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Thermalization of Strongly Coupled Field Theories

Vijay Balasubramanian  
*University of Pennsylvania, vijay@physics.upenn.edu*

Alice Bernamonti  
*Theoretische Natuurkunde, Vrije Universiteit Brussel, and International Solvay Institutes*

Johannes de Boer  
*University of Amsterdam*

Neil Copland  
*Theoretische Natuurkunde, Vrije Universiteit Brussel, and International Solvay Institutes*

Ben Craps  
*Theoretische Natuurkunde, Vrije Universiteit Brussel, and International Solvay Institutes*

*See next page for additional authors*

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Abstract
Using the holographic mapping to a gravity dual, we calculate 2-point functions, Wilson loops, and entanglement entropy in strongly coupled field theories in $d = 2, 3, \text{and } 4$ to probe the scale dependence of thermalization following a sudden injection of energy. For homogeneous initial conditions, the entanglement entropy thermalizes slowest and sets a time scale for equilibration that saturates a causality bound. The growth rate of entanglement entropy density is nearly volume-independent for small volumes but slows for larger volumes. In this setting, the UV thermalizes first.

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Author(s)
Vijay Balasubramanian, Alice Bernamonti, Johannes de Boer, Neil Copland, Ben Craps, Esko Keski-Vakkuri, Berndt O. Müller, Andreas Schäfer, Masaki Shigemori, and Wieland Staessens

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Thermalization of Strongly Coupled Field Theories

V. Balasubramanian,1 A. Bernamonti,2 J. de Boer,3 N. Copland,2 B. Craps,2 E. Keski-Vakkuri,4 B. Müller,5 A. Schäfer,6 M. Shigemori,7 and W. Staessens2

1David Rittenhouse Laboratory, University of Pennsylvania, Philadelphia, Pennsylvania 19104, USA
2Theoretische Natuurkunde, Vrije Universiteit Brussel, and International Solvay Institutes, B-1050 Brussels, Belgium
3Institute for Theoretical Physics, University of Amsterdam, 1090 GL Amsterdam, The Netherlands
4Helsinki Institute of Physics and Department of Physics, FIN-00014 University of Helsinki, Finland
5Department of Physics and CTMS, Duke University, Durham, North Carolina 27708, USA
6Institut für Theoretische Physik, Universität Regensburg, D-93040 Regensburg, Germany
7Kobayashi-Maskawa Institute for the Origin of Particles and the Universe, Nagoya University, Nagoya 464-8602, Japan

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Using the holographic mapping to a gravity dual, we calculate 2-point functions, Wilson loops, and entanglement entropy in strongly coupled field theories in $d = 2, 3, 4$ to probe the scale dependence of thermalization following a sudden injection of energy. For homogeneous initial conditions, the entanglement entropy thermalizes slowest and sets a time scale for equilibration that saturates a causality bound. The growth rate of entanglement entropy density is nearly volume-independent for small volumes but slows for larger volumes. In this setting, the UV thermalizes first.

It is widely believed that the observed nearly inviscid hydrodynamics of relativistic heavy ion collisions at collider energies is an indication that the matter produced in these nuclear reactions is strongly coupled [1]. Some such strongly coupled field theories can be studied by using the holographic duality between gravitational theories in asymptotically anti–de Sitter (AdS) space-times and quantum field theories on the boundary of AdS. The thermal state of the field theory is represented by a black brane in AdS, and near-equilibrium dynamics is studied in terms of perturbations of the black hole metric. A key remaining challenge is to understand the far from equilibrium process of thermalization. The AdS/CFT correspondence relates the approach to thermal equilibrium in the boundary theory to black hole formation in the bulk.

Recent works studied the gravitational collapse of energy injected into AdS3 and the formation of an event horizon [2]. These works started from locally anisotropic metric perturbations near the AdS boundary and studied the rate at which isotropic pressure was established by examining the evolution of the stress tensor. By studying gravitational collapse induced by a small scalar perturbation, the authors of Ref. [3] concluded that local observables behaved as if the system thermalized almost instantaneously. Here we model the equilibrating field configuration in AdS by an infalling homogeneous thin mass shell [4,5] and study how the rate of thermalization varies with spatial scale and dimension. We consider 2d, 3d, and 4d field theories dual to gravity in asymptotically AdS3, AdS4, and AdS5 space-times, respectively. Our treatment of 2d field theories is analytic.

