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Abstract

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Disciplines

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Comments

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Electric-magnetic duality and deformations of three-dimensional conformal field theories

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$SL(2,\mathbb{Z})$ duality transformations in asymptotically $\mathrm{AdS}_4 \times S^7$ act nontrivially on the three-dimensional superconformal field theory of coincident M2-branes on the boundary. We show how S-duality acts away from the IR fixed point. We develop a systematic method to holographically obtain the deformations of the boundary CFT and show how electric-magnetic duality relates different deformations. We analyze in detail marginal deformations and deformations by dimension 4 operators. In the case of massive deformations, the renormalization group flow relates S-dual CFT's. Correlation functions in the CFT are computed by varying magnetic bulk sources, whereas correlation functions in the dual CFT are computed by electric bulk sources. Under massive deformations, the boundary effective action is generically minimized by massive self-dual configurations of the $U(1)$ gauge field. We show that a self-dual choice of boundary conditions exists, and it corresponds to the self-dual topologically massive gauge theory in $2+1$ dimensions. Thus, self-duality in three dimensions can be understood as a consequence of electric-magnetic invariance in the bulk of AdS_4.

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I. INTRODUCTION AND SUMMARY OF THE RESULTS

Electric-magnetic duality has played an important role in understanding nonperturbative aspects of supersymmetric gauge theories and string theory. It was instrumental in the description of confinement by monopole condensation in $\mathcal{N}=2$ SYM theory by Seiberg and Witten [1]. In the case $\mathcal{N}=4$ SYM [2], it is related via AdS/CFT to the $SL(2,\mathbb{Z})$ invariance of type IIB string theory. The latter puts very strong constraints on the form of the coefficients or the effective action, as it relates the string tree-level and one-loop terms to instanton contributions [3]. The topologically twisted version of $\mathcal{N}=4$ has recently been shown to realize the basic geometric Langlands correspondence [4].

The question whether four-dimensional gauge theories coupled to gravity exhibit duality properties has a long history [5]. Gravity itself in four dimensions is known to be invariant under rotations of the linearized Riemann tensor and its dual [6]. For toroidal compactifications of M-theory, generalized electric-magnetic dualities are well known [5,7]. In this paper we will consider compactifications to asymptotically $\mathrm{AdS}_4 \times S^7$. Finding dualities in this case might mean significant progress towards understanding the physical properties of the theory describing interacting M2-branes at strong coupling.

An important source of motivation comes from [8], where it was argued that the $SL(2,\mathbb{Z})$ duality of Abelian gauge fields in the bulk of AdS_4 relates seemingly different CFT's to each other. The CFT's in question turn out to be relevant for the quantum Hall effect, where the S-duality of [8] had independently been found [9] and verified experimentally. In retrospect, this might be seen as an experimental prediction of AdS/CFT. These modular properties have recently been used to propose new predictions for the quantum Hall effect in graphene [10]. The S-duality of [8] is also related to the IR limit of three-dimensional mirror symmetry [11,12]. It implies that electrically charged particles in one theory correspond to vortices in the dual.

In this paper we develop the holographic map between $SL(2,\mathbb{Z})$ transformations in the bulk and dualities between three-dimensional effective CFT's away from the conformal fixed point. The massive deformations are induced by generalized boundary conditions in the bulk of AdS. We show that under RG flow towards the IR, the IR theory is described by a dimension 2 current which is S-dual to the UV current. The UV current-current correlator is given in terms of the dual gauge field. This is in agreement with the field theory predictions of [13].

Before we describe our results in more detail, let us give our third piece of motivation. It comes from the fact that conformal theories with instanton solutions in AdS have rather special properties. One can often write an effective boundary action for them, whose classical solutions reproduce the bulk instantons. The exact quantum effective action can be computed from the bulk at leading order in N. The fact that one is dealing with instantons implies that the stress-energy tensor is zero and there is no backreaction. In [14] it was shown in the context of a toy model that for a conformally coupled scalar field with a quartic interaction potential the classical bulk solutions can be matched exactly with classical solutions of a boundary effective field theory. The boundary field theory had appeared earlier in [15]. The conformally coupled scalar field was em-

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bedded in M-theory in [16], where it was also shown that the boundary effective action can be exactly computed using AdS/CFT.

In this paper we continue this program of conformal holography [14] for the case of Abelian gauge fields. Again, the boundary effective action can be exactly computed for the class of solutions under study. The large N effective boundary theory which reproduces the bulk instantons is the self-dual topologically massive three-dimensional theory of Pilch, Townsend and van Nieuwenhuizen [17,18]. Conformal holography is expected to play a role for higher spins also.

We now summarize our main results. At the level of classical bulk solutions, we will give the most general solution for a $U(1)$ gauge field in pure AdS_4, and compute the leading backreaction effects in the Fefferman-Graham expansion. The onset of backreaction is one order up from the Fefferman-Graham ambiguity, and for the purpose of computing two-point functions the effective action can be computed by the method of holographic renormalization [19] without using backreaction. We also obtain the regularity condition for Euclidean gauge fields, which does not seem to have appeared in the literature. This regularity condition relates the boundary value of the electric field to the gauge-invariant part of the gauge field. In momentum space it takes the form

$$f_i(p) = -|p|A_i^{\mathrm{T}}(p), \tag{1.1}$$

where $f_i(p)$ is the boundary value of the electric field (see notation in the next subsection). We explain the physical meaning of this boundary condition in Lorentzian signature: it is the condition that waves travel from the boundary into the bulk. In terms of the original M2-geometry, this is the condition that all matter falls into the M2-brane. We will show that self-dual bulk solutions are in one-to-one correspondence with a choice of what we call "self-dual boundary conditions":

$$f_i(x) = \pm\epsilon_{ijk}\partial_j A_k(x) \tag{1.2}$$

for (anti-)SD solutions, respectively. Combining (1.1) (or a massive version of it) and (1.2) gives the self-dual topogically massive theory of [17,18]. We show that, although in flat space there are no Abelian instantons, this is not necessarily true in AdS: configurations such as (1.1) and (1.2) can lead to finite Euclidean action. The theory of [17,18] has a single massive degree of freedom, and the on-shell value of the action is explicitly written in terms of it.

The bulk theory is electric-magnetic invariant only up to boundary terms. We will compute these explicitly for various choices of boundary conditions. Boundary conditions are imposed by the addition of boundary terms in the action, corresponding to deformations of the CFT. We will consider marginal operators and operators of classical dimension 4, and show how electric-magnetic duality maps different operators into each other. We identify configurations which are self-dual under duality, and these include the self-dual configurations of [17,18].

Scalars in the particular range of masses $-d^2/4 < m^2 < -d^2/4 + 1$ in AdS_{d+1} are well known to admit two kinds of normalizable solutions, which give rise to two possible quantization schemes and hence two different boundary interpretations, one in terms of an operator of dimension Δ_+ and the other in terms of an operator of dimension Δ_-. In the conformal case we will be interested in, and in $d = 3$, the dimensions are $\Delta_+ = 2$ and $\Delta_- = 1$. Four-dimensional gauge fields have the same property: both modes are normalizable, and two quantization schemes are available, corresponding to Dirichlet or Neumann boundary conditions. For Dirichlet boundary conditions where the boundary value of the gauge field (or, equivalently, the magnetic field) is held fixed, the holographic interpretation is the usual one: A is a source for a global symmetry current in the CFT. In the Neumann quantization scheme, the electric field is held fixed instead. The dual operator has dimension $\Delta_- = 1$ and is identified with a gauge field that corresponds to the boundary value of the gauge field A. This does not present a problem because one can always construct from it a conserved current of dimension 2 which saturates the unitarity bound $\Delta \geq 2$. Observables of the theory are then correlation functions of the new current. It is dual to the boundary value of the bulk magnetic field. Witten showed [8] that the bulk S-duality operation induces an S-duality operation on the CFT that consists of first gauging the global $U(1)$ symmetry, promoting A to a dynamical field, and coupling it to an external field v via a Chern-Simons coupling. This corresponds precisely to the above discussion of boundary conditions. A third choice of boundary condition is possible in this case, a mixed one, which corresponds to a multitrace deformation of the CFT. In this paper we will examine S-duality in the context of massive deformations of the CFT which induce a renormalization group flow. We will find an interesting interpretation of these dualities in terms of bulk sources. If we use the usual bulk magnetic source (one that couples to the electric field), the correlators that we find are the usual ones. On the other hand, if we use a bulk electric source (one that couples to the gauge field), we find the S-dual correlators. As we mentioned, RG flow interchanges these two. We also give a bulk proof of Witten's CFT argument that S-duality acts as a Legendre transform [8] (this was first shown in [20], where marginal deformations were also discussed). In fact, there are two types of Legendre transforms, one that changes the dimension of the operator and one that does not. We interpret these dualities as the particle-vortex duality of [9]. The generic IR effective action that we find is the one of QED in three dimensions at large N_f. Although this basically follows from conformal invariance, the fact that the quali-

tative behavior under massive deformations is similar indicates that QED may indeed play a role in the IR for the $U(1)$ gauge fields studied here. The supergravity configuration that we study is $N = 2$ supergravity in four dimensions, which is obtained as an M-theory compactification on an S^7 where all scalar fields have been projected out. The still unknown dual SCFT is the strong-coupling limit of $2 + 1$ dimensional SYM theory in the large N limit. We compute its effective action, and a relation with the large N_f limit of QED is suggested. Three-dimensional S-duality of $U(1)$ gauge theories including Chern-Simons terms has been discussed earlier in a somewhat different context in [21].

The plan of the paper is as follows. In Sec. II we review the discussion of scalar fields in the special range of masses mentioned above. We will use many of those properties in the gauge field case as well. In that section we also give the basic classical solutions that we will use and apply the method of holographic renormalization to compute the effective action. In Sec. III we discuss electric-magnetic duality in AdS_4. We give a novel proof of S-duality based on the first order formalism and show how the boundary terms change under duality. Section IV is devoted to the deformations of the boundary conditions and the deformations of the CFT that they induce. In Sec. V we show how electric-magnetic transformations act on the general boundary conditions. In Sec. VI we discuss the interplay of S-duality, Legendre transforms, and RG-flow. The results are summarized in Fig. 1. In that section we compare to the known CFT results and comment on the non-Abelian generalization. Some conclusions and future lines of investigation are presented in Sec. VII. In five appendices we present details concerning S-duality, the various solutions, holographic renormalization, and supersymmetry of the solutions.

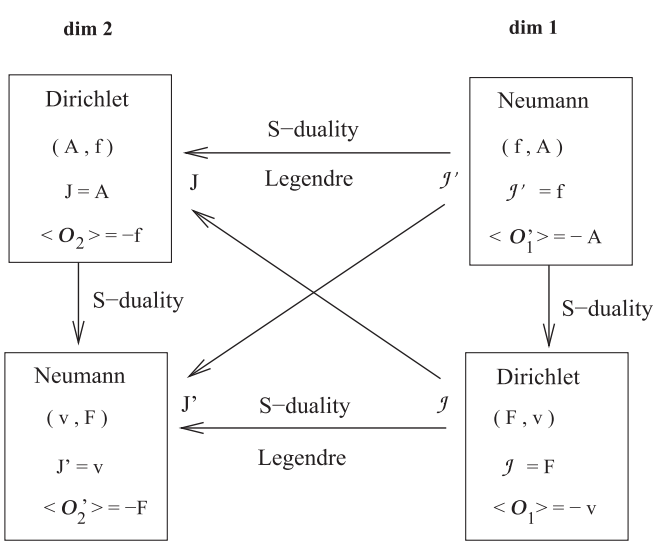

FIG. 1. Diagram of the various S-duality transformations.

Notation and conventions

$A_i(r, x)$ will denote the bulk gauge field and $A_i(x) = A_{(0)i}(x)$ its restriction to the boundary at $r = 0$. The indices $i = 0, 1, 2$ run over the boundary. The electric field will be denoted by $E_i(r, x) = F_{ri}(r, x)$ and its boundary component is $f_i(x) = E_i(0, x)$. We will use the following field redefinitions by a θ-term:

$$\mathsf{E} = E - \theta F \qquad \mathsf{f} = f - \theta F \qquad \mathsf{v} = v - \theta A. \tag{1.3}$$

We will use form notation for three-dimensional quantities, and will use a shorthand notation for contractions: $Af = A_i f^i$ etc. Boundary indices of r-dependent quantities are raised and lowered with $g_{ij}(r, x)$, indices of quantities evaluated at $r = 0$ are raised and lowered with $g_{(0)ij}$. We will use a projector onto the part of the gauge field transverse to the momentum:

$$\Pi_{ij} = g_{ij(0)} - \frac{\partial_i \partial_j}{\Box}, \qquad A_i^{\mathrm{T}} = \Pi_{ij} A_j = (\Pi A)_i. \tag{1.4}$$

We will denote $M = \Box^{1/2} = \sqrt{|\Box|}$ which both in Lorentzian and Euclidean signature is a positive definite quantity ($M = |p|$ in momentum space). The quantity t will be $+1$ in Euclidean signature, and -1 in Lorentzian signature.

The boundary magnetic field will be denoted by a vector F,

$$F_i = \tfrac{1}{2}\epsilon_{ijk} F_{jk} = \epsilon_{ijk} \partial_j A_k. \tag{1.5}$$

Twiddles denote Legendre transforms, tildes denote S-duality. So $\tilde{S}$ is the Legendre transformed action, S' denotes its S-dual.

Boundary operators will be denoted $\mathcal{O}_{\Delta i}(x)$, with $\Delta = 1$ for gauge fields and $\Delta = 2$ for currents. Boundary sources are denoted by $J_i(x)$.

II. THE BULK SOLUTION

A. Duality for scalar fields

In this section we review some facts known for scalar fields in AdS_4 which will set the stage of the discussion of gauge fields in the rest of the paper. It will also serve to fix notation.

Scalar fields in AdS_{d+1} with mass within the range

$$-\frac{d^2}{4} < m^2 < -\frac{d^2}{4} + 1 \tag{2.1}$$

are well known to admit two types of possible boundary conditions at the boundary preserving the asymptotic symmetry group and leading to finite energy [22–24]. For definiteness, consider the Fefferman-Graham form of the metric

$$ds^2 = \frac{\ell^2}{r^2}(dr^2 + g_{ij}(r, x) dx^i dx^j) \tag{2.2}$$

where $g_{ij}(r,x)$ has an expansion in the distance r to the boundary, such that at the boundary $r=0$ the metric is the Poincare metric of AdS. A scalar field of mass m will now have the asymptotic expansion

$$\phi = r^{\Delta_-}\phi_{(0)}(x) + \ldots + r^{\Delta_+}\phi_{(2\Delta-d)}(x) + \ldots \tag{2.3}$$

where $\Delta_\pm = \frac{d}{2} \pm \sqrt{\frac{d^2}{4}+m^2}$. $\phi_{(0)}$ and $\phi_{(2\Delta-d)}$ are not determined by the field equations. They are boundary conditions and have a holographic CFT interpretation as the source and 1-point function of an operator of dimension Δ_+, respectively, to which the source is coupled. For the range of masses (2.1), it was argued [25] that the two quantization schemes,

$$\begin{aligned} \delta\phi_{(0)}(x) &= 0 \qquad \text{Dirichlet} \\ \delta\phi_{(2\Delta-d)}(x) &= 0 \qquad \text{Neumann,} \end{aligned} \tag{2.4}$$

correspond to two possible boundary CFT's. The usual CFT is the one discussed above, where a source $\phi_{(0)}$ couples to an operator of dimension $\Delta = \Delta_+$. The usual AdS/CFT recipe can then be summarized as (up to terms that contribute contact terms in the two-point functions and which vanish identically in our case):

$$\begin{aligned} \phi_{(0)} &= J(x) \\ S_{\text{on-shell}}[\phi_{(0)}] &= -W[J] \\ \langle \mathcal{O}_\Delta(x)\rangle_J &= -\frac{\delta W}{\delta J} = \frac{\delta S_{\text{on-shell}}}{\delta\phi_{(0)}(x)} = (2\Delta-d)\phi_{(2\Delta-d)} \\ \langle \mathcal{O}_\Delta(x)\mathcal{O}_\Delta(x')\rangle_J &= -\frac{\delta^2 S_{\text{on-shell}}}{\delta\phi_{(0)}(x)\delta\phi_{(0)}(x')} \\ &\quad + \langle\mathcal{O}_\Delta(x)\rangle\langle\mathcal{O}_\Delta(x')\rangle, \end{aligned} \tag{2.5}$$

and likewise for the higher-point functions. The signature here is Euclidean. Our conventions are such that in the Lorentzian we have $S_{\text{on-shell}} = W$ and hence $\langle\mathcal{O}_\Delta\rangle = -(2\Delta-d)\phi_{(2\Delta-d)}$.

