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Typing Untyped Lambda-Terms,
or
Reducibility Strikes Again!

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Typing untyped λ-terms, or
Reducibility strikes again!

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Abstract. It was observed by Curry that when (untyped) λ-terms can be assigned types, for example, simple types, these terms have nice properties (for example, they are strongly normalizing). Coppo, Dezani, and Veneri, introduced type systems using conjunctive types, and showed that several important classes of (untyped) terms can be characterized according to the shape of the types that can be assigned to these terms. For example, the strongly normalizable terms, the normalizable terms, and the terms having head-normal forms, can be characterized in some systems $\mathcal{D}$ and $\mathcal{D}\Omega$. The proofs use variants of the method of reducibility. In this paper, we present a uniform approach for proving several meta-theorems relating properties of λ-terms and their typability in the systems $\mathcal{D}$ and $\mathcal{D}\Omega$. Our proofs use a new and more modular version of the reducibility method. As an application of our metatheorems, we show how the characterizations obtained by Coppo, Dezani, Veneri, and Pottinger, can be easily rederived. We also characterize the terms that have weak head-normal forms, which appears to be new. We conclude by stating a number of challenging open problems regarding possible generalizations of the realizability method.

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1 Introduction

In paper, we present a uniform approach for proving some general metatheorems relating properties of (pure) λ-terms and their typability in some type systems with conjunctive types $\mathcal{D}\Omega$ and $\mathcal{D}$, due to Coppo, Dezani, and Venneri [2, 3, 4]. As applications, we give simple proofs of the characterizations of the terms having head-normal forms, of the normalizable terms, and of the strongly normalizing terms. Versions of these results were first obtained by Coppo, Dezani, and Venneri [4], and Pottinger [17]. We are perfectly aware that many of the results of this paper are not original, but what we claim to be original is our restructuration of the method of reducibility. By separating sharply the conditions that a property of λ-terms needs to satisfy from the inductive conditions required for the reducibility method to go through, we were able to obtain a more modular version of the reducibility method. As a consequence, the proofs needed for the various classes of terms only need minor incremental changes.

Thus, the novel aspect of this paper is really in the development of a new version of the reducibility method rather than in the applications of this method. However, we find these applications particularly pretty, and thus, the paper can also be considered as a tutorial on conjunctive type systems and their use for studying properties of λ-terms. In this respect, we were very much inspired by Krivine's book [13]. As a matter of fact, at times, we follow Krivine's presentation rather closely [13], except that we use a new notion of reducibility, and that we prove more general meta-theorems (see below). An excellent survey on Curry-style type assignment systems can be found in Coppo and Cardone [1], where similar results are presented, and in some lecture notes on the λ-calculus by Gérard Huet [10]. We also give a characterization of the terms having weak head-normal forms. This last result appears to be new. The reducibility method presented in this paper is inspired from a proof of the Church-Rosser property given by Georges Koletsos [12].

The situation is that we have a unary predicate $\mathcal{P}$ describing a property of (untyped) λ-terms, and a type-inference system $\mathcal{S}$. For example, $\mathcal{P}$ could be the property of being head-normalizable, or normalizable, or strongly normalizing, and $\mathcal{S}$ could be the system $\mathcal{D}\Omega$ of the next section, or system $\mathcal{D}$ (see Krivine [13]). Our main goal is to find sufficient conditions on the predicate $\mathcal{P}$ so that every term M that type-checks in $\mathcal{S}$ with some "nice" type σ satisfies the predicate $\mathcal{P}$.

As an example of the above general schema, conditions (P1), (P2), (P3s) of definition 3.2 together with conditions (P4) and (P5n) of definition 3.6 are such conditions on $\mathcal{P}$ with respect to system $\mathcal{D}\Omega$ (see theorem 3.9). Since the property of being head-normalizable satisfies properties (P1)-(P5n), as a corollary, we have that every term that type-checks in $\mathcal{D}\Omega$ with a nontrivial type (see definition 2.3) is head-normalizable (see theorem 3.11). Another example is given by conditions (P1), (P2), (P3) of definition 5.2 together with conditions (P4) and (P5) of definition 5.6 with respect to system $\mathcal{D}$ (see theorem 5.9). Since the property of being strongly normalizing satisfies properties (P1)-(P5), as a corollary, we have that every term that type-checks in $\mathcal{D}$ is strongly normalizing.

The main technique involved is a kind of realizability argument known as *reducibility*. The crux of the reducibility method is to interpret every type σ as a set $[\![\sigma]\!]$ of λ-terms having certain closure properties (see Tait [18, 19], Girard [8, 9], Krivine [13], and Gallier [5, 6]). One of the crucial properties is that for a "nice" type σ, the terms in $[\![\sigma]\!]$ satisfy the predicate $\mathcal{P}$ (but this does not have to be the case for ugly types!). If the sets $[\![\sigma]\!]$ are defined right, then the following "realizability property" holds (for example, see lemma 3.8):

If $\mathcal{P}$ is a predicate satisfying conditions (P1)-(P5n), then for every term M that type-checks in $\mathcal{D}\Omega$ with type σ, for every substitution φ such that $\varphi(y) \in [\![\gamma]\!]$ for every $y{:}\gamma \in FV(M)$, we have $M[\varphi] \in [\![\sigma]\!]$.

Now, if the properties (P1)-(P5n) on the predicate $\mathcal{P}$ are right, every variable is in every $[\![\sigma]\!]$, and thus, by chosing φ to be the identity substitution, we get that $M \in [\![\sigma]\!]$ whenever M type-checks in $\mathcal{D}\Omega$ with type σ. Furthermore, when σ is a nice type (for example, nontrivial), properties (P1)-(P5n) imply that $[\![\sigma]\!] \subseteq \mathcal{P}$, and thus, we have shown that M satisfies the predicate $\mathcal{P}$ whenever M type-checks in $\mathcal{D}\Omega$ with a nice type σ.

Other examples of this schema are given by lemma 4.8 and lemma 5.8. In order for an argument of this kind to go through, the sets $[\![\sigma]\!]$ must satisfy some inductive invariant. In the literature, this is often referred to as being a *candidate*. Inspired by Koletsos [12], we use the notion of a $\mathcal{P}$-candidate defined in definition 3.3. This notion has the advantage of not requiring the terms to be strongly normalizing (as in Girard [8, 9]), or to involve rather strange looking terms such as $M[N/x]N_1 \ldots N_k$ (as in Tait [19], Mitchell [15], or Krivine [13]). By isolating the dual notions of I-terms and simple terms, we can give a definition that remains *invariant* no matter what the definition of the sets $[\![\sigma]\!]$ is. Also, the definition of a $\mathcal{P}$-candidate only requires that the predicate $\mathcal{P}$ be satisfied, but nothing to do with the properties (P1)-(P5) on $\mathcal{P}$. This separation is helpful in understanding how to derive sufficient properties on $\mathcal{P}$. In other presentations, properties of the predicate $\mathcal{P}$ are often incorporated in the definition of a candidate, and this tends to obscure the argument. Finally, our definition can be easily adapted to other type disciplines involving explicitly typed terms, or to higher-order types. Also, nice proofs of confluence can be obtained (see Koletsos [12], and Gallier [6]). We now proceed with the details.

2 Conjunctive Types and the System $\mathcal{D}\Omega$

The *conjunctive types*, due to Coppo, Dezani, and Venneri [2, 3, 4], are constructed from a countably infinite set of base types and the undefined type ω, using the type constructors $\rightarrow$ and $\wedge$. We follow Krivine [13] (the reader may also want to consult Coppo, Dezani, and Venneri [4], or Coppo and Cardone [1], for additional background). Let $\mathcal{T}$ denote the set of conjunctive types. As usual, a *context* (or *type assignment*) is a finite (possibly empty) set $\Gamma = x_1{:}\sigma_1, \ldots, x_n{:}\sigma_n$ of pairs $x_i{:}\sigma_i$, where x_i is a variable and σ_i is a type, and where $x_i \neq x_j$ for $i \neq j$.

Definition 2.1 The system $\mathcal{D}\Omega$ is defined by the following rules.

$$\Gamma, x{:}\sigma \rhd x{:}\sigma,$$

$$\frac{\Gamma, x{:}\sigma \rhd M{:}\tau}{\Gamma \rhd (\lambda x.\, M){:}\sigma \rightarrow \tau} \quad (abstraction)$$

$$\frac{\Gamma \rhd M{:}\sigma \rightarrow \tau \quad \Gamma \rhd N{:}\sigma}{\Gamma \rhd (M\, N){:}\tau} \quad (application)$$

$$\frac{\Gamma \rhd M{:}\sigma \quad \Gamma \rhd M{:}\tau}{\Gamma \rhd M{:}\sigma \wedge \tau} \quad (\wedge\text{-}intro)$$

$$\frac{\Gamma \rhd M : \sigma \wedge \tau}{\Gamma \rhd M : \sigma} \quad (\wedge\text{-}elim) \qquad \frac{\Gamma \rhd M : \sigma \wedge \tau}{\Gamma \rhd M : \tau} \quad (\wedge\text{-}elim)$$

$$\Gamma \rhd M : \omega,$$

where Γ and M are arbitrary.

We let Λ denote the set of all (untyped) λ-terms and Λ_σ denote the set of all λ-terms M such that $\vdash_{\mathcal{D}\Omega} \Gamma \rhd M : \sigma$ for some type σ and some context Γ. In this section, the only reduction rule considered is β-reduction:

$$(\lambda x . M)N \longrightarrow_\beta M[N/x].$$

The system $\mathcal{D}$, introduced by Coppo and Dezani [3], is obtained by restricting the types to be ω-free, and by by deleting the axiom

$$\Gamma \rhd M : \omega$$

involving the special type ω from the system $\mathcal{D}\Omega$. We let $\mathcal{SN}\Lambda_\sigma$ denote the set of all λ-terms M such that $\vdash_{\mathcal{D}} \Gamma \rhd M : \sigma$ for some type σ and some context Γ.

Definition 2.2 Given a term M, we let $FV(M)$ denote the set of free variables in M. We say that M is *closed* iff $FV(M) = \emptyset$. If $FV(M) = \{x_1, \ldots, x_m\}$, the *closure* of M is the (closed) term $\lambda x_1 \ldots \lambda x_m . M$.

We now define a class of types that will turn out to characterize the head-normalizable terms.

Definition 2.3 A type σ is *nontrivial* iff either σ is a base type and $\sigma \neq \omega$, or $\sigma = \gamma \to \tau$ where τ is nontrivial and γ is arbitrary, or $\sigma = \sigma_1 \wedge \sigma_2$ where σ_1 or σ_2 is nontrivial. If a type is not nontrivial, we call it *trivial*. A type σ is ω-*free* if ω does not occur in σ.

3 $\mathcal{P}$-Candidates for Head-Normalizing λ-Terms

It turns out that the behavior of a term depends heavily on the nature of the last typing inference rule used in typing this term. A term created by an introduction rule, or I-term, plays a crucial role, because when combined with another term, a new redex is created. On the other hand, for a term created by an elimination rule, or simple term, no new redex is created when this term is combined with another term. It should be noted that the rules ($\wedge$-*intro*) and ($\wedge$-*elim*) do not generate any new I-terms or simple terms, since the term M appearing in the conclusion is identical to the term(s) appearing in the premise(s). This motivates the following definition.

Definition 3.1 An *I-term* is a term of the form $\lambda x . M$. A *simple* term (or *neutral* term) is a term that is not an I-term. Thus, a simple term is either a variable x or an application MN. A term M is *stubborn* iff it is simple and, either M is irreducible, or M' is a simple term whenever $M \xrightarrow{+}_\beta M'$ (equivalently, M' is **not** an I-term).

Let $\mathcal{P} \subseteq \Lambda$ be a (nonempty) set of λ-terms. Actually, $\mathcal{P}$ is the set of λ-terms satisfying a given unary predicate. Our goal is to give sufficient conditions on $\mathcal{P}$ so that this predicate holds for certain sets of terms that type-check with types of a special form in system $\mathcal{D}\Omega$.

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Definition 3.2 Properties (P1)-(P3s) are defined as follows:

(P1) $x \in \mathcal{P}$, for every variable x.

(P2) If $M \in \mathcal{P}$ and $M \longrightarrow_\beta N$, then $N \in \mathcal{P}$.

(P3s) If M is simple, $M \in \mathcal{P}$, $N \in \Lambda$, and $(\lambda x. M')N \in \mathcal{P}$ whenever $M \stackrel{+}{\longrightarrow}_\beta \lambda x. M'$, then $MN \in \mathcal{P}$.

From now on, we only consider sets $\mathcal{P}$ satisfying conditions (P1)-(P3s) of definition 3.2.

Definition 3.3 A nonempty set C of (untyped) λ-terms is a $\mathcal{P}$-*candidate* iff it satisfies the following conditions:

(S1) $C \subseteq \mathcal{P}$.

(S2) If $M \in C$ and $M \longrightarrow_\beta N$, then $N \in C$.

(S3) If M is simple, $M \in \mathcal{P}$, and $\lambda x. M' \in C$ whenever $M \stackrel{+}{\longrightarrow}_\beta \lambda x. M'$, then $M \in C$.

(S3) implies that any $\mathcal{P}$-candidate C contains all variables. More generally, (S3) implies that C contains all stubborn terms in $\mathcal{P}$, and (P1) guarantees that variables are stubborn terms in $\mathcal{P}$.

By (P3s), if $M \in \mathcal{P}$ is a stubborn term and $N \in \Lambda$ is any term, then $MN \in \mathcal{P}$. Furthermore, MN is also stubborn since it is a simple term and since it can only reduce to an I-term (a λ-abstraction) if M itself reduces to a λ-abstraction, i.e. an I-term. Thus, if $M \in \mathcal{P}$ is a stubborn term and $N \in \Lambda$ is any term, then MN is a stubborn term in $\mathcal{P}$. As a consequence, since variables are stubborn, for any terms $N_1, \ldots, N_k$, for every variable x, the term $xN_1 \ldots N_k$ is a stubborn term in $\mathcal{P}$ (assuming appropriate types for x and $N_1, \ldots, N_k$). Instead of (S3), a condition that occurs frequently in reducibility arguments is the following:

(S2n) If $M[N/x]N_1 \ldots N_k \in C$, then $(\lambda x. M)NN_1 \ldots N_k \in C$.

It can be shown easily that (S2) and (S3) imply (S2n) (see the proof of lemma 3.7). Terms of the form $xN_1 \ldots N_k$ or $M[N/x]N_1 \ldots N_k$ are known to play a role in reducibility arguments (for example, by Tait, Mitchell, or Krivine), and it is no surprise that they crop up again. However, in contrast with other presentations, we do not have to deal with them explicitly.

Given a set $\mathcal{P}$, for every type σ, we define $[\![\sigma]\!] \subseteq \Lambda$ as follows.

Definition 3.4 The sets $[\![\sigma]\!]$ are defined as follows:

$$[\![\sigma]\!] = \mathcal{P}, \qquad \text{where } \sigma \neq \omega \text{ is a base type,}$$
$$[\![\sigma]\!] = \Lambda, \qquad \text{where } \sigma \text{ is a trivial type,}$$
$$[\![\sigma \to \tau]\!] = \{M \mid M \in \mathcal{P}, \text{ and for all } N, \text{ if } N \in [\![\sigma]\!] \text{ then } MN \in [\![\tau]\!]\},$$
$$\text{where } \sigma \to \tau \text{ is nontrivial,}$$
$$[\![\sigma \wedge \tau]\!] = [\![\sigma]\!] \cap [\![\tau]\!],$$
$$\text{where } \sigma \wedge \tau \text{ is nontrivial.}$$

By definition 2.3, a type is trivial if either it is ω, or it is of the form $\sigma \to \tau$ where τ is trivial, or it is of the form $\sigma \wedge \tau$ where both σ and τ are trivial. We could have defined $[\![\sigma]\!]$ by changing the second clause to $[\![\omega]\!] = \Lambda$, and by dropping the conditions $\sigma \to \tau$ nontrivial and $\sigma \wedge \tau$ nontrivial. However, it would no longer be true that $[\![\sigma]\!] = \Lambda$ for every trivial type, and this would be a serious obstacle to the proof of lemma 3.7. The following lemma shows that the property of being a $\mathcal{P}$-candidate is an inductive invariant.

Lemma 3.5   *If $\mathcal{P}$ is a set satisfying conditions (P1)-(P3s), then the following properties hold for every type σ: (1) $[\![\sigma]\!]$ contains all stubborn terms in $\mathcal{P}$ (and in particular, every variable); (2) $[\![\sigma]\!]$ satisfies (S2) and (S3); (3) If σ is a nontrivial type, then $[\![\sigma]\!]$ also satisfies (S1), and thus it is a $\mathcal{P}$-candidate.*

Proof. We proceed by induction on types. If σ is a base type, then by definition $[\![\sigma]\!] = \mathcal{P}$ if $\sigma \neq \omega$, and $[\![\omega]\!] = \Lambda$. Then, (1) and (2) are clear by (P1) and by (P2) (note that (S3) is trivial). If $\sigma \neq \omega$, then (S1) is trivial since $[\![\sigma]\!] = \mathcal{P}$.

We now consider the induction step.

(3) We prove that (S1) holds for nontrivial types. If $\sigma \to \tau$ is nontrivial, then τ is nontrivial, and by the definition of $[\![\sigma \to \tau]\!]$, we have $[\![\sigma \to \tau]\!] \subseteq \mathcal{P}$. If $\sigma = \sigma_1 \wedge \sigma_2$ is nontrivial, then σ_1 or σ_2 is nontrivial. Assume σ_1 is nontrivial, the case where σ_2 is nontrivial being similar. By the induction hypothesis, $[\![\sigma_1]\!] \subseteq \mathcal{P}$, and since $[\![\sigma_1 \wedge \sigma_2]\!] = [\![\sigma_1]\!] \cap [\![\sigma_2]\!]$, it is clear that $[\![\sigma_1 \wedge \sigma_2]\!] \subseteq \mathcal{P}$.

The verification of (1) and (2) is obvious for trivial types, since in this case, $[\![\sigma]\!] = \Lambda$. Thus, in the rest of this proof, we assume that we are considering nontrivial types.

(1) Given a type $\sigma \to \tau$, by the induction hypothesis, $[\![\tau]\!]$ contains all the stubborn terms in $\mathcal{P}$. Let $M \in \mathcal{P}$ be a stubborn term. Given any $N \in [\![\sigma]\!]$, obviously, $N \in \Lambda$. Since we have shown that MN is a stubborn term in $\mathcal{P}$ when $M \in \mathcal{P}$ is stubborn and N is arbitrary, we have $MN \in [\![\tau]\!]$. Thus, $M \in [\![\sigma \to \tau]\!]$. If $\sigma = \sigma_1 \wedge \sigma_2$, by the induction hypothesis, all stubborn terms in $\mathcal{P}$ are in $[\![\sigma_1]\!]$ and in $[\![\sigma_2]\!]$, and thus in $[\![\sigma_1 \wedge \sigma_2]\!] = [\![\sigma_1]\!] \cap [\![\sigma_2]\!]$.

(2) We prove (S2) and (S3).

