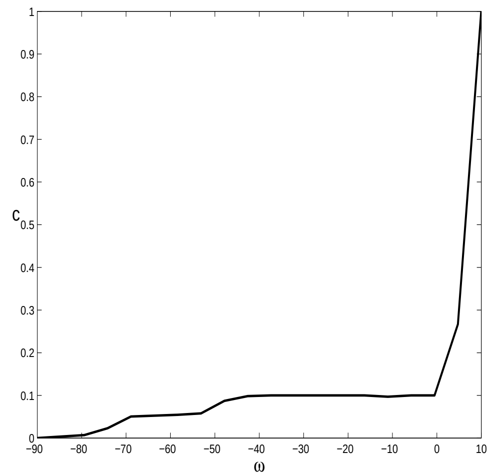

Figure 2.6: Consumption as a Function of ω

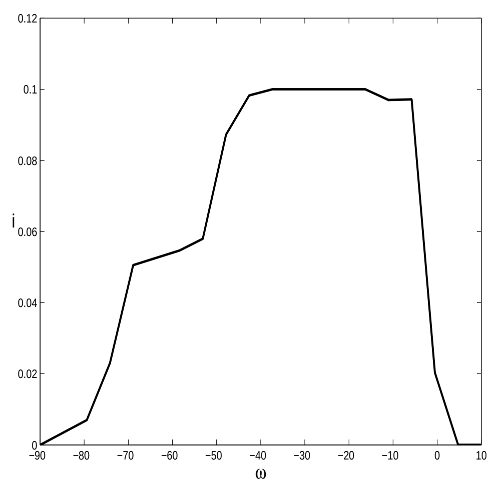

Figure 2.7: Physical Investment as a Function of ω

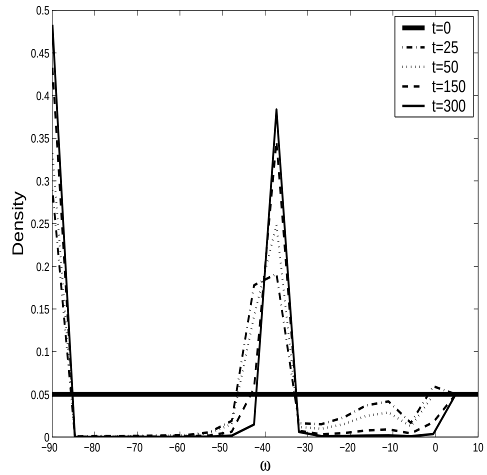

Figure 2.8: Density of ω Over Time

Figure 2.7, in which investment into skill is highest in the interior of Ω.

If the initial possible distribution of modified continuation promises is uniform across Ω_F, then Figure 2.8 tracks the evolution of continuation promises over various intervals. Notice that there is an absorbing state in the interior of Ω so that the agent does not necessarily tend toward the utility bounds as in other models of efficient insurance in environments of private information. That is, environments with hidden physical investment to human capital alongside hidden skill do not produce the immiseration result.

2.4 Omitted Proofs

Proposition 15 Proof. I show that the contract where the planner reoptimizes is also optimal. Consider some history h^τ and let $\mathcal{C}_\tau$ denote the optimal (auxiliary) contract for initial modified promise $\omega_0' = \omega_\tau(h^\tau, \mathcal{C}, \sigma_\tau^*)$. Consider now a contract $\mathcal{C}'$ which is equal to the original contract for all nodes other than h^τ and is equal to the reoptimized contract on that node. I must show that $\mathcal{C}'$ is also an optimal contract. On path utilities are the same at all nodes other than h^τ since the contract is identical on those nodes and on path utilities are the same along h^τ by construction.

Therefore, $\mathcal{C}'$ must satisfy the IR constraint.

Now we must show that $\mathcal{C}'$ satisfies the IC constraint. Suppose it does not, so that for some history h^k it is the case that:

$$U(\sigma_k, \mathcal{C}'|h^k) > U(\sigma_k^*, \mathcal{C}'|h^k)$$

Clearly, k cannot be greater than τ or else this would violate the fact that $\mathcal{C}$ ($\mathcal{C}_\tau$, respectively) is an optimal contract for histories not equal to (equal to) h^τ as it would violate IC constraints. Then it must be that $k < \tau + 1$.

Notice that it must be that for a history $\tilde{h}^\tau$ realized with positive probability it must be the case that

$$\omega(\tilde{h}^\tau, \mathcal{C}, \sigma_\tau^*, a) > \omega(\tilde{h}^\tau, \mathcal{C}, \sigma_\tau, a) \tag{2.2}$$

otherwise this would imply that the incentive compatibility constraint was violated for either contract $\mathcal{C}$ if $\tilde{h}^\tau \neq h^\tau$ or $\mathcal{C}_\tau$ if $\tilde{h}^\tau = h^\tau$

Combining this fact with 2.2 one can write

$$\begin{aligned}
\sum_{t=0}^{\tau-1} \beta^t \sum_{h^t} \left[u(a(\sigma^\theta(h^t)) - \sigma^i(h^t, a)) - \frac{v(y(\sigma^\theta(h^t, a)))}{\theta_t(\theta^t)} - \psi(\sigma^e(h^t, a)) \right] \pi(h^t|\sigma, \mathcal{C}') \\
+\beta^\tau \sum_{\tilde{h}^\tau} \left[\omega(\tilde{h}^\tau, \mathcal{C}', \sigma^*) - \frac{v(y(\sigma^\theta(\tilde{h}^\tau)))}{\theta_t(\tilde{\theta}^t)} \right] \pi(\tilde{h}^\tau|\sigma, \mathcal{C}') > \\
\sum_{t=0}^{\tau-1} \beta^t \sum_{h^t} \left[u(a(h^t) - i(h^t, a)) - \frac{v(y(\theta^t))}{\theta_t(\theta^t)} - \psi(e(h^t, a)) \right] \pi(h^t|\sigma^*, \mathcal{C}') \\
+\beta^\tau \sum_{\tilde{h}^\tau} \left[\omega(\tilde{h}^\tau, \mathcal{C}', \sigma^*) - \frac{v(y(\tilde{\theta}^\tau))}{\theta_t(\tilde{\theta}^t)} \right] \pi(\tilde{h}^\tau|\sigma^*, \mathcal{C}')
\end{aligned}$$

Since $\mathcal{C}$ and $\mathcal{C}'$ are identical up until time τ, the probabilities of a history path are identical for the two until that time. As noted before, it is also the case that the continuation values for the two contracts at time τ are identical. So, one can rewrite the above as:

$$\sum_{t=0}^{\tau-1}\beta^t\sum_{\theta^t}\left[u(a(\sigma^\theta(\theta^t))-\sigma^i(\theta^t))-\frac{v(y(\sigma^\theta(\theta^t)))}{\theta_t(\theta^t)}-\psi(\sigma^e(\theta^t))\right]\pi(\theta^t|\sigma,\mathcal{C})$$
$$+\beta^\tau\sum_{\tilde{\theta}^\tau}\left[\omega(\tilde{\theta}^\tau,\mathcal{C},\sigma^*)-\frac{v(y(\sigma^\theta(\tilde{\theta}^\tau)))}{\theta_t(\tilde{\theta}^t)}\right]\pi(\tilde{\theta}^\tau|\sigma,\mathcal{C}) >$$
$$\sum_{t=0}^{\tau-1}\beta^t\sum_{\theta^t}\left[u(a(\theta^t)-i(\theta^t))-\frac{v(y(\theta^t))}{\theta_t(\theta^t)}-\psi(e(\theta^t))\right]\pi(\theta^t|\sigma^*,\mathcal{C})$$
$$+\beta^\tau\sum_{\tilde{\theta}^\tau}\left[\omega(\tilde{\theta}^\tau,\mathcal{C},\sigma^*)-\frac{v(y(\tilde{\theta}^\tau))}{\theta_t(\tilde{\theta}^t)}\right]\pi(\tilde{\theta}^\tau|\sigma^*,\mathcal{C})$$

This is equivalent to saying that the agent could benefit in the original contract $\mathcal{C}$ by following some deviation strategy for the first $\tau-1$ periods, deviating by (possibly) misreporting in period τ and following the planners recommendations in period τ and following a truthtelling and obedience strategy thereafter. Thus, a contradiction. ■

Proposition 13 Proof. Suppose not. Consider an alternative contract in which, for state s and allocation a, the planner offers instead $y^*_{s,a}$ where $\frac{v(y^*_{s,a})}{\theta_s} = \mathbf{E}^{\mu(a,\cdot)}\left[\frac{v(y_{s,a})}{\theta_s}|\theta_s\right]$. The planner is equally well off with such a contract as by the convexity of v, $y^*_{s,a}\geq\mathbf{E}^{\mu(a,\cdot)}[y_{s,a}]$

Similarly, let $\omega^{*\prime}_{s,a}=\sum_{\omega'_s}\mu(a,y,\omega')$.

Note that the agent's optimal action under the alternative contract is unchanged, as conditional on his allocation a, his optimal action maximizes

$$u(a-i)+\beta\mathbf{E}^{\mu(a,\cdot)}\left[-\frac{v(y_{s,a})}{\theta_s}+\omega'_s\right]$$

which, by assumption, is equal to

$$u(a-i)+\mathbf{E}^{\mu(a,\cdot)}\left[-\frac{v(y^*_s)}{\theta_s}+\omega^{*\prime}_{s,a}\right]$$

. Now, the incentive compatibility requires that for all $\tilde{s}$,

$$\mathbf{E}^{\mu}\left[-\frac{v(y_s)}{\theta_s}|\theta_s+\beta\omega'_s\right]\geq\mathbf{E}^{\mu}\left[-\frac{v(y_{\tilde{s}})}{\theta_s}|\theta_s+\beta\omega'_{\tilde{s}}\right]$$

. Considering the alternative starred contract does not alter the evaluation of either side of the incentive compatibility constraint and is therefore also satisfied. ■

Proposition 14 Proof. First, it is straightforward to implement the lower (upper) utility bounds by promising the maximum (minimum) physical resources each period from today onwards and the minimum (maximum) output requirements from next period onward. Because the promises are independent of state, this plan satisfies IC trivially. However, the agent cannot be forced to take any particular action. Rather the agent will always choose (i, e) optimally. In the case of the upper bound, the agent will clearly choose these values to be their lower bounds. At the lower utility bound, these values are chosen to minimize the disutility of maximum forced output in the following period. As randomization is allowed, all interior utilities can be implemented by randomizing between the maximum and minimum utility levels. ■

Chapter 3

Honesty vs. White Lies

3.1 Introduction

Is honesty necessarily the best policy, or might a white lie be desirable when telling the truth hurts? This question is often raised as an ethical problem but relevant in several economic and political contexts. In politics, politicians often emphasize their honesty rather than their rationality or policy preferences.[1] In recruiting, candidates are assessed not only by their abilities and skills, but also by their characters, and presumably honesty is one of the most important virtues. The same concerns also apply to accountants, journalists, managers, and stock analysts.

Informal discussions abound on why we must be honest and why we must look for honest people. However, it is not at all obvious why honesty should be preferred over other characteristics. Whenever communication occurs between agents and one agent is uncertain over the motivations of the other, the former will necessarily discount the value of communication. It then seems reasonable to think that a white liar who takes into consideration his counterpart's discounting of communication and lies only to benefit his counterpart would be preferable to an honest agent who

[1] According to an Associated Press-Ipsos poll, "55% of those surveyed consider honesty, integrity and other values of character the most important qualities they look for in a presidential candidate *(USA Today, 3/12/2007).*"

does not make such considerations.[2] In this paper, we formally examine this simple argument and demonstrate that honesty is often preferable to white lies.

We study two variants of the standard cheap-talk game á la Crawford and Sobel (1982) (CS, hereafter), which is the standard model to address communication issues. Both of our models introduce another layer of incomplete information in the cheap talk game, which can be naturally interpreted as the receiver's uncertainty over the sender's motive or the sender's having an imperfect reputation.

In the honesty model, the sender is behavioral with a positive probability. The behavioral sender is committed to honestly reporting the state. That is, if the receiver were to face this sender (whom we call the honest sender) with certainty, then the receiver would always learn the true state of nature. With the complementary probability, the sender is of the type in CS, that is, he is strategic and has preferences that are not perfectly aligned with those of the receiver (we call this type the biased sender).

In the white lie model, the sender is always strategic, but there is uncertainty over the sender's bias. More precisely, the sender has perfectly aligned preferences with those of the receiver with a positive probability, and does not with the complementary probability. We call the strategic sender with no bias a white liar. This sender knows that the receiver is uncertain about his own motive and will discount the credibility of his recommendation. Due to this consideration, he may lie but this is only for the sake of the receiver.[3]

Intuitively, there are two opposing arguments as to which sender type the receiver would prefer between the honest sender and the white liar. On the one hand, the receiver may value the flexibility of the white liar. If the white liar is not honest in

[2]In the American Heritage Dictionary of the English Language, Fourth Edition, a "white lie" is defined as an often trivial, diplomatic or *well-intentioned* untruth. As in Erat and Gneezy (2009), we focus on the last aspect of white lies.

[3]We note that the more common understanding of white lies, that a white lie costs the liar nothing but makes the receiver feel better, can be accommodated within our model by interpreting the white liar as the one who is fully altruistic, that is, who cares only about the receiver's utility.

equilibrium, it is because the receiver is better off by doing so. On the other hand, the honest sender is committed to telling the truth, and the receiver may value this commitment.

We first characterize equilibria in each model. The key to characterization is to identify new equilibrium conditions, relative to those of CS, due to incomplete information over the sender's type. The conditions highlight the effects of the honest type and the strategic type with no bias on communication.