Thus, the renormalized bulk on-shell action is interpreted as minus the (renormalized) generating functional of connected correlation functions in the CFT, W, at leading order in N. We can also define the effective action of the boundary CFT, which is its Legendre transform:

$$\Gamma[\phi_+] = -\int d^dx\,\phi_-(x)\phi_+(x) + W[\phi_-]. \tag{2.6}$$

The second CFT is what we call the "dual CFT" [25], and it contains an operator of dimension Δ_-, which couples to a source of dimension Δ_+. This theory is defined as the Legendre transform of the usual CFT,

$$W[\phi_-] = \tilde{W}[\phi_+] + \int d^dx\,\phi_-\phi_+. \tag{2.7}$$

Its effective action $\tilde{\Gamma}$ is given by

$$\tilde{W}[\phi_+] = -\tilde{\Gamma}[\phi'] + \int d^dx\,\phi_+\phi'. \tag{2.8}$$

Comparing both equations, we see that the effective action of the dual CFT is identified with the bulk action, for $\phi' = -\phi_-$, and the generating functional of the dual CFT is identified with the effective action of the usual CFT:

$$\tilde{\Gamma}[-\phi_-] = -W[\phi_-] = S[\phi_-], \qquad \tilde{W}[\phi_+] = \Gamma[\phi_+]. \tag{2.9}$$

where ϕ_- is interpreted as the expectation value of an operator of dimension Δ_- and $\tilde{J} = \phi_+$ is the dual source. The operator[1] is given by:

$$\langle\mathcal{O}_{\Delta_-}\rangle_{\tilde{J}} = \frac{\delta\tilde{W}[\phi_+]}{\delta\phi_+}. \tag{2.10}$$

The fact that we are in the mass range (2.1) ensures that Δ_- is above the unitarity bound $\Delta_- > \frac{d-2}{2}$.

For spin-1 fields we have $\Delta_\pm = \frac{d}{2} \pm \sqrt{\frac{(d-2)^2}{4}+m^2}$, hence for massless spin-1 fields in AdS_4 we get $\Delta_- = 1$, $\Delta_+ = 2$ as in the case of conformally coupled scalar fields. As we will see, this fact implies that massless gauge fields and conformally coupled scalar fields share many important properties. For scalars in the range of masses (2.1), a generalized choice of boundary conditions is possible where the total energy is finite and conserved [27,28]:

$$\phi_+ = \phi_+(\phi_-) = \mathcal{F}'(\phi_-) \tag{2.11}$$

for some function $\mathcal{F}'$. Since ϕ_- is itself a gauge-invariant operator in the CFT, (2.11) deforms the boundary theory by a multiple-trace operator [29]. For example, in the case at hand, in order to obtain a generalized boundary condition that preserves all of the AdS isometries,

$$\phi_+ = \alpha\phi_-^2 \tag{2.12}$$

we need to deform the CFT by a marginal triple-trace operator $\int d^3x\phi_-^3(x)$.

In order to allow for generalized boundary conditions (2.11), the variational principle needs to be modified. Indeed, the usual variational principle implies either Dirichlet or Neumann boundary conditions, which correspond to the $\alpha=\infty$ and $\alpha=0$ limits of (2.12), respectively. In [30], and independently in [28], a recipe was given where the variational principle is well defined and automatically leads to boundary condition (2.11). This amounts to adding a boundary term to the action such that the boundary condition is enforced. Let us see this in some detail. Under usual Dirichlet/Neumann boundary conditions (2.4), the bulk path integral computes the CFT correlator with a fixed external source,

[1]For the conformal equivalence between ϕ_+ and ϕ_-, see [26].

$$Z[J] = \int_{\phi_\mp = J} \mathcal{D}\phi e^{-S[\phi]} = \langle e^{\int d^d x J(x) \mathcal{O}_{\Delta_\pm}(x)} \rangle_{\rm CFT}, \quad (2.13)$$

and integration over all the bulk fields should be understood here. On the other hand, the boundary condition (2.11) is achieved if we integrate over both the bulk and the boundary values of the scalar field:

$$\begin{aligned} Z &= \int \mathcal{D}\phi_- \int_{\phi|_{r=0}=\phi_-} \mathcal{D}\phi e^{-S[\phi]+S_{\rm bdy}[\phi_-]} \\ &= \int \mathcal{D}J e^{S_{\rm bdy}[J]} \langle e^{\int d^d x J(x) \mathcal{O}_{\Delta_+}(x)} \rangle_{\rm CFT}. \end{aligned} \quad (2.14)$$

Thus, the path integral is evaluated without any boundary conditions [30]. In general, the saddle point approximation for (2.14) gives two terms. The first one is the bulk equations of motion. The second one is proportional to

$$-\int d^3 x \phi_+ \delta\phi_- + \delta S_{\rm bdy}[\phi_-] \quad (2.15)$$

in other words, excluding the trivial case of Dirichlet boundary conditions, the general boundary condition is

$$\phi_+ = \frac{\delta S_{\rm bdy}}{\delta \phi_-}[\phi_-], \quad (2.16)$$

again for the relevant case of the conformally coupled scalar field in AdS_4. In [30] it was shown that, for quadratic $S_{\rm bdy}[\phi_-]$, the modified two-point functions extracted from this recipe agree with those of the field theory where an explicit double-trace deformation is introduced in the action. In this paper we will use a similar procedure for gauge fields.

Lorentzian signature

In Lorentzian signature there are a number of subtleties in the above picture. These have to do with the fact that propagating, normalizable modes now exist, and hence the solution is not uniquely determined by the boundary condition at the boundary plus regularity of the solution at infinity. In particular, the functional relation between ϕ_+ and ϕ_- is no longer fixed. The propagating modes correspond to a nontrivial choice of state in the boundary theory, and their wave functions have to be included in the path integral [31]. The solution that is analytically continued from the Euclidean corresponds to the vacuum state-in that case, the saddle point approximation to the bulk partition function gives the Euclidean CFT partition function, which when analytically continued gives the vacuum amplitude. In the Lorentzian, choosing any other solution than the regular one gives an amplitude between nontrivial states.

In this paper we will largely use the regular boundary condition. In Sec. II D we will derive this condition for gauge fields and give a purely Lorentzian interpretation of it: instead of as a boundary condition at $\Sigma_\pm$, it can be regarded as a condition for the flux to be purely incoming at the boundary. This is the natural boundary condition in the original M2-brane geometry. This implies that we can in fact impose this as an independent boundary condition at the boundary.

B. Gauge fields: solutions and dual interpretation

We will now solve the equations of motion for the gauge field. For later reference, we give the Lorentzian form of the action:

$$S[A] = \int d^4 x \sqrt{-g} \left(-\frac{1}{4g^2} F_{\mu\nu}^2 + \frac{\theta}{32\pi^2} e^{\mu\nu\lambda\sigma} F_{\mu\nu} F_{\lambda\sigma} \right). \quad (2.17)$$

The θ-term is the appropriate one for a spin manifold. It gives a Chern-Simons term at level $1/2$. If the manifold is not spin, the coefficient would have an additional factor of 2. The equations of motion that follow from the above matter part of the action are:

$$\nabla^\mu F_{\mu\nu} = 0, \quad (2.18)$$

In the Fefferman-Graham coordinate system (2.2), we can do the following asymptotic expansion for a massless field of spin 1:

$$\begin{aligned} A_r(r,x) &= A_r^{(0)}(x) + r A_r^{(1)}(x) + r^2 A_r^{(2)}(x) + \ldots \\ A_i(r,x) &= A_i^{(0)}(x) + r A_i^{(1)}(x) + r^2 A_i^{(2)}(x) + \ldots \end{aligned} \quad (2.19)$$

Solving Maxwell's equations then determines all the higher coefficients $A_r^{(n)}$, $A_i^{(n)}$, $n \geq 2$, in terms of the boundary conditions $A_i^{(0)}(x)$, $A_i^{(1)}$, $A_r^{(0)}$, $A_r^{(1)}$, as well as the metric. The radial components $A_r^{(0),(1)}$ are of course not real degrees of freedom as they can always be gauged away. Notice that $A_i^{(1)}$ is not left totally undetermined by the equations of motion. There is a constraint on its gauge-invariant part, which for future convenience we will denote $f_i(x)$:

$$\nabla_i^{(0)} f^i(x) = 0, \quad (2.20)$$

where the covariant derivative is defined with respect to the boundary metric $g_{(0)ij}$, boundary indices are raised and lowered with respect to this metric, and where we defined

$$f_i(x) = A_i^{(1)}(x) - \partial_i A_r^{(0)}(x) \qquad F_{ri}(r,x) = f_i(x) + \mathcal{O}(r), \quad (2.21)$$

with a conventional minus sign in Lorentzian signature. Of course, $f_i(x)$ is nothing but the boundary value of the electric field $E_i = F_{ri}$. The fact that the bulk equations only fix $f_i(x)$ up to an arbitrary conserved vector is called the Fefferman-Graham ambiguity.

Since $f_i(x)$ is conserved, in three dimensions and if $H^1(M) = 0$ where M is the boundary, we can introduce a vector v_i such that

$$f^i = \epsilon_{(0)}^{ijk} \partial_j v_k. \quad (2.22)$$

Again, the epsilon tensor is defined with respect to $g_{(0)ij}$. From now on, we will write $A_i(x)$ instead of $A_i^{(0)}(x)$ for the boundary value of the bulk field $A_i(r, x)$.

We can take advantage of conformal invariance to completely solve the Maxwell equations throughout the bulk. In Appendix B we do this for pure AdS. The general solution in that case is

$$A_i^{\mathrm{T}}(r, p) = A_i^{\mathrm{T}}(p)\cosh(|p|r) + \frac{1}{|p|} f_i(p)\sinh(|p|r), \tag{2.23}$$

in momentum space. Here A_i^{T} is the transverse, gauge-invariant part of the gauge field, $A_i^{\mathrm{T}}(p) = \Pi_{ij}A_j(p)$ with the projector defined in (1.4). By construction, $f_i(x)$ is also gauge invariant.

In this paper we will take advantage of the holographic interpretation in the dual CFT that is relevant to the Neumann and mixed boundary problem, as discussed in Secs. I and II A. In the standard CFT, the boundary values of the gauge field A (or alternately the magnetic field) are fixed in terms of a boundary source for a global $U(1)$ current. The boundary values of the electric field correspond to the conserved current. In the dual CFT, the global symmetry has been gauged and A has been promoted to a dynamical field that is integrated over in the path integral. Its one-point function corresponds holographically to the boundary value of the (transverse part of the) gauge field. It gives rise to a new conserved current $\tilde{f} = *dA$ which corresponds to the magnetic field and which is the new gauge-invariant observable of the theory saturating the unitarity bound $\Delta \geq 2$. Finally, a new background field $\boldsymbol{v}$ has been introduced which couples to this conserved current. This new background field corresponds to the bulk electric field via $f = *d\boldsymbol{v}$.

C. Holographic renormalization

We now summarize the analysis of the coupled gravity-Maxwell system and the holographic renormalization of the action, which is done in Appendix D.

The total Euclidean action is written in (D1). In the first order formalism, we simply replace the matter part of the action by the corresponding first order matter action (3.5):

$$S = -\frac{1}{16\pi G_N}\int d^4x\sqrt{g}(R - 2\Lambda) - \frac{1}{8\pi G_N}\int d^3x\sqrt{\gamma}K + S_{\mathrm{bulk}}[A, E] + S_{\mathrm{bdy}}[A, E]. \tag{2.24}$$

The components of the stress-energy tensor are

$$\begin{aligned} T_{\mu\nu} &= \frac{r^2}{\ell^2}\tau_{\mu\nu}; \\ \tau_{rr} &= \tfrac{1}{2}(E^2 - F^2), \\ \tau_{ri} &= E^j F_{ij} = \epsilon_{ijk}E_jF_k, \\ \tau_{ij} &= E_iE_j - F_iF_j - \tfrac{1}{2}g_{ij}(E^2 - F^2). \end{aligned} \tag{2.25}$$

The stress-energy tensor is manifestly invariant under electric-magnetic transformations (which are hyperbolic in Euclidean signature), as it should. In this formulation, it is obvious that the stress-energy tensor is zero if and only if

$$E_i = \varepsilon F_i, \tag{2.26}$$

where $\varepsilon^2 = 1$, i.e. the solution is (anti-)self-dual.

In Appendix D, Einstein's equations are solved in the coordinate system (2.2):

$$\begin{aligned} g(r, x) &= g_{(0)} + r^3 g_{(3)} + r^4 g_{(4)} + \cdots \\ \nabla^j g_{(3)ij} &= -\frac{16\pi G_N}{3\ell^2} f^j F_{ij} \\ g_{(4)ij} &= -\frac{4\pi G_N}{\ell^2}\left(f_i f_j - F_iF_j - \frac{1}{4}g_{ij}(f^2 - F^2)\right), \end{aligned} \tag{2.27}$$

where we have chosen a Ricci-flat boundary metric, hence $g_{(2)} = 0$. $g_{(3)}$ is otherwise undetermined by the field equations and has the interpretation as the stress-energy tensor of the boundary theory. This is the usual Fefferman-Graham ambiguity for the metric and it is fixed once one imposes regularity of the bulk solution. As explained in the Appendix, the middle equation above is a Ward identity that follows from the diffeomorphism invariance of the on-shell action. In fact, it is of the form

$$\nabla^j\langle T_{ij}\rangle = T_{ir} = -F_{(0)ij}f^j. \tag{2.28}$$

$\langle T_{ij}\rangle$ is also the quasilocal stress-energy tensor of Brown and York [32]. The above identity measures the matter flow through the boundary. From (2.25) we see that it vanishes if f_i is proportional to F_i, which are indeed AdS-preserving boundary conditions. Using the fact that the boundary current is

$$\langle J_i\rangle = -f_i, \tag{2.29}$$

the above is the expected field theory Ward identity [19,33].

In Appendix D we show that, after subtracting the divergent part of the effective action by adding the usual counterterm, there is no near-boundary contribution to the on-shell action coming from the gravity part of the action. Taking into account backreaction would result in triple and higher order terms in the boundary operators, which we may neglect since we are only interested in two-point functions. Therefore, in this approximation the only contribution is from the matter part of the action, which we analyze in the next subsection.