Expectation values of local gauge-invariant operators, including the energy-momentum tensor and its derivatives, provide valuable information about the applicability of viscous hydrodynamics but cannot be used to explore the scale dependence of deviations from thermal equilibrium. Equivalently, in the dual gravitational description these quantities are sensitive only to the metric close to the AdS boundary. Nonlocal operators, such as Wilson loops and 2-point correlators of gauge-invariant operators, probe the thermal nature of the quantum state on extended spatial scales. In the AdS language, these probes reach deeper into the bulk space-time, which corresponds to probing further into the infrared of the field theory. They are also relevant to the physics probed in relativistic heavy ion collisions, e.g., through the jet quenching parameter $q[6]$ and the color screening length.

A global probe of thermalization is the entanglement entropy $S_A[7,8]$ of a domain $A$, measured after subtraction of its vacuum value. In the strong coupling limit, it has been proposed that $S_A$ for a region $A$ with boundary $\partial A$ in the field theory is proportional to the area of the minimal surface $\gamma$ in AdS whose boundary coincides with $\partial A$: $S_A = \text{Area}(\gamma)/4G_N$, where $G_N$ is Newton’s constant [8]. Thus, for a $(d = 2)$-dimensional field theory, $S_A$ is the length of a geodesic curve in AdS3 (studied in Ref. [9]); for $d = 3$, $S_A$ is the area of a 2d sheet in AdS4 (studied in Ref. [10]); and for $d = 4$, $S_A$ is the volume of a 3d region in AdS5. In $d = 3$ the exponential of the area of the minimal surface that measures $S_A$ also computes the expectation value of the Wilson loop that bounds the minimal surface. Wilson loops in $d = 4$ correspond to 2d minimal surfaces as well.

First, we consider equal-time 2-point correlators of gauge-invariant operators $\mathcal{O}$ of large conformal dimension $\Delta$. In the dual supergravity theory this correlator can be expressed, in the semiclassical limit, in terms of the length...
of the bulk geodesic curve that connects the end points on the boundary: \( \langle \hat{O}(x, t)\hat{O}(0, t) \rangle \sim \exp[-\Delta \mathcal{L}(x, t)] \) [11]. When multiple such geodesics exist, one has to consider steepest descent contours to determine the contribution from each geodesic.

We consider a \((d + 1)\)-dimensional infalling shell geometry described in Poincaré coordinates by the Vaidya metric

\[
d s^2 = \frac{1}{z^2} \left(-[1 - m(v)z^d]d v^2 - 2dzdv + dx^2\right),
\]

where \(v\) labels ingoing null trajectories, and we set the AdS radius to 1. The boundary is at \(z = 0\), where \(v\) coincides with the observer time \(t\). The mass function of the infalling shell is

\[
m(v) = (M/2)[1 + \tanh(v/v_0)],
\]

where \(v_0\) determines the thickness of a shell falling along \(v = 0\). The metric interpolates between vacuum AdS inside the shell and an AdS black brane geometry with Hawking temperature \(T = dm^{1/2}/4\pi\) outside the shell. 2-point functions agree with those of a boundary field theory at thermal equilibrium only if they are dominated by geodesics that stay outside the shell.

The geodesic length \(L\) diverges due to contributions near the AdS boundary. We introduce an ultraviolet cutoff \(z_0\) and define a renormalized correlator \(\delta \mathcal{L} = \mathcal{L} + 2\ln(z_0/2)\) by removing the divergent part of the correlator in the vacuum state (pure AdS). The renormalized equal-time 2-point function is \(\langle \hat{O}(x, t)\hat{O}(0, t) \rangle_{\text{ren}} \sim \exp[-\Delta \delta \mathcal{L}(x, t)]\). We compute the renormalized correlator as a function of \(x\) and \(t\) in a state evolving towards thermal equilibrium and compare it to the corresponding thermal correlator. In the bulk, this amounts to computing geodesic lengths in a collapsing shell geometry and comparing them to geodesic lengths in the black brane geometry (\(\delta \mathcal{L}_{\text{thermal}}\)) resulting from the collapse.

We study geodesics with boundary separation \(\ell\) in the \(x\) direction in AdS\(_3\), AdS\(_4\), and AdS\(_5\) modified by the infalling shell. The end point locations are denoted as \((v, z, x) = (t_0, z_0, \pm \ell/2)\), where \(z_0\) is the UV cutoff. The lowest point of the geodesic in the bulk is the midpoint located at \((v, z, x) = (v_\ast, z_\ast, 0)\). Geodesics are obtained by solving differential equations for the functions \(v(x)\) and \(z(x)\) with these boundary conditions and are unique in the infalling shell background. The length of the geodesics is \(L(\ell, t_0) = 2\int_0^{\ell/2} dx z(x)^{d-2}\). In empty AdS, this gives the renormalized geodesic length \(\delta \mathcal{L}_{\text{AdS}} = 2\ln(\ell/2)\).