(S2). Let $M \in [\![\sigma \to \tau]\!]$ and assume that $M \longrightarrow_\beta M'$. Since $M \in \mathcal{P}$ by (S1), we have $M' \in \mathcal{P}$ by (P2). For any $N \in [\![\sigma]\!]$, since $M \in [\![\sigma \to \tau]\!]$ we have $MN \in [\![\tau]\!]$, and since $M \longrightarrow_\beta M'$ we have $MN \longrightarrow_\beta M'N$. Then, applying the induction hypothesis at type τ, (S2) holds for $[\![\tau]\!]$, and thus $M'N \in [\![\tau]\!]$. Thus, we have shown that $M' \in \mathcal{P}$ and that if $N \in [\![\sigma]\!]$, then $M'N \in [\![\tau]\!]$. By the definition of $[\![\sigma \to \tau]\!]$, this shows that $M' \in [\![\sigma \to \tau]\!]$, and (S2) holds at type $\sigma \to \tau$.

If $\sigma = \sigma_1 \wedge \sigma_2$, by the induction hypothesis, (S2) holds for $[\![\sigma_1]\!]$ and $[\![\sigma_2]\!]$, and thus for $[\![\sigma_1 \wedge \sigma_2]\!] = [\![\sigma_1]\!] \cap [\![\sigma_2]\!]$.

(S3). Let $M \in \mathcal{P}$ be a simple term, and assume that $\lambda x. M' \in [\![\sigma \to \tau]\!]$ whenever $M \overset{+}{\longrightarrow}_\beta \lambda x. M'$. We prove that for every N, if $N \in [\![\sigma]\!]$, then $MN \in [\![\tau]\!]$. The case where M is stubborn has already been covered in (1). Assume that M is not stubborn. First, we prove that $MN \in \mathcal{P}$, and for this, we use (P3s). If $M \overset{+}{\longrightarrow}_\beta \lambda x. M'$, then by assumption, $\lambda x. M' \in [\![\sigma \to \tau]\!]$, and for any $N \in [\![\sigma]\!]$, we have $(\lambda x. M')N \in [\![\tau]\!]$. Recall that we assumed $\sigma \to \tau$ nontrivial, and thus, τ is nontrivial. Then, by (S1), $(\lambda x. M')N \in \mathcal{P}$, and by (P3s), we have $MN \in \mathcal{P}$. Now, there are two cases.

If τ is a base type, then $[\![\tau]\!] = \mathcal{P}$ since $\tau \neq \omega$, and $MN \in [\![\tau]\!]$ (since $MN \in \mathcal{P}$).

If τ is not a base type, the term MN is simple. Thus, we prove that $MN \in [\![\tau]\!]$ using (S3) (which by induction, holds at type τ). The case where MN is stubborn is trivial. Otherwise, observe that if $MN \xrightarrow{+}_\beta Q$, where $Q = \lambda y.\, P$ is an I-term, then the reduction is necessarily of the form

$$MN \xrightarrow{+}_\beta (\lambda x.\, M')N' \longrightarrow_\beta M'[N'/x] \xrightarrow{*}_\beta Q,$$

where $M \xrightarrow{+}_\beta \lambda x.\, M'$ and $N \xrightarrow{*}_\beta N'$. Since by assumption, $\lambda x.\, M' \in [\![\sigma \to \tau]\!]$ whenever $M \xrightarrow{+}_\beta \lambda x.\, M'$, and by the induction hypothesis applied at type σ, by (S2), $N' \in [\![\sigma]\!]$, we conclude that $(\lambda x{:}\sigma.\, M')N' \in [\![\tau]\!]$. By the induction hypothesis applied at type τ, by (S2), we have $Q \in [\![\tau]\!]$, and by (S3), we have $MN \in [\![\tau]\!]$.

Since $M \in \mathcal{P}$ and $MN \in [\![\tau]\!]$ whenever $N \in [\![\sigma]\!]$, we conclude that $M \in [\![\sigma \to \tau]\!]$. $\square$

For the proof of the next lemma, we need to add two new conditions (P4) and (P5n) to (P1)-(P3s).

Definition 3.6 Properties (P4) and (P5n) are defined as follows:

(P4) If $M \in \mathcal{P}$, then $\lambda x.\, M \in \mathcal{P}$.

(P5n) If $M[N/x] \in \mathcal{P}$, then $(\lambda x.\, M)N \in \mathcal{P}$.

Lemma 3.7 *If $\mathcal{P}$ is a set satisfying conditions (P1)-(P5n), and $M[N/x] \in [\![\tau]\!]$ for every $N \in \Lambda$, then $\lambda x.\, M \in [\![\sigma \to \tau]\!]$.*

Proof. The lemma is obvious if $\sigma \to \tau$ is trivial, since in this case, $[\![\sigma \to \tau]\!] = \Lambda$. Thus, in the rest of this proof, we assume that $\sigma \to \tau$ is nontrivial. This implies that τ is nontrivial.

We prove that for every every N, if $N \in [\![\sigma]\!]$, then $(\lambda x.\, M)N \in [\![\tau]\!]$. We will need the fact that the sets of the form $[\![\sigma]\!]$ have the properties (S1)-(S3), but this follows from lemma 3.5, since (P1)-(P3s) hold. First, we prove that $\lambda x.\, M \in \mathcal{P}$.

By the assumption of lemma 3.7, $M[x/x] = M \in [\![\tau]\!]$ (by choosing $N = x$). Then, since τ is nontrivial, by (S1), $M \in \mathcal{P}$, and by (P4), we have $\lambda x.\, M \in \mathcal{P}$.

Next, we prove that for every every N, if $N \in [\![\sigma]\!]$, then $(\lambda x.\, M)N \in [\![\tau]\!]$. Let us assume that $N \in [\![\sigma]\!]$. Then, by the assumption of lemma 3.7, $M[N/x] \in [\![\tau]\!]$. Since τ is nontrivial, by (S1), we have $M[N/x] \in \mathcal{P}$. By (P5n), we have $(\lambda x.\, M)N \in \mathcal{P}$. Now, there are two cases.

If τ is a nontrivial base type, then $[\![\tau]\!] = \mathcal{P}$. Since we just showed that $(\lambda x.\, M)N \in \mathcal{P}$, we have $(\lambda x.\, M)N \in [\![\tau]\!]$.

If τ is not a base type, then $(\lambda x.\, M)N$ is simple. Thus, we prove that $(\lambda x.\, M)N \in [\![\tau]\!]$ using (S3). The case where $(\lambda x.\, M)N$ is stubborn is trivial. Otherwise, observe that if $(\lambda x.\, M)N \xrightarrow{+}_\beta Q$, where $Q = \lambda y.\, P$ is an I-term, then the reduction is necessarily of the form

$$(\lambda x.\, M)N \xrightarrow{*}_\beta (\lambda x.\, M')N' \longrightarrow_\beta M'[N'/x] \xrightarrow{*}_\beta Q,$$

where $M \xrightarrow{*}_\beta M'$ and $N \xrightarrow{*}_\beta N'$. But $M[N/x] \in [\![\tau]\!]$, and since

$$M[N/x] \xrightarrow{*}_\beta M'[N'/x] \xrightarrow{*}_\beta Q,$$

by (S2), we have $Q \in [\![\tau]\!]$. Since $(\lambda x.\, M)N \in \mathcal{P}$ and $Q \in [\![\tau]\!]$ whenever $(\lambda x.\, M)N \xrightarrow{+}_\beta Q$, by (S3), we have $(\lambda x.\, M)N \in [\![\tau]\!]$. $\square$

We now have the following main "realizability lemma".

Lemma 3.8   *If $\mathcal{P}$ is a set satisfying conditions (P1)-(P5n), then for every term $M \in \Lambda_\sigma$, for every substitution φ such that $\varphi(y) \in [\![\gamma]\!]$ for every $y\colon \gamma \in FV(M)$, we have $M[\varphi] \in [\![\sigma]\!]$.*

Proof. We proceed by induction on the proof $\vdash_{\mathcal{D}\Omega} \Gamma \triangleright M\colon \sigma$. The lemma is obvious if σ is a trivial type, since in this case, $[\![\sigma]\!] = \Lambda$. Thus, in the rest of this proof, we assume that we are considering nontrivial types.

In the case of an axiom $\Gamma, x\colon\sigma \triangleright x\colon\sigma$, we have $M = x$, and then $x[\varphi] = \varphi(x) \in [\![\sigma]\!]$ by the assumption on φ.

If the last rule is an application, then $M = M_1 N_1$, where M_1 has type $\sigma \to \tau$ and N_1 has type σ. By the induction hypothesis, $M_1[\varphi] \in [\![\sigma \to \tau]\!]$ and $N_1[\varphi] \in [\![\sigma]\!]$. By the definition of $[\![\sigma \to \tau]\!]$, we get $M_1[\varphi]N_1[\varphi] \in [\![\tau]\!]$, which shows that $(M_1 N_1)[\varphi] \in [\![\tau]\!]$, since $M_1[\varphi]N_1[\varphi] = (M_1 N_1)[\varphi]$.

If the last rule is an abstraction, then $M = \lambda x\colon\sigma.\, M_1$. By (P1) and (S3), $[\![\sigma]\!]$ is nonempty for every type σ. Consider any $N \in [\![\sigma]\!]$ and any substitution φ such that $\varphi(y) \in [\![\gamma]\!]$ for every $y\colon \gamma \in FV(\lambda x\colon\sigma.\, M_1)$. Thus, the substitution $\varphi[x\colon= N]$ has the property that $\varphi(y) \in [\![\gamma]\!]$ for every $y\colon \gamma \in FV(M_1)$. By suitable α-conversion, we can assume that x does not occur in any $\varphi(y)$ for every $y \in dom(\varphi)$, and that N is substitutable for x in M_1. Then, $M_1[\varphi[x\colon= N]] = M_1[\varphi][N/x]$. By the induction hypothesis applied to M_1 and $\varphi[x\colon= N]$, we have $M_1[\varphi[x\colon= N]] \in [\![\tau]\!]$, that is, $M_1[\varphi][N/x] \in [\![\tau]\!]$. Consequently, by lemma 3.7, $(\lambda x\colon\sigma.\, M_1[\varphi]) \in [\![\sigma \to \tau]\!]$, that is, $(\lambda x\colon\sigma.\, M_1)[\varphi] \in [\![\sigma \to \tau]\!]$, since $(\lambda x\colon\sigma.\, M_1[\varphi]) = (\lambda x\colon\sigma.\, M_1)[\varphi]$.

If the last rule is ($\wedge$-*intro*), by the induction hypothesis, $M[\varphi] \in [\![\sigma]\!]$ and $M[\varphi] \in [\![\tau]\!]$. Since $\sigma \wedge \tau$ is nontrivial, $[\![\sigma \wedge \tau]\!] = [\![\sigma]\!] \cap [\![\tau]\!]$, and thus, $M[\varphi] \in [\![\sigma \wedge \tau]\!]$.

If the last rule is ($\wedge$-*elim*), by the induction hypothesis, $M[\varphi] \in [\![\sigma \wedge \tau]\!]$, and since $\sigma \wedge \tau$ is nontrivial, $[\![\sigma \wedge \tau]\!] = [\![\sigma]\!] \cap [\![\tau]\!]$, and we have $M[\varphi] \in [\![\sigma]\!]$ and $M[\varphi] \in [\![\tau]\!]$. $\square$

As a corollary of lemma 3.8, we obtain the following general theorem for proving properties of terms that type-check in $\mathcal{D}\Omega$.

Theorem 3.9   *If $\mathcal{P}$ is a set of λ-terms satisfying conditions (P1)-(P5n), then $\Lambda_\sigma \subseteq \mathcal{P}$ for every nontrivial type σ (in other words, every term typable in $\mathcal{D}\Omega$ with a nontrivial type satisfies the unary predicate defined by $\mathcal{P}$).*

Proof. Apply lemma 3.8 to every term M in Λ_σ and to the identity substitution, which is legitimate since $x \in [\![\sigma]\!]$ for every variable of type σ (by lemma 3.5). Thus, $M \in [\![\sigma]\!]$ for every term in Λ_σ, that is $\Lambda_\sigma \subseteq [\![\sigma]\!]$. Finally, by lemma 3.5, if σ is nontrivial, (S1) holds for $[\![\sigma]\!]$, that is $\Lambda_\sigma \subseteq [\![\sigma]\!] \subseteq \mathcal{P}$. $\square$

As a corollary of theorem 3.9, we show that if a term M is typable in $\mathcal{D}\Omega$ with a nontrivial type, then the head reduction of M is finite (and so, M has a head-normal form, i.e. it is a solvable term (see definition 6.10). This result was first shown by Coppo, Dezani, and Venneri [4]. Our treatment is heavily inspired by Krivine [13], where we found the marvellous concept of a quasi-head reduction (which is actually due to Barendregt).

Definition 3.10 Given a term $M = \lambda x_1 \ldots \lambda x_m.\,((\lambda y.\,P)Q)N_1 \ldots N_k$, where $m \geq 0$ and $k \geq 0$, the term $(\lambda y.\,P)Q$ is the *head redex* of M. A *head reduction* is a reduction sequence in which every step reduces the head redex. A *quasi-head reduction* is a (finite or infinite) reduction sequence $s = \langle M_0, M_1, \ldots, M_i, \ldots \rangle$ such that, for every $i \geq 0$, if M_i is not the last term in the sequence s, there is some $j \geq i$ such that $M_j \longrightarrow_\beta M_{j+1}$ is a head-reduction step. A term is in *head-normal form* iff it has no head redex, that is, it is of the form $\lambda x_1 \ldots \lambda x_m.\,yN_1 \ldots N_k$, where $m \geq 0$ and $k \geq 0$. The variable y is called the *head variable*. A term is *head-normalizable* iff the head reduction from M is finite.

Note that the last step in a finite quasi-head reduction is necessarily a head-reduction step. Also, any suffix of a quasi-head reduction is a quasi-head reduction. The main advantage of quasi-head reductions over head-reductions is that (P2) obviously holds for terms for which every quasi-head reduction is finite.

Theorem 3.11 *If a term M is typable in $\mathcal{D}\Omega$ with a nontrivial type, then every quasi-head reduction from M is finite. As a corollary, the head reduction from M is finite (and so, M has a head-normal form).*

Proof. Let $\mathcal{P}$ be the set of λ-terms for which every quasi-head reduction is finite. To prove theorem 3.11, we apply theorem 3.9, which requires showing that $\mathcal{P}$ satifies the properties (P1)-(P5n). First, we make the following observation that will simplify the proof. Since there is only a finite number of redexes in any term, for any term M, the reduction tree[1] for M is finitely branching. Thus, if every quasi-head reduction sequence is finite, since the reduction tree is finite branching, by König's lemma, the subtree consisting of quasi-head reduction sequences is finite. Thus, for any term M from which every quasi-head reduction sequence is finite, the length of a longest quasi-head reduction path in the reduction tree from M is a natural number, and we will denote it as $l(M)$. Now, (P1) is trivial, and (P2) follows from the definition.

(P3s). Let M be simple, and assume that every quasi-head reduction from M is finite. We prove that every quasi-head reduction from MN is finite by induction on $l(M)$. Let $MN \longrightarrow_\beta Q$ be a reduction step. Because M is simple, MN is not a redex, and we must have $M \longrightarrow_\beta M_1$ or $N \longrightarrow_\beta N_1$. If M_1 is simple, since $l(M_1) < l(M)$, the induction hypothesis yields that every quasi-head reduction from $M_1 N$ is finite. If $N \longrightarrow_\beta N_1$, because we are considering quasi-head reductions from MN, there is a first step where a head reduction is applied, and it must be applied to M. Thus, we must have $MN \longrightarrow_\beta MN_1 \overset{*}{\longrightarrow}_\beta MN_i \longrightarrow_\beta M_1 N_i$. Since $l(M_1) < l(M)$, the induction hypothesis yields that every quasi-head reduction from MN_1 is finite. Otherwise, $M_1 = \lambda x.\,P$, and by assumption, every quasi-head reduction from $(\lambda x.\,P)N$ is finite. Thus every quasi-head reduction from MN is finite.

(P4). Assume that every quasi-head reduction from M is finite. It is immediate to prove by induction on $l(M)$ that every quasi-head reduction from $\lambda x.\,M$ is also finite.

(P5n). Let k be the index of the first head-reduction step in any quasi-head reduction from $(\lambda x.\,M)N$. We prove by induction on k that every quasi-head reduction from $(\lambda x.\,M)N$ is finite. If $k = 0$, then $(\lambda x.\,M)N$ is a head-redex. However, by the assumption, every quasi-head reduction from $M[N/x]$ is finite. Now, consider any quasi-head reduction s from $(\lambda x.\,M)N$ of index $k \geq 1$.

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[1] the tree of reduction sequences from M

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The first reduction step from $(\lambda x.\, M)N$ is either $(\lambda x.\, M)N \longrightarrow_\beta (\lambda x.\, M_1)N$ or $(\lambda x.\, M)N \longrightarrow_\beta (\lambda x.\, M)N_1$. In either case, the index of the first head-reduction step in the quasi-head reduction $tail(s)$ is $k-1$, and by the induction hypothesis, we get the desired result. $\square$

Note that we could have proved directly that (P2) holds using the following simple lemma.

Lemma 3.12 *If M is head-normalizable and $M \longrightarrow_\beta M'$, then M' is head-normalizable.*

Proof. We prove the following stronger property: If M is head-normalizable and M' is obtained from M by reducing in parallel any set of independant redexes in M (where the reduction applied to each redex is a one-step reduction), then M' is head-normalizable.

The above property is proved by induction on the length $l(M)$ of the head reduction from M. If $l(M) = 0$, then $M = \lambda x_1 \ldots \lambda x_m.\, y N_1 \ldots N_k$, and $M' = \lambda x_1 \ldots \lambda x_m.\, y N_1' \ldots N_k'$, where N_i' is obtained from N_i by performing reductions on independant redexes. We are done since $M' = \lambda x_1 \ldots \lambda x_m.\, y N_1' \ldots N_k'$ is a head-normal form. If $M = \lambda x_1 \ldots \lambda x_m.\, ((\lambda y.\, P)Q)N_1 \ldots N_k$, then either $M' = \lambda x_1 \ldots \lambda x_m.\, ((\lambda y.\, P')Q')N_1' \ldots N_k'$, or $M' = \lambda x_1 \ldots \lambda x_m.\, (P[Q/x])N_1' \ldots N_k'$. In the second case, letting $M_1 = \lambda x_1 \ldots \lambda x_m.\, (P[Q/x])N_1 \ldots N_k$ be the result of reducing the head redex in M, we have $l(M_1) < l(M)$, and since M' is obtained from M_1 by reducing independant redexes, we conclude by applying the induction hypothesis. In the first case, letting $M_1' = \lambda x_1 \ldots \lambda x_m.\, (P'[Q'/x])N_1' \ldots N_k'$ be the result of reducing the head redex in M', since M_1' is obtained from M_1 by reducing independant redexes, we also conclude by applying the induction hypothesis. $\square$

The converse of theorem 3.11 is true: if a λ-term is head-normalizable, then it is typable in $\mathcal{D}\Omega$ with a nontrivial type σ. The proof requires a careful analysis of type-ckecking in system $\mathcal{D}\Omega$. For the time being, we prove the following weaker result.