In the honesty model, the new condition (mass balance condition) concerns the lower bound of information transmission due to the behavior of the honest type and is, to our knowledge, unique to this paper. It is generated by the fact that a distinct message is sent at each state by the honest sender and, therefore, messages are endowed with an intrinsic meaning. To see this more clearly, suppose there are two states (high and low) and two messages (also high and low), with each state realized with equal probability. If the probability of the honest type is equal to 1/2, then it is possible for the strategic type to wash out the information from the honest sender by simply reporting high when the state is low and vice versa. A similar strategy can be used to wash out the honest sender's message whenever the probability of the honest type is less than or equal to 1/2. However, if the probability of the honest type is greater than 1/2, independently of the biased type's strategy, the receiver obtains at least some useful information from communication. In this case, whenever the receiver gets the high message, it must be the case that the state is also high with probability greater than 1/2. The mass balance condition is the generalization of this insight into the case with continuous state and message spaces. When the relevant spaces are continuous, the probability of the honest type and the mapping between the biased type's state space and the message space (the honest type's state space) together determine the lower bound of information transmission. The mass balance condition captures both effects in a simple fashion in the uniform-quadratic environment.[4]

In the white lie model, the analogous condition (no arbitrage condition for the white liar) is essentially identical to the no arbitrage condition in other cheap talk papers. It states that at the boundary state of two partition elements, the white liar must be indifferent between the two induced actions. This condition holds in equilibrium precisely because the white liar is a strategic player and thus will always be able to adjust his message. Combined with the corresponding condition for the biased sender, this condition imposes rather severe restrictions on the equilibrium outcome.

We compare the welfare consequences of the two models and demonstrate that the receiver is often better off in the honesty model than in the white lie model. To be more precise, let μ be the probability that the sender is honest or a white liar in each model. We show that when μ is sufficiently large, the receiver is strictly better off in the honesty model than in the white lie model. We also show that when μ is sufficiently small, for the majority of bias values, the receiver is better off in the honesty model.[5] Lastly, we explain by some numerical examples that the same conclusion would hold for intermediate cases.

Our welfare result is rather surprising, as the white liar chooses not to be honest in order to increase the receiver's utility. There are two main driving forces for the result. Both highlight the value of commitment in communication but emphasize different aspects of commitment.

The first reason relates to the fact that commitment simplifies communication protocol and thus reduces the loss due to strategic considerations. To see this point, suppose the receiver is certain that the sender is honest. In this case, there is a unique communication outcome. The receiver perfectly trusts the sender's recommendation and perfect communication results. This implies that, in the honesty model, when

[4] In the last section, we discuss how to generalize the condition into more general environment.

[5] We also show that for some parameter values, the receiver is instead better off in the white lie model.

there is a small probability that the sender is biased, any communication outcome will be close to that of perfect communication. Now suppose the sender is the white liar for sure. Perfect communication is still possible, but there are lots of other possibilities. For example, there may be no information transmission at all.[6] This multiplicity of equilibrium yields the following consequence in the white lie model: when there is a small probability that the sender is biased, the best communication outcome is far away from the perfect communication outcome. That is, the loss from imperfect communication does not vanish even when the sender is the white liar almost for sure.

The second reason that honesty is often preferable is that commitment has the effect of enriching language used in communication. In the honesty model each message might be sent by the honest sender, and thus the receiver interprets each message differently.[7] Therefore, different from CS and the white lie model, all messages are fully used in the honesty model, which allows freer communication between the receiver and the *biased* sender. To see how this can improve the receiver's welfare, recall the result by Dessein (2002). He shows that whenever communication is possible in equilibrium, both the receiver and the biased sender would benefit if the former is committed to following the latter's recommendation. This implies that they have a coordination incentive that is hindered by their selfishness in communication. The possibility of honesty, by enriching language used in communication, allows them to utilize the coordination incentive in a way that is not possible in the original cheap talk game and in the white lie model.[8]

Honesty is thought to be an important virtue in many contexts and our welfare

[6]If the sender does not provide any meaningful information, the receiver has no reason to pay attention to the sender's report, which in turn justifies the sender's behavior.

[7]In the honesty model, even though the receiver takes a constant action on a set of messages, her posterior over the set of states is not constant over those messages. Only her conditional expectation is constant.

[8]This effect is similar to the one generated by "noise" in communication. Blume, Board, and Kawamura (2007) showed that adding noise to communication can improve welfare. As in this paper, the interpretation of messages becomes crucial with noise, which is the key reason why welfare can improve.

result can be interpreted as a micro-foundation for the preference for honesty. In particular, the political economy literature has recognized the importance of character in politics and attempted to incorporate the preference for character in the analysis.[9] However, such preference has been assumed to be exogenously given. Our result provides a fundamental rationale for the preference for honesty in particular and character in general, focusing on the effect of honesty on communication.

In terms of analysis, this paper contributes to two branches of literature, each of which corresponds to the honesty model and the white lie model. The first strand examines the roles of behavioral (non-strategic) types in communication. Ottaviani and Squintani (2006) and Kartik, Ottaviani, and Squintani (2007) examine the cases where the receiver may be naive, that is, the receiver may simply follow the messages. They show that if the state and message spaces are unbounded, then there exists a fully revealing equilibrium (Kartik, Ottaviani, and Squintani (2007)), while if the state and message spaces are bounded, states are fully revealed in a low range and partitioned in the top range (Ottaviani and Squintani (2006)). Chen (2009) considers a cheap-talk game with both honest sender and naive receiver. The honest sender in her paper is in the same spirit as that of this paper, but only uses a finite number of messages: there is a finite partition in the state space and the honest type sends the same messages on each partition element. The finite message assumption was made because she restricted attention to message-monotone equilibria (in which the strategic sender sends weakly higher messages on higher states), which may not exist if the message space is continuous (see Example 2 in her paper). We focus on the case where the message space is continuous and do not restrict attention to message-monotone equilibria. The equilibria we characterize in this paper are not message-monotone, but are still intuitive and relatively tractable.

The second branch explores the implications of uncertain bias in the cheap-talk game. Morgan and Stocken (2003) (MS, hereafter) examine the white lie model in

[9] See, for example, Groseclose (2001), Aragones and Palfrey (2002), Diermeier, Keane, and Merlo (2005), and Kartik and McAfee (2007).

the context of stock analyst's problem where the stock analyst may or may not have an incentive to produce a favorable report to the firm's investment banking clients. They characterize two classes of equilibria, while we fully characterize the set of equilibria. The complete characterization is important in our paper because the new equilibria have an important welfare implication. If we had considered only the equilibria in MS, when μ is sufficiently close to 0, the receiver would be always better off in the honesty model than in the white lie model. In other words, the receiver is better off in the white lie model for some parameter values precisely because of the additional equilibria we identify. Li and Madarasz (2008) consider the case whether the sender has either high or low bias (the low bias can be negative), and examine whether requiring the sender to disclose his own bias is necessarily welfare-improving. Their characterization includes "categorical ranking system equilibria" in MS. In addition, they provide one example (Example 4 in their paper) which is one of "semiresponsive" equilibria in MS. Dimitrakas and Sarafidis (2005) consider the case where the sender's bias is drawn from a distribution over an interval. Their recursive characterization is similar to the one in this paper, but does not directly apply to the binary case. In addition, they briefly discuss the connection between their characterization and that of MS (Proposition 5 in their paper), but it is not clear how much the result can generalize beyond the case they examined.[10]

There are also a few papers that study dynamic cheap-talk games with uncertainty over the sender's motives. Sobel (1985) and Morris (2001) examine the white lie cases, while Benabou and Laroque (1992) and Olszewski (2004) consider the honesty cases. The focus of these works is the dynamic incentive of the sender to maintain reputation as well as to manipulate information. To highlight the intertemporal incentives, they consider simple stage games in which there are only a finite number of states. In order to focus on the effects of uncertain motives on communication itself, we consider static settings with continuous state and message spaces.

[10]They only consider the case where the sender has zero bias with probability 0.5 and the sender's bias is weakly greater than $2-\sqrt{2}$.

The remainder of the paper is organized as follows. The next section briefly reviews the standard cheap talk game. Then, we study the honesty model and the white lie model in Sections 3.3 and 3.4, respectively. We compare the two models in Section 3.5 and conclude in Section 3.6 by discussing two relevant issues.

3.2 Review of the Standard Cheap Talk Game

Our models and results can be best understood through comparison to the standard cheap talk model and results. We briefly review the standard cheap talk game in the uniform-quadratic environment.

3.2.1 Cheap talk game

There are two players, a receiver (she) and a sender (he). The sender observes a random variable θ and strategically transmits information on θ to the receiver. A random variable θ is drawn from a uniform distribution with support on $\Theta = [0, 1]$. The receiver takes an action, denoted by y, from the real line that affects utilities of both players. The receiver's utility function is $U^R(y, \theta) \equiv -(y - \theta)^2$, while the sender's is $U^S(y, \theta, b) \equiv -(y-(\theta+b))^2$ where $b \in (0, 1)$. As usual, b is interpreted as the "bias" of the sender. When the true state is θ, the most preferred action to the receiver is θ, while that of the sender is $\theta + b$. Without loss of generality, we assume that the set of messages, M, and the set of feasible actions, Y, are given by $[0, 1]$.

The receiver's strategy is her action choice rule $y : M \to Y$ where $y(m)$ is the action she takes after receiving message m. The sender's strategy is her reporting rule $r : \Theta \to \Delta(M)$ where $\Delta(M)$ is the set of probability measures over the set of messages, M, and $r(\theta)$ is her reporting policy conditional on observation of state θ.

Definition 2 *The strategy profile (r^*, y^*) constitutes an equilibrium if*

(1) given y^*, if m' is sent by the sender (m' is in the support of $r^*(\theta)$), then

$$m' \in \arg\max_{m \in M} U^S(y^*(m), \theta, b) = -(y^*(m) - (\theta + b))^2, \text{ and}$$

(2) given r^*, for all $m \in M$,

$$y^*(m) \in \arg\max_{y} E[U^R(y, \theta)|m].$$

Since the receiver has a quadratic utility function, the second requirement reduces to $y^*(m) = E[\theta|m]$. That is, the receiver's optimal strategy is always to choose the conditional expectation of θ.

3.2.2 Equilibrium Characterization

An equilibrium in the cheap talk game is characterized by a partition $\{\theta_0 = 0, \theta_1, ..., \theta_n = 1\}$ and a sequence $\{y_1, ..., y_n\}$. The sender sends an essentially identical message on each partition element, $[\theta_{k-1}, \theta_k]$. The receiver infers only in which partition element the true state, θ, lies, and thus takes only a finite number of actions, $\{y_1, ..., y_n\}$. The following two conditions are necessary and sufficient:

$$y_k = \tfrac{\theta_{k-1}+\theta_k}{2}, k = 1, ..., n, \qquad \text{(BR)}$$
$$(\theta_k + b) - y_k = y_{k+1} - (\theta_k + b), k = 1, ..., n-1. \qquad \text{(NA)}$$

BR (Best Response) corresponds to the receiver's optimality and NA (No Arbitrage) corresponds to the sender's. NA states that the sender must be indifferent between y_k and y_{k+1} at state θ_k for each $k = 1, ..., n-1$. This is necessary and sufficient for the sender's optimality because the sender's utility function is quadratic and thus the single crossing property holds.[11]

Figure 3.1 illustrates how the two conditions interact. We exploit this type of graphical representation throughout the rest of the paper. The bottom line represents the state space, Θ, and the top line represents the receiver's action space, Y. In

[11] If the sender prefers y to y' at θ where $y > y'$, then he prefers y to y' at any state θ' such that $\theta' > \theta$, and vice versa.

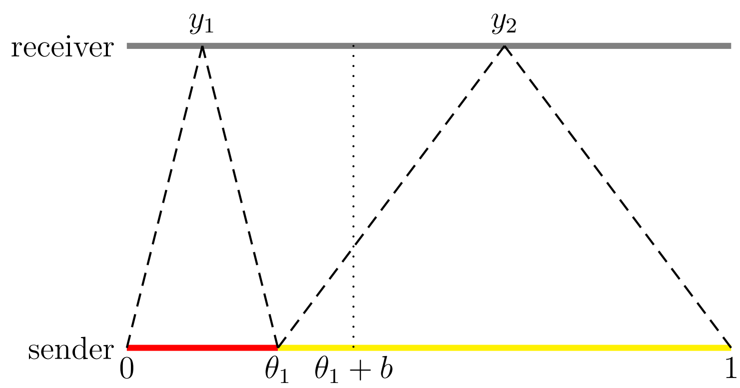

Figure 3.1: An equilibrium with two partition elements in the standard cheap talk game.

equilibrium, the sender partitions the state space and the receiver takes as many actions as the number of partition elements. When there are two partition elements, $[0, \theta_1]$ and $[\theta_1, 1]$, the receiver takes either y_1 or y_2. BR requires that the receiver's action (y_k) be the conditional expectation of θ, while NA states that the sender's most preferred action at state θ_1, $\theta_1 + b$, is the average of the two actions induced in equilibrium.

3.2.3 Incentive Compatibility Lemma

We present a lemma that is used throughout this paper. Consider a cheap talk game in which the set of sender types is given by T. Let h represent the honest type. Denote by B the subset of T that contains only strategic types. A typical element of B, denoted by b, represents the bias of the strategic sender. Then in the standard game $T = B = \{b\}$, in the honesty model $T = \{h, b\}$ and $B = \{b\}$, and in the white lie model $T = B = \{0, b\}$.

Fix an equilibrium of a cheap talk game. The outcome of the equilibrium is represented by a function $z : \Theta \times T \to Y$ where $z(\theta, t)$ is the action chosen by the receiver when the true state is θ and the sender's type is t. In addition, let

$V^S : \Theta \times B \to R$ be the indirect utility function of the equilibrium where $V^S(\theta, b)$ is the utility that the strategic sender with bias b obtains at state θ in the equilibrium.

A necessary condition for an equilibrium is that its outcome is interim incentive compatible for any strategic sender. As in the Bayesian mechanism design, this imposes certain restrictions on possible equilibrium outcomes, which are presented in the following lemma.