A numerical solution for the length of geodesics crossing the shell in the \(d = 2\) (AdS\(_3\)) case was obtained in Ref. [9]. We checked that physical results do not depend significantly on the shell thickness when \(v_0\) is small and then derived an analytical solution in the \(v_0 \to 0\) limit:

\[
\delta \mathcal{L}(\ell, t_0) = 2\ln\left[\frac{\sinh(\sqrt{M}t_0)}{\sqrt{Ms}(\ell, t_0)}\right].
\]

where \(s(\ell, t_0) \in [0, 1]\) is parametrically defined by

\[
\ell = \frac{1}{\sqrt{M}} \left[\frac{2c}{3\rho} + \ln\left(\frac{2(1 + c)^2 + 2s\rho - c}{2(1 + c)^2 + 2s\rho - c}\right)\right],
\]

\[
2\rho = \coth(\sqrt{M}t_0) + \coth^2(\sqrt{M}t_0) - \frac{2c}{c + 1},
\]

with \(c = \sqrt{1 - s^2}\) and \(\rho = (\sqrt{M}z_\ast)^{-1}\). Here \(z_\ast\) is the radial location of the intersection between the geodesic and the shell. For any given \(\ell\), at sufficiently late times, the geodesic lies entirely in the black brane background outside the shell. In this case the length is

\[
\delta \mathcal{L}_{\text{thermal}}(\ell) = 2\ln(1/\sqrt{M})\sinh(\sqrt{M}\ell/2),
\]

representing the result for thermal equilibrium.

We use these analytic relations in \(d = 2\) and find \(\delta \mathcal{L}(\ell, t_0)\) in \(d = 3, 4\) by numerical integration. We measure the approach to thermal equilibrium by comparing \(\delta \mathcal{L}\) at any given time with the late time thermal result (see Fig. 1). In any dimension, this compares the logarithm of the 2-point correlator at different spatial scales with the logarithm of the thermal correlator. For \(d = 2\), the same quantity measures by how much the entanglement entropy at a given spatial scale differs from the entropy at thermal equilibrium.

Various thermalization times can be extracted from Fig. 1. For any spatial scale we can ask for (a) the time \(\tau_{\text{dur}}\) until full thermalization (measured as the time when the geodesic between two boundary points just grazes the infalling shell), (b) the half-thermalization time \(\tau_{1/2}\), which measures the duration for the curves to reach half of their equilibrium value, and (c) the time \(\tau_{\text{max}}\) at which thermalization proceeds most rapidly, namely, the time for which the curves in Fig. 1 are steepest. These are plotted in Fig. 2. In \(d = 2\) we can analytically derive the linear relation \(\tau_{\text{dur}} = \ell/2\), as also observed in Ref. [9].

The linearity of \(\tau_{\text{dur}}(\ell)\) in \(2\ell\) is expected from general arguments in conformal field theory [7], and the coefficient is as small as possible under the constraints of causality. The thermalization times scales \(\tau_{1/2}\) and \(\tau_{\text{max}}\) for \(3d\) and \(4d\) field theories (Fig. 2, middle and right) are sublinear in the

![FIG. 1](color online). \(\delta \mathcal{L} - \delta \mathcal{L}_{\text{thermal}}(\mathcal{L} = L/\ell)\) as a function of boundary time \(t_0\) for \(d = 2, 3, 4\) (left, right, middle) for a thin shell \((v_0 = 0.01)\). The boundary separations are \(\ell = 1, 2, 3, 4\) (top to bottom curve). All quantities are given in units of \(M\). These numerical results match analytical results for \(d = 2\) as \(v_0 \to 0\).
spatial scale. In the range we study, the complete thermalization time $\tau_{\text{dur}}$ deviates slightly from linearity and is somewhat shorter than $\ell/2$. We will later discuss whether a rigorous causality bound for thermalization processes exists or not.

In $2d$ “quantum quenches” where a pure state prepared as the ground state of a Hamiltonian with a mass gap is followed as it evolves according to a different, critical Hamiltonian, a nonanalytic feature was found where thermalization at a spatial scale $\ell$ is completed abruptly at $\tau_{\text{dur}}(\ell)$ [7,9]. An analogous feature is evident in Fig. 1 (left) as a sudden change in the slope at $\tau_{\text{dur}}$, smoothed out only by the small nonzero thickness of the shell or, equivalently, by the intrinsic duration of the injection of energy. We find a similar (higher-order) nonanalyticity for $d = 3, 4$ (Fig. 1, middle and right) and expect this to be a general consequence of abrupt injection of energy in any dimension.