Lemma 3.13 *Given a term $M = \lambda x_1 \ldots \lambda x_m.\, y N_1 \ldots N_k$ in head-normal form, there are nontrivial types $\sigma = \sigma_1 \to \ldots \sigma_m \to \tau$ and γ, where τ is a base type, such that: if $y \neq x_i$ for all i, then $\vdash_{\mathcal{D}\Omega} y{:}\gamma \triangleright M{:}\sigma$ and the σ_i are arbitrary, else if $y = x_i$, then $\vdash_{\mathcal{D}\Omega} \triangleright M{:}\sigma$, $\sigma_i = \gamma$, and the σ_j are arbitrary for $j \neq i$.*

Proof. Let $\gamma = \omega \to \ldots \to \omega \to \tau$ with k occurrences of ω. Let $\Gamma = x_1{:}\sigma_1, \ldots, x_m{:}\sigma_m, y{:}\tau$ if $y \neq x_i$. It is easy to see that we have

$$\vdash_{\mathcal{D}\Omega} \Gamma, y{:}\gamma \triangleright y N_1 \ldots N_k{:}\tau,$$

and thus,

$$\vdash_{\mathcal{D}\Omega} y{:}\gamma \triangleright \lambda x_1 \ldots \lambda x_m.\, y N_1 \ldots N_k{:}\sigma,$$

where the σ_i are arbitrary. If $y = x_i$, let $\sigma_i = \gamma$ and $\Gamma = x_1{:}\sigma_1, \ldots, x_m{:}\sigma_m$. It is easy to see that we have

$$\vdash_{\mathcal{D}\Omega} \Gamma \triangleright y N_1 \ldots N_k{:}\tau,$$

and thus,

$$\vdash_{\mathcal{D}\Omega} \triangleright \lambda x_1 \ldots \lambda x_m.\, y N_1 \ldots N_k{:}\sigma,$$

where the σ_j are arbitrary for $j \neq i$. $\square$

Note that there are head-normalizable terms that are not normalizable. If $\delta = \lambda x.\, xx$, then $y(\delta\delta)$ is in head-normal form, but it is not normalizable since $\delta\delta$ is not.

4 $\mathcal{P}$-Candidates for Normalizable λ-Terms

In this section, we modify the definition of condition (P3s) in definition 3.2, so that our main theorem applies to the normalizable λ-terms. Although definition 3.1 is unchanged, we repeat it for the reader's convenience.

Definition 4.1 An *I-term* is a term of the form $\lambda x.\, M$. A *simple* term (or *neutral* term) is a term that is not an I-term. Thus, a simple term is either a variable x or an application MN. A term M is *stubborn* iff it is simple and, either M is irreducible, or M' is a simple term whenever $M \xrightarrow{+}_\beta M'$ (equivalently, M' is **not** an I-term).

Definition 4.2 Properties (P1)-(P3) are defined as follows:

(P1) $x \in \mathcal{P}$, for every variable x.

(P2) If $M \in \mathcal{P}$ and $M \longrightarrow_\beta N$, then $N \in \mathcal{P}$.

(P3) If M is simple, $M \in \mathcal{P}$, $N \in \mathcal{P}$, and $(\lambda x.\, M')N \in \mathcal{P}$ whenever $M \xrightarrow{+}_\beta \lambda x.\, M'$, then $MN \in \mathcal{P}$.

Note that the difference with (P3s) of definition 3.2 is that we now require that $N \in \mathcal{P}$. From now on, we only consider sets $\mathcal{P}$ satisfying conditions (P1)-(P3) of definition 4.2. Definition 3.3 is also unchanged, but we repeat it for convenience.

Definition 4.3 A nonempty set C of (untyped) λ-terms is a $\mathcal{P}$-*candidate* iff it satisfies the following conditions:

(S1) $C \subseteq \mathcal{P}$.

(S2) If $M \in C$ and $M \longrightarrow_\beta N$, then $N \in C$.

(S3) If M is simple, $M \in \mathcal{P}$, and $\lambda x.\, M' \in C$ whenever $M \xrightarrow{+}_\beta \lambda x.\, M'$, then $M \in C$.

(S3) implies that any $\mathcal{P}$-candidate C contains all variables. More generally, (S3) implies that C contains all stubborn terms in $\mathcal{P}$, and (P1) guarantees that variables are stubborn terms in $\mathcal{P}$.

By (P3), if $M \in \mathcal{P}$ is a stubborn term and $N \in \mathcal{P}$ is any term, then $MN \in \mathcal{P}$. Furthermore, MN is also stubborn since it is a simple term and since it can only reduce to an I-term (a λ-abstraction) if M itself reduces to a λ-abstraction, i.e. an I-term. Thus, if $M \in \mathcal{P}$ is a stubborn term and $N \in \mathcal{P}$ is any term, then MN is a stubborn term in $\mathcal{P}$. The difference with the previous section is that N **too must be in** $\mathcal{P}$ for MN to be stubborn if $M \in \mathcal{P}$ is stubborn. As a consequence, since variables are stubborn, for any terms $N_1, \ldots, N_k \in \mathcal{P}$, for every variable x, the term $x N_1 \ldots N_k$ is a stubborn term in $\mathcal{P}$ (assuming appropriate types for x and $N_1, \ldots, N_k$).

Given a set $\mathcal{P}$, for every type σ, we define $[\![\sigma]\!] \subseteq \Lambda$ as follows.

Definition 4.4 The sets $[\![\sigma]\!]$ are defined as follows:

$$[\![\sigma]\!] = \mathcal{P}, \qquad \text{where } \sigma \neq \omega \text{ is a base type,}$$
$$[\![\sigma]\!] = \Lambda, \qquad \text{where } \sigma \text{ contains } \omega,$$
$$[\![\sigma \to \tau]\!] = \{M \mid M \in \mathcal{P}, \text{ and for all } N, \text{ if } N \in [\![\sigma]\!] \text{ then } MN \in [\![\tau]\!]\},$$
$$\text{where } \sigma \to \tau \text{ is } \omega\text{-free,}$$
$$[\![\sigma \wedge \tau]\!] = [\![\sigma]\!] \cap [\![\tau]\!],$$
$$\text{where } \sigma \wedge \tau \text{ is } \omega\text{-free.}$$

Lemma 4.5 *If $\mathcal{P}$ is a set satisfying conditions (P1)-(P3), then the following properties hold for every type σ: (1) $[\![\sigma]\!]$ contains all stubborn terms in $\mathcal{P}$ (and in particular, every variable); (2) $[\![\sigma]\!]$ satisfies (S2) and (S3); (3) If σ is ω-free, then $[\![\sigma]\!]$ also satisfies (S1), and thus it is a $\mathcal{P}$-candidate.*

Proof. We proceed by induction on types. The proof is identical to that given in lemma 3.5 when σ is a base type.

We now consider the induction step.

(3) We prove that (S1) holds for ω-free types. If $\sigma \to \tau$ is ω-free, then by the definition of $[\![\sigma \to \tau]\!]$, we have $[\![\sigma \to \tau]\!] \subseteq \mathcal{P}$. If $\sigma = \sigma_1 \wedge \sigma_2$ is ω-free, then σ_1 and σ_2 are ω-free. By the induction hypothesis, $[\![\sigma_1]\!] \subseteq \mathcal{P}$ and $[\![\sigma_2]\!] \subseteq \mathcal{P}$, and since $[\![\sigma_1 \wedge \sigma_2]\!] = [\![\sigma_1]\!] \cap [\![\sigma_2]\!]$, it is clear that $[\![\sigma_1 \wedge \sigma_2]\!] \subseteq \mathcal{P}$.

The verification of (1) and (2) is obvious for types containing ω, since in this case, $[\![\sigma]\!] = \Lambda$. Thus, in the rest of this proof, we assume that we are considering ω-free types.

(1) Given a type $\sigma \to \tau$, by the induction hypothesis, $[\![\tau]\!]$ contains all the stubborn terms in $\mathcal{P}$. Let $M \in \mathcal{P}$ be a stubborn term. Given any $N \in [\![\sigma]\!]$, because $\sigma \to \tau$ is ω-free, so is σ, and by (S1), $N \in \mathcal{P}$. Since we have shown that MN is a stubborn term in $\mathcal{P}$ when $M \in \mathcal{P}$ is stubborn and $N \in \mathcal{P}$, we have $MN \in [\![\tau]\!]$. Thus, $M \in [\![\sigma \to \tau]\!]$. If $\sigma = \sigma_1 \wedge \sigma_2$, by the induction hypothesis, all stubborn terms in $\mathcal{P}$ are in $[\![\sigma_1]\!]$ and in $[\![\sigma_2]\!]$, and thus in $[\![\sigma_1 \wedge \sigma_2]\!] = [\![\sigma_1]\!] \cap [\![\sigma_2]\!]$.

(2) We prove (S2) and (S3).

(S2). The proof is identical to that given in lemma 3.5.

(S3). Let $M \in \mathcal{P}$ be a simple term, and assume that $\lambda x.M' \in [\![\sigma \to \tau]\!]$ whenever $M \stackrel{+}{\longrightarrow}_\beta \lambda x.M'$. We prove that for every N, if $N \in [\![\sigma]\!]$, then $MN \in [\![\tau]\!]$. The case where M is stubborn has already been covered in (1). Assume that M is not stubborn. First, we prove that $MN \in \mathcal{P}$, and for this, we use (P3). If $M \stackrel{+}{\longrightarrow}_\beta \lambda x.M'$, then by assumption, $\lambda x.M' \in [\![\sigma \to \tau]\!]$, and for any $N \in [\![\sigma]\!]$, we have $(\lambda x.M')N \in [\![\tau]\!]$. Recall that we assumed that $\sigma \to \tau$ is ω-free, and thus, both σ and τ are ω-free. Then, by (S1), $N \in \mathcal{P}$ and $(\lambda x.M')N \in \mathcal{P}$, and by (P3), we have $MN \in \mathcal{P}$. The rest of the proof is identical to that given in lemma 3.5. $\square$

Conditions (P4) and (P5n) of definition 3.6 are unchanged, but we repeat them for convenience.

Definition 4.6 Properties (P4) and (P5n) are defined as follows:

(P4) If $M \in \mathcal{P}$, then $\lambda x.M \in \mathcal{P}$.

(P5n) If $M[N/x] \in \mathcal{P}$, then $(\lambda x.M)N \in \mathcal{P}$.

Lemma 4.7   *If $\mathcal{P}$ is a set satisfying conditions (P1)-(P5n), and $M[N/x] \in [\![\tau]\!]$ for every $N \in \Lambda$, then $\lambda x . M \in [\![\sigma \to \tau]\!]$.*

Proof. The lemma is obvious if $\sigma \to \tau$ contains ω, since in this case, $[\![\sigma \to \tau]\!] = \Lambda$. Thus, in the rest of this proof, we assume that $\sigma \to \tau$ is ω-free. This implies that both σ and τ are ω-free.

We prove that for every every N, if $N \in [\![\sigma]\!]$, then $(\lambda x . M)N \in [\![\tau]\!]$. We will need the fact that the sets of the form $[\![\sigma]\!]$ have the properties (S1)-(S3), but this follows from lemma 4.5, since (P1)-(P3) hold. First, we prove that $\lambda x . M \in \mathcal{P}$.

By the assumption of lemma 4.7, $M[x/x] = M \in [\![\tau]\!]$ (by choosing $N = x$). Then, since τ is ω-free, by (S1), $M \in \mathcal{P}$, and by (P4), we have $\lambda x . M \in \mathcal{P}$.

Next, we prove that for every every N, if $N \in [\![\sigma]\!]$, then $(\lambda x . M)N \in [\![\tau]\!]$. Let us assume that $N \in [\![\sigma]\!]$. Then, by the assumption of lemma 4.7, $M[N/x] \in [\![\tau]\!]$. Since τ is ω-free, by (S1), we have $M[N/x] \in \mathcal{P}$. By (P5n), we have $(\lambda x . M)N \in \mathcal{P}$. The rest of the proof is identical to that of lemma 3.7. $\square$

Lemma 4.8   *If $\mathcal{P}$ is a set satisfying conditions (P1)-(P5n), then for every term $M \in \Lambda_\sigma$, for every substitution φ such that $\varphi(y) \in [\![\gamma]\!]$ for every $y : \gamma \in FV(M)$, we have $M[\varphi] \in [\![\sigma]\!]$.*

Proof. We proceed by induction on the proof $\vdash_{\mathcal{D}\Omega} \Gamma \triangleright M : \sigma$. This proof is identical to that of lemma 3.8, with "nontrivial type" replaced by "ω-free type". $\square$

Theorem 4.9   *If $\mathcal{P}$ is a set of λ-terms satisfying conditions (P1)-(P5n), then $\Lambda_\sigma \subseteq \mathcal{P}$ for every ω-free type σ (in other words, every term typable in $\mathcal{D}\Omega$ with an ω-free type satisfies the unary predicate defined by $\mathcal{P}$).*

Proof. Apply lemma 4.8 to every term M in Λ_σ and to the identity substitution, which is legitimate since $x \in [\![\sigma]\!]$ for every variable of type σ (by lemma 4.5). Thus, $M \in [\![\sigma]\!]$ for every term in Λ_σ, that is $\Lambda_\sigma \subseteq [\![\sigma]\!]$. Finally, by lemma 4.5, if σ is ω-free, (S1) holds for $[\![\sigma]\!]$, that is $\Lambda_\sigma \subseteq [\![\sigma]\!] \subseteq \mathcal{P}$. $\square$

As a consequence of theorem 4.9, if $\vdash_{\mathcal{D}\Omega} \Gamma M : \sigma$ where σ and all the types in Γ are ω-free, then $M \in \mathcal{P}$.

As a corollary of theorem 4.9, we show that if a term M is typable in $\mathcal{D}\Omega$ with an ω-free type, then M is normalizable. A version of this theorem was first shown by Coppo, Dezani, and Venneri [4]. Again, our treatment is heavily inspired by Krivine [13], where we found the concept of a quasi-leftmost reduction (which is actually due to Barendregt).

Definition 4.10 Given a term M, the *leftmost redex* in M is either the head-redex $(\lambda y . P)Q$ of M if $M = \lambda x_1 \ldots \lambda x_m . ((\lambda y . P)Q)N_1 \ldots N_k$, (where $m \geq 0$ and $k \geq 0$), or the leftmost redex in the leftmost reducible subterm N_i in M if $M = \lambda x_1 \ldots \lambda x_m . yN_1 \ldots N_k$, $1 \leq i \leq k$ (and thus, $N_1, \ldots, N_{i-1}$ are irreducible). A *leftmost reduction* is a reduction sequence in which every step reduces the leftmost redex. A *quasi-leftmost reduction* is a (finite or infinite) reduction sequence $s = \langle M_0, M_1, \ldots, M_i, \ldots \rangle$ such that, for every $i \geq 0$, if M_i is not the last term in the sequence s, there is some $j \geq i$ such that $M_j \longrightarrow_\beta M_{j+1}$ is a leftmost reduction step. A term is in *normal form* (or *irreducible*) iff it has no redex. A term is *normalizable* iff the leftmost reduction from M is finite.

It is immediate that M is in normal form iff it is of the form $\lambda x_1 \ldots \lambda x_m. y N_1 \ldots N_k$, where $N_1, \ldots, N_k$ are also in normal form ($m \geq 0$ and $k \geq 0$). Note that the last step in a finite quasi-leftmost reduction is necessarily a leftmost reduction step. Also, any suffix of a quasi-leftmost reduction is a quasi-leftmost reduction. The main advantage of quasi-leftmost reductions over leftmost reductions is that (P2) obviously holds for terms for which every quasi-leftmost reduction is finite.

Theorem 4.11   *If a term M is typable in $\mathcal{D}\Omega$ with an ω-free type, then every quasi-leftmost reduction from M in finite. As a corollary, the leftmost reduction from M is finite (and so, M has a normal form).*

Proof. Let $\mathcal{P}$ be the set of λ-terms for which every quasi-leftmost reduction is finite. To prove theorem 3.11, we apply theorem 3.9, which requires showing that $\mathcal{P}$ satifies the properties (P1)-(P5n). First, note that the observation made at the beginning of the proof of lemma 3.11 also applies. If every quasi-leftmost reduction sequence is finite, since the reduction tree is finite branching, by König's lemma, the subtree consisting of quasi-leftmost reduction sequences is finite. Thus, for any term M from which every quasi-leftmost reduction sequence is finite, the length of a longest quasi-leftmost reduction path in the reduction tree from M is a natural number, and we will denote it as $l(M)$. Now, (P1) is trivial, and (P2) follows from the definition.

(P3s). Let M be simple, and assume that every quasi-leftmost reduction from M or N is finite. We prove that every quasi-leftmost reduction from MN is finite by induction on $l(M) + l(N)$. Let $MN \longrightarrow_\beta Q$ be a reduction step. Because M is simple, MN is not a redex, and we must have $M \longrightarrow_\beta M_1$ or $N \longrightarrow_\beta N_1$. If M_1 is simple, since $l(M_1) + l(N) < l(M) + l(N)$, the induction hypothesis yields that every quasi-leftmost reduction from $M_1 N$ is finite. If $N \longrightarrow_\beta N_1$, since $l(M) + l(N_1) < l(M) + l(N)$, the induction hypothesis yields that every quasi-leftmost reduction from MN_1 is finite. Otherwise, $M_1 = \lambda x. P$, and by assumption, every quasi-leftmost reduction from $(\lambda x. P)N$ is finite. Thus every quasi-leftmost reduction from MN is finite.

(P4). Assume that every quasi-leftmost reduction from M is finite. It is immediate to prove by induction on $l(M)$ that every quasi-leftmost reduction from $\lambda x. M$ is also finite.

(P5n). Let k be the index of the first leftmost reduction step in any quasi-leftmost reduction from $(\lambda x. M)N$. We prove by induction on k that every quasi-leftmost reduction from $(\lambda x. M)N$ is finite. If $k = 0$, then $(\lambda x. M)N$ is a head-redex. However, by the assumption, every quasi-leftmost reduction from $M[N/x]$ is finite. Now, consider any quasi-leftmost reduction s from $(\lambda x. M)N$ of index $k \geq 1$. The first reduction step from $(\lambda x. M)N$ is either $(\lambda x. M)N \longrightarrow_\beta (\lambda x. M_1)N$ or $(\lambda x. M)N \longrightarrow_\beta (\lambda x. M)N_1$. In either case, the index of the first leftmost reduction step in the quasi-leftmost reduction $tail(s)$ is $k - 1$, and by the induction hypothesis, we get the desired result. $\square$

Actually, it is possible to prove directly that (P2) holds for leftmost reductions.

Lemma 4.12   *If M is normalizable and $M \longrightarrow_\beta M'$, then M' is normalizable.*

Proof. We prove the following stronger property: If M is normalizable and M' is obtained from M by reducing in parallel any set of independant redexes in M (where the reduction applied to each redex is a one-step reduction), then M' is normalizable.

