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Abstract
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Keywords
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Geometrical optics limit of stochastic electromagnetic fields

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I. INTRODUCTION

The behavior of monochromatic sources and fields at short wavelengths can be described, in many cases, by geometrical optics. The foundations of geometrical optics of scalar wave fields have been discussed by Sommerfeld and Runge in a classic paper [1], and their analysis has been generalized by Rytov [2] (see also [3], Chap. 3) to monochromatic electromagnetic fields.

A generalization of the results of Sommerfeld, Runge, and Rytov to broader classes of fields presents some difficulties, especially when these fields are stochastic. However, a solution for stochastic scalar fields was obtained [4] via the so-called coherent mode representation of scalar wave fields of any state of coherence (see [5] or [6], Sec. 4.1). In the present paper, that analysis is extended to stochastic, planar, secondary electromagnetic sources and to the fields that they generate.

In the high-frequency limit of electrodynamics, the vector fields are often approximated by scalar waves or by rays, depending on the application. The second-order correlations of fields in the scalar approximation have been extensively studied [6]. The average second-order correlation properties of electromagnetic fields can be described by correlation matrices in either the temporal or in the spectral domain. The time domain representation is often employed because detectors necessarily time integrate any signal received. The spectral domain representation, however, offers certain advantages, especially in connection with propagation in dispersive and absorptive media.

The electromagnetic cross-spectral density matrix of a planar source, which describes the spatial correlations in the spectral domain, may be expressed in terms of so-called coherent modes [7]. These modes are orthogonal and also mutually uncorrelated. The modes may be propagated, and though not necessarily mutually orthogonal, are also uncorrelated and thus the propagated cross-spectral density matrix can be reconstituted by adding the matrices formed by taking the outer product of each propagated mode with itself and multiplying by the appropriate weight. Propagation of electromagnetic fields can be described using dyadic Green function methods. However, the implementation of such methods may be prohibitively numerically intensive. Thus it is desirable to describe the propagation by approximate asymptotic methods. It will be shown that for certain systems, the propagation can be adequately carried out within the framework of geometrical optics, thus greatly reducing the computational complexity of the analysis and making available a wealth of computational tools for ray-tracing. The main part of this paper is organized as follows: In Sec. II, the scalar decomposition of planar, stochastic, electromagnetic sources is reviewed; in Sec. III, the propagation of the modes is considered within the high-frequency limit; finally, in Sec. IV, two examples are given to illustrate this geometrical method for computing the degree of polarization.

II. MODAL DECOMPOSITION

Consider a random, statistically stationary, secondary electromagnetic planar source represented by the mutual coherence matrix

$$\overleftrightarrow{\Gamma}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\tau) \equiv [\Gamma_{ij}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\tau)] \tag{1}$$

$$=[\langle E_i^*(\boldsymbol{\rho}_1,0)E_j(\boldsymbol{\rho}_2,\tau)\rangle], \tag{2}$$

where the $\boldsymbol{\rho}_k$ $(k=1,2)$, are position vectors of a point in the transverse plane, τ is a time delay, i and j label the x or y components, the asterisk denotes complex conjugation and the brackets denote an ensemble average over the fluctuating electric field. The Fourier transform of the mutual coherence matrix is the cross-spectral density matrix,

$$\overleftrightarrow{W}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\omega) \equiv \int d\tau \overleftrightarrow{\Gamma}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\tau)e^{i\omega\tau}. \tag{3}$$

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A random, planar, stochastic electromagnetic source considered in the space-frequency domain can be described by two sets of scalar modes [7]. The cross-spectral density at every point in the source plane can be written in terms of these modes. Propagation of the modes may be expressed by the method of dyadic Green functions. However, the implementation of this approach may be quite complicated in practice. In many cases of interest, methods of geometrical optics may be applied to each of the coherent modes which simplify the calculation. It is the purpose of this paper to develop such an approach.