D. Regularity and incoming boundary conditions

We now impose the usual normalizability conditions in AdS [22,23,34]. The situation for gauge fields is similar to the case of conformally coupled scalar fields. Let us denote the boundary momentum by $p = (\omega, k)$. In Euclidean

signature, as well as for Lorentzian tachyonic modes $\omega^2 < k^2$, the general solution of the wave equation is

$$A_i^{\mathrm{T}}(r,p) = \frac{1}{2}\left(A_i^{\mathrm{T}}(p) + \frac{1}{|p|}f_i(p)\right)e^{|p|r} + \frac{1}{2}\left(A_i^{\mathrm{T}}(p) - \frac{1}{|p|}f_i(p)\right)e^{-|p|r}. \tag{2.30}$$

The first solution blows up at $r = \infty$. It is clear that if we demand

$$A_i^{\mathrm{T}}(p) + \frac{1}{|p|}f_i(p) = 0, \tag{2.31}$$

then regularity is ensured. One can check [35] that the solutions are normalizable. In position space the above is a nonlocal relation:

$$A_i^{\mathrm{T}}(x) = -\frac{i}{\Box^{1/2}}f_i(x) = -\int d^3y\frac{f_i(y)}{|x-y|^4}. \tag{2.32}$$

Notice that this is simply the linear boundary condition (2.50) where $m = -i\Box^{1/2}$ is an operator. For later convenience we rewrite the gauge-invariant regularity condition as

$$f = -\sqrt{-\Box}\Pi A. \tag{2.33}$$

For regular bulk solutions, the gauge field and field strength now take the following form:

$$\begin{aligned} A_i(r,x) &= A_i(x)e^{-|p|r} \\ F_{ri}(r,p) &= -|p|\Pi_{ij}A_j(p)e^{-|p|r} \\ F_{ij}(r,p) &= i(p_iA_j(p) - p_jA_i(p))e^{-|p|r} \\ F_i(r,p) &= i\epsilon_{ijk}p_jA_k(p)e^{-|p|r}. \end{aligned} \tag{2.34}$$

Using the equations of motion, it is also easy to see that if we impose (2.31) at $r = 0$ then we also have

$$A_i^{\mathrm{T}}(r,p) + \frac{1}{|p|}E_i(r,p) = 0 \tag{2.35}$$

for all r. This will ensure that the contribution from $r = \infty$ to the on-shell action vanishes. The transverse part of the gauge field is defined in (B17), hence the above equation is gauge invariant with respect to boundary gauge transformations of the dual CFT (see end of Sec. II B).

In Lorentzian signature, all the modes satisfying $\omega^2 > k^2$ are normalizable and so there is no regularity condition for these modes. There is, however, a natural choice of boundary condition [36] which corresponds to imposing that all waves are ingoing at the boundary—that is, all waves travel from the boundary into the bulk. Notice that this is true both at $r = 0$ and $r = \infty$; since we are dealing with pure AdS, waves that are ingoing at the boundary remain ingoing near $r = \infty$. From the field theory point of view, this choice is natural because it corresponds to defining the vacuum in the Lorentzian field theory as the analytically continued Euclidean vacuum; with this choice of boundary conditions, a Lorentzian transition function in the vacuum corresponds to the Euclidean partition function. This is the choice that leads to *absorption* cross section by the M2-brane. Let us explain this point. When solving the wave equation in the original M2-brane geometry [37],

$$\begin{aligned} ds^2 &= f^{-2/3}(-dt^2 + d\vec{x}^2) + f^{1/3}(dR^2 + R^2d\Omega_7^2) \\ f &= 1 + \frac{2^5\pi^2N\ell_{\mathrm{Pl}}^6}{R^6}, \end{aligned} \tag{2.36}$$

with a nontrivial four-form flux, a choice of *incoming* waves must be made at the horizon-in other words, only the modes describing infalling matter are kept [38,39] (see [40] for the higher-spin extension). Taking the near-horizon limit of (2.36) we get

$$ds^2 = \frac{\ell^2}{r^2}(dr^2 - dt^2 + d\vec{x}^2) + 4\ell^2d\Omega_7^2 \tag{2.37}$$

with $2\ell = \ell_{\mathrm{Pl}}/(2^5\pi^2N)^{1/6}$ where $R \sim 1/r$. Hence, the modes absorbed by the M2-brane will be modes traveling from the boundary towards the bulk. Such modes have the form $e^{i\omega t + i|p|r}$: indeed; for such modes, as time elapses the wave front increases r and so they propagate to the bulk. Since we are doing classical field theory near the M2-brane, it is natural to take the waves to be ingoing at the horizon. Using the change of coordinates in [36], it is easy to check that this remains true in the nonextremal case. We will now see this for the gauge field.

We write the general form of the solution of Maxwell's equations (2.23) as

$$\begin{aligned} A_i^{\mathrm{T}}(r,p) &= A_i^{\mathrm{T}}(p)\cos(|p|r + \frac{1}{|p|}A_{i(1)}(p)\sin(|p|r) \\ &= \frac{1}{2}\left(A_i^{\mathrm{T}}(p) + \frac{1}{i|p|}A_{i(1)}(p)\right)e^{i|p|r} \\ &\quad + \frac{1}{2}\left(A_i^{\mathrm{T}}(p) + \frac{1}{i|p|}A_{i(1)}(p)\right)e^{-i|p|r} \end{aligned} \tag{2.38}$$

where now $|p| = \sqrt{\omega^2 - k^2}$. Analytic continuation from the Euclidean is done by $|p_E| \to -i|p_L|$, therefore the regularity condition (2.31) turns into

$$A_{(0)}^{\mathrm{T}}(p) + \frac{1}{i|p|}A_{(1)}(p) = 0 \tag{2.39}$$

whenever $\omega > 0$. This leaves only the first factor in (2.38). Taking into account the fact that $A_{(0)}$ contains a factor of $e^{-i\omega t}$, we get a wave $e^{-i\omega t + i|p|r}$. This agrees with the behavior of the natural boundary condition in [36].

One can check that, by construction, the bulk fields are purely negative frequency in the far past and purely positive frequency in the far future, as one would expect from the analytic continuation of the Euclidean 2-point function which is the Feynmann Green's function.

Ingoing vacuum and vanishing action

Incoming boundary conditions of the type just found in the Lorentzian describe a special vacuum that we will now analyze in more detail. As we will see, they have exactly zero action. Expand the bulk solutions again in plane waves:

$$\begin{aligned} A(r,x) &= \int d^3p a(p) e^{i|p|r+ip\cdot x} + \text{c.c.} \\ &= \int_0^\infty d\omega \int d^2k (a(p) e^{i(|p|r-\omega t)} \\ &\quad + a^*(-p) e^{-i(|p|r+\omega t)}) e^{ikx} + \text{c.c.} \end{aligned} \tag{2.40}$$

where k is a 2-vector that labels spatial momenta along the boundary. As before, the ingoing vacuum requires vanishing of the second factor, therefore

$$a(p) = a(\omega, k) = 0 \quad \text{if } \omega < 0. \tag{2.41}$$

Notice that only the sign of ω matters here, with respect to which positive frequencies are defined. We now look at the boundary expansion of (2.40):

$$\begin{aligned} A(r,x) &= \int_0^\infty d\omega \int d^2k [A_{(0)}(p) \cos(|p|r) \\ &\quad + iB_{(1)}(p) \sin(|p|r)] e^{-i\omega t + ikx} + \text{c.c.} \end{aligned} \tag{2.42}$$

where $A_{(0)}(p) = a(p) + a^*(-p)$, $B_{(1)}(p) = a(p) - a^*(-p)$, and $A_{(0)}(p)$ defines the natural field theory vacuum $A_{(0)}(p)|0\rangle = 0$, since at $r = 0$ $B_{(1)}$ disappears. The ingoing vacuum (2.41) gives $A_{(0)}(p) = \text{sgn}(\omega) B_{(1)}(p)$. Defining the vacuum in terms of $A_{(0)}$ makes sense in the dual CFT, where $A_{(0)}$ is interpreted as an operator.

A similar analysis for the electric field gives:

$$\begin{aligned} E(r,x) &= \int_0^\infty d\omega \int d^2k [G_{(1)}(p) \sin(|p|r) \\ &\quad + f(p) \cos(|p|r)] e^{-i\omega t + ikx} + \text{c.c.} \end{aligned} \tag{2.43}$$

where $f(p) = -i|p| B_{(1)}(p)$ and $G_{(1)}(p) = |p| A_{(0)}(p)$. The ingoing boundary condition now gives

$$f(p) = -i|p| \text{sgn}(\omega) A_{(0)}(p) \tag{2.44}$$

which is the generalization of (2.39) to arbitrary frequencies.

It is now straightforward to check that for such states the bulk action is zero. As we will prove in the next section, the on-shell action is proportional to $\int d^3x f A$. Expanding this in Fourier modes, and using (2.44), this is readily seen to be identically zero. On the other hand, the Hamiltonian is nonzero.

E. Self-duality and topologically massive theories

We will now analyze self-dual solutions of the Euclidean field equations. The Euclidean action is:

$$\begin{aligned} S &= \frac{1}{4g^2} \int d^4x \sqrt{g} F_{\mu\nu} F^{\mu\nu} \\ &\quad + \frac{\theta}{32\pi^2} \int d^4x \sqrt{g} \epsilon^{\mu\nu\lambda\sigma} F_{\mu\nu} F_{\lambda\sigma}. \end{aligned} \tag{2.45}$$

We will prove three main results concerning self-dual solutions. First, we will show that there is a one-to-one correspondence between self-dual solutions in the bulk and a choice of boundary conditions that we will call "self-dual boundary conditions." Second, we will show that bulk self-dual solutions together with a boundary condition that breaks conformal invariance leads to the self-dual topological theory of [17]. Finally, we will analyze regularity of the solutions and show that massive solutions are regular for negative value of the deformation parameter.

1. Self-duality and topologically massive theories

We start with (anti-)self-dual solutions in the bulk. These satisfy:

$$F_{\mu\nu} = -\frac{\varepsilon}{2} \epsilon_{\mu\nu\alpha\beta} F^{\alpha\beta}, \tag{2.46}$$

This equation is conformally invariant, therefore in the Poincare patch this is an equation on $\mathbb{R}^4_+$. ε can take the values ± 1, 0. We get:

$$F_{ri} = -\frac{\varepsilon}{2} \epsilon_{ijk} F_{jk}, \qquad F_{ij} = -\varepsilon \epsilon_{ijk} F_{rk}. \tag{2.47}$$

The condition (2.46) is a first order equation. Together with the Bianchi it implies the field equation. Therefore it determines one of the two boundary conditions. From (2.47):

$$f_i = -\varepsilon \epsilon_{ijk} \partial_j A_k. \tag{2.48}$$

This condition is what we will call the "self-dual boundary condition."

Next we will show that, given the self-dual boundary condition (2.48), any solution of the bulk equations is self-dual and hence satisfies (2.47) throughout the whole of AdS. This is easy to see. We use the solution of the equations of motion (2.23) to work out the field strength,

$$\begin{aligned} F_{ri}(r,p) &= |p| A_i^{\text{T}}(p) \sinh(|p|r) + f_i(p) \cosh(|p|r) \\ F_{ij}(r,p) &= i(p_i A_j - p_i A_j) \cosh(|p|r) \\ &\quad + \frac{i}{|p|} (p_i f_j - p_j f_i) \sinh(|p|r). \end{aligned} \tag{2.49}$$

Filling the boundary condition (2.48), it is easy to see that the self-duality condition (2.47) is satisfied.

This also implies that the self-dual boundary condition (2.48) is enough to ensure everywhere vanishing of the stress-energy tensor (see Sec. II C).

Now we will consider massive deformations of the boundary theory. This is analogous to the scalar field case, where a choice of massive boundary condition cor-

responds to adding relevant terms to the boundary theory. In this case, the relevant massive boundary condition is:

$$A_i^{\mathrm{T}} = \frac{1}{m} f_i \tag{2.50}$$

where m is a dimensionful parameter. We will call this the "massive boundary condition".

Combining (2.50) with (2.48), we see that the boundary value of the gauge field satisfies the following equation:

$$A_i^{\mathrm{T}} = -\frac{\varepsilon}{m} \epsilon_{ijk} \partial_j A_k^{\mathrm{T}}. \tag{2.51}$$

Surprisingly, this is the self-dual Abelian theory in three dimensions of [17]! Deser and Jackiw showed [18] that this theory is equivalent to topologically massive QED in three dimensions. We will come back to its interpretation later.

2. Regularity and massive solutions

Finally, we analyze regularity of the different solutions discussed here. Regularity relates the two boundary conditions of the solution in a nonlocal way, in this case $A_i(x)$ and $f_i(x)$, as we showed explicitly in (2.31). We will first construct a regular self-dual solution, that is we combine Eq. (2.31) with (2.48). We get the following nonlocal equation for the boundary gauge field:

$$A_i^{\mathrm{T}} = -\frac{i\varepsilon}{\Box^{1/2}} \epsilon_{ijk} \partial_j A_k^{\mathrm{T}}, \tag{2.52}$$

again a gauge-invariant equation. This gives the most general regular self-dual solution in the bulk. We solve this equation in Appendix C. In fact, this is the most general solution with vanishing stress-energy tensor.

Next we construct regular massive solutions. Thus we combine (2.31) and (2.50). This gives the following condition in momentum space:

$$|p| = -m. \tag{2.53}$$

Notice that in this case regularity does not act as a boundary condition on the boundary values of the fields. Instead, it puts the momentum of the boundary theory on the mass shell. Obviously, this is only possible if $m < 0$. This means that, for regular solutions, the deformation parameter of the boundary theory has negative sign. Thus, in this case we can still impose a second boundary condition. It is indeed possible to combine the self-dual boundary condition (2.48) with the massive one (2.50), subject to the regularity condition (2.53).

This fact is actually generic: the deformation parameter in the linear boundary condition, which corresponds to the deformation parameter of the double-trace deformation, is required to be negative in order for two-point functions of dual operators to be positive definite. This is reminiscent of the regularity analysis in [16] for triple-trace deformations and the constraints on the deformation parameters found there.

F. Abelian instantons in AdS_4

We have obtained regular self-dual solutions of the Euclidean equations of motion. We can now ask whether these solutions actually have finite action and correspond to instantons. In flat space there are of course no Abelian instantons because one cannot form a topological number. However, things may change in AdS due to the boundary terms.

The on-shell value of the action is computed in the next section:

$$S_{\text{on-shell}} = -\int d^3p A(p) f(-p), \tag{2.54}$$

where we have also included the contribution of explicit boundary terms [worked out in (5.13)]. We will now ask whether there are solutions such that this number is finite. We consider the regular instanton solutions of Sec. II E. The explicit solution of (2.52) is given in (C11) of the Appendix. First of all we remark that the action simply reduces to the Chern-Simons term $\int A \wedge dA$, which for these solutions can be written as:

$$S_{\text{on-shell}} = \frac{1}{g^2} \int d^3p |p| A_i^{\mathrm{T}}(p) A_i^{\mathrm{T}}(-p). \tag{2.55}$$

We should remember the constraints (C12). In the notation in the Appendix,

$$A_i^{\mathrm{T}}(p) A_i^{\mathrm{T}}(-p) = -\left(1 + \frac{4p_1^2 p_2^2}{(p_1^2 + p_2^2)^2}\right) f(p) f(-p). \tag{2.56}$$

Redefine now

$$F(p) = \sqrt{1 + \frac{4p_1^2 p_2^2}{(p_1^2 + p_2^2)^2}} f(p). \tag{2.57}$$

The action becomes

$$S_{\text{on-shell}} = \frac{2\sqrt{2}}{g^2} \int dp_1 \int dp_2 \sqrt{p_1^2 + p_2^2} |F(p)|^2 \tag{2.58}$$

and the integral is over the remaining $\mathbb{R}^2$. The following choice will obviously make the action finite:

$$F(p) = \frac{2^{1/4}\sqrt{\pi a \lambda}}{a^2 + p_1^2 + p_2^2}. \tag{2.59}$$

λ is an arbitrary dimensionless constant. a is an arbitrary scale and as usual for instantons the value of the action is independent of it. We get

$$S_{\text{on-shell}} = \frac{8\pi^2}{g^2} |\lambda|. \tag{2.60}$$

The value of $|\lambda|$ can be fixed by fixing, for example, the holonomy of the gauge field. Since λ must be independent of the coupling, it is remarkable that the action is in fact proportional to $1/g^2$.