The above property is proved by induction on the length $l(M)$ of the leftmost reduction from M. If $l(M) = 0$, then M is in normal form and the lemma is trivial. If $M = C[(\lambda y. P)Q]$ where $(\lambda y. P)Q$ is the leftmost redex in M, then either $M' = C'[(\lambda y. P')Q']$, or $M' = C'[P[Q/x]]$. In the second case, letting $M_1 = C[P[Q/x]]$ be the result of reducing the leftmost redex in M, we have $l(M_1) < l(M)$, and since M' is obtained from M_1 by reducing independant redexes, we conclude by applying the induction hypothesis. In the first case, letting $M_1' = C'[P'[Q'/x]]$ be the result of reducing the leftmost redex in M', since M_1' is obtained from M_1 by reducing independant redexes, we also conclude by applying the induction hypothesis. $\square$

The converse of theorem 4.11 is true: if a λ-term M is normalizable, then $\vdash_{\mathcal{D}\Omega} \Gamma \triangleright M : \sigma$ where σ and all the types in Γ are ω-free. For the time being, we prove that every term in normal form is typable in system $\mathcal{D}$. First, observe that because the first axiom in both systems $\mathcal{D}\Omega$ and $\mathcal{D}$ is of the form $\Gamma, x : \sigma \triangleright x : \sigma$, for any two contexts Γ and Δ, if $\Gamma \subseteq \Delta$ and $\vdash_{\mathcal{D}\Omega} \Gamma \triangleright M : \sigma$, then $\vdash_{\mathcal{D}\Omega} \Delta \triangleright M : \sigma$ (and similarly for $\vdash_{\mathcal{D}}$).

Lemma 4.13   If $\vdash_{\mathcal{D}\Omega} x : \sigma_1, \Gamma \triangleright M : \sigma$, then for any type τ_1, $\vdash_{\mathcal{D}\Omega} x : \sigma_1 \wedge \tau_1, \Gamma \triangleright M : \sigma$ (and similarly for $\vdash_{\mathcal{D}}$).

Proof. We proceed by induction on the proof. The only nonobvious case is the case where $x : \sigma_1, \Gamma \triangleright M : \sigma$ is an axiom, with $M = x$ and $\sigma = \sigma_1$. In this case, $x : \sigma_1 \wedge \tau_1, \Gamma \triangleright x : \sigma_1 \wedge \tau_1$ is also an axiom, and by $(\wedge\text{-}elim)$, we get $\vdash_{\mathcal{D}\Omega} x : \sigma_1 \wedge \tau_1, \Gamma \triangleright x : \sigma_1$. $\square$

Lemma 4.14   If $\vdash_{\mathcal{D}\Omega} \Gamma_1 \triangleright M : \sigma$ and $\vdash_{\mathcal{D}\Omega} \Gamma_2 \triangleright N : \tau$, then there is a context $\Gamma_1 \wedge \Gamma_2$ such that, $\vdash_{\mathcal{D}\Omega} \Gamma_1 \wedge \Gamma_2 \triangleright M : \sigma$ and $\vdash_{\mathcal{D}\Omega} \Gamma_1 \wedge \Gamma_2 \triangleright N : \tau$ (and similarly for $\vdash_{\mathcal{D}}$).

Proof. By the remark before lemma 4.13, Γ_1 and Γ_2 can be extended to contexts Γ_1' and Γ_2' which are of the form $\Gamma_1' = x_1 : \sigma_1, \ldots, x_m : \sigma_m$ and $\Gamma_2' = x_1 : \tau_1, \ldots, x_m : \tau_m$. Then, letting $\Gamma_1 \wedge \Gamma_2 = x_1 : \sigma_1 \wedge \tau_1, \ldots, x_m : \sigma_m \wedge \tau_m$, by lemma 4.13 (applied m times), we have $\vdash_{\mathcal{D}\Omega} \Gamma_1 \wedge \Gamma_2 \triangleright M : \sigma$ and $\vdash_{\mathcal{D}\Omega} \Gamma_1 \wedge \Gamma_2 \triangleright N : \tau$ $\square$

We can now prove the desired result.

Lemma 4.15   If M is in normal form, then there is a context Γ and a type σ (both ω-free) such that $\vdash_{\mathcal{D}} \Gamma \triangleright M : \sigma$. Furthermore, if M is not a λ-abstraction, the type σ can be chosen arbitrarily.

Proof. We proceed by induction on M. If $M = x$ is a variable, for every ω-free type σ, and any ω-free Γ, $x : \sigma, \Gamma \triangleright x : \sigma$ is an axiom.

If $M = \lambda x. M_1$, by the induction hypothesis, there is a context Γ and a type τ (both ω-free) such that $\vdash_{\mathcal{D}} \Gamma \triangleright M_1 : \tau$. If $x \notin dom(\Gamma)$, we can pick any ω-free type σ and extend Γ so that we still have $\vdash_{\mathcal{D}} x : \sigma, \Gamma \triangleright M_1 : \tau$. Thus, we assume that we are in the second case. But then, $\vdash_{\mathcal{D}} \Gamma \triangleright \lambda x. M_1 : \sigma \rightarrow \tau$.

If $M = M_1 M_2$, because M is in normal form, M_1 cannot be a λ-abstraction. By the induction hypothesis, there is a context Γ_2 and a type τ (both ω-free) such that $\vdash_{\mathcal{D}} \Gamma_2 \triangleright M_2 : \tau$, and for any arbitrary ω-free type σ, there is some ω-free context Γ_1 such that $\vdash_{\mathcal{D}} \Gamma_1 \triangleright M_1 : \tau \rightarrow \sigma$. By lemma 4.14, we have $\vdash_{\mathcal{D}} \Gamma_1 \wedge \Gamma_2 \triangleright M_1 : \tau \rightarrow \sigma$ and $\vdash_{\mathcal{D}} \Gamma_1 \wedge \Gamma_2 \triangleright M_2 : \tau$, and thus, $\vdash_{\mathcal{D}} \Gamma_1 \wedge \Gamma_2 \triangleright M_1 M_2 : \sigma$. $\square$

Note that there are normalizable terms that are not strongly normalizing. If $\delta = \lambda x. xx$, then $M = (\lambda x. y)(\delta\delta)$ is normalizable since $M \longrightarrow_\beta y$, but it is not strongly normalizing since $\delta\delta$ is not. There are even normalizable terms such that every subterm is SN that are not SN! For example, $M = [\lambda x. ((\lambda y. z)(x\delta))]\delta$ is such a term.

5 $\mathcal{P}$-Candidates for Strongly Normalizing λ-Terms

Although definition 4.1 is unchanged, we repeat it for convenience.

Definition 5.1 An *I-term* is a term of the form $\lambda x.\, M$. A *simple* term (or *neutral* term) is a term that is not an I-term. Thus, a simple term is either a variable x or an application MN. A term M is *stubborn* iff it is simple and, either M is irreducible, or M' is a simple term whenever $M \xrightarrow{+}_\beta M'$ (equivalently, M' is **not** an I-term).

   Similarly, although definition 4.2 is unchanged, we repeat it for convenience.

Definition 5.2 Properties (P1)-(P3) are defined as follows:

(P1) $x \in \mathcal{P}$, for every variable x.

(P2) If $M \in \mathcal{P}$ and $M \longrightarrow_\beta N$, then $N \in \mathcal{P}$.

(P3) If M is simple, $M \in \mathcal{P}$, $N \in \mathcal{P}$, and $(\lambda x.\, M')N \in \mathcal{P}$ whenever $M \xrightarrow{+}_\beta \lambda x.\, M'$, then $MN \in \mathcal{P}$.

   From now on, we only consider sets $\mathcal{P}$ satisfying conditions (P1)-(P3) of definition 5.2. Definition 4.3 is also unchanged, but we repeat it for convenience.

Definition 5.3 A nonempty set C of (untyped) λ-terms is a $\mathcal{P}$-*candidate* iff it satisfies the following conditions:

(S1) $C \subseteq \mathcal{P}$.

(S2) If $M \in C$ and $M \longrightarrow_\beta N$, then $N \in C$.

(S3) If M is simple, $M \in \mathcal{P}$, and $\lambda x.\, M' \in C$ whenever $M \xrightarrow{+}_\beta \lambda x.\, M'$, then $M \in C$.

   The remarks following definition 4.3 apply here too. Thus, (S3) implies that C contains all stubborn terms in $\mathcal{P}$, and (P1) guarantees that variables are stubborn terms in $\mathcal{P}$. Also, by (P3), if $M \in \mathcal{P}$ is a stubborn term and $N \in \mathcal{P}$ is any term, then $MN \in \mathcal{P}$ is stubborn. Instead of (S3), a condition that occurs frequently in reducibility arguments is the following:

(S2sn) If $N \in \mathcal{P}$ and $M[N/x]N_1 \ldots N_k \in C$, then $(\lambda x.\, M)N N_1 \ldots N_k \in C$.

   It can be shown easily that (S2) and (S3) imply (S2sn) (see the proof of lemma 5.7).

   Given a set $\mathcal{P}$, for every type σ, we define $[\![\sigma]\!] \subseteq \Lambda$ as follows.

Definition 5.4 The sets $[\![\sigma]\!]$ are defined as follows:

$$
\begin{aligned}
[\![\sigma]\!] &= \mathcal{P}, \qquad \text{where } \sigma \text{ is a base type,} \\
[\![\sigma \to \tau]\!] &= \{ M \mid M \in \mathcal{P}, \text{ and for all } N, \text{ if } N \in [\![\sigma]\!] \text{ then } MN \in [\![\tau]\!] \}, \\
[\![\sigma \wedge \tau]\!] &= [\![\sigma]\!] \cap [\![\tau]\!].
\end{aligned}
$$

Lemma 5.5 *If $\mathcal{P}$ is a set satisfying conditions (P1)-(P3), then the following properties hold for every type σ: (1) $[\![\sigma]\!]$ contains all stubborn terms in $\mathcal{P}$ (and in particular, every variable); (2) $[\![\sigma]\!]$ satisfies (S1), (S2), and (S3), and thus it is a $\mathcal{P}$-candidate.*

Proof. We proceed by induction on types. If σ is a base type, then by definition $[\![\sigma]\!] = \mathcal{P}$. Then, (1) and (2) are clear by (P1) and by (P2) (note that (S1) and (S3) are trivial).

We now consider the induction step.

(1) Given a type $\sigma \to \tau$, by the induction hypothesis, $[\![\tau]\!]$ contains all the stubborn terms in $\mathcal{P}$. Let $M \in \mathcal{P}$ be a stubborn term. Given any $N \in [\![\sigma]\!]$, by (S1), $N \in \mathcal{P}$. Since we have shown that MN is a stubborn term in $\mathcal{P}$ when $M \in \mathcal{P}$ is stubborn and $N \in \mathcal{P}$, we have $MN \in [\![\tau]\!]$. Thus, $M \in [\![\sigma \to \tau]\!]$. If $\sigma = \sigma_1 \wedge \sigma_2$, by the induction hypothesis, all stubborn terms in $\mathcal{P}$ are in $[\![\sigma_1]\!]$ and in $[\![\sigma_2]\!]$, and thus in $[\![\sigma_1 \wedge \sigma_2]\!] = [\![\sigma_1]\!] \cap [\![\sigma_2]\!]$.

(S1). By the definition of $[\![\sigma \to \tau]\!]$, we have $[\![\sigma \to \tau]\!] \subseteq \mathcal{P}$. If $\sigma = \sigma_1 \wedge \sigma_2$, by the induction hypothesis, $[\![\sigma_1]\!] \subseteq \mathcal{P}$ and $[\![\sigma_2]\!] \subseteq \mathcal{P}$, and since $[\![\sigma_1 \wedge \sigma_2]\!] = [\![\sigma_1]\!] \cap [\![\sigma_2]\!]$, it is clear that $[\![\sigma_1 \wedge \sigma_2]\!] \subseteq \mathcal{P}$.

(S2). The proof is identical to that of lemma 4.5.

(S3). Let $M \in \mathcal{P}$ be a simple term, and assume that $\lambda x.M' \in [\![\sigma \to \tau]\!]$ whenever $M \stackrel{+}{\longrightarrow}_\beta \lambda x.M'$. We prove that for every N, if $N \in [\![\sigma]\!]$, then $MN \in [\![\tau]\!]$. The case where M is stubborn has already been covered in (1). Assume that M is not stubborn. First, we prove that $MN \in \mathcal{P}$, and for this, we use (P3). If $M \stackrel{+}{\longrightarrow}_\beta \lambda x.M'$, then by assumption, $\lambda x.M' \in [\![\sigma \to \tau]\!]$, and for any $N \in [\![\sigma]\!]$, we have $(\lambda x.M')N \in [\![\tau]\!]$. By (S1), $N \in \mathcal{P}$ and $(\lambda x.M')N \in \mathcal{P}$, and by (P3), we have $MN \in \mathcal{P}$. The rest of the proof is identical to that of lemma 4.5. $\square$

Condition (P5n) of definition 4.6 is modified so that our main theorem applies to strongly normalizing terms.

Definition 5.6 Properties (P4) and (P5) are defined as follows:

(P4) If $M \in \mathcal{P}$, then $\lambda x.M \in \mathcal{P}$.

(P5) If $N \in \mathcal{P}$ and $M[N/x] \in \mathcal{P}$, then $(\lambda x.M)N \in \mathcal{P}$.

Note that the difference between (P5n) of definition 4.6 and (P5) is that we are now requiring that $N \in \mathcal{P}$.

Lemma 5.7 *If $\mathcal{P}$ is a set satisfying conditions (P1)-(P5) and for every N, ($N \in [\![\sigma]\!]$ implies $M[N/x] \in [\![\tau]\!]$), then $\lambda x.M \in [\![\sigma \to \tau]\!]$.*

Proof. We prove that for every every N, if $N \in [\![\sigma]\!]$, then $(\lambda x.M)N \in [\![\tau]\!]$. We will need the fact that the sets of the form $[\![\sigma]\!]$ have the properties (S1)-(S3), but this follows from lemma 5.5, since (P1)-(P3) hold. First, we prove that $\lambda x.M \in \mathcal{P}$.

By the assumption of lemma 5.7, $M[x/x] = M \in [\![\tau]\!]$, since by lemma 5.5, $x \in [\![\sigma]\!]$. Then, by (S1), $M \in \mathcal{P}$, and by (P4), we have $\lambda x.M \in \mathcal{P}$.

Next, we prove that for every every N, if $N \in [\![\sigma]\!]$, then $(\lambda x.M)N \in [\![\tau]\!]$. Let us assume that $N \in [\![\sigma]\!]$. Then, by the assumption of lemma 5.7, $M[N/x] \in [\![\tau]\!]$. By (S1), we have $N \in \mathcal{P}$ and $M[N/x] \in \mathcal{P}$. By (P5), we have $(\lambda x.M)N \in \mathcal{P}$. The rest of the proof is identical to that of lemma 4.7. $\square$

Lemma 5.8 *If $\mathcal{P}$ is a set satisfying conditions (P1)-(P5), then for every term $M \in \mathcal{SN}\Lambda_\sigma$, for every substitution φ such that $\varphi(y) \in [\![\gamma]\!]$ for every $y \colon \gamma \in FV(M)$, we have $M[\varphi] \in [\![\sigma]\!]$.*

Proof. We proceed by induction on the proof $\vdash_{\mathcal{D}} \Gamma \rhd M \colon \sigma$. The proof is actually identical to that of lemma 4.8, except that we don't even have to bother with types containing ω. $\square$

Theorem 5.9 *If $\mathcal{P}$ is a set of λ-terms satisfying conditions (P1)-(P5), then $\mathcal{SN}\Lambda_\sigma \subseteq \mathcal{P}$ for every type σ (in other words, every term typable in $\mathcal{D}$ satisfies the unary predicate defined by $\mathcal{P}$).*

Proof. Apply lemma 5.8 to every term M in $\mathcal{SN}\Lambda_\sigma$ and to the identity substitution, which is legitimate since $x \in [\![\sigma]\!]$ for every variable of type σ (by lemma 5.5). Thus, $M \in [\![\sigma]\!]$ for every term in $\mathcal{SN}\Lambda_\sigma$, that is $\mathcal{SN}\Lambda_\sigma \subseteq [\![\sigma]\!]$. Since by lemma 5.5, (S1) also holds for $[\![\sigma]\!]$, we have $\mathcal{SN}\Lambda_\sigma \subseteq [\![\sigma]\!] \subseteq \mathcal{P}$. $\square$

As a corollary of theorem 5.9, we show that if a term M is typable in $\mathcal{D}$, then M is strongly normalizing. This result was first proved by Pottinger [17].

Definition 5.10 A term M is *strongly normalizing* (or SN) iff every reduction sequence from M (w.r.t. $\longrightarrow_\beta$) is finite. The reduction relation $\longrightarrow_\beta$ is *strongly normalizing* (or SN) iff every term is normalizing (w.r.t. $\longrightarrow_\beta$).

Theorem 5.11 *If a term M is typable in $\mathcal{D}$, then M is strongly normalizing.*

Proof. Let $\mathcal{P}$ be the set of λ-terms that are strongly normalizing. To prove theorem 5.11, we apply theorem 5.9, which requires showing that $\mathcal{P}$ satifies the properties (P1)-(P5). First, note that the observation made at the beginning of the proof of lemma 3.11 also applies. If M is any strongly normalizing term, every path in its reduction tree is finite, and since this tree is finite branching, by König's lemma, this reduction tree is finite. Thus, for any SN term M, the depth[2] of its reduction tree is a natural number, and we will denote it as $d(M)$. We now check the conditions (P1)-(P5). (P1) and (P2) are obvious.

(P3) Since M and N are SN, $d(M)$ and $d(N)$ are finite. We prove by induction on $d(M) + d(N)$ that MN is SN. We consider all possible ways that $MN \longrightarrow_\beta P$. Since M is simple, MN itself is not a redex, and so $P = M_1 N_1$ where either $N = N_1$ and $M \longrightarrow_\beta M_1$, or $M = M_1$ and $N \longrightarrow_\beta N_1$.

If M_1 is simple or $M_1 = M$, $d(M_1) + d(N_1) < d(M) + d(N)$, and by the induction hypothesis, $P = M_1 N_1$ is SN. Otherwise, $M_1 = \lambda x.\, M'$, $N_1 = N$. By assumption, $(\lambda x.\, M')N$ is SN, and so P is SN. Thus, $P = M_1 N_1$ is SN in all cases, and MN is SN.

(P4) Any reduction from $\lambda x.\, M$ must be of the form $\lambda x.\, M \xrightarrow{+}_\beta \lambda x.\, M'$ where $M \xrightarrow{+}_\beta M'$. We use a simple induction on $d(M)$.

(P5) Since N and $M[N/x]$ are SN, the term M itself is SN. Thus, $d(M)$ and $d(N)$ are finite. We prove by induction on $d(M) + d(N)$ that $(\lambda x.\, M)N$ is SN. We consider all possible ways that $(\lambda x.\, M)N \longrightarrow_\beta P$. Either $P = (\lambda x.\, M_1)N$ where $M \longrightarrow_\beta M_1$, or $P = (\lambda x.\, M)N_1$ where $N \longrightarrow_\beta N_1$, or $P = M[N/x]$. In the first two cases, $d(M_1) + d(N) < d(M) + d(N)$, $d(M) + d(N_1) < d(M) + d(N)$, and by the induction hypothesis, P is SN. In the third case, by assumption $M[N/x]$ is SN. But then, P is SN in all cases, and so $(\lambda x.\, M)N$ is SN. $\square$

The converse of theorem 5.11 is true: if a λ-term M is strongly normalizing, then $\vdash_{\mathcal{D}} \Gamma \rhd M \colon \sigma$ for some Γ and some type σ.