Modal expansions have previously been expressed in terms of the solutions to two- or three-dimensional coupled Fredholm integral equations [8,9]. However, solving these integral equations is not often tractable. Instead, a different modal representation has been introduced in which the diagonal elements of the cross-spectral density are expressed in terms of two sets of independent coherent modes $\{\phi_n\}$ and $\{\psi_n\}$ which are solutions of uncoupled integral equations:

$$W_{xx}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\omega) = \sum_{n=0}^{\infty} \lambda_n(\omega)\phi_n^*(\boldsymbol{\rho}_1,\omega)\phi_n(\boldsymbol{\rho}_2,\omega), \tag{4}$$

$$W_{yy}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\omega) = \sum_{n=0}^{\infty} \gamma_n(\omega)\psi_n^*(\boldsymbol{\rho}_1,\omega)\psi_n(\boldsymbol{\rho}_2,\omega). \tag{5}$$

Here ϕ_n and λ_n are the eigenfunctions and the eigenvalues, respectively, of W_{xx}, and ψ_n and γ_n are the eigenfunctions and the eigenvalues of W_{yy}. The off-diagonal elements of the cross-spectral density matrix may be expressed in the form

$$W_{xy}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\omega) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \Lambda_{nm}(\omega)\phi_n^*(\boldsymbol{\rho}_1,\omega)\psi_m(\boldsymbol{\rho}_2,\omega), \tag{6}$$

$$W_{yx}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\omega) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \Lambda_{nm}^*(\omega)\psi_m^*(\boldsymbol{\rho}_1,\omega)\phi_n(\boldsymbol{\rho}_2,\omega). \tag{7}$$

In these two equations, the off-diagonal expansion coefficients Λ_{nm} are given by

$$\Lambda_{nm}(\omega) = \int_\Sigma d^2\rho_1 \int_\Sigma d^2\rho_2 \phi_n(\boldsymbol{\rho}_1,\omega) W_{xy}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\omega)\psi_m^*(\boldsymbol{\rho}_2,\omega), \tag{8}$$

where Σ is the source plane. As shown in the Appendix, when the modal expansion coefficients satisfy the equation $|\Lambda_{nm}|^2 = \lambda_n\gamma_m\delta_{nm}$, where δ_{nm} is the Kronecker δ symbol, the scalar mode expansion can be recast as an electromagnetic coherent mode expansion identical to the expansion previously introduced [8,9].

III. GEOMETRICAL OPTICS

It has been shown [4] that for scalar waves an eikonal approach to propagating the coherent modes of a field can be applied to certain classes of sources. In the electromagnetic case, the propagated modes are given by the expression

$$\boldsymbol{\Phi}_n(\mathbf{r},\omega) = -\int_\Sigma d^2\rho'\, \phi_n(\boldsymbol{\rho}',\omega)\hat{\mathbf{y}}\cdot\nabla'\times\overleftrightarrow{G}(\boldsymbol{\rho}',\mathbf{r}), \tag{9}$$

$$\boldsymbol{\Psi}_n(\mathbf{r},\omega) = \int_\Sigma d^2\rho'\, \psi_n(\boldsymbol{\rho}',\omega)\hat{\mathbf{x}}\cdot\nabla'\times\overleftrightarrow{G}(\boldsymbol{\rho}',\mathbf{r}), \tag{10}$$

where $\overleftrightarrow{G}(\mathbf{r},\mathbf{r}')$ is the dyadic Green function for the wave equation in the half-space [10]. This method of propagating the modes has been previously discussed [11] and the propagation of the correlation matrices using an approximation of the dyadic Green function has been treated elsewhere [12]. For sources that give rise to beams, the z component of the field can be neglected and the cross-spectral density matrix remains a 2×2 matrix. In general, though, the cross-spectral density matrix cannot be reduced to a 2×2 representation.