Lemma 15 *The outcome $z(\cdot, b)$ is interim incentive compatible for the strategic sender with bias b if and only if*

(i) $z(\cdot, b)$ is nondecreasing,

(ii) $V^S(\cdot, b)$ is continuous, and

(iii) if $z_1(\theta, b) = \partial z(\theta, b)/\partial\theta$ exists, then

$$\frac{\partial U^S(z(\theta, b), \theta, b)}{\partial y} \cdot \frac{\partial z(\theta, b)}{\partial \theta} = 0.$$

In this paper, we exploit, in particular, the implications of part (iii). In equilibrium, for almost all states, the sender must induce either a constant action around each state ($\partial z(\theta, b)/\partial\theta = 0$) or induce his most preferred action ($\partial U^S(z(\theta, b), \theta, b)/\partial y = 0$). To understand this in the context of the cheap talk literature, consider an equilibrium outcome in CS and the case where the receiver delegates the authority to choose an action to the sender in the sense of Dessein (2002).[12] The sender has no incentive to deviate in both cases, but for different reasons. In the latter, the sender is achieving the first-best payoff at each state, while in the former, a small deviation does not change the outcome and a large deviation lowers the sender's payoff. In other words, in CS, $\partial z(\theta)/\partial\theta = 0$ almost everywhere, while in Dessein, $\partial U^S(z(\theta, b), \theta, b)/\partial y = 0$ everywhere. Part (iii) of the lemma implies that if there exist any other kinds of equilibrium outcomes, then it must be that the structures are some combinations of partitional outcomes as in CS and outcomes in which the sender implements his most preferred actions as in Dessein.

[12]In other words, the receiver is committed to following the sender's recommendation.

3.3 Honesty Model

3.3.1 Setup

In this section, the sender is honest with probability $\mu \in (0,1)$, and is strategic with bias b with the complementary probability. The honest type is behavioral and his strategy is to send different messages for each state, so that the receiver knows the state for sure if the sender's type is known to be honest. Without loss of generality, we focus on the case where the honest type sends message θ when he observes state θ.[13] The strategies of the receiver and the biased type are the same as in the previous section. We denote by r_b the biased type's strategy.

Definition 3 *The strategy profile (r_b^*, y^*) constitutes an equilibrium if*

(1) given y^*, if m' is sent by the biased type (m' is in the support of $r_b^*(\theta)$), then

$$m' \in \arg\max_{m \in M} U^S(y^*(m), \theta, b) = -(y^*(m) - (\theta + b))^2, \text{ and}$$

(2) given r_b^*, for all $m \in M$,

$$\begin{aligned} y^*(m) &\in \arg\max_{y} E_{\mu, r_b^*}[U^R(y,\theta)|m], \\ &\Leftrightarrow y^*(m) = E_{\mu, r_b^*}[\theta|m], \end{aligned}$$

where E_{μ, r_b^*} is the conditional expectation operator generated by μ, the honest type's behavior, and r_b^*.

3.3.2 CS Equilibrium Outcomes

We first consider an equilibrium in which every message induces the receiver to take the same action. In CS, this equilibrium is called a "babbling" equilibrium because the sender essentially randomizes over the entire message space independently of his

[13] We consider any sender with a bijective mapping of states to messages as honest. So long a sender's strategy is such, then a receiver who knowingly faces this sender will know the state for sure.

private information. Since the honest type always reports the true state, a babbling equilibrium does not exist.

A no communication equilibrium still exists in a different form as long as $\mu \le 1/2$. Consider the following strategy profile.

$$\begin{aligned} r_b(\theta) &= \begin{cases} 1-\theta, & \text{with probability } \frac{\mu}{1-\mu}, \\ m \sim U[0,1], & \text{with probability } \frac{1-2\mu}{1-\mu}, \end{cases} \\ y(m) &= \frac{1}{2}, \forall m. \end{aligned}$$

In this profile, at state θ, the biased type sends message $1-\theta$ with probability $\mu/(1-\mu)$ and uniformly randomizes over the entire message space with the complementary probability. Having received any message m, the conditional probability that the message came from the honest type is equal to the unconditional probability of the honest type (μ). Similarly, the conditional probability that the message came from the biased type who is reporting $1-\theta$ and the conditional probability that it came from the biased type who is uniformly randomizing over $[0,1]$ are μ and $1-2\mu$, respectively. As such, for any m, the conditional expectation of the receiver on the true state is

$$E[\theta|m] = \mu m + (1-\mu)\left(\frac{\mu}{1-\mu}(1-m) + \frac{1-2\mu}{1-\mu}\int_0^1 \theta d\theta\right) = \frac{1}{2}.$$

Therefore, the receiver takes a single action independently of message. This, in turn, makes the biased type indifferent over all messages.

We note that although the receiver takes a single action, she updates her belief nontrivially. In the babbling equilibrium in CS, every posterior distribution is equal to the prior distribution. Under the strategy above, the receiver's posterior puts mass on the message she received, m, and its counterpart, $1-m$. Therefore, her posterior is different for each message even though her action is independent of message. It is only the conditional expectation of the true state that is constant across messages.

This equilibrium construction generalizes to all equilibrium outcomes in CS. We can use the same trick partition element by partition element: the biased type reports exactly the "opposite" state in each partition element with probability $\mu / (1-\mu)$ and randomizes over the interval with the remaining probability.

Proposition 16 *Fix an equilibrium outcome in CS that is characterized by a partition* $\{\theta_0 = 0, \theta_1, ..., \theta_n = 1\}$ *and a sequence* $\{y_1, ..., y_n\}$*. If* $\mu \leq 1/2$*, then there exists an equilibrium in the honesty model in which the receiver takes the action* y_k *whenever the true state lies in* $[\theta_{k-1}, \theta_k]$*, independently of the sender's type, for all* $k = 1, ..., n$*.*

Proof. Suppose $\mu \leq 1/2$. Consider the following strategy of the biased type: for each $k = 1, ..., n$ and each $\theta \in [\theta_{k-1}, \theta_k]$,

$$r_b(\theta) = \begin{cases} \theta_{k-1} + \theta_{k-1} - \theta, & \text{with probability } \frac{\mu}{1-\mu}, \\ m \sim U[\theta_{k-1}, \theta_k], & \text{with probability } \frac{1-2\mu}{1-\mu}, \end{cases} .$$

Given this strategy of the biased type and the behavior of the honest type, the conditional expectation of the receiver is equal to y_k whenever the message m belongs to the interval $[\theta_{k-1}, \theta_k]$. In turn, since we begin with an equilibrium in CS, in each state $\theta \in [\theta_{k-1}, \theta_k]$, the biased sender is indifferent over all messages in $[\theta_{k-1}, \theta_k]$ and strictly prefer those to the other messages. ■

3.3.3 Mass Balance Condition

In this subsection, we characterize a lower bound on the amount of information that can be transmitted in an equilibrium of the honest model. This lower bound depends on the probability of the honest sender μ. For example, if $\mu > 1/2$, the strategy profiles used in the previous subsection are not well defined and thus cannot be used to support CS equilibrium outcomes. The lower bound established in this subsection then implies that for $\mu > 1/2$ no other strategy profile induces any CS equilibrium outcome.

Suppose the receiver receives a message m in an interval $[m', m'']$. In addition, suppose the receiver knows that the biased type sends messages $[m', m'']$ if and only if the state is in an interval $[\theta', \theta'']$. Let $B(\mu, m', m'', \theta', \theta'')$ be the expectation of the state conditional on the event that the message belongs to the interval $[m', m'']$. Formally,

$$B(\mu, m', m'', \theta', \theta'') \equiv \frac{\mu(m''-m')}{\mu(m''-m')+(1-\mu)(\theta''-\theta')}\frac{m'+m''}{2} + \frac{(1-\mu)(\theta''-\theta')}{\mu(m''-m')+(1-\mu)(\theta''-\theta')}\frac{\theta'+\theta''}{2}.$$

Under what conditions is it possible that the receiver makes no further inference and implements only a constant action $B(\mu, m', m'', \theta', \theta'')$ over all messages in the interval $[m', m'']$? The following lemma establishes that it is possible if and only if it is sufficiently unlikely that the sender is honest (μ is small) or the biased type sends those messages on a sufficiently large range of states ($\theta'' - \theta'$ relative to $m'' - m'$). In other words, for the receiver to draw a single inference from a mass of messages, there must be sufficient mass of probability from the biased sender that washes out the inherent information in the messages.

Lemma 16 ***(Mass Balance Condition)*** *Suppose* $0 \le m' < m'' \le 1$ *and* $0 \le \theta' < \theta'' \le 1$. *There exists a collection of probability measures* $\{r(\theta), \theta \in [\theta', \theta'']\} \subset \Delta([m', m''])$ *such that*

$$E_{\mu,r}[\theta|\mathcal{M}] = B(\mu, m', m'', \theta', \theta'') \text{ for any Borel set } \mathcal{M} \text{ in } [m', m''],$$

if and only if

$$\mu(m''-m')^2 \le (1-\mu)(\theta''-\theta')^2,$$

where $E_{\mu,r}$ *is the conditional expectation operator generated by* μ *and* r.

This lemma relates the inherent informational content of messages to the biased type's strategy by providing a necessary and sufficient condition for the biased type to be able to induce a single inference (and, consequently, a single action for the receiver) over an interval of messages. The condition shows how the honest type's commitment to telling the truth imposes a lower bound on the amount of information transmission and thus highlights the effects of honesty on communication.

Intuitively, when μ is small or $m'' - m'$ is relatively smaller than $\theta'' - \theta'$, the biased sender can keep the receiver from drawing any further inference and thus induce her to take a constant action. However, when μ is large or $m'' - m'$ is relatively larger than $\theta'' - \theta'$, the biased type does not have enough latitude to stop more information being transmitted to the receiver. From the receiver's viewpoint, it is likely that the messages were sent by the honest type and thus she must take seriously into account the messages in her decision making.

The lemma implies that the converse of Proposition 16 is also true. That is, if $\mu > 1/2$ then no equilibrium outcome in CS can be supported as an equilibrium outcome in the honesty model.

Corollary 3 *Fix an equilibrium outcome in CS that is characterized by a partition* $\{\theta_0 = 0, \theta_1, ..., \theta_n = 1\}$ *and a sequence* $\{y_1, ..., y_n\}$. *If* $\mu > 1/2$, *then there does not exist an equilibrium in the honesty model in which the receiver takes the action* y_k *whenever the true state lies in* $[\theta_{k-1}, \theta_k]$, *independently of the sender's type, for all* $k = 1, ..., n$.

Proof. Suppose such equilibrium exists. Then, by the previous lemma, it must be that $\mu(\theta_k - \theta_{k-1})^2 \leq (1-\mu)(\theta_k - \theta_{k-1})^2$ for all $k = 1, ..., n$. But the inequalities do not hold if $\mu > 1/2$. ■

We will make use of the following reporting strategy of the biased sender throughout the paper:

$$r_b(\theta) = \begin{cases} \frac{m''-m'}{\theta''-\theta'}(\theta'' - \theta) + m', & \text{with probability } \frac{\mu}{1-\mu}\frac{(m''-m')^2}{(\theta''-\theta')^2}, \\ m \sim U[m', m''], & \text{with probability } 1 - \frac{\mu}{1-\mu}\frac{(m''-m')^2}{(\theta''-\theta')^2}. \end{cases}$$

For notational simplicity, we denote this strategy by "$r_b(\theta) = \tilde{r}([m', m''])$ if $\theta \in [\theta', \theta'']$."

3.3.4 Type I Equilibrium

Returning back to the least communication outcome, for $\mu > 1/2$, the following strategy profile is a natural extension of no communication equilibrium: for some $m_0 > 0$,

$$\begin{aligned} r_b(\theta) &= \tilde{r}([m_0, 1]), \forall\theta \\ y(m) &= \begin{cases} m, & \text{if } m < m_0, \\ B(\mu, m_0, 1, 0, 1), & \text{if } m \geq m_0. \end{cases} \end{aligned}$$

In this strategy profile, the biased type sends messages above m_0. The receiver believes that only the honest type sends messages below m_0, and so perfectly trusts their contents.

The strategy profile is specifically designed to overcome the binding mass balance condition. Notice that the mass balance condition is satisfied if m_0 is high enough: $\mu(1-m_0)^2 \leq 1-\mu$. Of course, we must ensure that the biased type does not want to deviate to some message below m_0. Therefore, it must be that

$$m_0 \leq b \text{ and } |B(\mu, m_0, 1, 0, 1) - b| \leq b - m_0.$$

The first inequality guarantees that the biased type cannot implement her most preferred action by deviating to below m_0 at state 0. The second inequality ensures that the biased type prefers $B(\mu, m_0, 1, 0, 1)$ to m_0 at state 0. By the single crossing property, the biased type does not deviate at any other state.

Example 1 *Suppose $\mu > 1/2$ and $b \geq \frac{1}{2}\left(\frac{\sqrt{\mu}-\sqrt{1-\mu}}{\sqrt{\mu}} + \frac{\sqrt{\mu}}{\sqrt{\mu}+\sqrt{1-\mu}}\right)$. Then the following strategy profile is an equilibrium.*

$$\begin{aligned} r_b(\theta) &= \tilde{r}\left(\left[\frac{\sqrt{\mu}-\sqrt{1-\mu}}{\sqrt{\mu}}, 1\right]\right), \forall\theta \\ y(m) &= \begin{cases} m, & \textit{if } m < \frac{\sqrt{\mu}-\sqrt{1-\mu}}{\sqrt{\mu}}, \\ \frac{\sqrt{\mu}}{\sqrt{\mu}+\sqrt{1-\mu}}, & \textit{if } m \geq \frac{\sqrt{\mu}-\sqrt{1-\mu}}{\sqrt{\mu}}. \end{cases} \end{aligned}$$

In this strategy profile, m_0 is chosen so that the mass balance condition is always binding, that is, $\mu(1-m_0)^2 = (1-\mu)$.