III. S-DUALITY IN AdS_4

We will now analyze the action of S-duality on the bulk action. Since electric-magnetic rotations are not canonical transformations, they act nontrivially on the canonically conjugated variables A and E. It is well known that, whereas electric-magnetic transformations are a manifest symmetry of the equations of motion, they are not a manifest symmetry of the action. They have to be realized at the level of the gauge potential, and this is nontrivial. The solution of this problem is well known (see for example [41–44]). Here we use the first order formulation in [45,46].

In AdS, electric-magnetic invariance is broken due to the presence of the boundary. Therefore, an electric-magnetic transformation changes the boundary terms in the action. We will first explicitly show the S-duality of the bulk part of the action and compute the boundary terms. We will then generalize this to the full $SL(2,\mathbb{R})$ ($SL(1,1)$ in the Euclidean case). Witten's proof of S-duality for Abelian theories involves introducing an additional two-form field in the action, together with an enlarged gauge symmetry [47]. S-duality is then showed after integrating out two different fields. We will instead give two elementary proofs that rely only on the first order formalism of the action. The advantage of our first proof is that it does not involve integrating out fields but only a simple field redefinition. The price we pay is that Lorentz invariance is not manifest. In this proof, we write the action in terms of first order fields E and A, where E is the redefined conjugate momentum. In terms of these variables, the action is manifestly invariant under S-duality including the θ-angle. The second proof, given in Appendix A, involves integrating out either E or A, to get two actions with the same form which are dual to each other. It presents some similarities to the methods in [47,48].

Then, using holographic renormalization, we will show how S-duality acts on the boundary effective potential, and on the boundary 1- and 2-point functions, and how the Legendre transform arises.

T-transformations simply act by $\theta \to \theta + 2\Pi$ in (2.17). These do not leave the action invariant but they do leave the partition function invariant.

A. The first order formalism

We first work out the first order form of the action (2.17) in Lorentzian signature.[2] We define canonical momenta:

$$\Pi^i = \frac{1}{\sqrt{-G}}\frac{\delta S}{\partial_r A_i} = -\frac{1}{g^2} g^{rr} g^{ij}(\partial_r A_i - \partial_i A_r). \tag{3.1}$$

It is easy to see that, in the Fefferman-Graham coordinate system 2.2, one gets the same Hamiltonian and action if one defines all quantities with respect to the conformally rescaled metric $\tilde{G}_{\mu\nu} = \frac{r^2}{l^2} G_{\mu\nu}$:

$$E^i = \frac{1}{\sqrt{-g}}\frac{\delta S}{\delta \partial_r A_i} = -\frac{1}{g^2} g^{ij}(\partial_r A_j - \partial_j A_r). \tag{3.2}$$

This amounts to removing the factors of g^{rr} from the formulas and raising and lowering indices with the boundary metric $g_{ij}(r,x)$. This is the procedure we will adopt in what follows, which of course is possible due to conformal invariance of the matter part of the action. From the analysis in Appendix D, Eq. (D6), it follows that for a Ricci-flat boundary metric the first correction to the boundary metric appears at order r^3 (this is the case, for instance, of the AdS black hole), therefore most boundary quantities we will be interested in are be unaffected by that.

The θ-term is a boundary term and does not contribute to the canonical momentum. The Hamiltonian now takes the following form:

$$H = \int d^4x\left(-\frac{g^2}{2}E^2 - \frac{1}{2g^2}F^2 + E^i\partial_i A_r\right) - \frac{\theta}{8\pi^2}\int d^3x AF. \tag{3.3}$$

Notice the unusual sign, which is due to the fact that we are doing radial quantization. The equations of motion give:

$$E_i = -\frac{1}{g^2}(\partial_r A_i - \partial_i A_r) \qquad \partial_r E = \frac{1}{g^2} * \mathrm{d}F \qquad \nabla^i E_i = 0. \tag{3.4}$$

The action, written in first order form, now gives

$$\begin{aligned} S &= \int d^4x\left(E^i(\partial_r A_i - \partial_i A_r) + \frac{g^2}{2}E^2 + \frac{1}{2g^2}F^2 - \frac{\theta}{4\pi^2}(\partial_r A)F\right) \\ &= \int d^4x\left(E^i(\partial_r A_i - \partial_i A_r) + \frac{g^2}{2}E^2 + \frac{1}{2g^2}F^2\right) + \frac{\theta}{8\pi^2}\int d^3x AF. \end{aligned} \tag{3.5}$$

The bulk part of the action has the obvious symmetry $E' = F$, $F' = -E$, $g' = 1/g$, $\theta' = -g^4\theta$. In the Euclidean signature action, the quadratic term in E_i has a $-$ sign instead.

A_r is a Lagrange multiplier. Its equation of motion gives the Gauss law

$$\nabla^i E_i = 0. \tag{3.6}$$

We can solve the latter by

$$E^i(r,x) = \epsilon^{ijk}\partial_j v_k(r,x) \tag{3.7}$$

where ϵ^{ijk} is the epsilon-tensor in the boundary metric $g_{ij}(r,x)$. Notice that E_i and v_i are r-dependent. We will denote

[2]For AdS/CFT in the Hamiltonian formalism for gravity and scalars and the role of the canonical momenta in correlation functions, see [49].

$$v_i = v_i(x) = v_i(0, x) \qquad f_i = f_i(x) = E_i(0, x). \tag{3.8}$$

One can fix a gauge $A_r = 0$, although this is not necessary. As we mentioned earlier, there is a residual gauge invariance that leaves this gauge fixed, namely $A_\mu \to A_\mu + \partial_\mu \varphi$ where $\varphi = \varphi(x)$ has no r-dependence. These residual gauge transformations will act on the boundary value of the gauge field $A_i(x)$ and account for the gauge invariance of the dual CFT.

We fill in the equations of motion and get for the on-shell action:

$$S_{\text{on-shell}} = \int d^3x\left(-\frac{1}{2}EA + \frac{\theta}{8\pi^2}AF\right). \tag{3.9}$$

B. Proof of S-duality

We will give a proof of S-duality which, to our knowledge, is new. It relies on recasting the first order form of the action in a way that is manifestly S-duality invariant. We first remark that, since E is a Lagrange multiplier, we can make the following redefinition:

$$\mathsf{E} = E - \frac{\theta}{4\pi^2}F. \tag{3.10}$$

Since F is conserved, this is compatible with a conservation constraint on E. In fact, written as a function of E the action takes the form:

$$S = \int d^4x\left(\mathsf{E}^i(\partial_r A_i - \partial_i A_r) + \frac{1}{2}g^2\mathsf{E}^2 + \frac{1}{2g^2}\left(1 + \frac{g^4\theta^2}{(4\pi^2)^2}\right)F^2 + \frac{\theta g^2}{4\pi^2}\mathsf{E}F\right). \tag{3.11}$$

With this rewriting, we have eliminated the boundary term. Indeed, the θ-angle in this form is not a boundary term.

As is easy to check, this form of the action has manifest S-duality invariance:

$$\mathsf{E}' = F \qquad F' = -\mathsf{E} \qquad \tau' = -\frac{1}{\tau}, \tag{3.12}$$

where

$$\begin{aligned} \tau &= \frac{\theta}{4\pi^2} + \frac{i}{g^2} = \tau_1 + i\tau_2 \\ \frac{\theta'}{4\pi^2} &= -\frac{\frac{\theta}{4\pi^2}}{\frac{\theta^2}{(4\pi^2)^2} + \frac{1}{g^4}} = \tau_1' = -\frac{\tau_2}{\tau_1^2 + \tau_2^2} \\ (g')^2 &= \frac{1}{g^2} + \frac{g^2\theta^2}{(4\pi^2)^2}, \\ \tau_2' &= \frac{\tau_2}{\tau_1^2 + \tau_2^2}. \end{aligned} \tag{3.13}$$

More precisely, we get

$$S[A', \mathsf{E}'] = S[A, \mathsf{E}] + \int d^3x\mathsf{E}A. \tag{3.14}$$

In other words, in terms of E the boundary term that we get from the variation is the *usual* one that we get in the absence of the θ-angle.

One can see S-duality directly in the original first form order of the action (3.5). We rewrite the S-duality transformations (3.12) in terms of the original variables

$$E' = \left(1 + \frac{\theta\theta'}{(4\pi^2)^2}\right)F - \frac{\theta'}{4\pi^2}E \qquad F' = -E + \frac{\theta}{4\pi^2}F \tag{3.15}$$

where θ' is the transformed coupling as in (3.13). Of course, this form of S-duality still satisfies $S^2 = -1$, as it should. It is now easy to check that this leaves the action unchanged, up to boundary terms:

$$S[A', E'] = S[A, E] + \int d^3x\left(E - \frac{\theta}{4\pi^2}F\right)A. \tag{3.16}$$

Of course, the extra terms are just the ones in (3.14), as it should. This gives a direct proof of S-duality that does not involve any field redefinitions or integrating out fields.

It is interesting to note that the above can be simply rewritten as:

$$S[A', E'] = S[A, E] - \int A' \wedge dA. \tag{3.17}$$

This is in agreement with the definition of S-duality in [8] from the CFT point of view. In fact, the above is like a Legendre transform between A and a field of the same dimension A'. Using the fact that S squares to -1, we get

$$S[A''] = S[A], \tag{3.18}$$

so we see that, whereas S is not an invariance of the boundary theory, S^2 is. The above agrees with Witten's definition of S-duality in CFT [8].

C. $SL(2, \mathbb{R})$ duality

We can generalize $SL(2, \mathbb{Z})$ invariance of the action to the full $SL(2, \mathbb{R})$ that is relevant classically and plays a role in M-theory compactifications. Thus, we will generalize S-duality to include a full $SO(2)$ transformation. Let us define $g^* = g^*(g, \theta)$ and $\theta^* = \theta^*(\theta, g)$ as the couplings that we get after the $SL(2, \mathbb{R})$ transformation. We will take E and F to transform as

$$E^* = aE + bF \qquad F^* = cE + dF. \tag{3.19}$$

Requiring invariance of the potential terms in (3.5) gives

$$\begin{aligned} g^*E^* &= \eta g E + \delta\sqrt{1 - \eta^2}F/g \\ F^*/g^* &= -\delta\sqrt{1 - \eta^2}gE + \eta F/g \end{aligned} \tag{3.20}$$

where $\delta^2 = 1$ is an arbitrary sign which simply corresponds to the choice of sign in the square root. There is another sign that we have fixed here. The sign in front of F on the right-hand side of the above two equations is not

fixed by invariance of the potential terms. It is fixed by invariance of the kinetic term (up to boundary terms), and the above choice is the one that renders the full bulk action invariant. η is arbitrary and parametrizes the rotation part of the transformation. The transformed action is

$$S[A^*, E^*] = S[A, E] + \int d^3x \Big[\Big(\frac{\delta}{2} g^2 \eta \sqrt{1-\eta^2} + g^2 g^{*2} \frac{\theta^*}{8\pi^2}(1-\eta^2) \Big) f v + \Big((1-\eta^2) - g^{*2} \frac{\theta^*}{4\pi^2} \delta \eta \sqrt{1-\eta^2} \Big) f A + \Big(-\frac{\delta \eta}{2g^2} \sqrt{1-\eta^2} + \frac{g^{*2}}{g^2} \frac{\theta^*}{8\pi^2} \eta^2 - \frac{\theta}{8\pi^2} \Big) F A \Big]. \tag{3.21}$$

It is easy to check that the general transformation (3.20) automatically preserves the stress-energy tensor, as it should. It is also easy to see that the transformation (3.20) has unit determinant. In fact, it is most convenient to parametrize the above $SO(2)$ transformation as

$$\begin{pmatrix} g^* E^* \\ F^*/g^* \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} gE \\ F/g \end{pmatrix}, \tag{3.22}$$

where without loss of generality we absorbed δ in the sign of α and $\eta = \cos\alpha$. This transformation has an analytical continuation to Euclidean signature:

$$\begin{pmatrix} g^* E^* \\ F^*/g^* \end{pmatrix} = \begin{pmatrix} \cosh\sigma & \sinh\sigma \\ \sinh\sigma & \cosh\sigma \end{pmatrix} \begin{pmatrix} gE \\ F/g \end{pmatrix}, \tag{3.23}$$

which is a symmetry of the Euclidean bulk action up to boundary terms. The Euclidean transformation again preserves the stress tensor and $dE \wedge dF$.

We also notice the regularity condition is invariant under the duality transformations. This can be seen as (2.33) implies

$$F = -t\sqrt{-\Box}\Pi v. \tag{3.24}$$

Under the transformation

$$E^* = cE + sF = -c\sqrt{-\Box}\Pi A - ts\sqrt{-\Box}\Pi v = -\sqrt{-\Box}\Pi A^* \tag{3.25}$$

where implicit factors of g, g^* are understood.

If we take $g^* = g'$ and $\theta^* = \theta'$ as in (3.13), we get back the S-duality transformation (3.15) if we set in addition

$$\cos\alpha = \frac{\frac{g^2\theta}{(4\pi)^2}}{\sqrt{1 + \frac{g^4\theta^2}{(4\pi^2)^2}}}, \qquad \sin\alpha = \frac{1}{\sqrt{1 + \frac{g^4\theta^2}{(4\pi^2)^2}}}. \tag{3.26}$$

Fill this in (3.21) we recover (3.16) as we should. If we set instead $g^* = 1/g$ and $\theta' = -g^4\theta$ then

$$\sin\alpha = 1 \tag{3.27}$$

in order to obtain the S-dual transformation of (3.5). Notice in both cases $g^{*2}\theta^* = -g^2\theta$, which implements $S^2 = -1$.

We note here that invariance of the bulk part of the action under the transformation (3.20) does not require the couplings to transform as (3.13). As we see, g can actually be rescaled away, and θ does not even enter (3.20). To find out the transformation of g, the rest of the fields in the supermultiplet must be taken into account. In our case, as we will see in section IV D the relevant theory is the $\mathcal{N} = 2$ supergravity in four dimensions, where $g \sim G_N/\ell 2 \sim 1/N^{3/2}$. Therefore, electric-magnetic duality will only be a symmetry of the action (up to boundary terms) if $g^* = g$. On the other hand, (3.13) relates theories in asymptotically $\mathrm{AdS}_4 \times S^7$ of different size. In such a situation, the duality of [8–10] which we prove here for any $g^*(g, \theta)$, $\theta^*(g, \theta)$ might actually be used to explore small AdS spaces.

Since $g^*(g, \theta)$ is in general an arbitrary function, we can in fact get the T-generator from (3.20) by choosing $g^* = g$, $\cos\alpha = 1$ and $\theta^* = \theta + 1$. Using (3.21), this indeed simply shifts the θ-term in the action. In this sense, our transformations are not $SO(2)$ but we in fact generate the full $SL(2, \mathbb{R})$.

IV. GENERALIZED BOUNDARY CONDITIONS

Our goal is to study massive deformations of three-dimensional CFT's and how to describe them from the bulk. We will now develop the formalism to impose general boundary conditions for gauge fields and compute their on-shell action.

A. Variation and consistent boundary conditions

In the first order formalism we vary the action with respect to E and A. The usual variational principle imposes on us a choice of Dirichlet or Neumann boundary conditions:

$$\delta_A S_{\text{bulk}}[A, E] = -\int d^3x E^i \delta A_i|_{r=0} + (\text{eom}). \tag{4.1}$$

so that we get for the two independent fields:

$$\delta A_i|_{r=0} = 0 \qquad \text{Dirichlet} \qquad E^i|_{r=0} \qquad \text{Neumann}. \tag{4.2}$$

Since we are assuming that the boundary topology is trivial and we integrate out A_r, E_i is expressed in terms of v_i as in (3.7) and there is no difference whether we vary with respect to E or v.

The variation of the action (4.1) also contains a contribution from the same boundary term evaluated at $r = \infty$. For regular solutions (2.34), using (2.35) this contribution is seen to be zero.