It is assumed that the propagated modes may be expanded in a series by asymptotic evaluation of the integrals in Eqs. (9) and (10) for sufficiently large values of the wave number $k=\omega/c$. The leading order term may be well approximated by the expression

$$\boldsymbol{\Phi}_n(\mathbf{r},\omega) = \boldsymbol{\Phi}_{E,n}(\mathbf{r})e^{ik\mathcal{S}_n^{(x)}(\mathbf{r})}, \tag{11}$$

where $\boldsymbol{\Phi}_{E,n}(\mathbf{r})$ and $\mathcal{S}_n^{(x)}(\mathbf{r})$ are the frequency-independent amplitude and the eikonal of $\boldsymbol{\Phi}_n(\mathbf{r},\omega)$, respectively, and the superscript x refers to a Cartesian component of the electric field. Upon substituting this expression into Maxwell's equations, one obtains three coupled first-order equations of different power in k (see [3], Chap. 3). The first is the eikonal equation, which takes the form

$$|\nabla\mathcal{S}_n^{(x)}(\mathbf{r})|^2 = n^2(\mathbf{r}), \tag{12}$$

where $n(\mathbf{r})$ is the refractive index. The second equation is the so-called transport equation for the field amplitudes,

$$\begin{aligned}[\nabla\mathcal{S}_n^{(x)}(\mathbf{r})\cdot\nabla]\boldsymbol{\Phi}_{E,n}(\mathbf{r}) &+ \frac{1}{2}\{\nabla^2\mathcal{S}_n^{(x)}(\mathbf{r}) \\ &- [\nabla\mathcal{S}_n^{(x)}(\mathbf{r})\cdot\nabla]\ln\mu(\mathbf{r})\}\boldsymbol{\Phi}_{E,n}(\mathbf{r}) \\ &+ [\boldsymbol{\Phi}_{E,n}(\mathbf{r})\cdot\nabla\ln n(\mathbf{r})]\nabla\mathcal{S}_n^{(x)}(\mathbf{r}) = 0,\end{aligned} \tag{13}$$

where μ is the magnetic permeability of the medium. The third equation may be disregarded at sufficiently high frequency. There is a similar set of equations involving $\boldsymbol{\Psi}_{E,n}(\mathbf{r})$ and $\mathcal{S}_n^{(y)}(\mathbf{r})$.

In the half-space into which the field propagates, the cross-spectral density matrix can be expressed in the form

$$\begin{aligned}\overleftrightarrow{W}(\mathbf{r}_1,\mathbf{r}_2,\omega) = &\sum_n \lambda_n(\omega)\boldsymbol{\Phi}_{E,n}^*(\mathbf{r}_1)\otimes\boldsymbol{\Phi}_{E,n}(\mathbf{r}_2)e^{ik\Delta\mathcal{S}_n^{(x)}(\mathbf{r}_1,\mathbf{r}_2)} \\ &+ \sum_m \gamma_m(\omega)\boldsymbol{\Psi}_{E,m}^*(\mathbf{r}_1)\otimes\boldsymbol{\Psi}_{E,m}(\mathbf{r}_2)e^{ik\Delta\mathcal{S}_m^{(y)}(\mathbf{r}_1,\mathbf{r}_2)} \\ &+ \sum_m\sum_n \Lambda_{nm}(\omega)\boldsymbol{\Phi}_{E,n}^*(\mathbf{r}_1) \\ &\otimes \boldsymbol{\Psi}_{E,m}(\mathbf{r}_2)e^{ik[\mathcal{S}_m^{(y)}(\mathbf{r}_2)-\mathcal{S}_n^{(x)}(\mathbf{r}_1)]} \\ &+ \sum_m\sum_n \Lambda_{nm}^*(\omega)\boldsymbol{\Psi}_{E,m}^*(\mathbf{r}_1)\end{aligned}$$