This strategy profile also constitutes an equilibrium when $\mu < 1/2$. Since the mass balance condition is satisfied vacuously, the strategy profile is an equilibrium as long as the incentive compatibility condition is satisfied.

Definition 4 ***(Type I Strategy Profile)*** *A type I strategy profile is represented by two partitions in a unit interval,* $\{0, m_0, m_1, ..., m_n = 1\}$ *and* $\{\theta_0 = 0, \theta_1, ..., \theta_n = 1\}$*, and a sequence,* $\{y_1, ..., y_n\}$*, such that*[14]

$$\begin{aligned} r_b(\theta) &= \tilde{r}([m_{k-1}, m_k]), \text{ if } \theta \in [\theta_{k-1}, \theta_k], \forall k = 1, .., n, \\ y(m) &= \begin{cases} m, & \text{if } m < m_0, \\ y_k, & \text{if } m \in [m_{k-1}, m_k], \forall k = 1, .., n. \end{cases} \end{aligned}$$

In this strategy profile, the biased type sends $[m_{k-1}, m_k]$ on $[\theta_{k-1}, \theta_k]$ and does not send any message below m_0. The following conditions are necessary and sufficient for a type I strategy profile to be an equilibrium:

$$|y_1 - b| \leq b - m_0 \text{ and } m_0 \leq b \text{ if } m_0 > 0, \quad \text{(IC)}$$

$$y_k + y_{k+1} = 2(\theta_k + b), \forall k = 1, ..., n-1, \quad \text{(NA)}$$

$$y_k = B(\mu, m_{k-1}, m_k, \theta_{k-1}, \theta_k), \forall k = 1, ..., n, \quad \text{(BR)}$$

$$\mu(m_k - m_{k-1})^2 \leq (1-\mu)(\theta_k - \theta_{k-1})^2, \forall k = 1, ..., n. \quad \text{(MB)}$$

IC is the incentive compatibility condition for the biased type to not deviate to below m_0. NA and BR are the same conditions as in CS. BR is modified to reflect uncertainty over the sender's type. MB is the mass balance condition in Lemma 16. As μ tends to 0, NA and BR converge to the equilibrium conditions in CS, and IC and MB become negligible. Notice that should there exist an equilibrium with n partition elements, there will typically exist a continuum of such equilibria due to the flexibility of MB.

Figure 3.2 shows an example of type I equilibrium. The biased type sends messages $[m_0, m_1]$ on $[0, \theta_1]$ and messages $[m_1, 1]$ on $[\theta_1, 1]$. The receiver believes that

[14] *We do not explicitly specify what the sender's reporting policies are at the boundary points of each partition element. They do not affect both players' ex ante utilities because the set has zero measure.*

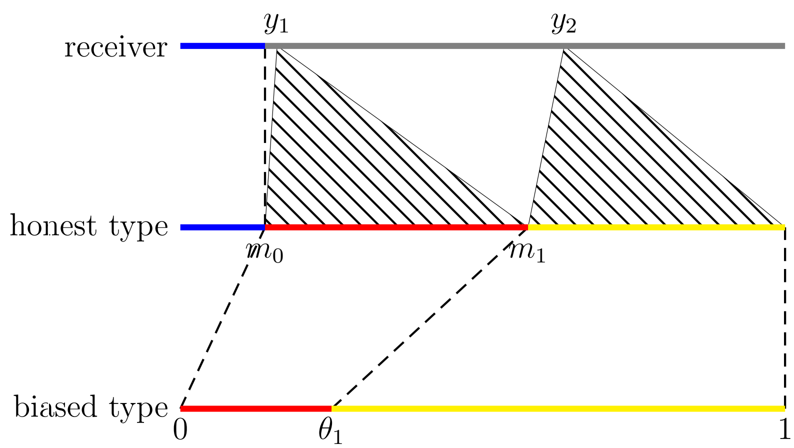

Figure 3.2: Type I equilibrium when $b = 0.15$ and $\mu = 0.1$.

if message m is below m_0, it was sent by the honest type and thus implements m. If the message is above m_0, then she chooses y_1 or y_2 depending on whether the message lies in $[m_0, m_1]$ or $[m_1, 1]$. The two policies induced by the biased type, y_1 and y_2, are conditional expectations of the true state (BR), and the biased type is indifferent between y_1 and y_2 at state θ_1 (NA). Though $m_1 - m_0 > \theta_1$, MB is satisfied because μ is small.

3.3.5 Type II Equilibrium

For μ sufficiently large, there cannot exist any type I equilibrium. As μ increases, m_0 must increase so that MB holds. But then IC binds because b is an upper bound of m_0 in type I equilibrium. Therefore, we need an alternative way to consume an excess of mass with the honest type in order to satisfy both IC and MB. The key to this issue is in Lemma 15. According to Lemma 15, $\Theta = [0, 1]$ (conditional on the event that the sender is strategic and has bias b) can be decomposed into the following three subsets:

$$\begin{aligned}\Theta_1 &= \{\theta \in \Theta : z(\theta, b) = \theta + b\} \\ &= \{\theta \in \Theta : \text{the sender implements his most preferred action}\}, \\ \Theta_2 &= \left\{\theta \in \Theta : z_1(\theta, b) = \frac{\partial z(\theta, b)}{\partial \theta} = 0\right\} \\ &= \{\theta \in \Theta : \text{the same action is implemented around } \theta\}, \\ \Theta_3 &= \{\theta \in \Theta : z_1(\theta, b) \text{ does not exist}\}.\end{aligned}$$

In many variations of the cheap talk game, Θ_2 has full measure; equilibrium features a partitioning of Θ with which a constant action is induced on each partition element. However, for μ sufficiently large, Θ_2 cannot have full measure because of the conflict between MB and IC. Then, Θ_1 is the only alternative of use, as the non-decreasing property of $z(\cdot, b)$ implies that Θ_3 has zero measure (see part (i) of Lemma 15). On Θ_1, the biased type induces his own optimal action. This possibility does not arise in CS and many other contexts. However, Θ_1 is *necessary* in the honesty model for μ sufficiently large.

Definition 5 ***(Type II Strategy Profile)*** *A type II strategy profile is represented by two partitions, $\{0, m_0, ..., m_n\}$ and $\{0, \theta_0, ..., \theta_n\}$, and a sequence, $\{y_1, ..., y_n\}$, such that*

$$\begin{aligned}r_b(\theta) &= \begin{cases} \theta + b/\mu, & \text{if } \theta \in [0, \theta_0], \\ \tilde{r}\left([m_{k-1}, m_k]\right), & \text{if } \theta \in [\theta_{k-1}, \theta_k], k = 1, ..., n, \end{cases} \\ y(m) &= \begin{cases} m, & \text{if } m \le b/\mu, \\ \mu m + (1-\mu)(m - b/\mu), & \text{if } b/\mu < m \le m_0, \\ y_k, & \text{if } m \in [m_{k-1}, m_k], \end{cases}\end{aligned}$$

In this strategy profile, the biased type induces his own optimal action on $[0, \theta_0]$. For example, suppose $\theta = 0$. Then the biased type sends message b/μ. When the receiver receives this message, her inference on θ is $\mu \cdot (b/\mu) + (1-\mu) \cdot 0 = b$, which

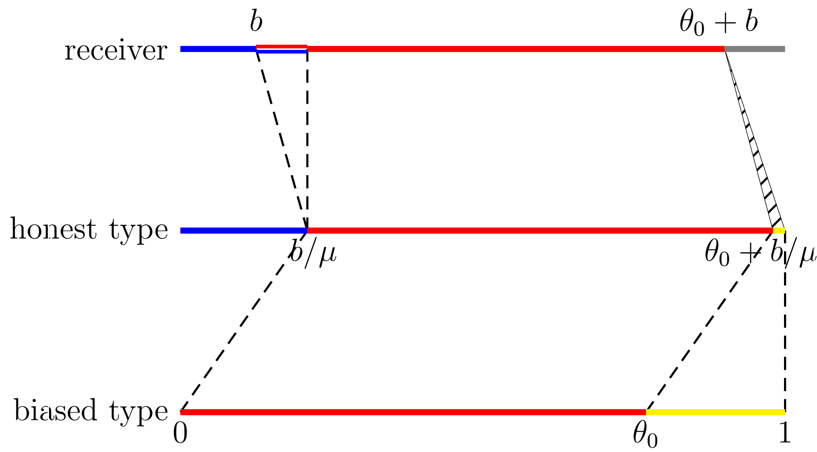

Figure 3.3: Type II equilibrium when $b = 1/8$ and $\mu = 0.6$.

is optimal to the biased type. The conditions required for this strategy profile to be an equilibrium are as follows:

$$
\begin{aligned}
& b/\mu \leq \theta_0 + b, && \text{(IC)},\\
& m_0 = \theta_0 + b/\mu, && \text{(EL)},\\
& y_1 = \theta_0 + b, && \text{(NA0)},\\
& y_k + y_{k+1} = 2(\theta_k + b), \forall k \geq 1, && \text{(NA)},\\
& y_k = B\left(\mu, m_{k-1}, m_k, \theta_{k-1}, \theta_k\right), \forall k \geq 1, && \text{(BR)},\\
& \mu(m_k - m_{k-1})^2 \leq (1-\mu)(\theta_k - \theta_{k-1})^2, \forall k \geq 1, && \text{(MB)}.
\end{aligned}
$$

NA, BR, and MB are the same as before. NA0 is the condition required to prevent the biased type from deviating to $[b/\mu, m_0]$ for $\theta > \theta_0$. IC guarantees that the deviation to $[0, b/\mu)$, where the receiver perfectly trusts messages, is not profitable. EL (Equal Length) is an obvious requirement from the structure of equilibrium.

Example 2 *For $\mu \geq 1/2$ and $b \leq \mu / \left(1 + \sqrt{\mu(1-\mu)}\right)$, the following strategy profile is a Type II equilibrium.*

$$r_b(\theta) = \begin{cases} \theta + b/\mu, & \text{if } 0 \leq \theta < \theta_0, \\ \tilde{r}([\theta_0 + b/\mu, 1]), & \text{if } \theta \in [\theta_0, 1], \end{cases}$$

$$y(m) = \begin{cases} m, & \text{if } m \leq b/\mu, \\ m - \frac{1-\mu}{\mu} b, & \text{if } b/\mu < m < \theta_0 + b/\mu, \\ \theta_0 + b, & \text{if } m \in [\theta_0 + b/\mu, 1], \end{cases}$$

where $\theta_0 = 1 - b - b\sqrt{(1-\mu)/\mu}$.

Figure 3.3 shows the structure of this equilibrium. The biased type sends message $\theta + b/\mu$ if $\theta < \theta_0$, and sends messages $[\theta_0 + b/\mu, 1]$ on $[\theta_0, 1]$. The receiver perfectly trust the sender's report if the message is below b/μ, while she believes that any message above b/μ might be sent by the biased type. The receiver discounts the sender's recommendation by the same amount, $(1-\mu)/\mu \cdot b$, on $[b/\mu, \theta_0 + b/\mu]$, while she takes a single action, $\theta + b$, on $[\theta_0 + b/\mu, 1]$. Notice that the biased type induces her most preferred actions on $[0, \theta_0]$, and actions $[b, b/\mu]$ are chosen both when $m \in [b, b/\mu]$ and when $m \in [b/\mu, 2b/\mu - b]$.

3.3.6 Other Possibilities

The two equilibrium structures do not exhaust all equilibria in the honesty model.[15] Still, they are the only equilibria that satisfy two natural properties of the strategy profiles, one for each player. We introduce each property and present an example of equilibrium that violates the property.

Definition 6 *(Convexity) The biased type's reporting strategy is convex if there exists $m_0 \in [0, 1]$ such that the biased type never sends messages below m_0 and sends*

[15]Either type I or type II equilibrium always exists for $b \leq 1/2$. For $\mu \leq 1/2$, equilibria that yield CS equilibrium outcomes, which are type I, exist, as shown in Proposition 16. For $\mu > 1/2$, Example 3.2 and Example 3.3 establish that at least one of the equilibria exists. The two examples also show that for $b > 1/2$ type I equilibrium exists when μ is not too large and type II equilibrium exists when μ is sufficiently large.

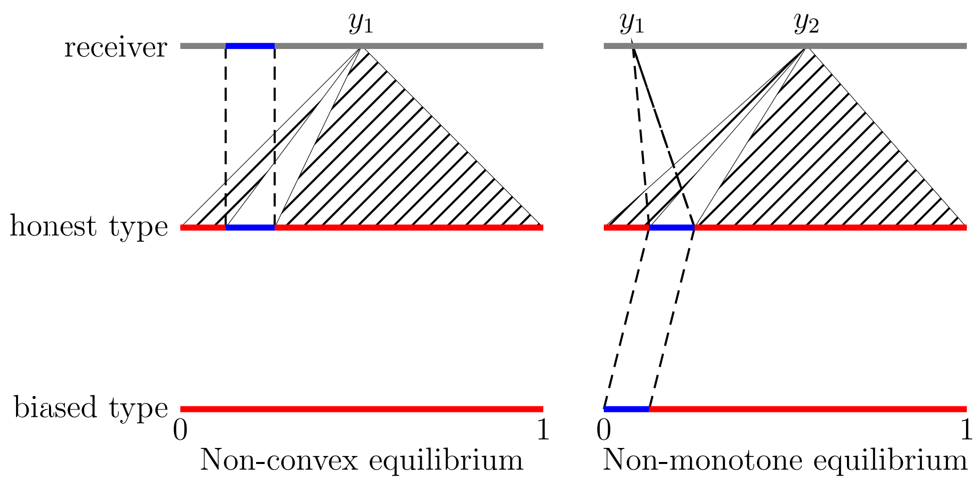

Figure 3.4: Equilibria that do not satisfy convexity or monotonicity

all messages above m_0*.*

That is, if the biased type's strategy is convex, then there is a cutoff point in the message space such that all messages above the point are contaminated by the biased type. This is natural because the biased type, due to his positive bias, is less willing to deviate to lower messages, but there are equilibria that violate this property.