As explained in Sec. II A for the scalar field case, in order to enforce general boundary conditions we can add a boundary term that will modify the stationarity conditions (4.2):

$$S[E, A] = S_{\rm bulk}[E, A] + S_{\rm bdy}[E, A]. \tag{4.3}$$

The Dirichlet condition $\delta A_i = 0$ is obviously always a possible boundary condition. We will consider the generalization of the Neumann boundary condition. Varying (4.3) and demanding that it be stationary, we find

$$\delta_E S_{\rm bdy}[E, A] = 0 \qquad \delta_A S_{\rm bdy}[E, A] = E^i(0, x). \tag{4.4}$$

In other words, in order for the action (4.3) to be stationary under variations we need to require

$$\frac{\delta S_{\rm bdy}}{\delta f_i}[f, A] = 0 \qquad \frac{\delta S_{\rm bdy}}{\delta A_i}[f, A] = f^i. \tag{4.5}$$

These two equations define our set of boundary conditions.

We can now obtain the on-shell effective action by adding the boundary term to (3.9):

$$S_{\rm on\text{-}shell}[f, A] = -\frac{1}{2}\int d^3x A_i f^i + S_{\rm bdy}[f, A], \tag{4.6}$$

and f and A satisfy (4.5). As usual, we are assuming that there are no boundary contributions from $|x| = \infty$, while we keep boundary terms at $r = 0$. As explained in Sec. II A (see also Sec. IV C), this can always be arranged by a choice of boundary terms or boundary conditions there [31]. Unless the regularity condition at $r = \infty$ is imposed, the above only describes the contribution to the effective action coming from $r = 0$. Notice that thanks to the classical conformal invariance of the bulk action, the on-shell action is completely finite at $r = 0$ without the need to include any counterterms for the matter fields.

B. General boundary action

General boundary couplings

We now construct the most general action with marginal operators and operators of dimension 4 that is gauge invariant, local, and covariant in v and A[3]:

$$S_{\rm bdy}[v, A] = \int d^3x\Big(\frac{1}{2}\alpha AF + \frac{1}{2}\beta vf + \gamma Af + \frac{1}{2}\delta F^2 + \frac{1}{2}\epsilon f^2 + \mu fF - A\tilde{\mathcal{J}} - fJ\Big) \tag{4.7}$$

(for notation, see Sec. I), where we leave the six coefficients arbitrary. $\tilde{\mathcal{J}}$ is necessarily conserved by gauge invariance. This means there is a $\tilde{J}$ such that $\tilde{\mathcal{J}} = *\mathrm{d}\tilde{J}$. We will also define $\tilde{\mathcal{J}} = *\mathrm{d}J$. The boundary conditions (4.5) derived from the above action take the form:

$$\begin{aligned} \beta\Pi v + \gamma\Pi A + \epsilon f + \mu F &= J \\ \alpha F + (\gamma - 1)f + \delta * dF + \mu * df &= \tilde{\mathcal{J}}, \end{aligned} \tag{4.8}$$

both in Euclidean and in Lorentzian signature.

Taking the derivative of the first equation, we get

$$\begin{aligned} \beta f + \gamma F + \epsilon * df + \mu * dF &= \mathcal{J} \\ \alpha F + (\gamma - 1)f + \delta * dF + \mu * df &= \tilde{\mathcal{J}}. \end{aligned} \tag{4.9}$$

The equations have an obvious symmetry under simultaneous exchange $f \leftrightarrow F$, $\alpha \leftrightarrow \beta$, $\varepsilon \leftrightarrow \delta$ and $\gamma \leftrightarrow \gamma - 1$. Notice that γ plays a special role since it couples both equations. The two limiting cases $\gamma = 0$ and $\gamma = 1$ are in fact each other's S-duals.

The above system can be reduced to

$$\begin{aligned} a_1 f + a_2 F + a_3 * df &= 0 \\ b_1 f + b_2 F - a_3 * dF &= 0, \end{aligned} \tag{4.10}$$

where the constants a_i, b_i are given in (C19) and we have set the sources to zero. It is possible to solve this system in general. One gets the following equation for A:

$$(1 + a\Box)A = bF. \tag{4.11}$$

The values of a, b are given in (C20). There is a similar equation for f, which for generic values of the constants α, β, γ, δ, ϵ, μ is independent of the above. Again, for nonzero b this is a (higher-derivative) massive equation for A_i. In fact, it is again the self-dual equation for the gauge field A, where the mass m is now an operator, namely $m = (1 + a\Box)/b\varepsilon$.

In Appendix C 4 we give one further example, corresponding to near-extremal solutions.

C. Symplectic flux through the boundary

In Sec. II D we derived the normalizability condition for Euclidean solutions. For nontachyonic Lorentzian solutions, all solutions are normalizable, and the Euclidean boundary condition turns into an ingoing boundary condition near the brane. In order to have a well-defined quantization problem, one also needs to ensure that the symplectic structure is finite and conserved [35]. Whereas this is automatically ensured for Neumann or Dirichlet boundary conditions, it needs to be checked for generalized boundary conditions.

Given either of the boundary conditions

$$f_i = J_i^f \qquad A_i = J_i^A, \tag{4.12}$$

vanishing of the symplectic flux in four dimensions requires that the matrices

$$\frac{\delta J_i^f(x)}{\delta A_j(x')} \quad \text{and} \quad \frac{\delta J_i^A(x)}{\delta f_j(x')} \tag{4.13}$$

[3] *A priori* there is no reason to require that the action be local in v and A. In fact, the effective action will be nonlocal, as we discuss in Sec. VI. The action described here corresponds to the simplest choice of local boundary conditions for the gauge field.

be symmetric. For our choice of boundary terms, the explicit expressions for J_i^f and J_i^A follow from (4.8). It is then straightforward to check that the symmetry condition on (4.13) is met. Evaluating (4.13), we encounter two types of structures: $\delta_{ij}\delta(x-x')$ and $(*d)_{ij}\delta(x-x')$ (or the inverse of the latter). Both terms are symmetric. Symmetry of the latter is shown using partial integration.

In the scalar field case, there is an ultralocality requirement on the sources (4.13) [35] in order to avoid partial derivatives contributing to the symplectic flux. For gauge fields, this requirement would seem to be spoiled by the presence of, for example, Chern-Simons terms. A modified symplectic structure can be defined [8,35] such that there is no net contribution to the symplectic flux through the boundary.

D. Supersymmetric boundary conditions

Another interesting aspect of the generalized boundary conditions is which ones give solutions preserving certain amount of supersymmetry asymptotically. This then tells us which deformations of the boundary CFT are supersymmetric deformations. The conclusion however usually does depend on how we embed the truncated bosonic field content in a bigger supergravity theory.

In our case the bosonic fields consist of only the graviton and Abelian gauge field in asymptotically AdS_4 space, in particular, we have no scalars. The minimal theory to embed these as a consistent truncation is the four dimensional gauged $\mathcal{N}=2$ supergravity [50] where our fields furnish the gravity multiplet. This is an extended supergravity theory with gauge symmetry $SO(2)=U(1)$ which can be further viewed as the consistent $SO(5)^V$ truncation of the gauged maximal $SO(8)$ supergravity [51]. This special $SO(5)$ invariance projects out all scalar fields and our theory is now indeed a sector of M-theory dimensionally reduced on S^7.

In order to find the supersymmetric boundary conditions, we follow the procedure of [52]. First, since fermionic fields contribute to the symplectic flux through AdS boundary, we impose conditions on the gravitini fields so that the latter vanishes. Second, we find boundary conditions for the gauge fields consistent with supersymmetry transformations. We relegate the detailed analysis to Appendix E and simply present the conclusion here. We find that the only supersymmetric boundary condition for the gauge field is the Dirichlet boundary condition:

$$A_i = 0, \tag{4.14}$$

which corresponds to the boundary action $S_{\rm bdy} = \int d^3x f A$, in Lorentzian signature.

V. THE FLOW OF THE BOUNDARY CFT

In this section we will analyze the electric-magnetic flow of a general CFT with an AdS_4 dual with $\mathcal{N}=2$ supersymmetry. This is done by studying how generalized boundary conditions transform under electric-magnetic transformations. For convenience we will work in the $SL(2,\mathbb{R})$ formulation ($SL(1,1)$ in the Euclidean) relevant at the level of supergravity. Of course, quantum mechanically only a discrete subgroup of this is a symmetry. The latter is obtained by evaluating our formulas at discrete points.

A. Flow of Dirichlet-Neumann boundary conditions

We now investigate the properties of the effective action under $SL(2,\mathbb{R})$ transformations. Since T-transformations simply act as translations of θ, in this section we will work with the compact part, i. e. electric-magnetic rotations. For this purpose, we may take $\theta^* = \theta = 0$. In the Lorentzian case, electric-magnetic duality acts as $SO(2)$ transformations. As seen above, the relevant pair is not (E,A), but (v,A) and their corresponding dual field strengths (E,F). In the Euclidean case, we get $SO(1,1)$ instead.

The relevant transformation is

$$g^* v_i^* = c g v_i + s\frac{A_i}{g} \qquad \frac{A_i^*}{g^*} = t s g v_i + c\frac{A_i}{g}. \tag{5.1}$$

Here we recall $t=-1$ for Lorentzian signature and 1 for Euclidean. Also c, s stands for $\cos\alpha$, $\sin\alpha$ in the former and $\cosh\sigma$, $\sinh\sigma$ in the latter. Taking the curl of this rotates electric and magnetic fields into each other. In the Euclidean case these are boosts, so we will loosely refer to them as the $SO(1,1)$ "flow" from now on. The first order bulk action is invariant under this up to a boundary term (3.21), which present below:

$$S_{\rm bulk}[v^*,A^*] - S_{\rm bulk}[v,A] = -t\int\left(s^2 A\wedge dv + \frac{1}{2}sc(g^2 v\wedge dv + \frac{t}{g^2}A\wedge dA)\right). \tag{5.2}$$

Of course, the boundary term $S_{\rm bdy}[v,A]$ of the effective action will transform as well, and its transformation properties need to be analyzed case by case. We will do this after introducing the general boundary conditions.

By minimizing the $SO(2)$-transformed action, it is not hard to check that the boundary conditions required for stationarity satisfy:

$$A_i + \frac{c}{s}v_i = 0, \tag{5.3}$$

This boundary condition is still linear, yet in contrast to (2.50) it preserves conformal invariance as it does not introduce a new scale. Thus, introducing the $SO(2)$ flow we probe a one-parameter family of conformally invariant boundary conditions. The $SO(2)$ flow corresponds to a 1-parameter deformation of boundary theories.

It is clear that for $c/s=\infty$ we get the Neumann boundary condition, and for $c/s=0$ the Dirichlet one. We

started with a situation where A was integrated over but v was fixed such that $F_{ij}[v]$ is a flat connection. After applying the S-dual transformation, we get a new boundary condition where A is fixed whereas v becomes dynamical. This precisely parallels the CFT discussion in [8].

There is a unique marginal purely three-dimensional action that is invariant under the electric-magnetic duality (5.1). It is the difference (Euclidean) or sum (Lorentzian) of two Chern-Simons terms:

$$S_{\text{invariant}}[v, A] = \int d^3x (v \wedge dv - tA \wedge dA). \tag{5.4}$$

B. $SL(2, \mathbb{R})$ transformed boundary conditions

We will now include the θ-term and analyze the transformation of general boundary conditions. The bulk action changes by [see (3.21)]

$$\begin{aligned} S[A^*, E^*] = S[A, E] + \int \mathrm{d}^3x \Big[& \Big(-\frac{t}{2} g^2 cs + g^2 g^{*2} \frac{\theta^*}{8\pi^2} s^2 \Big) f v \\ & - t \Big(s^2 - \frac{g^{*2}\theta^*}{4\pi^2} cs \Big) f A \\ & + \Big(-\frac{cs}{2g^2} + \frac{g^{*2}}{g^2} \frac{\theta^*}{8\pi^2} c^2 - \frac{\theta}{8\pi^2} \Big) F A \Big]. \end{aligned} \tag{5.5}$$

The corresponding equations of variation are

$$\begin{aligned} \frac{1}{g} \frac{\delta S^*_{\text{bdy}}}{\delta f_i} &= ts \Big(g^* v^* - \frac{g^{*2}\theta^*}{4\pi^2} \frac{A^*}{g^*} \Big) \\ g \frac{\delta S^*_{\text{bdy}}}{\delta A_i} &= c \Big(g^* f^* - \frac{g^{*2}\theta^*}{4\pi^2} \frac{F^*}{g^*} \Big). \end{aligned} \tag{5.6}$$

This can be reorganized in both signatures to

$$\frac{\delta S^*_{\text{bdy}}}{\delta f^*_i} = 0 \qquad \frac{\delta S^*_{\text{bdy}}}{\delta A^*_i} = f^* - \frac{\theta^*}{4\pi^2} F^* = \mathsf{f}^*. \tag{5.7}$$

Thanks to the fact that we are working off-shell, (5.7) says that extremization commutes with duality transformations.

We can further simplify our discussion by viewing the θ-term as a term in the boundary action. We may now keep factors of g implicit and use (5.2) to find

$$\frac{\delta S^*_{\text{bdy}}}{\delta f_i} = ts(cv + sA) \qquad \frac{\delta S^*_{\text{bdy}}}{\delta A_i} = c(cf + sF). \tag{5.8}$$

Since these transformations interpolate between different boundary effective actions, we will refer to this as the flow of the boundary theory (which is not to be confused with RG flow, although in the next section we will explain how they are connected). The flow starts at $\sigma = 0$, $s = 0$, $c = 1$, where it of course reduces to (4.5). The Lorentzian flow swaps the value of c, s at the end. In contrast, the Euclidean flow ends at $\sigma = -\infty$ where $s = -c$ blows up.

C. Massive deformation

We now come to the case of massive deformations, given by the linear boundary condition (2.50) which explicitly breaks conformal invariance. In fact, it adds a relevant operator to the action. It corresponds to the following choice of boundary term:

$$S_{\text{bdy}}[f, A] \int d^3x \Big(A_i f_i - \frac{1}{2m} f_i^2 \Big). \tag{5.9}$$

Without the deformation term, this action simply imposes the Dirichlet boundary condition $A = 0$. It is easy to check that after the flow we get again a single independent equation

$$cA_i + tsv = \frac{1}{m}(cf_i + sF_i), \tag{5.10}$$

which is of course nothing else than the flow of the linear boundary condition (2.50). In the Euclidean case, it is interesting to note that in the limit $\sigma \to \infty$ the above reduces to

$$\hat{A}_i = -\frac{1}{m} \epsilon_{ijk} \partial_j \hat{A}_k \tag{5.11}$$

where $\hat{A}_i = A_i - v_i$. Hence, in the limit $\sigma \to \infty$ we get the self-dual massive theory of [17]!

In Lorentzian signature, we can continue the flow to $c = 0$, $s = 1$ where the boundary condition becomes

$$v = -\frac{1}{m} F, \tag{5.12}$$

which is the linear boundary condition on the S-dual fields.

D. Self-dual boundary conditions

We will now study the self-dual case, which corresponds to the equations of motion of the topologically massive theory [17]. We look for a choice of boundary terms that combines the self-dual and massive boundary conditions leading to (2.51). This is achieved by the choice:

$$S_{\text{bdy}}[A, f] = -\int d^3x \Big(\frac{g^2}{2m} f^2 + \frac{\varepsilon}{m} fF \Big), \tag{5.13}$$

where we have reinstated factors of g left out in (2.48), and for simplicity we first consider $\theta = 0$. After some algebra, we find that the modified equations of motion combined together lead to

$$f = -\frac{\varepsilon}{g^2} F \qquad A = \frac{\varepsilon}{m} F, \tag{5.14}$$

that is, precisely the equations of motion of the "master action" for the self-dual theory! [18]. It is remarkable that all σ-dependence in (5.14) has disappeared. Another way to see this is to simply let the flow act directly on (5.14). It is easy to see these equations are invariant under it. So we find that the self-dual theory remains invariant during the whole flow! This is exactly what we expect: if we start with

the self-dual boundary condition (2.48) which is a solution of the bulk self-duality equation (2.51), the solution has vanishing bulk stress-energy tensor:

$$T_{\mu\nu} = 0. \tag{5.15}$$

Now since electric-magnetic duality preserves the form of the stress-energy tensor, (5.15) should be true throughout the flow. This is exactly what we find: (5.14) are valid throughout, and this ensures vanishing stress tensor. Of course, the above action again contains relevant deformations since the theory is massive.