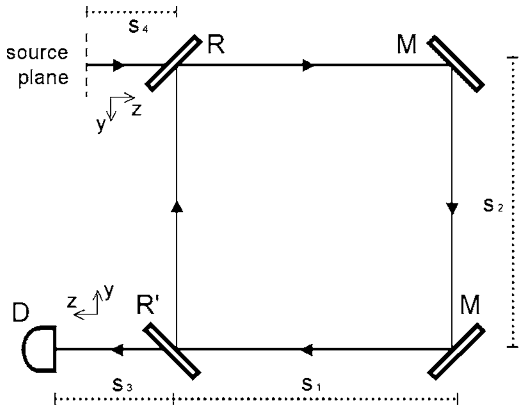

FIG. 1. A diagram of an external ring cavity. A ray that is only transmitted through the mirrors R and R' (not reflected) and travels a distance $d=2s_1+s_2+s_3+s_4$. The perimeter of the cavity is $l=2s_1+2s_2$.

$$\otimes \mathbf{\Phi}_{E,n}(\mathbf{r}_2)e^{ik[\mathcal{S}_n^{(x)}(\mathbf{r}_2)-\mathcal{S}_m^{(y)}(\mathbf{r}_1)]}, \qquad (14)$$

where $\Delta\mathcal{S}_m^{(i)}(\mathbf{r}_1,\mathbf{r}_2)=\mathcal{S}_m^{(i)}(\mathbf{r}_2)-\mathcal{S}_m^{(i)}(\mathbf{r}_1)$. The cross-spectral density matrix for the propagated field is thus the sum of matrices, each of which represents a coherent field. In the source plane, this decomposition has a scalar form. However, upon propagation, the modes must be described vectorially. Each coherent mode is made up of two factors: An outer product of vectors and a relative phase, both of which depend on two points.

IV. DEGREE OF POLARIZATION

Electromagnetic beams and electromagnetic plane waves make up special subsets of electromagnetic fields in which the field has a dominant direction of propagation, taken to be the z direction. In such a case, both the electric and the magnetic fields oscillate perpendicular to the propagation direction. The spectral degree of polarization of the field is given by the expression [13]

$$\mathcal{P}_\omega(\mathbf{r}) \equiv \sqrt{1-\frac{4\det \overleftrightarrow{W}(\mathbf{r},\mathbf{r},\omega)}{[\mathrm{Tr}\,\overleftrightarrow{W}(\mathbf{r},\mathbf{r},\omega)]^2}} = \frac{\lambda_1(\mathbf{r},\omega)-\lambda_2(\mathbf{r},\omega)}{\lambda_1(\mathbf{r},\omega)+\lambda_2(\mathbf{r},\omega)}, \qquad (15)$$

where $\lambda_1 \geq \lambda_2$ are the eigenvalues of the cross-spectral density matrix when both its arguments are evaluated at $\mathbf{r}$.

To illustrate the previous analysis, consider a laser beam incident on a ring cavity as shown in Fig. 1. Suppose that the mirror R' has reflection coefficients $r'_{te}(\hat{k})$ and $r'_{tm}(\hat{k})$ and transmission coefficients $t'_{te}(\hat{k})$ and $t'_{tm}(\hat{k})$ for the transverse electric and magnetic fields, respectively. There are likewise reflection and transmission coefficients for the mirror R. The other two mirrors are assumed to be perfectly reflecting.

Consider the case when the cross-spectral density matrix of the incident light has the form

$$\overleftrightarrow{W}(\mathbf{r}_1,\mathbf{r}_2,\omega) = \begin{bmatrix} W_{xx}(\mathbf{r}_1,\mathbf{r}_2,\omega) & 0 \\ 0 & W_{yy}(\mathbf{r}_1,\mathbf{r}_2,\omega) \end{bmatrix}. \qquad (16)$$