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[2] the length of a longest path in the tree, counting the number of edges

6 Typability in $\mathcal{D}\Omega$ and $\mathcal{D}$

We now prove the converse of each of the theorems 3.11, 4.11, and 5.11. Versions of these results were first obtained by Coppo, Dezani, and Venneri [4], and Pottinger [17]. Our treatment is basically that of Krivine [13]. The crucial property of system $\mathcal{D}\Omega$, and this is where essential use of conjunctive types and of the type ω is made, is the following: if $\vdash_{\mathcal{D}\Omega} \Gamma \triangleright N : \sigma$ and $M \longrightarrow_\beta N$, then we also have $\vdash_{\mathcal{D}\Omega} \Gamma \triangleright M : \sigma$. This property fails in general for system $\mathcal{D}$, but holds in the special case where $\vdash_{\mathcal{D}} \Gamma \triangleright M[N/x] : \sigma$ and $\vdash_{\mathcal{D}} \Gamma \triangleright N : \sigma_1$ for some σ_1. In that case, $\vdash_{\mathcal{D}} \Gamma \triangleright (\lambda x.\, M)N : \sigma$. We will need a number of preliminary results. First, we have the usual substitution lemma.

Lemma 6.1   Let $\mathcal{S} \in \{\mathcal{D}\Omega, \mathcal{D}\}$. If $\vdash_{\mathcal{S}} \Gamma, x : \sigma \triangleright M : \tau$ and $\vdash_{\mathcal{S}} \Gamma \triangleright N : \sigma$, then $\vdash_{\mathcal{S}} \Gamma \triangleright M[N/x] : \tau$. In particular, if $x \notin FV(M)$, then $\vdash_{\mathcal{D}\Omega} \Gamma \triangleright M : \tau$.

   Proof. An easy induction on typing derivations. $\square$

   We say that a type σ is *prime* iff $\sigma \neq \omega$ and σ is not of the form $\sigma_1 \wedge \sigma_2$. A type σ is a *prime factor* of a type τ iff it is a subtype of τ and it is prime. The following permutation lemma is technically very important.

Lemma 6.2   Let $\mathcal{S} \in \{\mathcal{D}\Omega, \mathcal{D}\}$, and let σ be a prime type. (1) If $\vdash_{\mathcal{S}} \Gamma \triangleright x : \sigma$, then there is a type σ' such that $x : \sigma' \in \Gamma$ and σ is a prime factor of σ'. (2) If $\vdash_{\mathcal{S}} \Gamma \triangleright MN : \sigma$, then either the last rule used in the proof is (application), or there is a type σ' such that σ is a prime factor of σ', $\vdash_{\mathcal{S}} \Gamma \triangleright MN : \sigma'$, and the last rule used in the proof is (application). (3) Given a proof $\vdash_{\mathcal{S}} \Gamma \triangleright \lambda x.\, M : \sigma$ then there is a proof in which the last rule is (abstraction), and given a proof $\vdash_{\mathcal{S}} \Gamma \triangleright \lambda x.\, M : \sigma_1 \wedge \sigma_2$, then there is a proof in which the last rule applied is ($\wedge$-intro).

   Proof. (1) We prove the slightly more general fact that (1) holds for any type σ, where σ is a factor of σ', provided that the last step in the proof is not ($\wedge$-*intro*), by induction on the depth k of the derivation. Since σ is prime, the last rule in $\vdash_{\mathcal{S}} \Gamma \triangleright x : \sigma$ cannot be ($\wedge$-*intro*). If $\vdash_{\mathcal{S}} \Gamma \triangleright x : \sigma$ is not an axiom, then the last rule must be ($\wedge$-*elim*) and either $\vdash_{\mathcal{S}} \Gamma \triangleright x : \tau \wedge \sigma$ or $\vdash_{\mathcal{S}} \Gamma \triangleright x : \sigma \wedge \tau$ is a proof of depth $k - 1$. If the last step is ($\wedge$-*intro*), then we have a proof $\vdash_{\mathcal{S}} \Gamma \triangleright x : \sigma$ of depth $k - 2$, and we conclude by applying the induction hypothesis. Otherwise, by the induction hypothesis, there is some σ' such that either $\tau \wedge \sigma$ is a factor of σ' or $\sigma \wedge \tau$ is a factor of σ', and $x : \sigma' \in \Gamma$. In either case, σ is a prime factor of σ'.

   (2) We prove the slightly more general fact that (2) holds for any type σ, where σ is a factor of σ', provided that the last step in the proof is not ($\wedge$-*intro*), by induction on the depth k of the derivation. Since σ is prime, the last rule in in $\vdash_{\mathcal{S}} \Gamma \triangleright MN : \sigma$ cannot be ($\wedge$-*intro*). If the last rule in $\vdash_{\mathcal{S}} \Gamma \triangleright MN : \sigma$ is not (*application*), it must be ($\wedge$-*elim*), and either $\vdash_{\mathcal{S}} \Gamma \triangleright MN : \sigma \wedge \tau_1$ or $\vdash_{\mathcal{S}} \Gamma \triangleright MN : \tau_1 \wedge \sigma$ is a proof of depth $k - 1$. If the last step is ($\wedge$-*intro*), then we have a proof $\vdash_{\mathcal{S}} \Gamma \triangleright MN : \sigma$ of depth $k - 2$, and we conclude by applying the induction hypothesis. Otherwise, by the induction hypothesis, there is some σ' such that either $\sigma \wedge \tau_1$ is a factor of σ' and $\vdash_{\mathcal{S}} \Gamma \triangleright MN : \sigma'$, or $\tau_1 \wedge \sigma$ is a factor of σ' and $\vdash_{\mathcal{S}} \Gamma \triangleright MN : \sigma'$, and the last rule applied is (*application*). In either case, σ is a prime factor of σ'.

   (3) We prove that given a proof $\vdash_{\mathcal{S}} \Gamma \triangleright \lambda x.\, M : \sigma$ of depth k, then there is a proof of depth at most k in which the last rule is (*abstraction*), and given a proof $\vdash_{\mathcal{S}} \Gamma \triangleright \lambda x.\, M : \sigma_1 \wedge \sigma_2$ of depth k, then there is a proof of depth at most k in which the last rule applied is ($\wedge$-*intro*). Since σ is

prime, the last rule in $\vdash_{\mathcal{S}} \Gamma \triangleright \lambda x.\, M\!:\!\sigma$ cannot be ($\wedge$-$intro$). If the last rule in $\vdash_{\mathcal{S}} \Gamma \triangleright \lambda x.\, M\!:\!\sigma$ is not ($abstraction$), then it must be ($\wedge$-$elim$), and either $\vdash_{\mathcal{S}} \Gamma \triangleright \lambda x.\, M\!:\!\sigma \wedge \tau_1$ or $\vdash_{\mathcal{S}} \Gamma \triangleright \lambda x.\, M\!:\!\tau_1 \wedge \sigma$ is a proof of depth $k-1$. By the induction hypothesis, there is a proof of depth at most $k-1$ in which the last rule is ($\wedge$-$intro$). But then, we have a proof $\vdash_{\mathcal{S}} \Gamma \triangleright \lambda x.\, M\!:\!\sigma$ of depth at most $k-2$, and we conclude by applying the induction hypothesis.

If the last rule in $\vdash_{\mathcal{S}} \Gamma \triangleright \lambda x.\, M\!:\!\sigma_1 \wedge \sigma_2$ is not ($\wedge$-$intro$), then it must be ($\wedge$-$elim$). So, either $\vdash_{\mathcal{S}} \Gamma \triangleright \lambda x.\, M\!:\!\tau_1 \wedge (\sigma_1 \wedge \sigma_2)$ or $\vdash_{\mathcal{S}} \Gamma \triangleright \lambda x.\, M\!:\!(\sigma_1 \wedge \sigma_2) \wedge \tau_1$ is a proof of depth $k-1$. By the induction hypothesis, there is a proof of depth at most $k-1$ in which the last rule in ($\wedge$-$intro$). But then, we have a proof $\vdash_{\mathcal{S}} \Gamma \triangleright \lambda x.\, M\!:\!(\sigma_1 \wedge \sigma_2)$ of depth at most $k-2$, and we conclude by applying the induction hypothesis. $\square$

We can now prove that β-reduction preserves typing. This property is often known as "subject-reduction" property.

Lemma 6.3   Let $\mathcal{S} \in \{\mathcal{D}\Omega, \mathcal{D}\}$. If $\vdash_{\mathcal{S}} \Gamma \triangleright M\!:\!\sigma$ and $M \longrightarrow_\beta N$, then $\vdash_{\mathcal{S}} \Gamma \triangleright N\!:\!\sigma$. As a corollary, if $\vdash_{\mathcal{S}} \Gamma \triangleright M\!:\!\sigma$ and $M \stackrel{*}{\longrightarrow}_\beta N$, then $\vdash_{\mathcal{S}} \Gamma \triangleright N\!:\!\sigma$.

Proof. We proceed by induction on the typing derivation. Since $M \longrightarrow_\beta N$, the last rule used in the proof $\vdash_{\mathcal{S}} \Gamma \triangleright M\!:\!\sigma$ cannot be an axiom.

If the last rule is ($abstraction$), then $M = \lambda x.\, M_1$ and $N = \lambda x.\, N_1$, where $M_1 \longrightarrow_\beta N_1$, and we have

$$\vdash_{\mathcal{S}} \Gamma, x\!:\!\gamma \triangleright M_1\!:\!\delta$$

with $\gamma \rightarrow \delta = \sigma$. By the induction hypothesis, we have

$$\vdash_{\mathcal{S}} \Gamma, x\!:\!\gamma \triangleright N_1\!:\!\delta,$$

and thus $\vdash_{\mathcal{S}} \Gamma \triangleright \lambda x.\, N_1\!:\!\gamma \rightarrow \delta$.

If the last rule is ($application$), then $M = M_1 M_2$ and we have

$$\vdash_{\mathcal{S}} \Gamma \triangleright M_1\!:\!\gamma \rightarrow \sigma \quad \text{and} \quad \vdash_{\mathcal{S}} \Gamma \triangleright M_2\!:\!\gamma.$$

There are three cases depending on the reduction $M \longrightarrow_\beta N$.

If $M = M_1 M_2$ and $N = N_1 M_2$, where $M_1 \longrightarrow_\beta N_1$, then by the induction hypothesis, we have

$$\vdash_{\mathcal{S}} \Gamma \triangleright N_1\!:\!\gamma \rightarrow \sigma,$$

and thus, $\vdash_{\mathcal{S}} \Gamma \triangleright N_1 M_2\!:\!\sigma$.

If $M = M_1 M_2$ and $N = M_1 N_2$, where $M_2 \longrightarrow_\beta N_2$, then by the induction hypothesis, we have

$$\vdash_{\mathcal{S}} \Gamma \triangleright N_2\!:\!\gamma,$$

and thus, $\vdash_{\mathcal{S}} \Gamma \triangleright M_1 N_2\!:\!\sigma$.

If $M = (\lambda x.\, M_1) N_1$ and $N = M_1[N_1/x]$, since

$$\vdash_{\mathcal{S}} \Gamma \triangleright \lambda x.\, M_1\!:\!\gamma \rightarrow \sigma,$$

by lemma 6.2 (3), we have

$$\vdash_{\mathcal{S}} \Gamma, x{:}\gamma \triangleright M_1{:}\sigma.$$

Since we also have $\vdash_{\mathcal{S}} \Gamma \triangleright N_1{:}\gamma$, by lemma 6.1, we have

$$\vdash_{\mathcal{S}} \Gamma \triangleright M_1[N_1/x]{:}\sigma.$$

The cases where the last rule is ($\wedge$-*intro*) or ($\wedge$-*elim*) are trivial. The corollary is obtained by induction on the number of steps in the reduction $M \xrightarrow{\;*\;}_{\beta} N$. $\square$

We now show a crucial lemma about type-checking in the systems $\mathcal{D}\Omega$ and $\mathcal{D}$. It is in this lemma that the power of conjunctive types is really used. Again, we follow Krivine [13].

Lemma 6.4   *(1) If $\vdash_{\mathcal{D}\Omega} \Gamma \triangleright M[N/x]{:}\tau$, then there is a type σ such that $\vdash_{\mathcal{D}\Omega} \Gamma, x{:}\sigma \triangleright M{:}\tau$ and $\vdash_{\mathcal{D}\Omega} \Gamma \triangleright N{:}\sigma$.*

(2) If $\vdash_{\mathcal{D}} \Gamma \triangleright M[N/x]{:}\tau$ and $\vdash_{\mathcal{D}} \Gamma \triangleright N{:}\gamma$, then there is a type σ that $\vdash_{\mathcal{D}} \Gamma, x{:}\sigma \triangleright M{:}\tau$ and $\vdash_{\mathcal{D}} \Gamma \triangleright N{:}\sigma$.

Proof. We proceed by induction on $\langle |M|, |\tau| \rangle$, where $|M|$ is the size of M and $|\tau|$ is the size of τ.

(1) The case where $\tau = \omega$ is trivial, we take $\sigma = \omega$.

If $\tau = \tau_1 \wedge \tau_2$, since $\vdash_{\mathcal{D}\Omega} \Gamma \triangleright M[N/x]{:}\tau_1 \wedge \tau_2$, by ($\wedge$-*elim*), we have

$$\vdash_{\mathcal{D}\Omega} \Gamma \triangleright M[N/x]{:}\tau_1 \quad \text{and} \quad \vdash_{\mathcal{D}\Omega} \Gamma \triangleright M[N/x]{:}\tau_2.$$

Since $|\tau_1| < |\tau|$ and $|\tau_2| < |\tau|$, by the induction hypothesis, there are types σ_1 and σ_2 such that $\vdash_{\mathcal{D}\Omega} \Gamma, x{:}\sigma_1 \triangleright M{:}\tau_1$ and $\vdash_{\mathcal{D}\Omega} \Gamma \triangleright N{:}\sigma_1$, and $\vdash_{\mathcal{D}\Omega} \Gamma, x{:}\sigma_2 \triangleright M{:}\tau_2$ and $\vdash_{\mathcal{D}\Omega} \Gamma \triangleright N{:}\sigma_2$. Taking $\sigma = \sigma_1 \wedge \sigma_2$, by lemma 4.13, we have $\vdash_{\mathcal{D}\Omega} \Gamma, x{:}\sigma \triangleright M{:}\tau_1$ and $\vdash_{\mathcal{D}\Omega} \Gamma, x{:}\sigma \triangleright M{:}\tau_2$, and by ($\wedge$-*intro*), we get $\vdash_{\mathcal{D}\Omega} \Gamma, x{:}\sigma \triangleright M{:}\tau_1 \wedge \tau_2$. From $\vdash_{\mathcal{D}\Omega} \Gamma \triangleright N{:}\sigma_1$ and $\vdash_{\mathcal{D}\Omega} \Gamma \triangleright N{:}\sigma_2$, by ($\wedge$-*intro*), we get $\vdash_{\mathcal{D}\Omega} \Gamma \triangleright N{:}\sigma$.

From now on, we can assume that τ is prime.

If $M = x$, then $M[N/x] = x[N/x] = N$, and $\vdash_{\mathcal{D}\Omega} \Gamma \triangleright N{:}\tau$. Take $\sigma = \tau$, and then $\vdash_{\mathcal{D}\Omega} \Gamma, x{:}\tau \triangleright x{:}\tau$ is an axiom.

If $M = y$ with $y \neq x$, then $M[N/x] = y[N/x] = y$, and $\vdash_{\mathcal{D}\Omega} \Gamma \triangleright y{:}\tau$. Take $\sigma = \omega$, and then $\vdash_{\mathcal{D}\Omega} \Gamma, x{:}\omega \triangleright y{:}\tau$ and $\vdash_{\mathcal{D}\Omega} \Gamma \triangleright N{:}\omega$.

If $M = M_1 M_2$, then $M[N/x] = (M_1 M_2)[N/x] = M_1[N/x]M_2[N/x]$, and we have $\vdash_{\mathcal{D}\Omega} \Gamma \triangleright M_1[N/x]M_2[N/x]{:}\tau$ where τ is prime. By lemma 6.2 (2), there is a type τ' such that τ is a prime factor of τ', $\vdash_{\mathcal{D}\Omega} \Gamma \triangleright M_1[N/x]M_2[N/x]{:}\tau'$, and the last rule used in the proof is (*application*). Then, we have $\vdash_{\mathcal{D}\Omega} \Gamma \triangleright M_1[N/x]{:}\gamma \to \tau'$, and $\vdash_{\mathcal{D}\Omega} \Gamma \triangleright M_2[N/x]{:}\gamma$, for some type γ. Since $|M_1| < |M|$ and $|M_2| < |M|$, by the induction hypothesis, there are types σ_1 and σ_2 such that,

$$\vdash_{\mathcal{D}\Omega} \Gamma, x{:}\sigma_1 \triangleright M_1{:}\gamma \to \tau', \qquad \vdash_{\mathcal{D}\Omega} \Gamma \triangleright N{:}\sigma_1,$$

$$\vdash_{\mathcal{D}\Omega} \Gamma, x{:}\sigma_2 \triangleright M_2{:}\gamma, \quad \text{and} \quad \vdash_{\mathcal{D}\Omega} \Gamma \triangleright N{:}\sigma_2.$$

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Then, taking $\sigma = \sigma_1 \wedge \sigma_2$, by lemma 4.13, we have $\vdash_{\mathcal{D}\Omega} \Gamma, x{:}\sigma \rhd M_1{:}\gamma \rightarrow \tau'$ and $\vdash_{\mathcal{D}\Omega} \Gamma, x{:}\sigma \rhd M_2{:}\gamma$. Then, by ($application$), we have $\vdash_{\mathcal{D}\Omega} \Gamma, x{:}\sigma \rhd M_1 M_2{:}\tau'$. Since σ is a prime factor of τ', by application(s) of ($\wedge$-$elim$), we have $\vdash_{\mathcal{D}\Omega} \Gamma, x{:}\sigma \rhd M_1 M_2{:}\tau$. Since $\vdash_{\mathcal{D}\Omega} \Gamma \rhd N{:}\sigma_1$ and $\vdash_{\mathcal{D}\Omega} \Gamma \rhd N{:}\sigma_2$, by ($\wedge$-$intro$), we also have $\vdash_{\mathcal{D}\Omega} \Gamma \rhd N{:}\sigma$. This concludes this case.