We should notice here that, although the equations of motion (5.14) are invariant under $SO(1,1)$, the action (5.13) (or, for that matter, the full action including the bulk term) is not invariant under it. In Appendix C 3 we give two other choices of boundary terms that give the same boundary conditions. These choices should be continuously related to the one in (5.13) by the flow.

This discussion should shed more light on the "self-duality" property of this three-dimensional theory: this is the boundary theory that corresponds to a choice of boundary conditions that leaves the action invariant under four-dimensional electric-magnetic transformations. Notice that the bulk part of the action is essential in checking this invariance. It is also easy to check that none of the purely three-dimensional actions [18] are invariant under (5.1). They are only invariant under the transformation in [53], which is different from electric-magnetic duality in the bulk. In fact, the unique boundary invariant at this order was written in Eq. (5.4), which is simply two noninteracting copies of Abelian Chern-Simons theory. The point is that the effective action receives a boundary contribution coming from the variation of the bulk term $\int d^4x E_i \partial_r A_i$.

E. General case

We are now ready to study the more general boundary action (4.7). For simplicity we will set $g = 1$. The transformed equations (4.8) read

$$\begin{aligned} \beta v' + \gamma A' + \epsilon f' + \mu F' &= 0 \\ \alpha F' + (\gamma - 1) f' + \delta * dF' + \mu * df' &= 0. \end{aligned} \tag{5.16}$$

Performing linear combinations as in (4.10) leads to a

$$\begin{aligned} \alpha_1' f + a_2' F + a_3' * df &= 0 \\ b_1' f + b_2' F - a_3' * dF &= 0, \end{aligned} \tag{5.17}$$

where we have the transformed coefficients

$$\begin{aligned} a_1' &= a_1 c^2 + t(a_2 + t b_1) cs + t b_2 s^2 \\ a_2' &= a_2 c^2 + (a_1 + b_2) cs + b_1 s^2 \\ b_1' &= b_1 c^2 + t(a_1 + b_2) cs + a_2 s^2 \\ b_2' &= b_2 c^2 + t(a_2 + t b_1) cs + t a_1 s^2 \qquad a_3' = a_3. \end{aligned} \tag{5.18}$$

Remarkably it can be shown that the following are invariants under the flow (5.18)

$$\begin{aligned} a_1' - b_2' = a_1 - b_2, \qquad a_2' - t b_1' = a_2 - t b_1, \\ a_3' = a_3, \qquad a_1' b_2' - a_2' b_1' = a_1 b_2 - a_2 b_1, \\ c_1' d_2' - c_2' d_1' = 0. \end{aligned} \tag{5.19}$$

Using the definition of a, b in (C20) we see that these are invariants as well. This means under the flow, the equation for A remains unchanged

$$(1 + a\Box) A = bF. \tag{5.20}$$

From this, we conclude that for generic values of coupling constants, the Eqs. (4.8) and hence the boundary conditions are invariant under the flow! We should stress though that the effective action (4.7) itself is in general not invariant and hence the invariance of the equation is not a consequence of the invariance of the action. This is reminiscent of the realization of EM duality in the 4d theory!

This would seem to contradict what we found above where there were nontrivial flows of boundary condition. However recall a, b are well defined only if $a_1 b_2 - a_2 b_1 \neq 0$. In fact, it is easy to check that actions given in (5.9), (5.13), and (C18), as well as the trivial case $S_{\rm bdy} = 0$, all have $a_1 b_2 - a_2 b_1 = 0$.

VI. RG-FLOWS, S-DUALITY, AND PARTICLE-VORTEX DUALITY

Now we come to the interplay between S-duality and RG-flows. We will first compute the one- and two-point functions using the method of holographic renormalization, and then explain how S-duality acts on them. We will then discuss the physical interpretation, which requires considering both electric and magnetic sources in the AdS/CFT dictionary. Finally, we will discuss massive deformations leading to RG-flows from one theory to its S-dual, and compute the effective action.

A. S-duality of the two-point function and Legendre transforms

1. Legendre transformed one-point function

We can now easily compute the 1- and 2-point functions. We use the method of holographic renormalization [19] (see also [54]). After solving the bulk equations of motion, one regularizes the action and adds counterterms. To compute the 1-point function, one needs to take the derivative of the action with respect to the source $A_{i(0)}(x)$. Doing this in general is nontrivial, as the dependence of $f_i(x)$ on $A_{i(0)}$ depends on the particular solution. The way to do this in general is to write the boundary action in terms of the regulated field $A_i(\epsilon, x)$. We find for the dimension-2 current:

$$\delta S = \int d^3x \left[\frac{1}{g^2} (\partial_r A_i(r,x) - \partial_i A_r(r,x)) + \frac{\theta}{4\pi^2} F_i(r,x) \right] \delta A_i(r,x)|_{r=\epsilon}, \tag{6.1}$$

where we have used the equations of motion. The above variation is completely finite, as it should. It is shown in Sec. II C how to renormalize the action, including the coupling to gravity.[4] We are left with:

$$\begin{aligned} \langle \mathcal{O}_{2i}(x) \rangle_A &= \lim_{\epsilon \to 0} \frac{\delta S[A]}{\delta A_i(\epsilon, x)} \\ &= \left(-E_i(r,x) + \frac{\theta}{4\pi^2} F_i(r,x) \right) \Big|_{r=0} \\ &= -f_i(x) + \frac{\theta}{4\pi^2} F_i(x) = -\mathsf{f}_i(x). \end{aligned} \tag{6.2}$$

The first term is the usual one, see (2.5). The second term is the contribution from the θ-term. The θ-term does not modify the equations of motion in the bulk. However, it does modify the on-shell action, and therefore it contributes to the expectation value of the conserved current.

In the theory where we have the operator of dimension 2, we have shown that the generating functional of connected correlation functions is: $W = -S_{\text{on-shell}} = \frac{1}{2} \int d^3x \mathsf{f} A$. One can check that the effective action is minus this $\Gamma = S_{\text{on-shell}}$.

Now as we discussed in Sec. II A, we can get to the dual CFT if we do a Legendre transform. The roles of the effective action and the generating functional are then interchanged. In particular, for a linear choice of boundary conditions we have

$$\tilde{\Gamma} = W \qquad \tilde{W} = \Gamma, \tag{6.3}$$

where the right-hand side should be viewed as a function of the source A, and the left-hand side as a function of the dual source $\tilde{f}$. It is easy to check that the one-point function is independent of the linear choice of boundary condition. That is basically what the generic holographic result (6.1) tells us. For any such choice we get:

$$\langle \tilde{\mathcal{O}}_1 \rangle_{\mathsf{f}} = -\frac{\delta \tilde{W}[\mathsf{f}]}{\delta \mathsf{f}} = -\Pi A \tag{6.4}$$

as we would expect. This is a gauge field of dimension 1, projected onto its transverse part. From this we now get the conserved current of the dual theory,

$$\langle \tilde{\mathcal{O}}_2 \rangle_{\mathsf{f}} = -F. \tag{6.5}$$

2. S-duality of the one- and two-point functions

We saw that the action $S[A, E]$ and its S-dual $S[A', E']$ are each other's Legendre transforms, (3.17). This explains the minus sign in comparing $S[A]$ (A2) with its S-dual (A5). The generating functional of connected correlation functions of the S-dual theory thus coincides with the Legendre transformed functional, with the S-dual couplings:

$$W'[\mathsf{f}'] = \tilde{W}[\mathsf{f}'; g', \theta']. \tag{6.6}$$

Let us spell this out in more detail in this case. Recall the definition of the generator of connected correlation functions: $-W[A] = S_{\text{on-shell}}[A] = -\frac{1}{2} \int \mathsf{f} A$, where $\mathsf{f} = \mathsf{f}[A]$. Define now the Legendre transform:

$$\tilde{W}[\tilde{A}] = W[A] + \int \tilde{A} F, \tag{6.7}$$

therefore

$$\frac{\delta \tilde{W}}{\delta \tilde{A}} = F, \qquad \frac{\delta W}{\delta A} = -\tilde{F}. \tag{6.8}$$

Using (6.8) and filling the value of $W[A]$ in (6.7), we find in this case $\tilde{W}[\tilde{A}] = -\frac{1}{2} \int \mathsf{f} A = S[A]$. From (6.8) we then precisely find the definition of S-duality (3.12), as it should. This justifies (6.6) and agrees with what we found in (3.17). We rewrite the dual potential as:

$$\tilde{W}[\tilde{A}] = \frac{1}{2} \int \tilde{\mathsf{f}} \tilde{A} = -\tilde{S}[\tilde{A}]. \tag{6.9}$$

This is just what we expect—the dual CFT is related to the dual bulk action in the standard way.

The one-point function of the S-dual conserved current is

$$\langle \mathcal{O}'_{2i} \rangle_{A'} = \frac{\delta S'}{\delta A'} = -F = -\mathsf{f}'. \tag{6.10}$$

We now compute the two-point function and its S-dual. The two-point function is defined as

$$\begin{aligned} \langle \mathcal{O}_{2i}(x) \mathcal{O}_{2j}(x') \rangle_{A=0} &= \frac{\delta}{\delta A_i(x)} \frac{\delta}{\delta A_j(x')} S_{\text{on-shell}}|_{A=0} \\ &= \frac{\sqrt{-\Box}}{g^2} \Pi_{ij}(x - x') \\ &\quad + \frac{\theta}{4\pi^2} (*d)_{ij} \delta(x - x'). \end{aligned} \tag{6.11}$$

In momentum space,

[4] In usual holographic renormalization, one should also use the induced boundary metric $\gamma_{ij}(\epsilon, x) = 1/\epsilon^2 g_{ij}(\epsilon, x)$. Thanks to conformal invariance of the matter part of the action the powers of ϵ drop out. See Sec. III A.

$$\langle \mathcal{O}_{2i}(p)\mathcal{O}_{2i}(-p)\rangle_{A=0} = \frac{1}{g^2}|p|\Pi_{ij} - \frac{\theta}{(4\pi^2)^2} i\epsilon_{ijk}p_k. \tag{6.12}$$

Since the right-hand side only depends on the couplings, in order to compute the S-dual of this we might just apply (3.13) and get the S-dual result in [8,13]. However, our formalism allows us to perform a check from first principles, which is what we will now do. This will be useful both in order to understand the structure of S-duality, and apply it in the general case.

The definition of the two-point function in the S-dual theory is simply:

$$\langle \mathcal{O}'_{2i}\mathcal{O}'_{2j}\rangle = \frac{\delta}{\delta A'_i}\frac{\delta}{\delta A'_j} S'_{\text{on-shell}}. \tag{6.13}$$

We use the transformations (3.15) to compute the S-dual of the on-shell action

$$\begin{aligned} S'_{\text{on-shell}} &= -\frac{1}{2}\int d^3x \mathsf{E}'A' = \frac{1}{2}\int d^3x EA - \frac{\theta}{4\pi^2}AF \\ &= \frac{1}{2}\int d^3x\left(-\frac{1}{g^2}AM\Pi A - \frac{\theta}{4\pi^2}A * dA\right), \end{aligned} \tag{6.14}$$

where in the last line we have used the ingoing boundary condition. This is, as announced, minus the on-shell action.

We now solve (3.15) and write the functional derivative as follows:

$$\frac{\delta}{\delta A'_i} = -\left(\frac{\theta'}{4\pi^2} + g^2\left(1 + \frac{\theta\theta'}{(4\pi^2)^2}\right)M^{-1} * d\right)_{ij}\frac{\delta}{\delta A_j}, \tag{6.15}$$

where we have used the first of the identities (B35).

Since Π is noninvertible, these identities should be understood as follows. One adds a gauge-fixing term to the action and inverting the resulting modified propagator. The final result does not contain any inverse propagators, and one can safely take the limit that the gauge-fixing terms go to zero. We get:

$$\begin{aligned} \langle \mathcal{O}'_{2i}\mathcal{O}'_{2j}\rangle &= \frac{\delta}{\delta A'_i}\frac{\delta}{\delta A'_j}S' = \left(-\frac{\theta'}{4\pi^2} + g^2\left(1 + \frac{\theta\theta'}{(4\pi^2)^2}\right)M^{-1} * d\right)_{ik}\left(-\frac{\theta'}{4\pi^2} + g^2\left(1 + \frac{\theta\theta'}{(4\pi^2)^2}\right)M^{-1} * d\right)_{jl}\frac{\delta}{\delta A_k}\frac{\delta}{\delta A_l}S \\ &= \left(\frac{\theta'^2}{(4\pi^2)^2 g^2} + g^2\left(1 + \frac{\theta\theta'}{(4\pi^2)^2}\right)^2\right)M\Pi - \left(\frac{\theta\theta'^2}{(4\pi^2)^3} + \frac{\theta}{4\pi^2}g^4\left(1 + \frac{\theta\theta'}{(4\pi^2)^2}\right)^2\right) * d \\ &= \frac{g^2}{1 + g^2\frac{\theta^2}{(4\pi^2)^2}}M\Pi - \frac{\frac{g^4\theta}{4\pi^2}}{1 + \frac{g^4\theta^2}{(4\pi^2)^2}} * d \end{aligned} \tag{6.16}$$

which is indeed the S-dual transform in the usual sense.

B. Physical interpretation of S-duality

We have seen how S-duality acts in the bulk and on the boundary, and we now come to the physical interpretation.

In this section we will deal with the pure Dirichlet/Neumann problems, therefore it will not be necessary to add any additional boundary terms. More general cases will be worked out in the next section.

We saw in section IV B that the parameter γ interpolates between mutually S-dual solutions. This is easy to see from (4.8): the case $\gamma = 1$ corresponds to the Dirichlet boundary condition, whereas $\gamma = 0$ is the Neumann boundary condition. Inclusion of the sources J and $\tilde{\mathcal{J}}$ will clarify what this S-duality actually means for the boundary CFT. Indeed, (4.8) is not quite symmetric with respect to J and $\tilde{\mathcal{J}}$: J has dimension 1, whereas $\tilde{\mathcal{J}}$ has dimension 2. Therefore, there exists a $\tilde{\mathcal{J}}$ such that $\tilde{\mathcal{J}} = *d\tilde{J}$. We can then rewrite the source terms in the following symmetric form:

$$S_{\text{source}} = -\int d^3x(F\tilde{J} + fJ) = -\int d^3x(A\tilde{\mathcal{J}} + fJ). \tag{6.17}$$

Although $\tilde{J}$ is not necessarily conserved, this expression is gauge invariant and hence well defined. In a theory with gauge fields, this expression has a natural interpretation. Let us promote J and $\tilde{J}$ to bulk sources $J(r, x)$, $\tilde{J}(r, x)$ localized at the boundary, with a delta-function at $r = 0$. $\mathcal{J}$ is then the magnetic source that couples to the bulk electric field $E(r, x)$. $\tilde{J}(r, x)$, on the other hand, acts as a source for the magnetic field in the directions transverse to it. This is easy to see: if the sources would contain any bulk contributions, they would modify the equations of motion as

$$E = -\frac{1}{g^2}\partial_r A + \frac{1}{g^2}J \qquad \partial_r E = \frac{1}{g^2} * dF - \tilde{J}. \tag{6.18}$$

Clearly then, $\tilde{J}$ will contribute to the curl of the magnetic field, and $*dJ$ will contribute to the curl of the electric field. We write $(J_e, J_m) = (\tilde{\mathcal{J}}, \mathcal{J})$. In this paper we will not

consider any bulk contributions coming from the sources. When restricted to the boundary, both sources have dimension 2. However, since they are conserved we can write $(\tilde{\mathcal{J}}, \mathcal{J}) = *d(\tilde{J}, J)$ where $(\tilde{J}, J)$ have dimension 1 in the Legendre transformed theory. Bulk electric-magnetic duality interchanges them: $(\tilde{J}', J') = (J, \tilde{J})$.