In the geometrical limit, the modal decomposition takes the form

$$W_{xx}(\mathbf{r}_1,\mathbf{r}_2,\omega) = \sum_n \lambda_n(\omega)\phi^*_{E,n}(\boldsymbol{\rho}_1)e^{-ikz_1}\phi_{E,n}(\boldsymbol{\rho}_2)e^{ikz_2}, \qquad (17)$$

$$W_{yy}(\mathbf{r}_1,\mathbf{r}_2,\omega) = \sum_n \gamma_n(\omega)\psi^*_{E,n}(\boldsymbol{\rho}_1)e^{-ikz_1}\psi_{E,n}(\boldsymbol{\rho}_2)e^{ikz_2}, \qquad (18)$$

$$W_{xy}(\mathbf{r}_1,\mathbf{r}_2,\omega) = 0. \qquad (19)$$

Each mode can be represented by a set of rays that propagate in the z direction. At the two mirrors R and R', every incident ray creates two outgoing rays: One that continues on the straight line path with an amplitude t_{te} (t_{tm}) times the original amplitude and one that continues in a direction governed by the law of reflection with amplitude r_{te} (r_{tm}). Hence, the field at the detector is comprised of a series of collections of rays for each mode: A collection of rays that did not reflect at either R or R', a collection that reflected once at each of these mirrors, etc. In the detection plane, the effect of the series of reflections and transmissions can be described by the formulas

$$\phi^{(D)}_{E,n}(\boldsymbol{\rho}) = T_{te}(kl)\phi_{E,n}(\boldsymbol{\rho}), \qquad (20)$$

$$\psi^{(D)}_{E,n}(\boldsymbol{\rho}) = T_{tm}(kl)\psi_{E,n}(\boldsymbol{\rho}), \qquad (21)$$

where $\phi^{(D)}_{E,n}$ and $\psi^{(D)}_{E,n}$ are the propagated coherent modes at the detector and

$$T_{te}(kl) = \frac{t_{te}t'_{te}e^{ikd}}{1-r_{te}r'_{te}e^{ikl}}, \qquad (22)$$

$$T_{tm}(kl) = \frac{t_{tm}t'_{tm}e^{ikd}}{1-r_{tm}r'_{tm}e^{ikl}}. \qquad (23)$$

Here d is the single pass length around the cavity to the detector, and l is the length of the cavity. Note that the total transmission coefficients are independent of the mode index. For this reason, the cross-spectral density matrix elements are proportional to the original cross-spectral density matrix.

In the detection plane, the cross-spectral density takes the form

$$\overleftrightarrow{W}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\omega) = \begin{bmatrix} W^{(D)}_{xx}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\omega) & 0 \\ 0 & W^{(D)}_{yy}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\omega) \end{bmatrix}, \qquad (24)$$

where

$$W^{(D)}_{xx}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\omega) = W_{xx}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\omega)|T_{te}|^2, \qquad (25)$$

$$W^{(D)}_{yy}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\omega) = W_{yy}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\omega)|T_{tm}|^2. \qquad (26)$$

Because the TE and TM coefficients for reflection and transmission are, in general, different, the degree of polarization