If $M = \lambda y.\, M_1$, by suitable α-renaming, we can assume that $y \notin FV(N)$. Then, $M[N/x] = (\lambda y.\, M_1)[N/x] = \lambda y.\, M_1[N/x]$, and $\vdash_{\mathcal{D}\Omega} \Gamma \rhd \lambda y.\, M_1[N/x]{:}\tau$ where τ is prime. By lemma 6.2 (3), there is a proof $\vdash_{\mathcal{D}\Omega} \Gamma \rhd \lambda y.\, M_1[N/x]{:}\tau$ where the last rule used is ($abstraction$). Then, we have $\vdash_{\mathcal{D}\Omega} \Gamma, y{:}\gamma \rhd M_1[N/x]{:}\delta$ for some types γ and δ such that $\tau = \gamma \rightarrow \delta$. Since $|M_1| < |M|$, by the induction hypothesis, there is some type σ such that

$$\vdash_{\mathcal{D}\Omega} \Gamma, y{:}\gamma, x{:}\sigma \rhd M_1{:}\delta \quad \text{and} \quad \vdash_{\mathcal{D}\Omega} \Gamma, y{:}\gamma \rhd N{:}\sigma.$$

Since $y \notin FV(N)$, by lemma 6.1, we have $\vdash_{\mathcal{D}\Omega} \Gamma \rhd N{:}\sigma$. Since $\vdash_{\mathcal{D}\Omega} \Gamma, y{:}\gamma, x{:}\sigma \rhd M_1{:}\delta$, we have $\vdash_{\mathcal{D}\Omega} \Gamma, x{:}\sigma \rhd \lambda y.\, M_1{:}\gamma \rightarrow \delta$, that is, $\vdash_{\mathcal{D}\Omega} \Gamma, x{:}\sigma \rhd \lambda y.\, M_1{:}\tau$. This concludes the proof of (1).

(2) The proof is similar to that of (1), but we have to be careful not to use any type containing ω. A careful inspection reveals that this only happens when $\tau = \omega$, which is ruled out in system $\mathcal{D}$, or in the case where $M = y$ and $y \neq x$. But in the second case, since we assumed that $\vdash_{\mathcal{D}} \Gamma \rhd N{:}\gamma$, we can take $\sigma = \gamma$. $\square$

As a consequence of lemma 6.4 we obtain the following important lemma.

Lemma 6.5 (1) If $\vdash_{\mathcal{D}\Omega} \Gamma \rhd M[N/x]{:}\tau$, then $\vdash_{\mathcal{D}\Omega} \Gamma \rhd (\lambda x.\, M)N{:}\tau$.

(2) If $\vdash_{\mathcal{D}} \Gamma \rhd M[N/x]{:}\tau$ and $\vdash_{\mathcal{D}} \Gamma \rhd N{:}\gamma$, then $\vdash_{\mathcal{D}} \Gamma \rhd (\lambda x.\, M)N{:}\tau$.

Proof. (1) By lemma 6.4 (1), if $\vdash_{\mathcal{D}\Omega} \Gamma \rhd M[N/x]{:}\tau$, then there is a type σ such that

$$\vdash_{\mathcal{D}\Omega} \Gamma, x{:}\sigma \rhd M{:}\tau \quad \text{and} \quad \vdash_{\mathcal{D}\Omega} \Gamma \rhd N{:}\sigma.$$

Then, by ($abstraction$), we have $\vdash_{\mathcal{D}\Omega} \Gamma \rhd (\lambda x.\, M){:}\sigma \rightarrow \tau$, and since $\vdash_{\mathcal{D}\Omega} \Gamma \rhd N{:}\sigma$, by ($application$), we get

$$\vdash_{\mathcal{D}\Omega} \Gamma \rhd (\lambda x.\, M)N{:}\tau.$$

(2) By lemma 6.4 (2), if $\vdash_{\mathcal{D}} \Gamma \rhd M[N/x]{:}\tau$ and $\vdash_{\mathcal{D}} \Gamma \rhd N{:}\gamma$, then there is a type σ that $\vdash_{\mathcal{D}} \Gamma, x{:}\sigma \rhd M{:}\tau$ and $\vdash_{\mathcal{D}} \Gamma \rhd N{:}\sigma$. The rest of the proof is as in (1). $\square$

The following lemma generalizes lemma 6.5, and will be needed to prove that every strongly normalizing term is typable in system $\mathcal{D}$.

Lemma 6.6 (1) If $\vdash_{\mathcal{D}\Omega} \Gamma \rhd M[N/x]N_1 \ldots N_k{:}\tau$, then $\vdash_{\mathcal{D}\Omega} \Gamma \rhd ((\lambda x.\, M)N)N_1 \ldots N_k{:}\tau$.

(2) If $\vdash_{\mathcal{D}} \Gamma \rhd M[N/x]N_1 \ldots N_k{:}\tau$ and $\vdash_{\mathcal{D}} \Gamma \rhd N{:}\gamma$, then $\vdash_{\mathcal{D}} \Gamma \rhd ((\lambda x.\, M)N)N_1 \ldots N_k{:}\tau$.

Proof. We proceed by induction on $\langle k, |\tau| \rangle$.

(1) If $k = 0$, we conclude by lemma 6.5 (1). If $\tau = \tau_1 \wedge \tau_2$, by ($\wedge$-$elim$), we have

$$\vdash_{\mathcal{D}\Omega} \Gamma \rhd M[N/x]N_1 \ldots N_k{:}\tau_1 \quad \text{and} \quad \vdash_{\mathcal{D}\Omega} \Gamma \rhd M[N/x]N_1 \ldots N_k{:}\tau_2.$$

By the induction hypothesis, we have

$$\vdash_{\mathcal{D}\Omega} \Gamma \rhd ((\lambda x.\, M)N)N_1 \ldots N_k{:}\tau_1 \quad \text{and} \quad \vdash_{\mathcal{D}\Omega} \Gamma \rhd ((\lambda x.\, M)N)N_1 \ldots N_k{:}\tau_2,$$

and thus, $\vdash_{\mathcal{D}\Omega} \Gamma \triangleright ((\lambda x.\, M)N)N_1 \ldots N_k\!:\!\tau$.

We can now assume that τ is prime and $k \geq 1$. Since $\vdash_{\mathcal{D}\Omega} \Gamma \triangleright M[N/x]N_1 \ldots N_k\!:\!\tau$, by lemma 6.2 (2), there are types γ and τ' where τ is a prime factor of τ' such that,

$$\vdash_{\mathcal{D}\Omega} \Gamma \triangleright M[N/x]N_1 \ldots N_{k-1}\!:\!\gamma \to \tau' \quad \text{and} \quad \vdash_{\mathcal{D}\Omega} \Gamma \triangleright N_k\!:\!\gamma.$$

By the induction hypothesis, we have

$$\vdash_{\mathcal{D}\Omega} \Gamma \triangleright ((\lambda x.\, M)N)N_1 \ldots N_{k-1}\!:\!\gamma \to \tau',$$

and thus, $\vdash_{\mathcal{D}\Omega} \Gamma \triangleright ((\lambda x.\, M)N)N_1 \ldots N_k\!:\!\tau'$. Since τ is a prime factor of τ', by application(s) of $(\wedge\text{-}elim)$, we have $\vdash_{\mathcal{D}\Omega} \Gamma \triangleright ((\lambda x.\, M)N)N_1 \ldots N_k\!:\!\tau$.

(2) In the base case $k = 0$, we use lemma 6.5 (2). The rest of the proof is identical to that of (1). $\square$

The following lemma will be needed in showing that a term has a head-normal form iff it is solvable (see definition 6.10).

Lemma 6.7 *If the term $M = \lambda x.\, M_1$ or the term $M = M_1 N_1$ is typable in system $\mathcal{D}\Omega$ with a nontrivial type, then M_1 itself is typable in system $\mathcal{D}\Omega$ with a nontrivial type.*

Proof. Assume $\vdash_{\mathcal{D}\Omega} \Gamma \triangleright \lambda x.\, M_1\!:\!\sigma$ or $\vdash_{\mathcal{D}\Omega} \Gamma \triangleright M_1 N_1\!:\!\sigma$. We proceed by induction on the typing derivation. The last rule cannot be an axiom since the terms involved are not variables and $\sigma \neq \omega$.

If the last rule is (*abstraction*), then we must have

$$\vdash_{\mathcal{D}\Omega} \Gamma, x\!:\!\gamma \triangleright M_1\!:\!\delta,$$

with $\sigma = \gamma \to \delta$, and since σ is nontrivial, δ is nontrivial.

If the last rule is (*application*), then we must have

$$\vdash_{\mathcal{D}\Omega} \Gamma \triangleright M_1\!:\!\gamma \to \sigma \quad \text{and} \quad \vdash_{\mathcal{D}\Omega} \Gamma \triangleright N_1\!:\!\gamma.$$

Since σ is nontrivial, $\gamma \to \sigma$ is nontrivial.

If the last rule is ($\wedge$-*intro*), we have

$$\vdash_{\mathcal{D}\Omega} \Gamma \triangleright M\!:\!\sigma_1 \quad \text{and} \quad \vdash_{\mathcal{D}\Omega} \Gamma \triangleright M\!:\!\sigma_2,$$

and $\sigma = \sigma_1 \wedge \sigma_2$. Since σ is nontrivial, either σ_1 or σ_2 is nontrivial. The result follows from the induction hypothesis.

If the last rule is ($\wedge$-*elim*), we have

$$\vdash_{\mathcal{D}\Omega} \Gamma \triangleright M\!:\!\sigma_1 \wedge \sigma_2,$$

and either $\sigma = \sigma_1$ or $\sigma = \sigma_2$. Since σ is nontrivial, in either case, $\sigma_1 \wedge \sigma_2$ is nontrivial. The result follows from the induction hypothesis. $\square$

We can now prove the following fundamental theorem about type-checking in system $\mathcal{D}\Omega$. It is a dual of lemma 6.3, in the sense that it shows that in system $\mathcal{D}\Omega$, typing is preserved under reverse β-reduction. This theorem first proved by Coppo, Dezani, and Venneri [4], also appears in Krivine [13].

Theorem 6.8   *(1) If $\vdash_{\mathcal{D}\Omega} \Gamma \triangleright N\!:\!\tau$ and $M \longrightarrow_\beta N$, then $\vdash_{\mathcal{D}\Omega} \Gamma \triangleright M\!:\!\tau$.*

(2) If $\vdash_{\mathcal{D}\Omega} \Gamma \triangleright M\!:\!\tau$ and $M \stackrel{*}{\longleftrightarrow}_\beta N$, then $\vdash_{\mathcal{D}\Omega} \Gamma \triangleright N\!:\!\tau$.

Proof. Assume that $M \longrightarrow_\beta N$ and $\vdash_{\mathcal{D}\Omega} \Gamma \triangleright N\!:\!\tau$. We proceed by induction on $\langle |M|, |\tau| \rangle$, where $|M|$ is the size of M and $|\tau|$ is the size of τ.

(1) The case where $\tau = \omega$ is trivial.

If $\tau = \tau_1 \wedge \tau_2$, since $\vdash_{\mathcal{D}\Omega} \Gamma \triangleright N\!:\!\tau_1 \wedge \tau_2$, by ($\wedge$-*elim*), we have

$$\vdash_{\mathcal{D}\Omega} \Gamma \triangleright N\!:\!\tau_1 \quad \text{and} \quad \vdash_{\mathcal{D}\Omega} \Gamma \triangleright N\!:\!\tau_2.$$

Since $|\tau_1| < |\tau|$ and $|\tau_2| < |\tau|$, by the induction hypothesis,

$$\vdash_{\mathcal{D}\Omega} \Gamma \triangleright M\!:\!\tau_1 \quad \text{and} \quad \vdash_{\mathcal{D}\Omega} \Gamma \triangleright M\!:\!\tau_2,$$

and by ($\wedge$-*intro*), we have $\vdash_{\mathcal{D}\Omega} \Gamma \triangleright M\!:\!\tau_1 \wedge \tau_2$.

Thus, from now on, we can assume that τ is prime. The case where M is a variable is impossible.

If $M = \lambda x.\, M_1$, then we must have $N = \lambda x.\, N_1$ where $M_1 \longrightarrow_\beta N_1$, and $\vdash_{\mathcal{D}\Omega} \Gamma \triangleright \lambda x.\, N_1\!:\!\tau$ where τ is prime. By lemma 6.2 (3), there are some types γ and δ such that $\tau = \gamma \to \delta$, and we have

$$\vdash_{\mathcal{D}\Omega} \Gamma, x\!:\!\gamma \triangleright N_1\!:\!\delta.$$

Since $|M_1| < |M|$, by the induction hypothesis, we have

$$\vdash_{\mathcal{D}\Omega} \Gamma, x\!:\!\gamma \triangleright M_1\!:\!\delta,$$

and by (*abstraction*), we get $\vdash_{\mathcal{D}\Omega} \Gamma \triangleright \lambda x.\, M_1\!:\!\gamma \to \delta$, that is, $\vdash_{\mathcal{D}\Omega} \Gamma \triangleright M\!:\!\tau$.

If $M = M_1 M_2$, there are three cases. Either $N = N_1 M_2$ where $M_1 \longrightarrow_\beta N_1$, or $N = M_1 N_2$ where $M_2 \longrightarrow_\beta N_2$, or $M = (\lambda x.\, M_1) N_1$ and $N = M_1[N_1/x]$.

If $N = N_1 M_2$ where $M_1 \longrightarrow_\beta N_1$, we have $\vdash_{\mathcal{D}\Omega} \Gamma \triangleright N_1 M_2\!:\!\tau$ where τ is prime. By lemma 6.2 (2), there are some types γ and τ' where τ is a prime factor of τ' such that

$$\vdash_{\mathcal{D}\Omega} \Gamma \triangleright N_1\!:\!\gamma \to \tau' \quad \text{and} \quad \vdash_{\mathcal{D}\Omega} \Gamma \triangleright M_2\!:\!\gamma.$$

Since $|M_1| < |M|$, by the induction hypothesis, we have

$$\vdash_{\mathcal{D}\Omega} \Gamma \triangleright M_1\!:\!\gamma \to \tau',$$

and since $\vdash_{\mathcal{D}\Omega} \Gamma \triangleright M_2\!:\!\gamma$, we get

$$\vdash_{\mathcal{D}\Omega} \Gamma \triangleright M_1 M_2\!:\!\tau'.$$

Since τ is a prime factor of τ', by application(s) of ($\wedge$-*elim*), we get

$$\vdash_{\mathcal{D}\Omega} \Gamma \triangleright M_1 M_2\!:\!\tau.$$

The case where $N = M_1 N_2$ and $M_2 \longrightarrow_\beta N_2$ is similar to the previous case.

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If $M = (\lambda x.\, M_1)N_1$ and $N = M_1[N_1/x]$, since $\vdash_{\mathcal{D}\Omega} \Gamma \triangleright M_1[N_1/x]\colon\tau$, by lemma 6.5 (1), we have

$$\vdash_{\mathcal{D}\Omega} \Gamma \triangleright (\lambda x.\, M_1)N_1\colon\tau.$$

(2) is obtained by induction on the number of steps in $M \stackrel{*}{\longleftrightarrow}_\beta N$ using lemma 6.3 and theorem 6.8 (1). $\square$

Theorem 6.8 fails for system $\mathcal{D}$, even for terms M that type-check in $\mathcal{D}$, as shown next. Let $M = \lambda y.\,((\lambda x.\, y)(yy))$. We have $M \longrightarrow_\beta N = \lambda y.\, y$, and clearly $N = \lambda y.\, y$ type-checks in $\mathcal{D}$ with type $\tau \to \tau$, where τ is a base type. However, we prove that M **does not** type-check in $\mathcal{D}$ with the type $\tau \to \tau$, even though M type-checks in $\mathcal{D}$ with type $\sigma \wedge (\sigma \to \tau) \to \sigma \wedge (\sigma \to \tau)$.

Indeed, if $\vdash_{\mathcal{D}} \triangleright \lambda y.\,((\lambda x.\, y)(yy))\colon\tau \to \tau$, by lemma 6.2 (3), we must have

$$\vdash_{\mathcal{D}} y\colon\tau \triangleright (\lambda x.\, y)(yy)\colon\tau.$$

Since τ is prime, by lemma 6.2 (2), we must have

$$\vdash_{\mathcal{D}} y\colon\tau \triangleright (yy)\colon\sigma$$

for some type σ. Now, σ is not necessarily prime, but since σ is a type in $\mathcal{D}$, σ is a conjunction of prime types different from ω, and thus, by application(s) of ($\wedge$-$elim$), we can assume that $\vdash_{\mathcal{D}} y\colon\tau \triangleright (yy)\colon\sigma$ where σ is prime. Again, by lemma 6.2 (3), we must have

$$\vdash_{\mathcal{D}} y\colon\tau \triangleright y\colon\gamma \to \sigma'$$

where σ is a prime factor of σ'. But now, $\gamma \to \sigma'$ is not a prime factor of τ since τ is a base type, which contradicts lemma 6.2 (1). Thus, M does not type-check in $\mathcal{D}$ with the type $\tau \to \tau$.

We now prove that every strongly normalizing term M is typable in system $\mathcal{D}$. This theorem first proved by Pottinger [17], also appears in Krivine [13].

Lemma 6.9   *If a term M is strongly normalizing, then it is typable in system $\mathcal{D}$.*

Proof. We proceed by induction on $\langle d(M), |M| \rangle$, where $d(M)$ is the depth of the reduction tree from M and $|M|$ is the size of M. There are two cases, the first one being the case where M is in head-normal form, the second one where it is not.

If M is in head-normal form, it is of the form $M = \lambda x_1 \ldots \lambda x_m.\, yN_1 \ldots N_k$, and the proof is similar to that of lemma 3.13. Since $|N_i| < |M|$ and $d(N_i) \le d(M)$, by the induction hypothesis, each N_i is typable in $\mathcal{D}$, and by lemma 4.14, we can assume that they are typable in the same context, that is,

$$\vdash_{\mathcal{D}} \Gamma, x_1\colon\sigma_1, \ldots, x_m\colon\sigma_m, y\colon\gamma \triangleright N_i\colon\tau_i,$$

if $y \ne x_i$ for all i, or

$$\vdash_{\mathcal{D}} \Gamma, x_1\colon\sigma_1, \ldots, x_m\colon\sigma_m \triangleright N_i\colon\tau_i,$$

if $y = x_i$. Now, letting

$$\sigma = \gamma \wedge (\tau_1 \to \ldots \to \tau_k \to \delta),$$

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for any base type δ, with $\gamma = \sigma_i$ if $y = x_i$, it is immediate (using lemma 4.13) that we have

$$\vdash_{\mathcal{D}} \Gamma, y{:}\sigma \rhd \lambda x_1 \ldots \lambda x_m. \, yN_1 \ldots N_k{:}\tau,$$

with $\tau = (\sigma_1 \rightarrow \ldots \rightarrow \sigma_m \rightarrow \delta)$ if $y \neq x_i$ for all i, or

$$\vdash_{\mathcal{D}} \Gamma \rhd \lambda x_1 \ldots \lambda x_m. \, yN_1 \ldots N_k{:}\tau,$$

with $\tau = (\sigma_1 \rightarrow \ldots \rightarrow \sigma_m \rightarrow \delta)$ and $\sigma_i = \sigma$ if $y = x_i$.