In AdS/CFT with the usual boundary conditions, J is interpreted as the source in the boundary CFT. However, the above makes clear that, in the presence of both electric and magnetic charges, the theory can also be deformed by $\tilde{J}$, hence it ought to have a boundary interpretation. Since it has dimension 1, this suggests that it is dual to an operator of dimension 2, and in fact this operator should be S-dual to $\mathcal{O}_2$. Furthermore, from the existence of an alternative quantization scheme where the boundary operator has dimension 1, one would guess that the bulk sources $\tilde{\mathcal{J}}$ and $\mathcal{J}$ themselves (restricted to the boundary) have an interpretation in terms of dual operators of dimension 1. This is in fact very natural, as $\mathcal{J}$ and $\tilde{\mathcal{J}}$ are conserved so they might give rise to a boundary operator which is the gauge field itself. We will now verify that this is indeed the case.

We start with the on-shell action $S_{\text{on-shell}} = -\frac{1}{2}\int d^3x \mathsf{f}A$. Now instead of fixing the gauge field A, we will fix the electric field by a source of dimension 1:

$$\tilde{J} = -J' = -A' = \mathsf{v}. \tag{6.19}$$

The prime suggests that this is the S-dual to J, however at this stage we are not making use of S-duality and we regard the above purely as a boundary problem for v. Clearly, the dual operator will have dimension 2. We compute:

$$\langle \mathcal{O}_2' \rangle = -\frac{\delta W'}{\delta \mathsf{v}} = -F, \tag{6.20}$$

where we have used $W' = S'_{\text{on-shell}} = \frac{1}{2}\int d^3x \mathsf{f}A$. This result agrees with (6.10). We now compute the two-point function. To do that, we use regularity to write $F = M\Pi\upsilon$ and take a further derivative with respect to v. In order to do that, we invert $\mathsf{v} = \upsilon - \theta A$. We get:

$$\upsilon = \frac{1}{1 + \frac{\theta^2}{(4\pi^2)^2}}\left(\mathsf{v} - \frac{\theta}{4\pi^2 M} * d\mathsf{v}\right). \tag{6.21}$$

So we get

$$\langle \mathcal{O}_2' \mathcal{O}_2' \rangle = \frac{M}{1 + \frac{\theta^2}{(4\pi^2)^2}}\Pi - \frac{\frac{\theta}{4\pi^2}}{1 + \frac{\theta^2}{(4\pi^2)^2}} * d. \tag{6.22}$$

This is exactly the S-dual (6.16) of the two-point function of a conserved current! (we have set $g = 1$). Notice that we have not assumed any knowledge of S-duality here: it simply follows from the fact that we identify $\tilde{J}$ as the source. This describes the lower-left corner of Fig. 1.

The upper-right corner of Fig. 1 is the (equivalent) description in terms of the dual gauge field: this is obtained by taking the dual source $\mathcal{J}' = -\tilde{\mathcal{J}}$ and fixing:

$$\tilde{\mathcal{J}} = \mathsf{f}. \tag{6.23}$$

We find:

$$\begin{aligned} \langle \mathcal{O}_1' \rangle_{\mathcal{J}} &= \Pi A \\ \langle \mathcal{O}_1' \mathcal{O}_1' \rangle &= \frac{1}{M(1 + \frac{\theta^2}{(4\pi^2)^2})}\Pi - \frac{\frac{\theta}{4\pi^2}}{M^2(1 + \frac{\theta^2}{(4\pi^2)^2})} * d. \end{aligned} \tag{6.24}$$

This is indeed the S-dual of the two-point function of a gauge field and agrees with (6.4). Of course, taking the derivative of the above we recover the correlation functions of the dual current. This is another check that the method of dual sources gives correct results.

We now complete the Dirichlet case on the main diagonal of 1. The upper-left corner is the usual Dirichlet boundary condition $A = J$. The lower-right case is obtained identifying $F = \mathcal{J}$. The one- and two-point functions are then:

$$\langle \mathcal{O}_1 \rangle = -\Pi\mathsf{v} \qquad \langle \mathcal{O}_1 \mathcal{O}_1 \rangle = \frac{1}{M}\Pi - \frac{\theta}{4\pi^2 M^2} * d. \tag{6.25}$$

There is no S-duality transform here, as it should. Once more, taking the derivative we get the usual current.

We can now rephrase physically the situation in Fig. 1. First of all, notice that the Legendre transform and S-duality transformations that translate us vertically or horizontally are the same. In both cases, the term that we add to the action is $\int \mathsf{f}A$. Only in one case (the horizontal one) this is regarded as a coupling between the source and the current, whereas in the vertical case it is a Chern-Simons coupling between the background field and the new dynamical gauge field. Obviously, the diagonal action simply corresponds to the action of $*d$ and ensures that the theory is unitary. The difference between S-dual theories lies in what we interpret as a source and as an operator. In the upper-left corner, we fix as a source the bulk gauge field A. The corresponding CFT operator that couples to it is then an induced current of dimension 2. In the upper-right, on the other hand, A has become a dynamical field with corresponding conserved current F, and the background field υ is fixed via its curl f. This is the particle-vortex duality. In the next subsection we will see how this duality is achieved by an RG-flow. From the bulk point of view, particle-vortex duality corresponds to interchanging J and $\mathcal{J}'$ as we have shown. The latter fact may have applications to other situations, such as the non-Abelian case (see Sec. VI E).

C. RG-flow of the two-point function and S-duality

It is known from field theory that massive deformations of three-dimensional CFT's by double-trace operators lead to RG flows from the UV towards the IR fixed point. In [13] it was shown using field theory arguments that if the IR dimension of a conserved current is 2, in the UV it corresponds to an operator that is S-dual to a gauge field of

dimension 1. In this section we will show how this result comes about from the bulk point of view, and how it fits in the general picture of Fig. 1.

We will modify the massive boundary condition to include a θ-angle:

$$A + \frac{1}{m}\mathsf{f} = J, \tag{6.26}$$

in other words

$$A - \frac{\theta}{m}F + \frac{1}{m}f = J. \tag{6.27}$$

As our second boundary condition, we take

$$f = cM\Pi A. \tag{6.28}$$

We will choose $c = -1$, which corresponds (for the tachyonic case) to the regular configurations, as explained in Sec. II D. The value of c only results in an overall rescaling of the two-point function, however c has to be negative in order for the two-point function to be positive definite, as explained in Sec. II E. We obtain the above boundary conditions if we choose $\gamma = 1$, $\epsilon = 1/m$, $\mu = -\theta/m$, $\alpha = -M\theta/m$ in (4.8), and the other coefficients zero (in the rest of this section we rescale θ by a factor of $4\pi^2$ compared to our previous formulas). The on-shell action is:

$$S_{\text{on-shell}} = \frac{1}{2}\int\left(-\frac{M}{m}(m-M)A\Pi A + \frac{M\theta}{m}A\wedge dA + 2MJA\right). \tag{6.29}$$

Now we solve for A in (6.27) in terms of the source. We get:

$$\begin{aligned} A &= \frac{m}{\Gamma}((m-M)J + \theta * dJ) \\ \Gamma &= (m-M)^2 - t\theta^2 M^2. \end{aligned} \tag{6.30}$$

Plugging this back in the action, we get

$$S_{\text{on-shell}} = \int \frac{mM}{2\Gamma}(J(m-M)\Pi J + \theta J\wedge dJ). \tag{6.31}$$

The one-point function is easily computed and found to be $\langle \mathcal{O}_2\rangle = MA = -f$. It is clear that the IR limit $|p| \ll m$ of the two-point function is the usual one for a conserved current of dimension 2, (6.12). On the other hand, it is quite remarkable that in the UV limit we get

$$\langle \mathcal{O}_{2i}\mathcal{O}_{2j}\rangle = \frac{m}{1+\theta^2}\left(\Pi_{ij} - \theta\frac{i\epsilon_{ijk}p_k}{|p|}\right) + \frac{m^2}{|p|(1+\theta^2)^2}\left((1-\theta^2)\Pi_{ij} - 2\theta\frac{i\epsilon_{ijk}p_k}{|p|}\right). \tag{6.32}$$

The second term in (6.32) was also found in [13] and corresponds to the (S-dual) two-point function of an operator of dimension 1—*a* gauge field. This is what we expect. Notice that, once again, there is no assumption about S-duality made; the massive deformation automatically leads us to the upper-right part of the diagram 1.

The first term in (6.32) is a contact term [13]. Naively it would correspond to an operator with UV dimension $3/2$. However, a spin one operator cannot have dimension $3/2$ in a conformal theory, therefore it cannot be a chiral primary and it must be a descent operator. There is indeed such an operator in QED in 3 dimensions [11,12]. In that case, the current gets dimension $3/2$ and it can be written as $f_i = \partial_i\phi$ where ϕ is a scalar field. However, the presence of such a term without its primary field in the OPE breaks conformal invariance in the UV.

The result can be given a different interpretation that agrees with what we said earlier if we take $\mathcal{J} = *dJ$ as the source. In that case, we find the two-point function of $\Pi\mathsf{v}$ with itself, and we are in the lower-right part of 1! In particular, the S-duality transform is absent here and will instead appear in the IR.

It is straightforward to check that the method to get the S-dual two-point functions outlined in Sec. VI B works in this case as well. If we fix the boundary value of the gauge field, $J = A$, we easily find the two-point function from (6.29). If instead we couple the theory to an electric source $\tilde{\mathcal{J}}$ as in (6.17) and take derivatives of (6.29) with respect to $\tilde{\mathcal{J}}$, we get the S-dual two-point function. This explains the S-duality in the flow when we turn on the mixed boundary condition (6.27): in the IR limit we are fixing A and J is the usual magnetic source; in the UV, the second term dominates and J becomes an electric source! Thus, (6.27) interpolates between the two.

D. The effective action

We now present an application that motivated this work, namely, finding the effective action on a stack of M2-branes using AdS/CFT for the particular bulk configurations that we have considered. This effective action gives the response of the boundary theory to the action of an external source. Our Abelian configuration corresponds either to a Higgs or a Coulomb phase, depending on the operator that gets a vev. As we have discussed, there are two possible meanings to this effective action; the usual theory and the dual one where the on-shell bulk action is identified with the boundary effective action. Since the main focus of this paper has been the dual CFT, we will consider the latter.

In the case of instantons there is no backreaction, hence we are ensured that the asymptotic expansion of the on-shell action gives the full quantum effective action at large N. For a particular solution with the sources set to zero this was shown in section II F to be a pure number, $8\pi^2|\lambda|/g^2$. In the case of nonzero backreaction, since the backreaction sets in only at order r^4, gravity and the matter fields decouple near the boundary and we can solve the asymptotic equations in a fixed background. At order r^4 the backreaction sets in as in (2.27), but this does not affect the two-point functions. The effective action up to this order reduces in the IR to:

$$S_{\rm eff} = \frac{1}{2}\int (F_{ij}[A]\Box^{-1/2}F_{ij}[A] + \theta\epsilon^{ijk}A_i\partial_j A_k). \quad (6.33)$$

A is the one-point function of the gauge field as a response to the source J. The above is in fact the on-shell action for the Dirichlet boundary problem, and it is the generic behavior that we found under massive deformations: the theory flows towards the Dirichlet problem in the IR. We get a similar result if we start with the massive case (6.31):

$$S_{\rm eff} = \frac{1}{2}\int (F_{ij}[A]\Box^{-1/2}F_{ij}[A] + \theta m\epsilon^{ijk}A_i\partial_j\Box^{-1/2}A_k). \quad (6.34)$$

We remark here that for zero theta angle, the above is the effective action of three-dimensional QED at large N_f. As we noted in the previous section, there is another point where our results seem to agree with QED. The naive UV dimension of the current is $3/2$ rather than 2; the two-point function of such an operator in momentum space would be a contact term; this is exactly what we find in (6.32). As discussed, if such a term is to be interpreted as an operator then it must be a nonprimary operator. The present computation also predicts that the UV and IR couplings are each other's S-duals.[5] The subleading correction is the two-point function of the dual gauge field.

E. Non-Abelian case

Because of interactions, classical electric-magnetic duality does not straightforwardly generalize to non-Abelian gauge theories [41] (but see [55] where electric-magnetic duality is achieved at the cubic level). The key point is that, as explained earlier, a formal replacement $E \leftrightarrow \pm B$ in the Lagrangian does not suffice. Duality has to be realized at the level of the gauge potential. Duality in non-Abelian theories involves quantum effects in an essential way, which goes beyond the scope of this work. We will suffice with a few comments and suggestions for further work.

Instanton configurations that minimize the energy will satisfy:

$$E_i^a = F_{ri}^a = -\tfrac{1}{2}\varepsilon\epsilon_{ijk}F^{ajk}. \quad (6.35)$$

Restricted to the boundary, this gives again the same boundary condition as before. If in addition we also choose the boundary condition $E_i^a = mA_i^a$, we get the non-Abelian version of the self-dual equation:

$$A_i^a = \frac{1}{2m}\epsilon_{ijk}F_{jk}^a. \quad (6.36)$$

[5]The existence of the two-point function with the operator of dimension $1/2$ suggests that in the UV the gauge field is dualized to a scalar: $F_i = \partial_i\phi$, in agreement with [12].

This is the non-Abelian generalization of [18]. We have checked that usual 't Hooft instantons do not satisfy this condition but a modification of it. It would be interesting to see what boundary action gives rise to those solutions.

In Sec. VI B we outlined a method to compute the two-point function at any of the points in the diagram in Fig. 1. In particular, to compute the S-dual two-point functions we use electric rather than magnetic bulk sources, that is we couple the source to the gauge field rather than to the electric field. This method can be generalized to the non-Abelian case. That is why we expect that some of our results may easily generalize to that case.

As we have noted, the particular compactification we have used has a non-Abelian extension where the gauge symmetry is $SU(2)$, with $\mathcal{N} = 2$ supersymmetry. As mentioned in the introduction, it is an interesting question whether this theory still has some sort of electric-magnetic duality and what is the holographic image of it. One may be able to shed light on this question by analyzing instanton solutions considered above. These have $T_{\mu\nu} = 0$, and therefore gravity and the gauge fields basically decouple. It would be interesting to see whether the theory around such configurations exhibits definite electric-magnetic properties of the type in [1].

VII. DISCUSSION

The off-shell bulk electromagnetic action is invariant under electric-magnetic duality up to boundary terms. In this paper we have analyzed the way these terms transform for arbitrary choice of boundary conditions. The fact that the action is invariant off-shell implies that the boundary conditions transform as well. In the simplest example, Neumann and Dirichlet boundary conditions for the gauge field are interchanged by S-duality. Classically, there is a one-parameter family of deformations connected by bulk duality. Quantum mechanically only a discrete subgroup survives. It would be interesting to check whether the latter statement also follows from quantization of Chern-Simons couplings. Operators of higher dimension are also mapped into each other by electric-magnetic transformations. Again, quantum mechanically the coefficients within a particular $SL(2,\mathbb{Z})$ orbit do not appear in arbitrary but only in specific combinations. In this way, electric-magnetic duality gives a way of probing the moduli space of deformations of effective CFT actions for operators of given dimension. This agrees with interesting experimental results in quantum Hall systems [9,10]. In the massive case, S-duality can change dimension.

We presented a bulk computation of the finite renormalization of dimensions of operators under RG flow of the deformed three-dimensional SCFT. The conserved current has IR dimension 2 but the expansion of its two-point function contains an S-dual dimension 1 gauge field in the UV. In other words, the 2-point function contains a contribution of the 2-point function of the dual current.