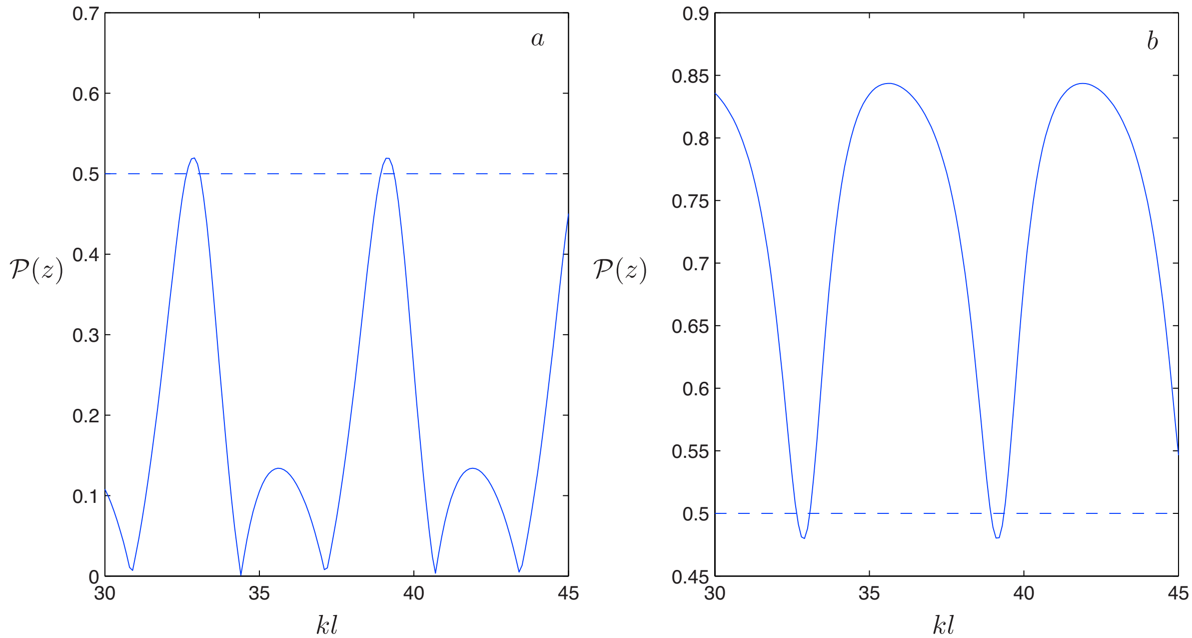

FIG. 2. (Color online) The degree of polarization, calculated by the geometrical optics model, at the output as a function of cavity size parameter kl. In panel (a), the incident field is more TE polarized, that is, the spectral density for the TE component of the field is 3 times larger than the spectral density for the TM component of the field, and the incident field has degree of polarization $\mathcal{P}=0.5$. In panel (b), the incident field is more TM polarized, that is, the spectral density for the TM component of the field is 3 times larger than the spectral density for the TE component of the field, and the incident field has degree of polarization $\mathcal{P}=0.5$. The mirrors R and R' are identical dielectric mirrors with thickness $t=31\lambda$, $\epsilon=11.34\epsilon_0$, and $\mu=\mu_0$.

at the output can be altered by changing the path length around the cavity, i.e., changing the eikonal of the output field. In Fig. 2, the degree of polarization at the detector is plotted against the size of the cavity. As the cavity size becomes larger, the degree of polarization changes periodically. In panel (a), the degree of polarization falls off rapidly from the initial value of $\mathcal{P}=0.5$. In panel (b), the degree of polarization increases well above the initial value. The specific choices of mirror parameters make the cavity preferentially favor the TM polarization. Mirrors can be designed with different parameters to favor the TE polarization or to lessen the change in the state of polarization.

As another example, consider a planar source with a uniform cross-spectral density for all pairs of points. The cross-spectral density matrix takes the form

$$\overleftrightarrow{W}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\omega)=\begin{bmatrix}\lambda(\omega) & \Lambda(\omega)\\ \Lambda^*(\omega) & \gamma(\omega)\end{bmatrix}. \tag{27}$$

This source generates a z-directed plane wave. Suppose the propagated field is incident upon a biaxial medium having the fast axis in the x direction and slow axis in the y direction. The eikonal along the fast axis is $\mathcal{S}^{(x)}=n_f z$ and the eikonal along the slow axis is $\mathcal{S}^{(y)}=n_s z$. From Eq. (14), the cross-spectral density of the field in the media takes the form (up to a nonessential prefactor)

$$\overleftrightarrow{W}(\mathbf{r}_1,\mathbf{r}_2,\omega)=\begin{bmatrix}\lambda(\omega)e^{ik_0 n_f(z_2-z_1)} & \Lambda(\omega)e^{ik_0(n_s z_2-n_f z_1)}\\ \Lambda^*(\omega)e^{ik_0(n_f z_2-n_s z_1)} & \gamma(\omega)e^{ik_0 n_s(z_2-z_1)}\end{bmatrix}, \tag{28}$$

where $k_0=\omega/c$, and n_s and n_f are the refractive indices along the slow axis and fast axis, respectively.