Example 3 *Suppose* $\mu = 1/8$ *and* $b = 2/5$*. The following strategy profile is an equilibrium but does not satisfy convexity. See the left panel of Figure 3.4 for the equilibrium structure.*

$$\begin{aligned} r_b(\theta) &= m \sim U[0,1/8] \cup [1/4,1], \forall\theta, \\ y(m) &= \begin{cases} m, & \text{if } m \in [1/8,1/4], \\ 509/1008, & \text{otherwise.} \end{cases} \end{aligned}$$

Definition 7 *(Monotonicity) The receiver's strategy* y *is monotone if for all* $m' > m$ *such that* $m \in supp\ r_b(\theta)$ *and* $m' \in supp\ r_b(\theta')$ *for some* θ *and* θ'*,* $y(m') \geq y(m)$*.*

In words, for those messages that the biased type may send, the higher message the receiver gets, the weakly higher action she implements. This restriction makes the biased type's strategy weakly monotone in the sense that a strictly higher action can be induced only by sending a message higher than any message that would induce a lower action.

Example 4 *Suppose $\mu = 1/8$ and $b = 87/448$. The following strategy profile is an equilibrium but does not satisfy monotonicity. See the right panel of Figure 3.4 for the equilibrium structure.*

$$r_b(\theta) = \begin{cases} m \sim U[1/8, 1/4], & \text{if } \theta \in [0, 1/8] \\ m \sim U[0, 1/8] \cup [1/4, 1], & \text{otherwise,} \end{cases}$$

$$y(m) = \begin{cases} 5/64, & \text{if } m \in [1/8, 1/4], \\ 251/448,, & \text{otherwise.} \end{cases}$$

Proposition 17 *Any equilibrium in which the biased type's strategy is convex and the receiver's strategy is monotone is either Type I or II.*

3.4 White Lie Model

3.4.1 Setup

In this section, the receiver is uncertain about the sender's bias. The sender has no bias with probability μ, and has bias $b > 0$ with the complementary probability. We denote by r_0 and r_b the white liar's and the biased type's strategies, respectively.

Definition 8 *The strategy profile (r_0^*, r_b^*, y^*) constitutes an equilibrium if*

(1) given y^*, if m' is sent by the biased type at state θ (in the support of $r_b^*(\theta)$), then

$$m' \in \arg\max_{m \in M} U^S(y^*(m), \theta, b) = -(y^*(m) - (\theta + b))^2,$$

(2) given y^*, if m' is sent by the white liar at state θ (in the support of $r_0^*(\theta)$), then

$$m' \in \arg\max_{m \in M} U^S(y^*(m), \theta, 0) = -(y^*(m) - \theta)^2, \text{ and}$$

(3) given (r_0^*, r_b^*), for all $m \in M$,

$$\begin{aligned} y^*(m) &\in \arg\max_y E_{\mu, r_0^*, r_b^*}[U^R(y, \theta)|m] \\ &\Leftrightarrow y^*(m) = E_{\mu, r_0^*, r_b^*}[\theta|m]. \end{aligned}$$

where E_{μ, r_0^*, r_b^*} is the conditional expectation operator generated by μ, r_0^* and r_b^*.

3.4.2 CS Equilibrium Outcomes

Consider the babbling equilibrium in CS. Such equilibrium always exists in the white lie model. Suppose both sender types uniformly randomize over the entire message space independently of state. Then the receiver cannot make any meaningful inference and thus takes a single action independently of message. This in turn makes both sender types indifferent over all messages independently of state.

Unlike in the honesty model, any other equilibrium outcome in CS cannot be supported as an equilibrium outcome in the white lie model.

Proposition 18 *Suppose $\mu > 0$ and fix an equilibrium outcome in CS that is characterized by a partition $\{\theta_0 = 0, \theta_1, ..., \theta_n = 1\}$ and a sequence $\{y_1, ..., y_n\}$. If $n > 1$, then there does not exist an equilibrium in the white lie model in which the receiver takes the action y_k whenever the true state lies in $[\theta_{k-1}, \theta_k]$, independently of the sender's type, for all $k = 1, ..., n$.*

Proof. Suppose such equilibrium exists in the white lie model. The biased type is indifferent between y_k and y_{k+1} at state θ_k. But the white liar wants to induce a lower action than the biased type at any state and thus strictly prefers y_k to y_{k+1} at state θ_k. Consequently, the white liar deviates. ■

3.4.3 No Arbitrage Condition for the White Liar

The proof of Proposition 18 suggests a new equilibrium condition in the white lie model, no arbitrage condition for the white liar (NAW): at the boundary state of two partition elements, the white liar must be indifferent between the two induced actions. This condition is a consequence of the white liar's flexibility: whenever this condition does not hold, the white liar can increase the receiver's utility by adjusting his report. Together with no arbitrage condition for the biased type (NAB), this implies that if two actions, and no action in between, are induced by both types in equilibrium, then both white liar and biased type must be indifferent between the two actions at their own boundary states.[16]

3.4.4 Equilibrium Characterization

The following proposition characterizes the set of *all* equilibria in the white lie model.

Proposition 19 *Any equilibrium in the white lie model is characterized by two finite partitions,* $\{0, \theta_0^0, \theta_1^0, ..., \theta_n^0 = 1\}$ *and* $\{\theta_0^b = 0, \theta_1^b, ..., \theta_n^b = 1\}$*, a finite sequence,* $\{y_1, ..., y_n\}$*, and* $n_0 \in N \cup \{\infty\}$ *if* $\theta_0^0 > 0$ *such that*[17]

$$\begin{aligned}
&y_1 + \tfrac{2n_0-1}{2n_0}\theta_0^0 \leq 2b \text{ if } \theta_0^0 > 0, && \text{(ICB)}\\
&y_1 = \tfrac{2n_0+1}{2n_0}\theta_0^0 \text{ if } \theta_0^0 > 0, && \text{(ICW)}\\
&y_k + y_{k+1} = 2(\theta_k^b + b), && \text{(NAB)}\\
&y_k + y_{k+1} = 2\theta_k^0, && \text{(NAW)}\\
&y_k = B(\mu, \theta_{k-1}^0, \theta_k^0, \theta_{k-1}^b, \theta_k^b). && \text{(BR)}
\end{aligned}$$

[16] This insight immediately generalizes into the case where there are more than two types. See Dimitrakas and Sarafidis (2005) for the case with a continuum of sender types.

[17] *Morgan and Stocken (2003) consider the same model and characterize the cases where* $n_0 = 0$ *(categorial ranking system equilibria) or* $n_0 = \infty$ *(semiresponsive equilibria). The subsequent paragraph explains the importance of the additional equilibria we characterize here.*

We first explain the role of n_0.[18] Unlike in the honesty model, the receiver does not simply follow the messages even though she knows that those were sent by the white liar. To see this, consider the case where $\mu = 1$. In that case, there exists a perfect communication equilibrium as well as a babbling equilibrium. In addition, there exist infinitely many equilibria that lie in between. Those equilibria have the following form: for each natural number n_0, the white liar evenly partitions the state space into n_0 subintervals, and the receiver takes actions that are at the average of each partition element. The even size of partition elements is due to NAW. The babbling equilibrium and the perfect communication equilibrium can be interpreted as the limits of these equilibria as n_0 tends to 1 and infinity, respectively. When $\mu < 1$ and for some values of b, there may exist an interval of Θ on which the white liar is revealed (below θ_0^0). That is, the conditional probability that the receiver is facing the white liar is equal to 1. A similar logic then holds and this is why $n_0 \in N$ appears in Proposition 19.

Figure 3.5 illustrates the equilibrium structure. Type i sender induces the receiver to take an action y_k when she observes θ in $[\theta_{k-1}^i, \theta_k^i]$. The white liar separates from the biased type when he observes θ in $[0, \theta_0^0]$. If $n_0 < \infty$ then the white liar induces the receiver to take an action $\frac{2k-1}{2n_0}\theta_0^0$ when he observes $\theta \in [\frac{k-1}{n_0}\theta_0^0, \frac{k}{n_0}\theta_0^0]$ (see the left panel of Figure 3.5). If $n_0 = \infty$ then the white liar induces the receiver to take their most preferred action when the true state is in $[0, \theta_0^0]$ (see the right panel of Figure 3.5). The latter case is similar to type I equilibrium in the honesty model.

Among the equilibrium conditions, BR and NAB are straightforward generalizations of the equilibrium conditions in CS. Also, as we explained, NAW must hold due to the rationality of the white liar. ICB guarantees that the biased type does not deviate to any action below y_1. Similarly, ICW ensures that the white liar does not deviate to any action below y_1 at state above θ_0^0.[19]

[18] The number n_0 need not be unique. As n_0 increases, so too does the highest action induced only by the honest sender. Whether equilibrium allows for larger n_0 then depends on the incentive compatibility constraint for the biased sender which depends on the value of b.

[19] ICW can be understood as another NAW.

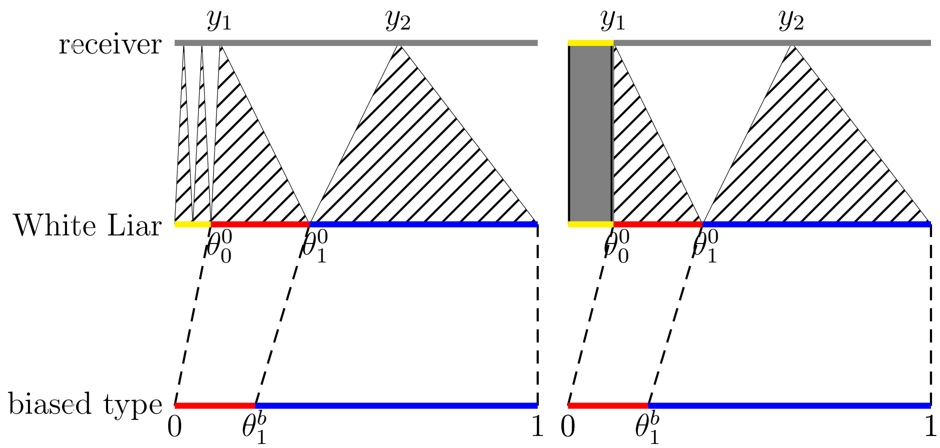

Figure 3.5: Equilibrium structure in the white lie model. There are two partition elements below θ_0^0 in the left panel ($n_0 = 2$), while the white liar induces her most preferred action on $[0, \theta_0^0]$ in the right panel ($n_0 = \infty$).

Example 5 *Consider the following strategy profile.*

$$\begin{aligned}
r_0(\theta) &= \begin{cases} m \sim U[\frac{k-1}{n_0}\theta^*, \frac{k}{n_0}\theta^*], & \textit{if } \theta \in [\frac{k-1}{n_0}\theta^*, \frac{k}{n_0}\theta^*], \\ m \sim U[\theta^*, 1], & \textit{if } \theta \in [\theta^*, 1], \end{cases} \\
r_b(\theta) &= \quad m \sim U[\theta^*, 1], \quad \textit{if } \theta \in [0, 1], \\
y(m) &= \begin{cases} \frac{2k-1}{2n_0}, & \textit{if } m \in [\frac{k-1}{n_0}\theta^*, \frac{k}{n_0}\theta^*], \\ \frac{2n_0+1}{2n_0}\theta^*, & \textit{if } m \in [\theta^*, 1], \end{cases}
\end{aligned}$$

where

$$\theta^* = \frac{2n_0}{2n_0 + 1 + \sqrt{4(1-\mu)n_0^2 + 4(1-\mu)n_0 + 1}}.$$

This strategy profile is an equilibrium if and only if

$$\frac{4n_0 + 1}{2n_0}\theta^* \leq 2b.$$

Now we present a way to find the set of equilibria. First, given μ and b, suppose there exists an equilibrium with n partition elements and $\theta_0^0 = 0$. From all equality conditions, we find

$$
\begin{aligned}
\theta_k^b &= k\theta_1^b + b(k-1)\left[2(1-\mu)k + \mu - \frac{\mu(1-\mu)b}{\theta_1^b + \mu b}\right], k = 2, .., n-1, \quad (3.1)\\
\theta_k^0 &= \theta_k^b + b, \forall k = 1, ..., n-1,\\
3\theta_{n-1}^b &- \theta_{n-2}^b + 2b(2-\mu) = 2B(\mu, \theta_{n-1}^0, \theta_n^0, \theta_{n-1}^b, \theta_n^b).
\end{aligned}
$$

Combined with $\theta_n^b = \theta_n^0 = 1$, this system of equations has a unique solution for each n. The necessary condition for the existence of this equilibrium is $\theta_1^b > 0$. As in CS, this condition imposes an upper bound on the possible number of partition elements.

Now suppose $\theta_0^0 > 0$ and there are n_0 partition elements below θ_0^0. By a similar algebra, we get

$$
\begin{aligned}
\theta_k^b &= 2k\theta_1^b + \\
&\quad (k-1)\left[2(1-\mu)bk - \frac{\mu(b+\theta_0^0)(\theta_1^b + b - \theta_0^0)}{\theta_1^b + \mu(b - \theta_0^0)}\right], k = 2, .., n-1 \quad (3.2)\\
\theta_0^0 &= \frac{2n_0}{2n_0+1} B(\mu, \theta_0^0, \theta_1^0, 0, \theta_1^b),\\
\theta_k^0 &= \theta_k^b + b, \forall k = 1, ..., n-1.\\
3\theta_{n-1}^b &- \theta_{n-2}^b + 2b(2-\mu) = 2B(\mu, \theta_{n-1}^0, \theta_n^0, \theta_{n-1}^b, \theta_n^b)
\end{aligned}
$$

Again using $\theta_n^b = \theta_n^0 = 1$, we can identify all equilibrium values. In this equilibrium, there are two restrictions on the number of partition elements above θ_0^0. The first one is the same as above: $\theta_1^b > 0$. As in CS, this condition imposes an upper bound on the possible number of partition elements. The second condition is the incentive compatibility for the biased type: $\theta_0^0 = \frac{2n_0}{2n_0+1} y_1 \leq b$. This condition imposes a lower bound on n. Different from the case with $\theta_0^0 = 0$, an equilibrium with $\theta_0^0 > 0$ exists only when there are enough number of partition elements above θ_0^0.