If $M = \lambda x_1 \ldots \lambda x_m. \, ((\lambda y. \, P)Q)N_1 \ldots N_k$ has head-redex $(\lambda y. \, P)Q$, then

$$N = \lambda x_1 \ldots \lambda x_m. \, P[Q/x]N_1 \ldots N_k$$

is such that $d(N) < d(M)$, and clearly we also have $d(P[Q/x]N_1 \ldots N_k) \leq d(N)$ and $d(Q) \leq d(N)$. By the induction hypothesis,

$$\vdash_{\mathcal{D}} \Gamma', x_1{:}\sigma'_1, \ldots, x_m{:}\sigma'_m \rhd P[Q/x]N_1 \ldots N_k{:}\delta,$$

and

$$\vdash_{\mathcal{D}} \Gamma'', x_1{:}\sigma''_1, \ldots, x_m{:}\sigma''_m \rhd Q{:}\gamma,$$

and by lemma 4.14, letting $\sigma_i = \sigma'_i \wedge \sigma''_i$, there is a context Γ such that

$$\vdash_{\mathcal{D}} \Gamma, x_1{:}\sigma_1, \ldots, x_m{:}\sigma_m \rhd P[Q/x]N_1 \ldots N_k{:}\delta,$$

and

$$\vdash_{\mathcal{D}} \Gamma, x_1{:}\sigma_1, \ldots, x_m{:}\sigma_m \rhd Q{:}\gamma.$$

By lemma 6.6 (2), we have

$$\vdash_{\mathcal{D}} \Gamma, x_1{:}\sigma_1, \ldots, x_m{:}\sigma_m \rhd ((\lambda y. \, P)Q)N_1 \ldots N_k{:}\delta,$$

and thus,

$$\vdash_{\mathcal{D}} \Gamma \rhd \lambda x_1 \ldots \lambda x_m. \, ((\lambda y. \, P)Q)N_1 \ldots N_k{:}\tau,$$

with $\tau = (\sigma_1 \rightarrow \ldots \rightarrow \sigma_m \rightarrow \delta)$. $\square$

We are now ready to prove the fundamental theorems characterizing the terms that have head-normal forms, the terms that are normalizable, and the terms that are strongly normalizing, in terms of typability in the systems $\mathcal{D}\Omega$ and $\mathcal{D}$. These theorems are proved in Krivine [13]. Before we do so, we define the notion of a solvable term, a notion that turns out to be equivalent to the property of having a head-normal form (a result due to Wadsworth).

Definition 6.10 A closed term M is *solvable* iff there are terms $N_1, \ldots, N_k$, where $k \geq 0$, such that, $MN_1 \ldots N_k \overset{*}{\longleftrightarrow}_\beta \lambda x. \, x$. A nonclosed term M is solvable iff its closure is solvable.

If a term M is not closed and $FV(M) = \{x_1, \ldots, x_m\}$, its closure is $\lambda x_1 \ldots \lambda x_m. \, M$, and M solvable means that there are terms $N_1, \ldots, N_k$ such that

$$(\lambda x_1 \ldots \lambda x_m. \, M)N_1 \ldots N_k \overset{*}{\longleftrightarrow}_\beta \lambda x. \, x.$$

Thus, if $k < m$, this means that

$$\lambda x_{k+1} \ldots \lambda x_m. M[N_1/x_1, \ldots, N_k/x_k] \overset{*}{\longleftrightarrow}_\beta \lambda x. x,$$

and if $k \geq m$, this means that

$$M[N_1/x_1, \ldots, N_m/x_m]N_{m+1} \ldots N_k \overset{*}{\longleftrightarrow}_\beta \lambda x. x.$$

Thus, solvability can also be defined by saying that a term (closed or open) is solvable iff there is a substitution φ for some of the free variables of M and some terms $N_1, \ldots, N_k$ such that, $M[\varphi]N_1 \ldots N_k \overset{*}{\longleftrightarrow}_\beta \lambda x. x$.

It is also easy to see that M is solvable iff for every term Q, there is a substitution φ for some of the free variables in M and some terms $N_1, \ldots, N_k$ such that, $M[\varphi]N_1 \ldots N_k \overset{*}{\longleftrightarrow}_\beta Q$. Indeed, this second definition implies the first by picking $Q = \lambda x. x$. Conversely, if $M[\varphi]N_1 \ldots N_k \overset{*}{\longleftrightarrow}_\beta \lambda x. x$, then $M[\varphi]N_1 \ldots N_k Q \overset{*}{\longleftrightarrow}_\beta Q$. Finally, we prove our three major theorems. A version of the next theorem was first obtained by Coppo, Dezani, and Venneri [4].

Theorem 6.11 *For any term M of the (untyped) λ-calculus, the following properties are equivalent.*

(1) M is solvable;

(2) M has a head-normal form (i.e., there is some head-normal form N such that $M \overset{*}{\longleftrightarrow}_\beta N$).

(3) M is typable in system $\mathcal{D}\Omega$ with a nontrivial type;

(4) Every quasi-head reduction from M is finite. In particular, the head-reduction from M is finite.

Proof. $(1) \Rightarrow (3)$. If M is solvable, then there are terms $N_1, \ldots, N_k$ such that

$$(\lambda x_1 \ldots \lambda x_m. M)N_1 \ldots N_k \overset{*}{\longleftrightarrow}_\beta \lambda x. x,$$

where $m = 0$ if M is closed. Since $\lambda x. x$ is typable with the type $\tau \to \tau$ where τ is any nontrivial type, by theorem 6.8, $(\lambda x_1 \ldots \lambda x_m. M)N_1 \ldots N_k$ is also typable in $\mathcal{D}\Omega$ with the nontrivial type $\tau \to \tau$. Then, by application(s) of lemma 6.7, M itself is typable in $\mathcal{D}\Omega$ with a nontrivial type.

$(3) \Rightarrow (4)$. This follows from theorem 3.11.

$(4) \Rightarrow (2)$. This is trivial.

$(2) \Rightarrow (1)$. If M is equivalent to a head-normal form, clearly its closure is equivalent to a head-normal form, and thus we assume that M is closed. By assumption,

$$M \overset{*}{\longleftrightarrow}_\beta \lambda x_1 \ldots \lambda x_m. x_i Q_1 \ldots Q_k,$$

where $\lambda x_1 \ldots \lambda x_m. x_i Q_1 \ldots Q_k$ is a closed head-normal form. Let

$$N_i = \lambda y_1 \ldots \lambda y_k \lambda z. z,$$

and N_j any arbitrary term for $j \neq i$, $1 \leq j \leq m$. Then, it is immediate that $M N_1 \ldots N_m \overset{*}{\longleftrightarrow}_\beta \lambda z.z$, and M is solvable. $\square$

It should be noted that the implication $(2) \Rightarrow (3)$ follows directly from lemma 3.13 and theorem 6.8, and no detour via the solvable terms is necessary. Furthermore, this implication shows that every head-normalizable term is typable in $\mathcal{D}\Omega$ with a nontrivial type of a rather special kind (since the types arising in lemma 3.13 are quite special). Next we consider normalizable terms. A version of the next theorem was first obtained by Coppo, Dezani, and Venneri [4].

Theorem 6.12   *For any term M of the (untyped) λ-calculus, the following properties are equivalent.*

(1) M is normalizable;

(2) There exist a context Γ and a type σ, both ω-free, such that $\vdash_{\mathcal{D}\Omega} \Gamma \triangleright M : \sigma$;

(3) Every quasi-leftmost reduction from M is finite. In particular, the leftmost reduction from M is finite.

Proof. $(1) \Rightarrow (2)$. This follows from lemma 4.15 and theorem 6.8.

$(2) \Rightarrow (3)$. This follows from theorem 4.11.

$(3) \Rightarrow (1)$. This is trivial. $\square$

The implication $(1) \Rightarrow (2)$ shows that every normalizable term is typable in $\mathcal{D}\Omega$ with an ω-free (context and) type of a rather special kind (since the types arising in lemma 4.15 are quite special). Finally, we consider strongly normalizing terms. A version of the next theorem was first obtained by Pottinger [17].

Theorem 6.13   *For any term M of the (untyped) λ-calculus, the following properties are equivalent.*

(1) M is strongly normalizing;

(2) M is typable in system $\mathcal{D}$.

Proof. $(1) \Rightarrow (2)$. This follows from lemma 6.9.

$(2) \Rightarrow (1)$. This follows from theorem 5.11. $\square$

Other interesting results can be obtained, for example the finite developments theorem (see Krivine [13]). In the next section, we characterize the terms that have a weak head-normal form. This result appears to be new.

7 $\mathcal{P}$-Candidates for Weakly Head-Normalizing λ-Terms

In this section, we generalize theorem 3.9 and theorem 6.11 to the terms that are weakly head-normalizable. First, we need to adapt definition 2.3 so that our results apply to weakly head-normalizable λ-terms. We thank Mariangiola Dezani for suggesting a simplification in the definition of a weakly nontrivial type. The difference between head-normalizable λ-terms and weakly head-normalizable λ-terms is that **any** λ-abstraction $\lambda x. M$ is considered a weak head-normal form, even if M has a head redex.

Definition 7.1 A type σ is ω-*free* iff ω does not occur in σ. A type is *weakly nontrivial* iff either σ is a base type and $\sigma \neq \omega$, or $\sigma = \gamma \to \tau$ where τ is weakly nontrivial and γ is arbitrary, or $\sigma = \sigma_1 \wedge \sigma_2$ where σ_1 or σ_2 is weakly nontrivial, or $\sigma = \omega \to \omega$. A type is *weakly trivial* iff it is not weakly nontrivial.[3]

Definition 3.1 remains unchanged, as well as definition 3.2, but we repeat definition 3.2 for convenience.

Definition 7.2 Properties (P1)-(P3s) are defined as follows:

(P1) $x \in \mathcal{P}$, for every variable x.

(P2) If $M \in \mathcal{P}$ and $M \longrightarrow_\beta N$, then $N \in \mathcal{P}$.

(P3s) If M is simple, $M \in \mathcal{P}$, $N \in \Lambda$, and $(\lambda x.\, M')N \in \mathcal{P}$ whenever $M \stackrel{+}{\longrightarrow}_\beta \lambda x.\, M'$, then $MN \in \mathcal{P}$.

From now on, we only consider sets $\mathcal{P}$ satisfying conditions (P1)-(P3s) of definition 7.2. Definition 3.3 remains unchanged, as well as the remarks on stubborn terms following this definition. However, we need to modify definition 3.4. Given a set $\mathcal{P}$, for every type σ, we define $[\![\sigma]\!] \subseteq \Lambda$ as follows.

Definition 7.3 The sets $[\![\sigma]\!]$ are defined as follows:

$$[\![\sigma]\!] = \mathcal{P}, \qquad \text{where } \sigma \neq \omega \text{ is a base type,}$$
$$[\![\sigma]\!] = \Lambda, \qquad \text{where } \sigma \text{ is a weakly trivial type,}$$
$$[\![\sigma \to \tau]\!] = \{M \mid M \in \mathcal{P}, \text{ and for all } N, \text{ if } N \in [\![\sigma]\!] \text{ then } MN \in [\![\tau]\!]\},$$
$$\text{where } \sigma \to \tau \text{ is weakly nontrivial,}$$
$$[\![\sigma \wedge \tau]\!] = [\![\sigma]\!] \cap [\![\tau]\!],$$
$$\text{where } \sigma \wedge \tau \text{ is weakly nontrivial.}$$

By definition 7.1, a type is weakly trivial if either it is ω, or it is of the form $\sigma \to \tau$ where τ is weakly trivial (except for $\omega \to \omega$), or it is of the form $\sigma \wedge \tau$ where both σ and τ are weakly trivial. We could have defined $[\![\sigma]\!]$ by changing the second clause to $[\![\omega]\!] = \Lambda$, and by dropping the conditions $\sigma \to \tau$ weakly nontrivial and $\sigma \wedge \tau$ weakly nontrivial. However, it would no longer be true that $[\![\sigma]\!] = \Lambda$ for every weakly trivial type, and this would be a serious obstacle to the proof of lemma 7.6. The following lemma shows that the property of being a $\mathcal{P}$-candidate is an inductive invariant.

Lemma 7.4   *If $\mathcal{P}$ is a set satisfying conditions (P1)-(P3s), then the following properties hold for every type σ: (1) $[\![\sigma]\!]$ contains all stubborn terms in $\mathcal{P}$ (and in particular, every variable); (2) $[\![\sigma]\!]$ satisfies (S2) and (S3); (3) If σ is weakly nontrivial, then $[\![\sigma]\!]$ also satisfies (S1), and thus it is a $\mathcal{P}$-candidate.*

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[3]In an earlier version, we were also considering types $\sigma \to \omega$ where σ is ω-free, among the weakly nontrivial types. However, as suggested by Mariangiola Dezani, it is simpler to use the type $\omega \to \omega$.

Proof. We proceed by induction on types. If σ is a base type, then by definition $[\![\sigma]\!] = \mathcal{P}$ if $\sigma \neq \omega$, and $[\![\omega]\!] = \Lambda$. Then, (1) and (2) are clear by (P1) and by (P2) (note that (S3) is trivial). If $\sigma \neq \omega$, then (S1) is trivial since $[\![\sigma]\!] = \mathcal{P}$.

We now consider the induction step.

(3) We prove that (S1) holds for weakly nontrivial types. If $\sigma \to \tau$ is weakly nontrivial, then there are two cases: (a) the type τ is weakly nontrivial, and by the definition of $[\![\sigma \to \tau]\!]$, we have $[\![\sigma \to \tau]\!] \subseteq \mathcal{P}$. (b) $\sigma = \omega \to \omega$. In this case, since $[\![\omega]\!] = \Lambda$, it is clear from definition 7.3 that $[\![\omega \to \omega]\!] = \mathcal{P}$.

If $\sigma = \sigma_1 \wedge \sigma_2$ is weakly nontrivial, then σ_1 or σ_2 is weakly nontrivial. Assume σ_1 is weakly nontrivial, the case where σ_2 is weakly nontrivial being similar. By the induction hypothesis, $[\![\sigma_1]\!] \subseteq \mathcal{P}$, and since $[\![\sigma_1 \wedge \sigma_2]\!] = [\![\sigma_1]\!] \cap [\![\sigma_2]\!]$, it is clear that $[\![\sigma_1 \wedge \sigma_2]\!] \subseteq \mathcal{P}$.

The verification of (1) and (2) is obvious for weakly trivial types, since in this case, $[\![\sigma]\!] = \Lambda$. Thus, in the rest of this proof, we assume that we are considering weakly nontrivial types.

(1) Given a type $\sigma \to \tau$, by the induction hypothesis, $[\![\tau]\!]$ contains all the stubborn terms in $\mathcal{P}$. Let $M \in \mathcal{P}$ be a stubborn term. Given any $N \in [\![\sigma]\!]$, obviously, $N \in \Lambda$. Since we have shown that MN is a stubborn term in $\mathcal{P}$ when $M \in \mathcal{P}$ is stubborn and N is arbitrary, we have $MN \in [\![\tau]\!]$. Thus, $M \in [\![\sigma \to \tau]\!]$. If $\sigma = \sigma_1 \wedge \sigma_2$, by the induction hypothesis, all stubborn terms in $\mathcal{P}$ are in $[\![\sigma_1]\!]$ and in $[\![\sigma_2]\!]$, and thus in $[\![\sigma_1 \wedge \sigma_2]\!] = [\![\sigma_1]\!] \cap [\![\sigma_2]\!]$.

(2) We prove (S2) and (S3).

(S2). Let $M \in [\![\sigma \to \tau]\!]$ and assume that $M \longrightarrow_\beta M'$. Since $M \in \mathcal{P}$ by (S1), we have $M' \in \mathcal{P}$ by (P2). For any $N \in [\![\sigma]\!]$, since $M \in [\![\sigma \to \tau]\!]$ we have $MN \in [\![\tau]\!]$, and since $M \longrightarrow_\beta M'$ we have $MN \longrightarrow_\beta M'N$. Then, applying the induction hypothesis at type τ, (S2) holds for $[\![\tau]\!]$, and thus $M'N \in [\![\tau]\!]$. Thus, we have shown that $M' \in \mathcal{P}$ and that if $N \in [\![\sigma]\!]$, then $M'N \in [\![\tau]\!]$. By the definition of $[\![\sigma \to \tau]\!]$, this shows that $M' \in [\![\sigma \to \tau]\!]$, and (S2) holds at type $\sigma \to \tau$.

If $\sigma = \sigma_1 \wedge \sigma_2$, by the induction hypothesis, (S2) holds for $[\![\sigma_1]\!]$ and $[\![\sigma_2]\!]$, and thus for $[\![\sigma_1 \wedge \sigma_2]\!] = [\![\sigma_1]\!] \cap [\![\sigma_2]\!]$.

(S3). Let $M \in \mathcal{P}$ be a simple term, and assume that $\lambda x.M' \in [\![\sigma \to \tau]\!]$ whenever $M \overset{+}{\longrightarrow}_\beta \lambda x.M'$. If $\sigma \to \tau = \omega \to \omega$, then we saw that $[\![\omega \to \omega]\!] = \mathcal{P}$. In this case, (S3) is trivial. Thus, we now assume that $\sigma \to \tau$ is weakly nontrivial and not $\omega \to \omega$.

We prove that for every N, if $N \in [\![\sigma]\!]$, then $MN \in [\![\tau]\!]$. The case where M is stubborn has already been covered in (1). Assume that M is not stubborn. First, we prove that $MN \in \mathcal{P}$, and for this, we use (P3s). If $M \overset{+}{\longrightarrow}_\beta \lambda x.M'$, then by assumption, $\lambda x.M' \in [\![\sigma \to \tau]\!]$, and for any $N \in [\![\sigma]\!]$, we have $(\lambda x.M')N \in [\![\tau]\!]$. Recall that we assumed that $\sigma \to \tau$ is weakly nontrivial and not $\omega \to \omega$. This implies that τ is weakly nontrivial. Then, by (S1), $(\lambda x.M')N \in \mathcal{P}$, and by (P3s), we have $MN \in \mathcal{P}$. Now, there are two cases.

If τ is a base type, then $[\![\tau]\!] = \mathcal{P}$ since $\tau \neq \omega$, and $MN \in [\![\tau]\!]$ (since $MN \in \mathcal{P}$).

If τ is not a base type, the term MN is simple. Thus, we prove that $MN \in [\![\tau]\!]$ using (S3) (which by induction, holds at type τ). The case where MN is stubborn is trivial. Otherwise,

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observe that if $MN \xrightarrow{+}_\beta Q$, where $Q = \lambda y. P$ is an I-term, then the reduction is necessarily of the form

$$MN \xrightarrow{+}_\beta (\lambda x. M')N' \longrightarrow_\beta M'[N'/x] \xrightarrow{*}_\beta Q,$$

where $M \xrightarrow{+}_\beta \lambda x. M'$ and $N \xrightarrow{*}_\beta N'$. Since by assumption, $\lambda x. M' \in [\![\sigma \to \tau]\!]$ whenever $M \xrightarrow{+}_\beta \lambda x. M'$, and by the induction hypothesis applied at type σ, by (S2), $N' \in [\![\sigma]\!]$, we conclude that $(\lambda x{:}\sigma. M')N' \in [\![\tau]\!]$. By the induction hypothesis applied at type τ, by (S2), we have $Q \in [\![\tau]\!]$, and by (S3), we have $MN \in [\![\tau]\!]$.