Figure 2 shows that complete thermalization of the equal-time correlator is first observed at short length scales or large momentum scales (see also [5]). While this behavior follows directly in our setup with a shell falling in from the (“UV”) boundary of AdS, this “top-down” thermalization contrasts with the behavior of weakly coupled gauge theories even with energy injected in the UV. In the “bottom-up” scenario [12] applicable to that case, hard quanta of the gauge field do not equilibrate directly by randomizing their momenta but gradually degrade their energy by radiating soft quanta, which fill up the thermal phase space and equilibrate by collisions among themselves. This bottom-up scenario is linked to the infrared divergence of the splitting functions of gauge bosons and fermions in perturbative gauge theory. It contrasts with the “democratic” splitting properties of excitations in strongly coupled super Yang-Mills theory that favor an approximately equal sharing of energy and momentum [13].

The thermal limit of the Wightman function that we studied above is a necessary but not a sufficient condition for complete thermalization. To examine whether thermalization proceeds similarly for other probes, we also studied entanglement entropy and spacelike Wilson loop expectation values in $3d$ (following [10]) and $4d$ field theories. Entanglement entropy in $3d$ field theories is holographically related to minimal surfaces in AdS$_4$ and hence to the logarithm of the expectation value of Wilson loops.

We considered circular loops of radius $R$ in $d = 3, 4$. The minimal spacelike surface in $d = 3, 4$ is this circular loop extends into the bulk space radially and into the past. The tip occurs at $(v, z, x = 0)$. The cross section at fixed $z$ and $v$ is a circle, and thus the surface is parameterized in terms of the radii $\rho$ of these circles. The overall shape minimizes the action for the two functions $z(\rho)$ and $v(\rho)$:

$$A[R] = 2\pi \int_0^R d\rho \frac{\rho}{z} \sqrt{1 - \left[1 - m(v)z^3\right]v^2 - 2z'v'},$$  \hspace{1cm} (6)

where $z'(\rho) = d(z/d\rho)$, etc. The resulting Euler-Lagrange equations can be numerically integrated. We regularize the area by subtracting the divergent piece of the area in “empty” AdS: $\delta A[R] = A[R] - (R/z_0)$. Entanglement entropy of spherical volumes in $d = 4$ is similarly computed in terms of minimal volumes in AdS$_4$ by minimizing an equation similar to (6) and defining $\delta V[R]$ by subtracting the divergent volume in empty AdS.

The deficit area $\delta A = \delta A_{\text{thermal}}$ for Wilson loops in $d = 3, 4$ and the deficit volume $\delta V = \delta V_{\text{thermal}}$ are plotted in Fig. 3 for several boundary radii $R$ as a function of the boundary time $t_0$. By subtracting the thermal values, we can observe the deviation from equilibrium for each spatial scale at a time $t_0$. Comparing the three thermalization times defined earlier as a function of the loop diameter (Fig. 4), we find that for the entanglement entropy in $d = 3, 4$, the complete thermalization time $\tau_{\text{dur}}(R)$ is close to being a straight line with unit slope over the range of scales that we

![FIG. 2](color online). Thermalization times ($\tau_{\text{dur}}$, top line; $\tau_{\text{max}}$, middle line; $\tau_{1/2}$, bottom line) as a function of spatial scale for $d = 2$ (left), $d = 3$ (middle), and $d = 4$ (right) for a thin shell ($t_0 = 0.01$). All thermalization time scales are linear in $\ell$ for $d = 2$ and deviate from linearity for $d = 3, 4$.

![FIG. 3](color online). $\delta A = \delta A_{\text{thermal}}$ ($\delta A = A/\pi R^2$; left and middle panels) and $\delta V = \delta V_{\text{thermal}}$ [$V = V/(4\pi R^3/3)$; right panel] as a function of $t_0$ for radii $R = 0.5, 1, 1.5, 2$ (top curve to bottom curve) and mass shell parameters $\nu_0 = 0.01, M = 1$, in $d = 3$ (left panel) and $d = 4$ (middle and right panel) field theories.

![FIG. 4](color online). Thermalization times ($\tau_{\text{dur}}$, top line; $\tau_{\text{max}}$, middle line; $\tau_{1/2}$, bottom line) as a function of the diameter for circular Wilson loops in $d = 3, 4$ (left, middle) and for entanglement entropy of spherical regions in $d = 4$ (right).
Equivalently, the entropy has a growth rate that approaches a constant limiting value for large $\ell$ [Fig. 5(c)] and thus cannot arise from a local phenomenon. This behavior suggests that entanglement entropy and coarse grained entropy have different dynamical properties.

We have investigated the scale dependence of thermalization following a sudden injection of energy in $2d$, $3d$, and $4d$ strongly coupled field theories with gravity duals. The entanglement entropy sets a time scale for equilibration that saturates a causality bound. The relationship between the entanglement entropy growth rate and the KS entropy growth rate defined by coarse graining of the phase space distribution raises interesting questions.

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