3.5 Welfare Comparison

Now we compare the maximal expected utilities the receiver can achieve in the two models.[20]

When μ is large

The following proposition establishes that when the sender is honest or a white liar in each model with sufficiently high probability, the receiver is strictly better off in the honesty model than in the white lie model.

Proposition 20 *Given $\varepsilon > 0$, there exists $\mu(b)$ such that if $\mu > \mu(b)$ then the expected payoff of the receiver in any equilibrium in the honesty model is ε-close to that under perfect communication. On the other hand, there exists $\delta > 0$ such that the maximal expected payoff of the receiver in the white lie model is less than that under perfect communication at least by δ.*

Proof. Suppose μ is close to 1 and consider the honesty model. Fix *any* strategy for the biased sender and suppose the receiver simply follows the recommendation of the sender. That is, suppose the receiver's strategy is $y(m) = m$. Then, the receiver takes her most preferred action with at least probability μ. Since the receiver can do no worse than this strategy, her ex-ante expected utility is close to that under perfect communication.

Now consider the white lie model. From Equation (3.1), for $\theta_0^0 = 0$, the number of partition elements is bounded by $1/b + 2$, because the left-hand side is less than 1, while the right-hand side is greater than $b(n-2)$ in the limit as μ tends to 1. Similarly, from Equation (3.2), the size of partition elements $(\theta_k^0 - \theta_{k-1}^0)$ is bounded

[20] Since there are typically multiple equilibria in each model, unless μ is sufficiently close to 1, our welfare results do not imply that the receiver will be better off in one model than in the other for sure. However, we think comparing the maximal utilities is still a meaningful exercise. Not only is it the standard choice in the literature when there are multiple equilibria without definite Pareto rankings, but also does it highlight the difference between honesty and white lies in communication, which is part of our motivation.

strictly above 0 and so the number of partition elements in $[\theta_0^0, 1]$ is bounded above also when $\theta_0^0 > 0$. These immediately establish the second part of the proposition. ■

The intuitions behind this result are as follows. When μ is equal to 1, in the honesty model, there is a unique equilibrium in which the first-best outcome is achieved. Therefore, when μ is close to 1, any equilibrium will be close to the perfect communication. On the other hand, in the white lie model, when μ is equal to 1, as discussed before, there are infinitely many equilibria, from no communication to perfect communication. It turns out that the perfect communication outcome in the white lie model when μ is equal to 0 is not lower hemi-continuous. That is, even a small probability introduces a non-trivial noise in the white lie model, and thus the loss from imperfect communication does not vanish even when the sender is the white liar almost for sure.

When μ is small

Now we consider the case where μ is small, that is, the sender is biased with a high probability. The following result shows that if μ is sufficiently close to 0, for the majority of bias values, the receiver is better off in the honesty model than in the white lie model.[21]

Proposition 21 *Whenever $b \in \left(\frac{1}{2n(n+1)}, \frac{1}{2n(n+0.5)}\right) \cup \left(\frac{1}{2n^2}, \frac{1}{2n(n-1)}\right)$ for some natural number n, there exists $\underline{\mu}(b) > 0$ such that if $\mu \leq \underline{\mu}(b)$ then the receiver's maximal utility is weakly greater in the honesty model than in the white lie model.*

The key idea of the proof is as follows. For μ small, equilibrium is Type I in the honesty model. Then, the honesty model and the white lie model (for the cases where $n_0 = 0$ or $n_0 = \infty$) share equilibrium conditions other than MB for the former and NAW for the latter. As μ tends to 0, MB becomes negligible and thus imposes

[21]These intervals cover approximately 79.1% of bias values.

no restriction on equilibrium outcome. To the contrary, NAW is independent of μ and thus still binds. Consequently, when μ is sufficiently close to 0, any equilibrium outcome in the white lie model with the same structure as that of type I equilibrium can be replicated in the honesty model.[22]

The following example shows that the receiver can be *strictly* better off in the honesty model than in the white lie model.

Example 6 *Consider the case where $b \in \left(\frac{1}{4}, \frac{1}{3}\right)$. When μ is sufficiently close to 0, the babbling equilibrium is the unique equilibrium in the white lie model (see Lemma 19 in the omitted proofs). To the contrary, in the honesty model, there exists an equilibrium in which the biased type does not send messages below $m_0 \in (0, 2b - 0.5)$. The latter yields a strictly greater receiver utility than the former.*

The following proposition is a partial converse to Proposition 21.

Proposition 22 *For μ sufficiently close to 0, if $b \in \left(\frac{1}{2n(n+0.5)}, \frac{1}{2n^2}\right)$ for some natural number n, there is an equilibrium in the white lie model whose outcome cannot be replicated in the honesty model. Also, for a fixed (small) μ there exists a $\bar{b}_n(\mu)$ such that for $b \in [\frac{1}{2n(n+1)}, \bar{b}_n(\mu)]$ there exists an equilibrium in the white model whose outcome cannot be replicated in the honesty model.*

The two cases have different reasons. For μ positive, if b is slightly greater than one of CS critical values, the number of partition elements increases, but such increase is faster in the white lie model than in the honesty model. Therefore, there may be an $(n+1)$-partition-element (above θ_0^0) equilibrium in the white lie model, while the maximal number of partition elements is still n in the honesty model.

[22]In the omitted proofs, we show that if $b \in \left(\frac{1}{2n(n+1)}, \frac{1}{2n(n+0.5)}\right)$, when μ is close to 0, there does not exist an equilibrium in the white lie model in which $n_0 > 1$ (Lemma 19). In addition, if $b \in \left(\frac{1}{2n^2}, \frac{1}{2n(n-1)}\right)$, then it is possible that $n_0 > 1$. However, we show in this region of b, for a finite n_0, there exists a corresponding equilibrium in the honesty model in which the biased sender's behavior is unchanged and the fully revealing subintervals of the white liar are replaced by perfect communication with the honest sender. Such an equilibrium yields strictly higher utility to the principal (Lemma 20).

For $b \in \left(\frac{1}{2n(n+0.5)}, \frac{1}{2n^2}\right)$, there is an equilibrium with a finite number of partition elements below $\theta_0^0 > 0$ whose outcome cannot be replicated in the honesty model. The reason for this is that in the honesty model, for all states in which the honest sender is revealed, there is full communication. Meanwhile, in the white lie model, in equilibria where $n_0 < \infty$, the states in which the white liar is revealed exhibit imperfect communication. The full communication with the honest sender below θ_0^0 implies more equilibrium actions and imposes greater restrictions on the biased sender to not deviate. To see this, notice that $\theta_0^0 = \frac{2n_0}{2n_0+1}y_1$ in an equilibrium with n_0 positive and finite. For this to be an equilibrium, the biased type must prefer to induce y_1 to $\frac{2n_0-1}{2n_0+1}y_1$ at state 0. If we try to replicate the equilibrium outcome in the honesty model by setting $m_0 = \theta_0^0$, the biased type must now prefer y_1 to $\theta_0^0 = \frac{2n_0}{2n_0+1}y_1 > \frac{2n_0-1}{2n_0+1}y_1$, which cannot be the case when $b \in \left(\frac{1}{2n(n+0.5)}, \frac{1}{2n^2}\right)$ and n_0 takes its largest possible value.

White lies are sometimes preferred

Proposition 22 does not establish that when b satisfies conditions in the proposition the receiver is necessarily better off in the white lie model. We show for the special case of $n = 1$ that white lies may or may not be preferable to honesty in such conditions and explain its underlying reason.[23]

Consider the case where $b \in \left(\frac{1}{3}, \frac{1}{2}\right)$. Given μ and b, let $U_W(\mu, b)$ and $U_H(\mu, b)$ be the maximal receiver utilities in the white lie model and in the honesty model, respectively. When μ is sufficiently close to 0, there can exist only one partition

[23] Numerical analysis suggests that the considerations for the $n = 1$ yield the same conclusions for the case where $n > 1$, though we are unable to prove the result analytically.

element above θ_0^0 and m_0 in each model.[24] Therefore,

$$\begin{aligned} U_W(\mu,b) &= -\mu\left[\sum_{k=1}^{n_0}\int_{\frac{(k-1)}{n_0}\theta_0^0}^{\frac{k}{n_0}\theta_0^0}\left(\theta-\frac{(2k-1)}{2n}\theta_0^0\right)^2 d\theta+\int_{\theta_0^0}^1(\theta-y_1)^2 d\theta\right] \\ &\quad -(1-\mu)\int_0^1(\theta-y_1)^2 d\theta, \end{aligned}$$

where n_0 is the maximal number of partition elements below θ_0^0. Also,

$$U_H(\mu,b) = -\mu\int_{m_0}^1(\theta-y_1)^2 d\theta-(1-\mu)\int_0^1(\theta-y_1)^2 d\theta,$$

where m_0 is the largest value subject to the incentive compatibility of the biased sender. Then,[25]

$$\begin{aligned} \lim_{\mu\to 0}\frac{\partial U_W(\mu,b)}{\partial\mu} &= -\left[n_0\int_0^{\frac{\theta_0^0}{n_0}}\left(\theta-\frac{\theta_0^0}{2n}\right)^2 d\theta+\int_{\theta_0^0}^1(\theta-y_1)^2 d\theta\right]-\int_0^1(\theta-y_1)^2 d\theta, \\ \lim_{\mu\to 0}\frac{\partial U_H(\mu,b)}{\partial\mu} &= -\int_{2b-y_1}^1(\theta-y_1)^2 d\theta-\int_0^1(\theta-y_1)^2 d\theta. \end{aligned}$$

As μ tends to 0, y_1 converges to $\frac{1}{2}$, n_0 to the largest integer that is weakly smaller than $\frac{b}{1-2b}$, θ_0^0 to $\frac{n_0}{2n_0+1}$, and m_0 to $2b-y_1$. Hence,

$$\lim_{\mu\to 0}\frac{\partial U_W(\mu,b)}{\partial\mu}-\lim_{\mu\to 0}\frac{\partial U_H(\mu,b)}{\partial\mu}=-n_0\int_0^{\frac{\theta_0^0}{n_0}}\left(\theta-\frac{\theta_0^0}{2n_0}\right)^2 d\theta+\int_{2b-y_1}^{\theta_0^0}(\theta-y_1)^2 d\theta. \tag{3.3}$$

The two terms in Equation (3.3) show that when μ is sufficiently close to 0, the difference between equilibrium welfare in the two models arises from the outcomes when the sender is not biased and the state is in $[0,\theta_0^0]$. In the honesty model, when the sender is honest, perfect communication occurs on $[0,2b-y_1]$ and y_1 is taken on $[2b-y_1,\theta_0^0]$ (see the right panel of Figure 3.6). In the white lie model, a finite

[24] One can easily show that when the biased type induces a single action in the honesty model, the welfare maximizing equilibrium strategy profile satisfies both convexity and monotonicity. Therefore, the restriction to such equilibria in the welfare comparison is without loss of generality.

[25] The terms involving $\frac{\partial y_1}{\partial\mu}$ vanish as $\mu\to 0$.

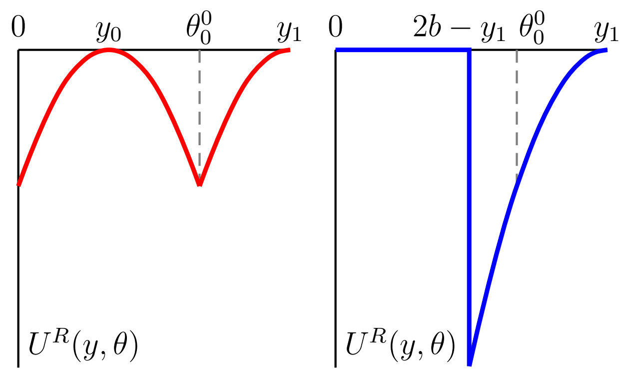

Figure 3.6: The receiver's ex post utilities in the white lie model (left) and in the honesty model (right) when $b \in \left(\frac{1}{2n(n+0.5)}, \frac{1}{2n(n+0.2)}\right)$, the sender is either honest or a white liar, and μ is sufficiently small.

number of actions that are uniformly distributed are taken on $[0, \theta_0^0]$ (see the left panel of Figure 3.6). The first term represents the losses in the white lie model due to imperfect communication between the white liar and the receiver, while the second term represents the excessive losses in the honesty model due to the honest sender's inflexibility. Perfect communication must occur whenever the receiver knows that the sender is honest. Therefore, it is harder in the honesty model to deter the biased sender from deviating to lower messages than in the white lie model. As shown in Figure 3.6, the latter can be relatively large because of the concavity of quadratic utility function.

Figure 3.7 shows when white lies are preferable to honesty for μ sufficiently close to 0. Each jump in welfare corresponds to a jump in the maximum possible n_0. As b increases, the maximum n_0 is locally constant and consequently, the maximum welfare in the white lie model is fixed. However, the maximum welfare in the honesty model increases, as higher b allows for a larger region of perfect communication with the honest sender. Thus, the negative welfare effect in the honesty model arising from the quadratic utility loss shrinks as b increases. For some ranges of b, this allows for a higher maximum welfare under the honesty model.