Since $M \in \mathcal{P}$ and $MN \in [\![\tau]\!]$ whenever $N \in [\![\sigma]\!]$, we conclude that $M \in [\![\sigma \to \tau]\!]$. $\square$

For the proof of the next lemma, we need to add two new conditions (P4w) and (P5n) to (P1)-(P3s).

Definition 7.5 Properties (P4w) and (P5n) are defined as follows:

(P4w) If $M \in \Lambda$, then $\lambda x. M \in \mathcal{P}$.

(P5n) If $M[N/x] \in \mathcal{P}$, then $(\lambda x. M)N \in \mathcal{P}$.

Note that by (P4w), terms of the form $\lambda x. M$ are automatically in $\mathcal{P}$, no matter what M is.

Lemma 7.6 *If $\mathcal{P}$ is a set satisfying conditions (P1)-(P5n), and $M[N/x] \in [\![\tau]\!]$ for every $N \in \Lambda$, then $\lambda x. M \in [\![\sigma \to \tau]\!]$.*

Proof. The lemma is obvious if $\sigma \to \tau$ is weakly trivial, since in this case, $[\![\sigma \to \tau]\!] = \Lambda$. If $\sigma \to \tau = \omega \to \omega$, by (P4w), $\lambda x. M \in \mathcal{P}$, and since $[\![\omega \to \omega]\!] = \mathcal{P}$, the result holds. Thus, in the rest of this proof, we assume that $\sigma \to \tau$ is weakly nontrivial and not $\omega \to \omega$. This implies that τ is weakly nontrivial.

We prove that for every every N, if $N \in [\![\sigma]\!]$, then $(\lambda x. M)N \in [\![\tau]\!]$. We will need the fact that the sets of the form $[\![\sigma]\!]$ have the properties (S1)-(S3), but this follows from lemma 7.4, since (P1)-(P3s) hold. By (P4w), we have $\lambda x. M \in \mathcal{P}$.

Next, we prove that for every every N, if $N \in [\![\sigma]\!]$, then $(\lambda x. M)N \in [\![\tau]\!]$. Let us assume that $N \in [\![\sigma]\!]$. Then, by the assumption of lemma 7.6, $M[N/x] \in [\![\tau]\!]$. Since τ is weakly nontrivial, by (S1), we have $M[N/x] \in \mathcal{P}$. By (P5n), we have $(\lambda x. M)N \in \mathcal{P}$. The rest of the proof is identical to that of lemma 3.7. $\square$

Lemma 7.7 *If $\mathcal{P}$ is a set satisfying conditions (P1)-(P5n), then for every term $M \in \Lambda_\sigma$, for every substitution φ such that $\varphi(y) \in [\![\gamma]\!]$ for every $y{:}\gamma \in FV(M)$, we have $M[\varphi] \in [\![\sigma]\!]$.*

Proof. We proceed by induction on the proof $\vdash_{\mathcal{D}\Omega} \Gamma \triangleright M{:}\sigma$. The lemma is obvious if σ is a weakly trivial type, since in this case, $[\![\sigma]\!] = \Lambda$. Thus, in the rest of this proof, we assume that we are considering weakly nontrivial types. The rest of the proof is identical to that of lemma 3.8, with "nontrivial" replaced by "weakly nontrivial". $\square$

Theorem 7.8 *If $\mathcal{P}$ is a set of λ-terms satisfying conditions (P1)-(P5n), then $\Lambda_\sigma \subseteq \mathcal{P}$ for every weakly nontrivial type σ (in other words, every term typable in $\mathcal{D}\Omega$ with a weakly nontrivial type satisfies the unary predicate defined by $\mathcal{P}$).*

Proof. Apply lemma 7.7 to every term M in Λ_σ and to the identity substitution, which is legitimate since $x \in [\![\sigma]\!]$ for every variable of type σ (by lemma 7.4). Thus, $M \in [\![\sigma]\!]$ for every term in Λ_σ, that is $\Lambda_\sigma \subseteq [\![\sigma]\!]$. Finally, by lemma 7.4, if σ is weakly nontrivial, (S1) holds for $[\![\sigma]\!]$, that is $\Lambda_\sigma \subseteq [\![\sigma]\!] \subseteq \mathcal{P}$. $\square$

As a corollary of theorem 7.8, we show that if a term M is typable in $\mathcal{D}\Omega$ with a weakly nontrivial type, then the weak head reduction from M is finite (and so, M has a weak head-normal form).

Definition 7.9 Given a term $M = ((\lambda y.P)Q)N_1 \ldots N_k$, where $m \geq 0$ and $k \geq 0$, the term $(\lambda y.P)Q$ is the *weak head redex* of M. A *weak head reduction* is a reduction sequence in which every step reduces the weak head redex. A *weak quasi-head reduction* is a (finite or infinite) reduction sequence $s = \langle M_0, M_1, \ldots, M_i, \ldots \rangle$ such that, for every $i \geq 0$, if M_i is not the last term in the sequence s, there is some $j \geq i$ such that $M_j \longrightarrow_\beta M_{j+1}$ is a weak head-reduction step. A term is in *weak head-normal form* iff it has no weak head redex, that is, either it is a λ-abstraction $\lambda x. M_1$, or it is of the form $yN_1 \ldots N_k$, where $k \geq 0$. The variable y is called the *head variable*. A term is *weak head-normalizable* iff the weak head reduction from M is finite.

Note that the last step in a finite weak quasi-head reduction is necessarily a weak head-reduction step. Also, any suffix of a weak quasi-head reduction is a weak quasi-head reduction. The main advantage of weak quasi-head reductions over weak head-reductions is that (P2) obviously holds for terms for which every weak quasi-head reduction is finite.

Theorem 7.10 *If a term M is typable in $\mathcal{D}\Omega$ with a weakly nontrivial type, then every weak quasi-head reduction from M is finite. As a corollary, the weak head reduction from M is finite (and so, M has a weak head-normal form).*

Proof. Let $\mathcal{P}$ be the set of λ-terms for which every weak quasi-head reduction is finite. To prove theorem 7.10, we apply theorem 7.8, which requires showing that $\mathcal{P}$ satifies the properties (P1)-(P5n). The remark made at the beginning of the proof of lemma 3.11 also applies here. If every weak quasi-head reduction sequence is finite, since the reduction tree is finite branching, by König's lemma, the subtree consisting of weak quasi-head reduction sequences is finite. Thus, for any term M from which every weak quasi-head reduction sequence is finite, the length of a longest weak quasi-head reduction path in the reduction tree from M is a natural number, and we will denote it as $l(M)$. Now, (P1) is trivial, and (P2) follows from the definition.

(P3s). Let M be simple, and assume that every weak quasi-head reduction from M is finite. We prove that every weak quasi-head reduction from MN is finite by induction on $l(M)$. Let $MN \longrightarrow_\beta Q$ be a reduction step. Because M is simple, MN is not a redex, and we must have $M \longrightarrow_\beta M_1$ or $N \longrightarrow_\beta N_1$. If M_1 is simple, since $l(M_1) < l(M)$, the induction hypothesis yields that every weak quasi-head reduction from M_1N is finite. If $N \longrightarrow_\beta N_1$, because we are considering weak quasi-head reductions from MN, there is a first step where a weak head reduction is applied, and it must be applied to M. Thus, we must have $MN \longrightarrow_\beta MN_1 \overset{*}{\longrightarrow}_\beta MN_i \longrightarrow_\beta M_1N_i$. Since $l(M_1) < l(M)$, the induction hypothesis yields that every weak quasi-head reduction from MN_1 is finite. Otherwise, $M_1 = \lambda x. P$, and by assumption, every weak quasi-head reduction from $(\lambda x.P)N$ is finite. Thus every weak quasi-head reduction from MN is finite.

(P4w). Assume that every weak quasi-head reduction from M is finite. By definition, $\lambda x. M$ is a weak head normal form, and the result is trivial.

(P5n). Let k be the index of the first weak head-reduction step in any weak quasi-head reduction from $(\lambda x. M)N$. We prove by induction on k that every weak quasi-head reduction from $(\lambda x. M)N$ is finite. If $k = 0$, then $(\lambda x. M)N$ is a weak head-redex. However, by the assumption, every weak quasi-head reduction from $M[N/x]$ is finite. Now, consider any weak quasi-head reduction s from $(\lambda x. M)N$ of index $k \geq 1$. The first reduction step from $(\lambda x. M)N$ is either $(\lambda x. M)N \longrightarrow_\beta (\lambda x. M_1)N$ or $(\lambda x. M)N \longrightarrow_\beta (\lambda x. M)N_1$. In either case, the index of the first weak head-reduction step in the weak quasi-head reduction $tail(s)$ is $k - 1$, and by the induction hypothesis, we get the desired result. $\square$

The converse of theorem 7.10 is true: if a λ-term is weak head-normalizable, then it is typable in $\mathcal{D}\Omega$ with a weakly nontrivial type σ. First, we prove the following weaker result.

Lemma 7.11   *Given a term $M = yN_1 \ldots N_k$, there are nontrivial types σ and γ, where σ is a base type, such that $\vdash_{\mathcal{D}\Omega} y{:}\gamma \triangleright M{:}\sigma$. Given a term $M = \lambda x. M_1$, for any type σ, we have $\vdash_{\mathcal{D}\Omega} \triangleright M{:}\sigma \to \omega$.*

Proof. Let $\gamma = \omega \to \ldots \to \omega \to \sigma$ with k occurrences of ω. It is easy to see that we have

$$\vdash_{\mathcal{D}\Omega} y{:}\gamma \triangleright yN_1 \ldots N_k{:}\sigma.$$

If $M = \lambda x. M_1$, for any type σ, by the ω-axiom, we have

$$\vdash_{\mathcal{D}\Omega} x{:}\sigma \triangleright M_1{:}\omega,$$

and thus $\vdash_{\mathcal{D}\Omega} \triangleright \lambda x. M_1{:}\sigma \to \omega$. $\square$

Note that there are weakly head-normalizable terms that are not head-normalizable. If $\delta = \lambda x. xx$, then $\lambda x. (\delta\delta)$ is in weak head-normal form, but it is not head normalizable since $\delta\delta$ is not.

We are now ready to prove the theorem characterizing the λ-terms that are weakly head-normalizable in terms of type-checking in $\mathcal{D}\Omega$. However, we do not have a notion of "weak solvability".

Theorem 7.12   *For any term M of the (untyped) λ-calculus, the following properties are equivalent.*

(1) *M has a weak head-normal form (i.e., there is some weak head-normal form N such that $M \overset{*}{\longleftrightarrow}_\beta N$).*

(2) *M is typable in system $\mathcal{D}\Omega$ with a weakly nontrivial type;*

(3) *Every weak quasi-head reduction from M is finite. In particular, the weak head-reduction from M is finite.*

Proof. $(1) \Rightarrow (2)$. This follows from lemma 7.11 and theorem 6.8.

$(2) \Rightarrow (3)$. This follows from theorem 7.10.

$(3) \Rightarrow (1)$. This is trivial. $\square$

It should be noted that the implication $(1) \Rightarrow (2)$ shows that every weakly head-normalizable term is typable in $\mathcal{D}\Omega$ with a weakly nontrivial type of a rather special kind (since the types arising in lemma 7.11 are quite special).

8 Conclusion, Open Problems, and Challenges

We have shown four metatheorems (theorems 3.9, 4.9, 5.9, and 7.8) about interesting classes of λ-terms, using a fairly generic version the reducibility method. Obviously, the proofs do not differ very much, but even though we have made some progress in isolating some of their common ingredients (for example, the $\mathcal{P}$-candidate conditions (S1), (S2), (S3)), we have not yet succeeded in extracting what they really share in common. Thus, we have our first challenge:

Challenge 1: Find a common generalization of the four proofs of the theorems 3.9, 4.9, 5.9, and 7.8.

The method of $\mathcal{P}$-candidates can also be applied to various typed λ-calculi, including system F, and we worked out a generalized version of reducibility for such typed calculi (see Gallier [6] and [7]). To define this version of realizability, it was necessary to define a new class of applicative structures, called pre-applicative structures, in which the carriers are equipped with preorders, and the various inductive conditions on candidates of reducibility can be viewed as *sheaf conditions*. Families of realizers are *sheaves* w.r.t. a suitable notion of *cover* (see Gallier [7]). It is worth noting that pre-applicative structures are models of *reduction* rather than models of *convertibility*. There is a preorder $\succeq$ on each carrier, to model reduction. Although models of convertibility have been studied extensively (starting with some seminal work of Dana Scott and Gordon Plotkin), we feel that the surface has been barely scratched when it comes to models of reduction.

Our work seems to indicate that the notion of cover is very robust. In the next paragraphs, which assume some familiarity with Gallier [7], we clarify this previous statement. Given a pre-applicative structure $\mathcal{A} = (A^\sigma)_{\sigma \in \mathcal{T}}$ (with preorder $\succeq$), given a family $\mathcal{S} = (S_\sigma)_{\sigma \in \mathcal{T}}$, where $S_\sigma \subseteq A^\sigma$, the family $\mathcal{S}$ is a $\mathcal{P}$-*sheaf* iff

(S1) $S_\sigma \subseteq P_\sigma$.

(S2) If $M \in S_\sigma$ and $M \succeq N$, then $N \in S_\sigma$.

(S3) If $\mathrm{Cov}_\sigma(C, M)$, and $C \subseteq S_\sigma$, then $M \in S_\sigma$.

The family $\mathcal{S} = (S_\sigma)_{\sigma \in \mathcal{T}}$ can be viewed as a functor

$$\mathcal{S}: \mathcal{A}^{op} \to \mathtt{Sets},$$

by letting $\mathcal{S}(M) = \{\sigma \mid M \in S_\sigma\}$. Then, (S3) can be written as:

(S3) If $\mathrm{Cov}_\sigma(C, M)$, and $\sigma \in \mathcal{S}(N)$ for every $N \in C$, then $\sigma \in \mathcal{S}(M)$.

It can be verified that $\mathcal{S}$ is a sheaf with respect to the cover algebra $\mathtt{Cov}$ on $\mathcal{A}$ (see Gallier [7]).

This brings us to our second challenge:

Challenge 2: Is there a notion of pre-applicative structure applying to both untyped terms and typed terms?

Close examination of the approach in this paper and in Gallier [7], shows that there seems to be six parameters in reducibility proofs:

(1) The class of λ-terms

(2) The type system $\mathcal{T}$

(3) The property $\mathcal{P}$ to be proved.

(4) The class of pre-applicative structures $\mathcal{A}$.

(5) The notion Cov of cover.

(6) The definition of realizability (the sets of realizers $[\![\sigma]\!]$).

We now come to our bigest challenge:

Main Challenge 3: Is there a generalization of the reducibility method applying to untyped terms and typed terms, and to various type systems and properties?

We conjecture that covers will play a central role, but their definition may need adjustements. Finally, as if we did not have enough trouble already, one more nagging questions remains:

What about *dependent* types? (this seems hard!)

In a recent paper, McAllester, Kučan, and Otth [14], prove various strong normalization results using another variation of the reducibility method. Although we see their approach as much less fundamental and too restrictive (it only seems to deal with strong normalization), it would be interesting to understand how this method relates to the method presented in this paper or in Gallier [7]. The papers by Hyland and Ong [11] and by Michel Parigot [16], also present proofs of strong normalization, using new variants of the reducibility method. The technical details are very different, and we are unable to make a precise comparison at this point. Clearly, further work is needed to clarify the connection between these approaches and ours.

References

[1] M. Coppo and F. Cardone. Two extensions of Curry's type inference system. In P. Odifreddi, editor, *Logic And Computer Science*, pages 19–75. Academic Press, London, New York, May 1990.

[2] M. Coppo and M. Dezani. A new type-assignment for λ-terms. *Archiv. Math. Logik*, 19:139–156, 1978.

[3] M. Coppo and M. Dezani. An extension of basic functionality theory for lambda-calculus. *Notre Dame J. Formal Log.*, 21:685–693, 1980.

[4] M. Coppo, M. Dezani, and B. Venneri. Functional characters of solvable terms. *Z. Math. Log. Grund. Math.*, 27:45–58, 1981.

[5] Jean H. Gallier. On Girard's "candidats de reductibilité". In P. Odifreddi, editor, *Logic And Computer Science*, pages 123–203. Academic Press, London, New York, May 1990.

[6] Jean H. Gallier. On the correspondence between proofs and λ-terms. In P. DeGroote, editor, *The Curry-Howard Isomorphism*, Cahiers du Centre de Logique, No. 8, pages 55–138. Université Catholique de Louvain, 1995.

[7] Jean H. Gallier. Proving properties of typed λ-terms using realizability, covers, and sheaves. *Theoretical Computer Science*, 142:299–368, 1995.

[8] Jean-Yves Girard. Une extension de l'interprétation de Gödel à l'analyse, et son application à l'élimination des coupures dans l'analyse et la théorie des types. In J.E. Fenstad, editor, *Proc. 2nd Scand. Log. Symp.*, pages 63–92. North-Holland, 1971.

[9] Jean-Yves Girard. *Interprétation fonctionnelle et élimination des coupures de l'arithmétique d'ordre supérieur*. PhD thesis, Université de Paris VII, June 1972. Thèse de Doctorat d'Etat.

[10] Gérard Huet. Initiation au λ-calcul. Technical report, Université Paris VII, Paris, 1991. Lectures Notes.

[11] J. M. E. Hyland and C.-H. L. Ong. Modified realizability topos and strong normalization proofs. In M Bezem and J.F. Groote, editors, *Typed Lambda Calculi and Applications*, volume 664 of *Lecture Notes in Computer Science*, pages 179–194. Springer Verlag, 1993.

[12] G. Koletsos. Church-Rosser theorem for typed functional systems. *J. Symbolic Logic*, 50(3):782–790, 1985.

[13] J.L. Krivine. *Lambda-Calcul, types et modèles*. Etudes et recherches en informatique. Masson, 1990.

[14] D. McAllester, J. Kučan, and D. F. Otth. A proof of strong normalization for F_2, F_ω, and beyond. Technical report, MIT, Boston, Mass, 1993. Draft.

[15] J. C. Mitchell. A type-inference approach to reduction properties and semantics of polymorphic expressions. In *ACM Conference on LISP and Functional Programming*, pages 308–319. ACM, 1986. Reprinted in *Logical Foundations of Functional Programming*, G. Huet, Ed., Addison Wesley, 1990, 195-212.

[16] M. Parigot. Strong normalization for second-order classical natural deduction. In *Eighth Annual IEEE Symposium on Logic In Computer Science*, pages 39–46. IEEE, 1993.

[17] G. Pottinger. A type assignment to the strongly normalizable λ-terms. In J. P. Seldin and J. R. Hindley, editors, *To H. B. Curry: Essays on Combinatory Logic, Lambda Calculus and Formalism*, pages 561–577, London, 1980. Academic Press.

[18] W.W. Tait. Intensional interpretation of functionals of finite type I. *J. Symbolic Logic*, 32:198–212, 1967.

[19] W.W. Tait. A realizability interpretation of the theory of species. In R. Parikh, editor, *Logic Colloquium*, volume 453 of *Lecture Notes in Math.*, pages 240–251. Springer Verlag, 1975.