Assume that the source is broadband with $\lambda(\omega)=\gamma(\omega)=\Lambda(\omega)=\exp(-\frac{(\omega-\omega_c)^2}{2\sigma_\omega^2})$. It is clear that at any point in the half-space, the spectral degree of polarization $\mathcal{P}_\omega=1$. For broadband light, it is appropriate to define the degree of polarization in terms of the mutual coherence matrix ([14], pp. 174ff) rather than in terms of the cross-spectral density matrix as

$$\mathcal{P}(\mathbf{r})=\sqrt{1-\frac{4\det\overleftrightarrow{\Gamma}(\mathbf{r},\mathbf{r},0)}{[\mathrm{Tr}\,\overleftrightarrow{\Gamma}(\mathbf{r},\mathbf{r},0)]^2}}. \tag{29}$$

For any plane $z=z_p$ in the crystal, the equal-time mutual coherence matrix takes the form

$$\overleftrightarrow{\Gamma}(z_p,z_p,0)=\begin{bmatrix}\tilde{\lambda}(0) & \tilde{\Lambda}\left(\frac{n_f-n_s}{c}z_p\right)\\ \tilde{\Lambda}^*\left(\frac{-n_f+n_s}{c}z_p\right) & \tilde{\gamma}(0)\end{bmatrix}, \tag{30}$$

where tilde denotes Fourier transform. It is apparent that the degree of polarization of the field in the crystal changes as a function of propagation distance, viz.,

$$\mathcal{P}(z)=\exp\left(-\frac{\sigma_\omega^2 n_{\mathrm{dif}}^2 z^2}{2c^2}\right), \tag{31}$$

where $n_{\mathrm{dif}}=n_s-n_f$ and the approximation $\omega_c/\sigma_\omega\gg 1$ has been used.

In Fig. 3, the degree of polarization as a function of axial distance z is shown. The values for bandwidth and difference in refractive index are typical of those in a polarization sensitive optical coherence tomography imaging experiment

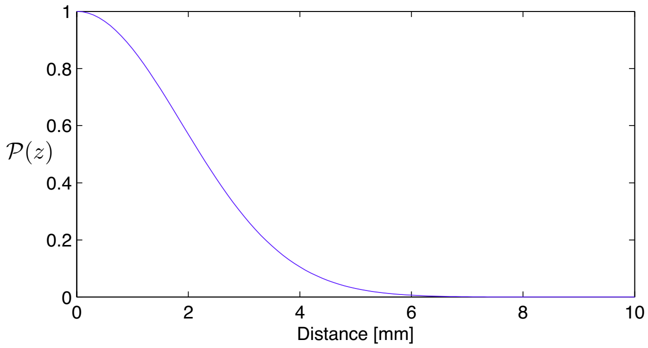

FIG. 3. (Color online) The degree of polarization, $\mathcal{P}(z)$, calculated from the present theory, as a function of distance traversed through the biaxial media with $n_{\mathrm{dif}}=0.0019$ and $\sigma_\omega=8.37\times10^{13}$ rad/s. The centerline frequency is $\omega_c=1.45\times10^{15}$ rad/s.

[15]. After the beam travels a distance 4 mm through the biaxial medium, its degree of polarization changes from the initial value of 1 to a value of 0.1.

Although in the preceding case the spectral degree of polarization is invariant on propagation, it is clear that the temporal degree of polarization changes drastically. The change is a consequence of the different phase that each spectral component accumulates on propagation.

V. CONCLUSION

Modal decomposition of planar electromagnetic, secondary, partially coherent sources has been developed and the propagation from such sources has been considered in the short wavelength limit. The electric cross-spectral density matrix of the propagated field has also been studied; specifically, a geometrical interpretation of changes in the degree of polarization due to propagation has been considered. The examples make it clear that a geometrical model can be useful for analysis in either the space-time or in the space-frequency domains.