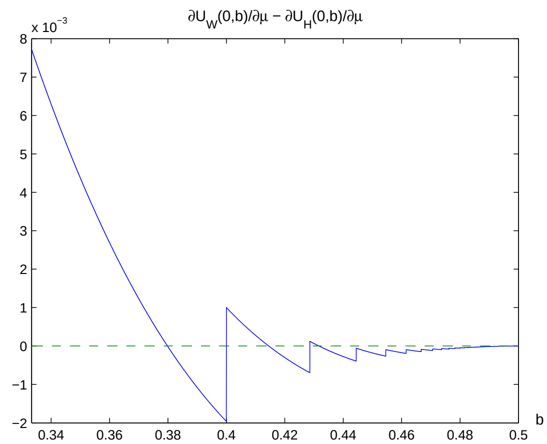

Figure 3.7: Welfare comparison for $b \in (1/3, 1/2)$ when μ is sufficiently small.

Intermediate Cases

For intermediate values of μ, we explain the underlying reasons why the receiver can be better off in the honesty model than in the white lie model. We support our arguments by providing some numerical examples.[26]

In the intermediate case, the welfare gain in the honesty model mainly comes from reducing the excessive losses in communication due to strategic considerations of players. To see this, recall that in the original CS game, partition element size increases for higher θ. This is true in the white lie model as well, though the increase in partition element size is decreasing in μ. In the honesty model, however, this is not necessarily true. The inequality property of MB does not require that partition element sizes increase in the honesty model as they must in the white lie model. Partition elements can be adjusted (as long as all MB constraints are satisfied) so that higher partition elements need not be larger than lower partition elements. This has the consequence of allowing better communication in the honesty model at

[26]The analytical difficulty for intermediate cases lies in characterizing the set of equilibria in the honesty model. The set of equilibria is very large (due to the inequalities of MB), not convex (IC is necessary only when $m_0 > 0$), and possibly not closed (due to $m_0 < m_1 < .. < m_n$). Furthermore, we cannot fix the dimension of control variables, because the maximal number of partition elements is not analytically available.

higher θ values where the welfare losses were largest in the white lie model.

The first consequence of this is that such a freedom in the honesty model allows for more uniform partition element sizes by manipulating the partitioning of the message space. While this may reduce the expected welfare conditional on the sender being honest, it also makes the biased type's partitioning more uniform, which is directly beneficial to the receiver due to the concavity of quadratic utility function. This effect is highlighted in Example 7.

Example 7 *Suppose $b = 0.15$ and $\mu = 0.1$. In the white lie model, there exist 2-partition-element equilibria for each $n_0 \in \mathbf{N} \cup \infty$ and the equilibrium yielding the maximal receiver utility, -0.0392, is characterized by*

$$\theta^0 = \{0.126, 0.371, 1\}, \theta^b = \{0, 0.221, 1\}, y = \{0.126, 0.617\}, \textit{and } n_0 = \infty.$$

The ex-ante expected loss from communication with the white liar and biased sender are -0.0029 and -0.0363, respectively. In the honesty model, there exists a 2-partition-element equilibrium that is characterized by

$$m = \{0.138, 0.575, 1\}, \theta^b = \{0, 0.2478, 1\}, y = \{0.162, 0.634\}.$$

In this equilibrium, the receiver achieves utility -0.0374 where the ex-ante loss from communication with the honest and biased senders are -0.004 and -0.0335, respectively.

The second consequence of allowing for more uniform partition element sizes is that as μ increases, the maximal number of partition elements increases faster in the honesty model than in the white lie model. In the white lie model, the increasing partition element sizes for both white liar and biased sender inhibit the ability for larger number of partition elements. In the honesty model, smaller mass from the honest sender may be placed on higher messages, raising the higher actions, shrinking the largest partition elements, and allowing for greater number of intermediate equilibrium actions. This is illustrated in Example 8.

Example 8 *Consider the case where $b = 1/12$ and $\mu = 1/4$. The largest number of partition elements (above θ_0^0) in the white lie model is 3 and any $n_0 \in \mathbf{N} \cup \infty$ is possible, with $n_0 = \infty$ yielding the maximal receiver utility. There exists a 5-partition equilibrium in the honesty model that is characterized by*

$$m = \{0.083, 0.228, 0.539, 0.756, 0.796, 1\},$$
$$\theta^b = \{0, 0.083, 0.263, 0.387, 0.565, 1\},$$
$$y = \{0.083, 0.250, 0.443, 0.498, 0.798\}.$$

Note that the partition element sizes for the biased sender are $0.083, 0.180, 0.124, 0.178$ *and* 0.435*, respectively. If we consider any 5-partition equilibrium in the white lie model, then it must be that the third and fourth partition elements increase in size by $4b(1-\mu) = 1/4$ from the preceding element. Relative to honesty, this is a constraining factor in allowing partition equilibria with greater number of elements as μ increases. The maximal receiver utility in the white lie model is* -0.0198*, while it is* -0.0147 *in the honest model.*

3.6 Discussion

3.6.1 Generalization

All our analyses have been performed under a convenient uniform-quadratic environment. We discuss to what extent our results can generalize.

All of our qualitative results, including our analytical welfare results, immediately generalize to the class of utility functions that satisfy the following three properties:

1. $U^R(\cdot, \theta)$ and $U^S(\cdot, \theta, b)$ are symmetric and strictly concave around $y^R(\theta)$ and $y^S(\theta, b)$,

2. $y^S(\theta, b) - y^R(\theta) = b$ for all θ,

3. $U^R(y,\theta) = U^R(y + (\theta' - \theta), \theta'), U^S(y,\theta,b) = U^S(y + (\theta' - \theta), \theta', b), \forall y, \forall \theta,$

where $y^R(\theta) = \arg\max_y U^R(y,\theta)$ and $y^S(\theta, b) = \arg\max_y U^S(y,\theta,b)$. One can check that we have used only these properties in the equilibrium characterization and analytical welfare comparison.

Regarding the distribution over the state space, the only non-trivial problem is how to generalize the mass balance condition. A closed-form mass balance condition is not available for the general distribution, but a partial characterization is still possible.

Suppose $\theta \in [0,1]$ is drawn from a distribution function F with a positive and continuous density f. Fix $\mu \in (0,1)$ and $[\theta', \theta''], [m', m''] \subseteq [0,1]$. Let

$$\begin{aligned}\overline{y} &= \frac{\mu(F(m'') - F(m'))}{\mu(F(m'') - F(m')) + (1-\mu)(F(\theta'') - F(\theta'))} E[\theta|\theta \in [m', m'']] \\ &+ \frac{(1-\mu)(F(\theta'') - F(\theta'))}{\mu(F(m'') - F(m')) + (1-\mu)(F(\theta'') - F(\theta'))} E[\theta|\theta \in [\theta', \theta'']].\end{aligned}$$

We want to know under what conditions there exists a collection of probability measures $\{r(\theta), \theta \in [\theta', \theta'']\} \subset \Delta([m', m''])$ such that

$$E_{\mu,r}[\theta|\mathcal{M}] = \overline{y}, \forall m \in (m', m''), \text{ for any Borel set } \mathcal{M} \text{ in } [m', m''].$$

Suppose such a collection $\{r(\cdot)\}$ exists. Define $\gamma : \Delta([m', m'']) \to [0,1]$ by

$$\gamma(\mathcal{M}) = \int_{\theta'}^{\theta''} r(\theta)(\mathcal{M})\, dF(\theta), \text{ for all Borel set } \mathcal{M} \subset [m', m''].$$

In addition, define $\gamma_1, \gamma_2 : [m', m''] \to [\theta', \theta'']$ so that

$$\begin{aligned}\overline{y} &= \frac{\mu(F(m) - F(m'))}{\mu(F(m) - F(m')) + (1-\mu)(F(\theta'') - F(\gamma_1(m)))} E[\theta|\theta \in [m', m]] \\ &+ \frac{(1-\mu)(F(\theta'') - F(\gamma_1(m)))}{\mu(F(m) - F(m')) + (1-\mu)(F(\theta'') - F(\gamma_1(m)))} E[\theta|\theta \in [\gamma_1(m), \theta'']],\end{aligned}$$

and

$$\begin{aligned}\overline{y} &= \frac{\mu(F(m'') - F(m))}{\mu(F(m'') - F(m)) + (1-\mu)(F(\gamma_2(m)) - F(\theta'))} E[\theta|\theta \in [m, m'']] \\ &+ \frac{(1-\mu)(F(\gamma_2(m)) - F(\theta'))}{\mu(F(m'') - F(m)) + (1-\mu)(F(\gamma_2(m)) - F(\theta'))} E[\theta|\theta \in [\theta', \gamma_2(m)]].\end{aligned}$$

Both γ_1 and γ_2 must be well-defined if a collection $\{r(\cdot)\}$ exists. In addition, $\gamma([m', m]) \geq \gamma_1(m)$ and $\gamma([m, m'']) \geq \gamma_2(m)$ for all $m \in [m', m'']$. Since $\gamma([m', m]) + \gamma([m, m'']) = F(\theta'') - F(\theta')$, this condition implies that

$$\gamma_1(m) + \gamma_2(m) \leq F(\theta'') - F(\theta'), \forall m \in [m', m''].$$

Of particular interest is when this condition holds with equality for all $m \in [m', m'']$, that is, $\gamma_1(m) + \gamma_2(m) = F(\theta'') - F(\theta'), \forall m \in [m', m'']$. In this case, $r(\theta)$ is a degenerate random variable and r^{-1} coincides with γ_1. Furthermore, γ_1 satisfies the following first-order ordinary differential equation:

$$(1-\mu) f(\gamma_1(m)) (\gamma_1(m) - \overline{y}) \gamma_1'(m) = \mu f(m)(m - \overline{y}), \forall m \in [m', m''],$$

with boundary conditions, $\gamma_1(m') = \theta''$ and $\gamma_1(m'') = \theta'$. For the uniform distribution, this happens when $\mu(m'' - m')^2 = (1-\mu)(\theta'' - \theta')^2$, and

$$\gamma_1(m) = \theta'' - \frac{\theta'' - \theta'}{m'' - m'}(m - m'),$$

which was used in Section 3.3.

3.6.2 Equilibrium Selection

Chen, Kartik, and Sobel (2008) show that the limit of message-monotone equilibria as the probabilities of the honest sender and the naive receiver vanish satisfies NITS (Proposition 5 in their paper), and thus the behavioral perturbation provides a way to select the most informative equilibrium in CS. They note that the monotonicity restriction on equilibrium strategies is necessary for the result. Our analysis (in particular, Proposition 16) suggests that the selection through behavioral perturbation may fail once other equilibria are considered. Of course, the answer is not definite because the naive receiver is absent in our model, while the presence of the naive receiver introduces an intrinsic cost of lying and thus may dramatically change the set of equilibria. One can attempt to examine this issue by building upon this paper and other relevant papers (in particular, Ottaviani and Squintani (2006), Kartik, Ottaviani, and Squintani (2007), and Chen (2009)).

3.7 Omitted Proofs

Proof of Lemma 15. Let $z^+(\theta', b) = \lim_{\theta \to \theta'+} z(\theta, b)$, $z^-(\theta', b) = \lim_{\theta \to \theta'-} z(\theta, b)$.

($\Rightarrow$) (i) $z(\cdot, b)$ is nondecreasing.

Suppose $z(\cdot, b)$ is strictly decreasing on (θ^1, θ^2) with $\theta^1 < \theta^2$. For $z(\cdot, b)$ to be incentive compatible, $U_S(z(\theta^1, b), \theta^1, b) \geq U^S(z(\theta, b), \theta^1, b)$ and $U^S(z(\theta^2, b), \theta^2, b) \geq U^S(z(\theta, b), \theta^2, b)$ for all $\theta \in \Omega$. Since $z(\cdot, b)$ is strictly decreasing on (θ^1, θ^2), $z(\cdot, b)$ is continuous except countably many points. Pick some $\theta \in (\theta^1, \theta^2)$ at which $z(\cdot, b)$ is continuous. If $\partial U^S(z(\theta, b), \theta, b)/\partial y \neq 0$, then the strategic sender has a profitable deviation (If $\partial U^S(z(\theta, b), \theta, b)/\partial y > (<) 0$ then the strategic sender deviates to $\theta' < (>) \theta$). Hence $\partial U^S(z(\theta, b), \theta, b)/\partial y = 0$ almost everywhere on (θ^1, θ^2). But this is a contradiction because we assumed that $z(\cdot, b)$ is strictly decreasing on (θ^1, θ^2).

(ii) $V^S(\cdot, b)$ is continuous.

Suppose $V^S(\cdot, b)$ is not continuous at $\theta' \in (0, 1)$. Then $z(\cdot, b)$ cannot be continuous at θ'. Since $z(\cdot, b)$ is nondecreasing, this means $z(\cdot, b)$ has jump at θ'. Pick θ^- and θ^+ sufficiently close to θ' where $\theta^- < \theta' < \theta^+$. We have the following three cases: (1) $z(\theta^+, b) \leq \theta' + b$, (2) $z(\theta^-, b) < \theta' + b < z(\theta^+, b)$ and (3) $\theta' + b \leq z(\theta^-, b)$. In case (1), the strategic sender has an incentive to deviate at θ^-, while he does at θ^+ in case (3). In case (3), no type has an incentive to deviate only when $\lim_{\theta \to \theta'+} V^S(\theta, b) = \lim_{\theta \to \theta'-} V^S(\theta, b)$. Hence $V^S(\cdot, b)$ is continuous.

(iii) If $z_1(\theta, b)$ exists, then $U_1^S(z(\theta, b), \theta, b) \cdot z_1(\theta, b) = 0$.

The differentiability of z at θ implies the differentiability of $V^S(\cdot, b)$ at θ, because $V^S(\theta, b) = U^S(z(\theta, b), \theta, b)$. By the Envelope theorem,

$$\frac{\partial U^S(z(\theta, b), \theta, b)}{\partial y} \frac{\partial z(\theta, b)}{\partial \theta} = 0.$$

($\Leftarrow$) We want to show that $U^S(z(\theta'',b),\theta'',b) \geq U^S(z(\theta',b),\theta'',b)$ for all $\theta',\theta'' \in \Omega$.

$$\begin{aligned}
U^S(z(\theta'',b),\theta'',b) - U^S(z(\theta',b),\theta'',b) &= [U^S(z(\theta'',b),\theta'',b) - U^S(z(\theta',b),\theta',b)] \\
&\quad -[U^S(z(\theta',b),\theta'',b) - U^S(z(\theta',b),\theta',b)] \\
&= \int_{\theta'}^{\theta''} \frac{\partial V^S(\theta,b)}{\partial\theta} d\theta - \int_{\theta'}^{\theta''} U_2^S(z(\theta',b),\theta,b)d\theta \\
&= \int_{\theta'}^{\theta''} U_2^S(z(\theta,b),\theta,b)d\theta - \\
&\qquad \int_{\theta'}^{\theta''} U_2^S(z(\theta',b),\theta,b)d\theta \\
&= \int_{\theta'}^{\theta''} \left[\int_{z(\theta',b)}^{z(\theta,b)} U_{12}^S(z(t,b),\theta,b)dz\right] d\theta.
\end{aligned}$$

V^S is absolutely continuous via application of an Envelope theorem for this environment (see Milgrom and Segal (2002)).

If $\theta'' > \theta'$, then $z(\theta,b) \geq z(\theta',b), \forall\theta \in [\theta',\theta'']$, and so $U^S(z(\theta'',b),\theta'',b) - U^S(z(\theta',b),\theta'',b) \geq 0$. Similarly, if $\theta'' < \theta'$, then $z(\theta,b) \leq z(\theta',b), \forall\theta \in [\theta',\theta'']$, and again $U^S(z(\theta'',b),\theta'',b) - U^S(z(\theta',b),\theta'',b) \geq 0$. **Q.E.D.**

Proof of Lemma 16. ($\Leftarrow$) For each θ, consider the following probability measure.

$$r(\theta) = \begin{cases} \frac{m''-m'}{\theta''-\theta'}(\theta''-\theta)+m', & \text{with probability } \frac{\mu}{1-\mu}\frac{(m''-m')^2}{(\theta''-\theta')^2}, \\ m \sim U[m',m''], & \text{with probability } 1-\frac{\mu}{1-\mu}\frac{(m''-m')^2}{(\theta''-\theta')^2}. \end{cases}$$

This probability measure is well-defined when $\mu(m''-m')^2 \leq (1-\mu)(\theta''-\theta')^2$. Then for any $m \in [m',m'']$,

$$\begin{aligned}
&E_{\mu,r}[\theta|m] = \frac{\mu(m''-m')}{\mu(m''-m')+(1-\mu)(\theta''-\theta')}m \\
&+\frac{(1-\mu)(\theta''-\theta')}{\mu(m''-m')+(1-\mu)(\theta''-\theta')}\frac{\mu}{1-\mu}\frac{(m''-m')^2}{(\theta''-\theta')^2}\left(\theta''-\frac{\theta''-\theta'}{m''-m'}(m-m')\right) \\
&+\frac{(1-\mu)(\theta''-\theta')}{\mu(m''-m')+(1-\mu)(\theta''-\theta')}\left(1-\frac{\mu}{1-\mu}\frac{(m''-m')^2}{(\theta''-\theta')^2}\right)\frac{\theta''+\theta'}{2} \\
&= B(\mu,m',m'',\theta',\theta'').
\end{aligned}$$

($\Rightarrow$) Let $m^* \in [m', m'']$ be the value such that $m^* - m' = m'' - m^*$. Let θ^* be the value such that

$$B(\mu, m', m^*, \theta^*, \theta'') = B(\mu, m', m'', \theta', \theta'').$$

Such θ^* is well-defined if and only if $\mu\left(m'' - m'\right)^2 \leq (1-\mu)\left(\theta'' - \theta'\right)^2$.

Suppose there exists a collection of probability measures, $\{r(\theta), \theta \in [\theta', \theta'']\}$, that satisfies the given property. Then by construction,

$$\int_{\theta'}^{\theta''} r(\theta)([m', m^*])d\theta \geq \theta'' - \theta^*.$$

Therefore, θ^* must be well-defined, which is true only when

$$\mu\left(m'' - m'\right)^2 \leq (1-\mu)\left(\theta'' - \theta'\right)^2.$$

Q.E.D.

Proof of Proposition 17: Given that we excluded both non-convex and non-monotone strategy profiles, the only equilibrium structure that may not be either type I or type II is the one with more than one intervals of states on which the biased type induces his more preferred action. The following lemma establishes that it cannot be the case and thus completes the proof.

Lemma 17 *Fix $\theta_0 \in (0, 1 - b/\mu)$ and $m_0 = \theta_0 + b/\mu$. There cannot exist $\theta \in [0, \theta_0)$ and $m \in [0, m_0)$ such that $B(\mu, m, m_0, \theta, \theta_0) = \theta_0 + b$.*

Proof. Suppose not. Then for some θ and m,

$$\theta_0 + b = \frac{\mu(m_0 - m)}{\mu(m_0 - m) + (1-\mu)(\theta_0 - \theta)}\frac{m + m_0}{2} + \frac{(1-\mu)(\theta_0 - \theta)}{\mu(m_0 - m) + (1-\mu)(\theta_0 - \theta)}\frac{\theta + \theta_0}{2}$$

Re arranging terms,

$$\theta_0^2 - 2\left(\mu m + (1-\mu)\theta - b\right)\theta_0 + 2b\left(b - \mu m - (1-\mu)\theta\right) - \frac{b^2}{\mu} + \mu m^2 + (1-\mu)\theta^2$$
$$= \left[\theta_0^2 - \left(\mu m + (1-\mu)\theta - b\right)\right]^2 + \mu(1-\mu)\left[(m-\theta)^2 - \left(\frac{b}{\mu}\right)^2\right] = 0$$

For the solution to exist, the right-hand side should be not more than 0 when $\theta_0 = \mu m + (1-\mu)\theta - b$. But in that case,

$$(m-\theta)^2 - \left(\frac{b}{\mu}\right)^2 = \frac{1}{\mu^2}\left[(\theta_0-\theta)^2 + 2b(\theta_0-\theta)\right] > 0.$$

Hence there cannot exist such θ and m. ■

Q.E.D.

Proof of Proposition 19: Take any equilibrium in the white lie model. Due to the flexibility of the white liar, the biased sender cannot induce his most preferred action on a positive measure of states.

Now let Θ^* be the set of states on which the white liar induces actions that are not induced by the biased sender and let $\overline{\theta}$ be the supremum of Θ^*. $\overline{\theta}$ is less than or equal to any equilibrium action induced by the biased sender, otherwise there would be some state at which the biased sender would deviate to induce it. In addition, Θ^* is convex. This is because by part (i) of Lemma 15, the white liar induces a (weakly) lower action than $\overline{\theta}$ at any state $\theta < \overline{\theta}$.

All other arguments are straightforward from equilibrium necessary conditions. **Q.E.D.**

Proof of Proposition 21:

The following three lemmas establish the proof of the proposition.

Lemma 18 *Suppose $b \neq \frac{1}{2n(n-1)}$. If μ is sufficiently close to 0, for any equilibrium in the white lie model in which $\theta_0^0 = 0$ (that is, $n_0 = 0$) or the white liar induces his more preferred actions on $[0, \theta_0^0]$ (that is, $n_0 = \infty$), there exists a corresponding equilibrium in the honesty model which yields the same outcome.*

Proof. Notice that all partition element sizes for the white liar are less than or equal to the partition element sizes for the biased sender except the first partition element, because $\theta_k^0 = \theta_k^b + b, \forall k = 1, .., n-1$. Therefore, given an equilibrium in the white lie model, MB trivially holds for all but the first partition element as long as $\mu \leq 1/2$.

Suppose that as $\mu \to 0$, $\theta_1^b \to \bar{\theta} > 0$. Then the proof is immediate as any equilibrium outcome in the white lie model $\{\theta^b, \theta^0, y\}$ can be replicated as an equilibrium in the honest model for μ sufficiently small.

Now consider the case in which $\theta_1^b \to 0$. First, suppose $\theta_0^0 = 0$. It is straightforward to show that $\theta_1^b \to \frac{1-2n(n-1)b}{n}$. Therefore, $\theta_1^b \to 0$ only when b is a critical CS value. Second, suppose $\theta_0^0 > 0$. Let $\theta_0^0 \to \bar{\theta}_0$ and consider the case where $\theta_1^b \to 0$. Then $\theta_n^b \to 2(nb - \bar{\theta}_0)(n-1) = 1$ so that $\bar{\theta}_0 = \frac{2(n-1)nb-1}{2(n-1)}$. Respecting $\theta_0^0 \leq y_2$ and noting that $\lim_{\mu\to 0} y_2 = \theta_2^b/2$ and $\lim_{\mu\to 0} \theta_2^b = \frac{1-2n(n-1)b}{n}$ it must be that $\frac{2n(n-1)b-1}{2(n-1)} \leq \frac{1-2n(n-1)b}{2n}$. Combined with the fact that $\bar{\theta}_0 \geq 0$, this implies that, $b = \frac{1}{2n(n-1)}$. ■

Lemma 19 *For $b \in \left(\frac{1}{2n(n+1)}, \frac{1}{2n(n+0.5)}\right)$, if μ is sufficiently small, there is no equilibrium in the white lie model in which there are n partition elements in the biased sender's strategy and the white liar separates from the biased sender on a positive measure of states, that is, $\theta_0^0 = 0$ in any equilibrium with n partition elements above θ_0^0 in the white lie model.*

Proof. Suppose $\theta_0^0 > 0$ and let y_1 be the lowest action induced by the biased sender. By the incentive compatibility of the biased sender, $\theta_0^0 = \frac{2n_0}{2n_0+1} y_1 \leq b$. This condition does not hold for any $n_0 \geq 1$ if and only if $y_1 > 1.5b$. When μ is sufficiently close to 0, the strategies of the biased sender and the receiver are close to those of CS, and thus $y_1 \simeq 1/(2n) + b(1-n)$ (see Crawford and Sobel (1982), Section 4). Applying this result to $y_1 > 1.5b$, we get the condition in the lemma. ■

Lemma 20 *For $b \in \left(\frac{1}{2n^2}, \frac{1}{2n(n-1)}\right)$, if μ is sufficiently small, for any equilibrium in the white lie model, there exists a corresponding equilibrium in the honesty model which yields a weakly better utility to the receiver.*

Proof. If $b \in \left(\frac{1}{2n^2}, \frac{1}{2n(n-1)}\right)$, for μ sufficiently small, then the lowest action induced by the biased sender, y_1, is smaller than b. Then, the biased sender has no incentive to deviate to below θ_0^0, even though the receiver perfectly trusts messages below θ_0^0. Then consider a type I strategy profile that inherits all the properties of the original white lie equilibrium except that the receiver follows the sender's recommendation whenever the message is below θ_0^0 (that is, set $m_0 = \theta_0^0$). This is obviously an equilibrium in the honesty model and yields at least as much utility to the receiver as the original white lie equilibrium.[27] ■

Q.E.D.

Proof of Proposition 22: We first prove the first half of the statement. Suppose μ is arbitrarily close to 0 and $\frac{2n_0+3}{2n_0+2}b < y_1 \leq \frac{2n_0+1}{2n_0}b$ for some natural number n_0. Then there exists an equilibrium in which there are n_0 partition elements below θ_0^0. In such an equilibrium, $\theta_0^0 = \frac{2n_0}{2n_0+1}y_1$. This equilibrium outcome cannot be replicated in the honesty model, because $m_0 \leq 2b - y_1$ due to the incentive compatibility of the biased sender, while $\theta_0^0 > 2b - y_1$ by construction for the case with $n_0 = 1$). Since this argument holds for any natural number n_0, there is an equilibrium with $\theta_0^0 > 0$ whose outcome cannot be replicated in the honesty model when $b < y \leq 1.5b$, whose condition coincides with $b \in \left(\frac{1}{2n(n+0.5)}, \frac{1}{2n^2}\right)$.

Next, we prove the case where b is greater but sufficiently close to $\frac{1}{2n(n-1)}$. For small but positive μ, there exists an n-partition equilibrium in the white lie model with θ_1^b small but positive. MB requires that $\mu(\theta_1^b + b - \theta_0^0)^2 \leq (1-\mu)(\theta_1^b)^2$. We show that this condition does not hold when $b = \frac{1}{2n(n-1)}$. By continuity, the condition does not hold for b is greater but sufficiently close to $\frac{1}{2n(n-1)}$ as well.

Fix μ close to 0. From Equation (3.1), we know that θ_1^b satisfies

$$1 = n\theta_1^b + (n-1)\left[2(1-\mu)nb + \mu b - \frac{\mu(1-\mu)b^2}{\theta_1^b + \mu b}\right].$$

[27] If $n_0 < \infty$ ($n_0 = \infty$), then the new equilibrium yields a strictly greater (the same) utility to the receiver.

Since $b = \frac{1}{2n(n-1)}$,

$$1 = n\theta_1^b + (1-\mu) + \frac{\mu}{2n} - \frac{\mu\,(1-\mu)}{2n\,(n-1)\,\theta_1^b + \mu}\frac{1}{2n}.$$

Arranging terms,

$$\mu\left[2n\,(n-1)\,(2n-1)\,\theta_1^b + \mu\,(2n-1) + (1-\mu) - 2n^2\theta_1^b\right] = 4n^3\,(n-1)\left(\theta_1^b\right)^2.$$

When μ is close to 0, θ_1^b is also close to 0 because b is near a critical value, and

$$\mu \approx 4n^3\,(n-1)\left(\theta_1^b\right)^2.$$

Therefore,

$$\begin{aligned}\mu(\theta_1^b + b - \theta_0^0)^2 &\approx 4n^3\,(n-1)\left(\theta_1^b\right)^2\frac{1}{4n^2\,(n-1)^2} = \frac{n}{n-1}\left(\theta_1^b\right)^2 \\ &> \left(\theta_1^b\right)^2 > (1-\mu)(\theta_1^b)^2.\end{aligned}$$

Q.E.D.

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