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APPENDIX

In this appendix, it is shown that under certain circumstances, the two kinds of coherent mode decompositions [7,8] may be expressed into the same form. The electromagnetic coherent mode decomposition developed previously [8,9] is expressed in terms of solutions to the matrix integral equation

$$\int d^2\rho' \overleftrightarrow{W}(\boldsymbol{\rho},\boldsymbol{\rho}',\omega)\cdot\mathbf{e}_n^*(\boldsymbol{\rho}',\omega)=\chi_n(\omega)\mathbf{e}_n^*(\boldsymbol{\rho},\omega). \tag{A1}$$

The $\mathbf{e}_n$ and $\chi_n(\omega)$ are the right eigenvectors and eigenvalues of $\overleftrightarrow{W}(\boldsymbol{\rho},\boldsymbol{\rho}',\omega)$. When the off-diagonal elements of $\overleftrightarrow{W}$ are nonzero, Eq. (A1) represents a set of coupled equations. When the off-diagonal elements are identically zero, the two equations become uncoupled. The cross-spectral density matrix may then be expressed in the form

$$\overleftrightarrow{W}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\omega)=\sum_n \chi_n(\omega)\mathbf{e}_n^*(\boldsymbol{\rho}_1,\omega)\otimes\mathbf{e}_n(\boldsymbol{\rho}_2,\omega). \tag{A2}$$

The recently introduced scalar mode decomposition of a stochastic source [7] can be expressed in terms of solutions to two uncoupled scalar equations,

$$\int d^2\rho W_{xx}(\boldsymbol{\rho},\boldsymbol{\rho}',\omega)\phi_n^*(\boldsymbol{\rho}',\omega)=\lambda_n(\omega)\phi_n^*(\boldsymbol{\rho},\omega), \tag{A3}$$

$$\int d^2\rho W_{yy}(\boldsymbol{\rho},\boldsymbol{\rho}',\omega)\psi_n^*(\boldsymbol{\rho}',\omega)=\gamma_n(\omega)\psi_n^*(\boldsymbol{\rho},\omega). \tag{A4}$$

The diagonal elements of the cross-spectral density matrix are then expressed as in the form of Eqs. (4) and (5), and the expansion of the off-diagonal elements is given by Eqs. (6)–(8). When the expansion coefficients take the form

$$\Lambda_{mn}(\omega)=\sqrt{\lambda_n(\omega)}\sqrt{\gamma_m(\omega)}e^{i\alpha_n(\omega)}\delta_{nm}, \tag{A5}$$

a set of vector modes $\boldsymbol{\mathcal{E}}_n(\boldsymbol{\rho},\omega)$ can be constructed as follows:

$$\boldsymbol{\mathcal{E}}_n(\boldsymbol{\rho},\omega)=\sqrt{\lambda_n(\omega)}\phi_n(\boldsymbol{\rho},\omega)\hat{\mathbf{x}}+\sqrt{\gamma_n(\omega)}e^{i\alpha_n(\omega)}\psi_n(\boldsymbol{\rho},\omega)\hat{\mathbf{y}}. \tag{A6}$$

Unlike the vectors in Eq. (A1), these vectors, although orthogonal, are not normalized. Their normalized form is

$$\boldsymbol{\mathcal{E}}_n'(\boldsymbol{\rho},\omega)=\frac{\boldsymbol{\mathcal{E}}_n(\boldsymbol{\rho},\omega)}{\sqrt{\beta_n(\omega)}}, \tag{A7}$$

where

$$\beta_n(\omega)=\int d^2\rho\boldsymbol{\mathcal{E}}_n^*(\boldsymbol{\rho},\omega)\cdot\boldsymbol{\mathcal{E}}_n(\boldsymbol{\rho},\omega). \tag{A8}$$

It is then not difficult to show that the expression

$$\overleftrightarrow{W}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\omega)=\sum_n \beta_n(\omega)\boldsymbol{\mathcal{E}}_n'^*(\boldsymbol{\rho}_1,\omega)\otimes\boldsymbol{\mathcal{E}}_n'(\boldsymbol{\rho}_2,\omega) \tag{A9}$$

represents the original cross-spectral density matrix and is also in the form of Eq. (A2).

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