This is in agreement with the field theory expectations [13]. We also discussed the similarity of the effective action and the flow with those of $2+1$ dimensional QED at large N_f. It may turn out that the high-energy limit of this theory is relevant to the SCFT in a phase where the non-Abelian gauge symmetry is broken, or the theory has become classical due to the large N limit. We notice, however, that whenever we have bulk instanton solutions we will get Chern-Simons terms on the boundary, which as stressed in [56] are not present in SYM theory. It would be interesting to study this further.

The form of the S-transformation worked out in [8] depends on the choice of boundary terms. As it turns out, it corresponds to pure Dirichlet and Neumann boundary conditions. In this paper we wrote down the S-generator for arbitrary boundary conditions. We found that generic choices of boundary conditions are actually invariant under electric-magnetic duality. A particularly interesting case is the self-dual massive bulk solution. In this case, the boundary conditions do not entirely determine the solution, and there is a single remaining boundary degree of freedom. This degree of freedom in fact corresponds to the self-dual topologically massive theory in three dimensions [17,18]. This should shed some new light on the meaning of self-duality of topologically massive theories in three dimensions: it in fact corresponds to electric-magnetic duality in the bulk of AdS_4! It would be interesting to extend this analysis to the non-Abelian case, and we offered some thoughts to analyze this question. In particular, it seems that the method of doing AdS/CFT with both electric and magnetic sources should work also in that case. One could then test to what extent the effective non-Abelian CFT's have dual properties.

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Note added.—While this paper was being completed, [58] appeared, where some related questions are studied.

APPENDIX A: ANOTHER PROOF OF S-DUALITY

We now give the second proof, which relies on integrating out some fields (see also [47]). The equations of motion of (3.11) are:

$$\begin{aligned}
&\delta_E\colon\ \partial_r A_i - \partial_i A_r + g^2 \mathsf{E}_i + \frac{g^2\theta}{4\pi^2} F_i = 0,\\
&\delta_A\colon\ -\partial_r E + \frac{1}{g^2}\left(1 + \frac{g^4\theta^2}{(4\pi^2)^2}\right) * dF + \frac{g^2\theta}{4\pi^2} * d\mathsf{E} = 0,\\
&\delta_{A_r}\colon\ \partial^i \mathsf{E}_i = 0.
\end{aligned} \tag{A1}$$

We first integrate out E using its equation of motion and the fact that it is conserved, getting the usual second-order action for the boundary component of the gauge field A:

$$S[A; g, \theta] = \int d^4x \left(-\frac{1}{2g^2}(\partial_r A)^2 + \frac{1}{2g^2} F^2 - \frac{\theta}{4\pi^2}(\partial_r A) F \right). \tag{A2}$$

This is the usual second-order action (2.17).

Next we use the equation of the gauge field to integrate out A instead. We make use of conservation of E to introduce a gauge field v:

$$\mathsf{E} = *d\mathsf{v} \qquad \mathsf{v} = v - \frac{\theta}{4\pi^2} A. \tag{A3}$$

Then the second equation in (A1) is integrated to:

$$F = \frac{g^2}{1 + \frac{g^4\theta^2}{(4\pi^2)^2}} \partial_r \mathsf{v} - \frac{g^4 \frac{\theta}{4\pi^2}}{1 + \frac{g^4\theta^2}{(4\pi^2)^2}} \mathsf{E}. \tag{A4}$$

Observe that this is nothing but the S-dual of the equation of motion for E! We get:

$$\begin{aligned}
S[\mathsf{E}] = -\int d^4x \Bigg(&-\frac{1}{2} \frac{g^2}{1 + \frac{g^4\theta^2}{(4\pi^2)^2}} (\partial_r \mathsf{v})^2 + \frac{1}{2} \frac{g^2}{1 + \frac{g^4\theta^2}{(4\pi^2)^2}} \mathsf{E}^2 \\
&+ \frac{g^4 \frac{\theta}{4\pi^2}}{1 + \frac{g^4\theta^2}{(4\pi^2)^2}} (\partial_r \mathsf{v}) \mathsf{E} \Bigg).
\end{aligned} \tag{A5}$$

This is the S-dual of (A2)!

In conclusion, the gauge fields A and v are each other's S-duals: integrating out v we get the usual second form of the action; integrating out A gives its S-dual version where the gauge field is v. The first order form of the action, on the other hand, (3.11) or equivalently (3.5), interpolate between both. In particular, (3.11) has manifest S-duality.

The on-shell action now is:

$$S_{\text{on-shell}} = -\frac{1}{2} \int d^3x \mathsf{E}\mathsf{A}, \tag{A6}$$

which is obviously the same as (3.9). Notice however that in terms of the redefined E, the on-shell action takes the same form as the on-shell action without the θ-term.

APPENDIX B: REGULARITY OF THE SOLUTIONS

In this appendix we study the regularity condition on bulk solutions and give some other formulas that were used in the main text.

1. Gauge invariance

We first discuss gauge invariance. The longitudinal, pure gauge part $\varphi = -\frac{i}{p^2} p^i A_i$ is not fixed by the equations of motion.

$$A_i(r,x) = A_i^{\mathrm{T}} + \partial_i\varphi \qquad A_r(r,x) = \partial_r\varphi$$
$$F_{ri}(r,x) = \partial_r A_i^{\mathrm{T}}. \tag{B1}$$

From this, the radial gauge $A_r = 0$ is simply $\partial_r\varphi = 0$, leaving the residual r-independent gauge freedom $\varphi(x)$. $\varphi(x)$ realizes the gauge symmetry of the dual CFT, where the background field has been gauged and a dual current is constructed from the dual gauge field A. By construction, A_i^{T} and f_i are invariant under such transformations.

2. Regularity in two different bulk gauges

As the topological term does not effect equations of motion, we will take the Maxwell action without θ-term

$$S_{\mathrm{bulk}}(A) = \frac{1}{2}\int d^4x\sqrt{g}F_{\mu\nu}F^{\mu\nu} \tag{B2}$$

We will study two different gauges. The first is $\nabla_\mu A^\mu = 0$, where the bulk equations are

$$\nabla_\mu F^{\mu\nu} = 0, \qquad \nabla_\mu A^\mu = 0 \tag{B3}$$

Making use of the following identities

$$\partial_r^2 - \frac{2}{r}\partial_r + \frac{2}{r^2} = \left(\partial_r - \frac{1}{r}\right)^2 \qquad \left(\partial_r - \frac{1}{r}\right)^n (r\phi) = r\partial_r^n\phi \tag{B4}$$

and denoting $\mathcal{A} = A_0(r,\bar{x})/r$, we rewrite the equations

$$(\partial_r^2 + \bar{\partial}^2)\mathcal{A} = 0 \qquad (\partial_r^2 + \bar{\partial}^2)A_i = 2\partial_i\mathcal{A} \tag{B5}$$

which are solved by

$$\mathcal{A}(r,\bar{p}) = \mathcal{A}_0(\bar{p})\cosh(|\bar{p}|r) + \frac{1}{|\bar{p}|}\mathcal{A}_1(\bar{p})\sinh(|\bar{p}|r)$$
$$A_i(r,\bar{p}) = A_i^{(0)}\cosh(|\bar{p}|r) + \frac{1}{|\bar{p}|}A_i^{(1)}\sinh(|\bar{p}|r) + ir\frac{\bar{p}_i}{|\bar{p}|}\left(\mathcal{A}_0\sinh(|\bar{p}|r) + \frac{1}{|\bar{p}|}\mathcal{A}_1\cosh(|\bar{p}|r)\right) \tag{B6}$$

The gauge fixing $\nabla_\mu A^\mu = 0$ implies

$$\mathcal{A}_0(\bar{p}) = i\bar{p}_i A_i^{(0)}(\bar{p}), \qquad \mathcal{A}_1(\bar{p}) = i\bar{p}_i A_i^{(1)}(\bar{p}) \tag{B7}$$

The two boundary functions $A_i^{(0)}$, $A_i^{(1)}$ completely determine the bulk solutions.

Near the boundary $r \to 0$ we find

$$A_i(r,\bar{p}) = A_i^{(0)}(\bar{p}) + r\Pi_{ij}A_j^{(1)}(\bar{p}) + O(r^2)$$
$$E_i = F_{0i} = \Pi_{ij}A_j^{(1)}(\bar{p}) + r\bar{p}_i\bar{p}_j A_j^{(0)} + O(r^2) \tag{B8}$$

where

$$\Pi_{ij} = \delta_{ij} - \frac{p_i p_j}{p^2} \tag{B9}$$

By construction, this matrix has a null eigenvector p_i. It is clear that $f_i = E_i^{(0)}$ is the transverse part of $A_i^{(1)}$. In position space

$$A_i(r,\bar{x}) = A_i^{(0)}(\bar{x}) + rE_i^{(0)}(\bar{x}) + \ldots \tag{B10}$$

where

$$f_i(\bar{x}) = E_i^{(0)}(\bar{x}) = \int d^3\bar{y}\frac{A_j^{(0)}(\bar{y})}{|\bar{x}-\bar{y}|^4}\left(\delta_{ij} - \frac{(\bar{x}_i-\bar{y}_i)(\bar{x}_j-\bar{y}_j)}{|\bar{x}-\bar{y}|^2}\right) \tag{B11}$$

To find regular solutions, we expand the general solution (B6)

$$A_i = \frac{1}{2}\left(A_i^{(0)} + \frac{1}{|\bar{p}|}A_i^{(1)}\right)e^{|\bar{p}|r} + \frac{1}{2}\left(A_i^{(0)} - \frac{1}{|\bar{p}|}A_i^{(1)}e^{-|\bar{p}|r}\right. + i\frac{\bar{p}_i}{|\bar{p}|}\left[\frac{1}{2}\left(\mathcal{A}_0 + \frac{1}{|\bar{p}|}\mathcal{A}_1\right)re^{|\bar{p}|r} - \frac{1}{2}\left(\mathcal{A}_0 - \frac{1}{|\bar{p}|}\mathcal{A}_1\right)re^{-|\bar{p}|r}\right] \tag{B12}$$

To remove the divergent terms, it is enough to require

$$A_i^{(0)}(\bar{p}) + \frac{1}{|\bar{p}|}A_i^{(1)}(\bar{p}) = 0 \tag{B13}$$

Next we study the radial gauge $A_r = 0$, with bulk equations

$$\nabla_\mu F^{\mu\nu} = 0, \qquad A_r = 0 \tag{B14}$$

These reduce to

$$(\partial_r^2 + \bar{\partial}^2)A_i(r,\bar{x}) = \partial_i\partial_j A_j(r,\bar{x}), \qquad \partial_r\partial_i A_i = 0 \tag{B15}$$

which is simply a rewriting of (B5) if one replaces $\mathcal{A}$ by $\partial_j A_j$. Notice this replacement is consistent with (B7). The second equation in (B20) imposes the Gauss law $\partial^i E_i = 0$.

In summary, we found regularity requires in either gauge

$$\Pi_{ij}A_j(p) + \frac{1}{|p|}f_i(p) = 0 \tag{B16}$$

in terms of the boundary fields A_i and f_i. In momentum space, we denote the transverse part of the boundary field

$$A_i^{\mathrm{T}}(p) = \Pi_{ij}A_j(p). \tag{B17}$$

The regularity condition is then

$$A_i^{\mathrm{T}}(p) + \frac{1}{|p|}f_i(p) = 0 \tag{B18}$$

3. Regularity without gauge fixing

We take advantage of the conformal invariance of the Maxwell equations. In Poincare coordinates they take the form

$$\nabla_\mu F^{\mu\nu} = 0, \tag{B19}$$

where the metric is the flat Euclidean metric. These reduce to

$$(\partial_r^2 + \Box)A_i(r, \bar{x}) = \partial_i \partial^j A_j + \partial_i \partial_r A_r \qquad \Box A_r = \partial_r \partial^j A_j, \tag{B20}$$

where $\Box = \partial^j \partial_j$. The latter is the conservation equation for the electric field $E_i = F_{ri}$, $\partial^i E_i = 0$.

To separate the different degrees of freedom, we decompose the gauge field in the following way

$$A_i = \partial_i \varphi + A_i^{\mathrm{T}} \qquad \Box\varphi = \partial^i A_i \tag{B21}$$

Clearly, A^{T} is the transverse part of the gauge field $\partial^i A_i^{\mathrm{T}} = 0$, which carries the physical polarizations. Indeed, $A_i^{\mathrm{T}} = A_i - \Box^{-1}\partial_i \partial^j A_j$ is invariant under gauge transformations. We find

$$E_i = \partial_r A_i^{\mathrm{T}}. \tag{B22}$$

In momentum space, we have

$$A_i^{\mathrm{T}}(r, p) = \Pi_{ij} A_j(r, p). \tag{B23}$$

The longitudinal, pure gauge part of the gauge field is described by φ. In momentum space, $\varphi = -\frac{i}{p^2} p^j A_j$..

Fill the above decomposition into the equations of motion. We get

$$A_r = \partial_r \varphi, \tag{B24}$$

and so A_r is not determined, since φ is not. The remaining equation of motion reads

$$(\partial_r^2 + \Box)A_i^{\mathrm{T}} = 0. \tag{B25}$$

This is solved by

$$A_i^{\mathrm{T}}(r, \bar{p}) = A_{(0)i}^{\mathrm{T}} \cosh(|\bar{p}|r) + \frac{1}{|\bar{p}|} A_{i(1)}^{\mathrm{T}} \sinh(|\bar{p}|r). \tag{B26}$$

We recognize this as the homogeneous piece of the solution in (B6), as the other piece there is pure gauge.

We will now impose regularity of the solution. Rewrite the solution as

$$A_i^{\mathrm{T}} = \frac{1}{2}\left(A_{i(0)}^{\mathrm{T}} + \frac{1}{|\bar{p}|} A_{i(1)}^{\mathrm{T}}\right) e^{|\bar{p}|r} + \frac{1}{2}\left(A_{i(0)}^{\mathrm{T}} - \frac{1}{|\bar{p}|} A_{i(1)}^{\mathrm{T}}\right) e^{-|\bar{p}|r}. \tag{B27}$$

As before, regularity at $r \to \infty$ requires

$$A_{i(0)}^{\mathrm{T}}(\bar{p}) + \frac{1}{|\bar{p}|} A_{i(1)}^{\mathrm{T}}(\bar{p}) = 0. \tag{B28}$$

We can rewrite this in terms of the full gauge field as:

$$\begin{aligned} \Pi_{ij}\left(A_{(0)j} + \frac{1}{|p|} A_{(1)j}\right) &= 0 \\ A_{(0)i} + \frac{1}{|p|} A_{(1)i} &= i p_i \left(\varphi_{(0)} + \frac{1}{|p|}\varphi_{(1)}\right), \end{aligned} \tag{B29}$$

where we expanded φ in the usual way. Hence, regularity relates $A_{(0)}$ and $A_{(1)}$ up to an arbitrary gauge transformation.

The self-duality boundary condition (2.48) can be rewritten as

$$A_{(1)i}^{\mathrm{T}} = i\varepsilon \epsilon_{ijk} p_j A_{(0)k} \tag{B30}$$

which is again rewritten as

$$\Pi_{ij}(A_{(1)j} - i\varepsilon \epsilon_{jkl} p_k A_{(0)l}) = 0. \tag{B31}$$

Combining regularity (B29) and self-duality, we get

$$\Pi_{ij}\left(A_{(0)j} + \frac{i\varepsilon}{|p|}\epsilon_{jkl} p_k A_{(0)l}\right) = 0. \tag{B32}$$

Again, the combination between brackets is set to zero up to a gauge transformation.

Imposing the gauge $A_r = 0$ sets $\partial_r \partial^j A_j = 0$, which at lowest order simply sets the longitudinal part of $A_{(1)}$ to zero, $\varphi_{(1)} = 0$. The longitudinal part of $A_{(0)}$ is left undetermined. This corresponds to the residual gauge symmetry of (B20) corresponding to r-independent gauge transformations. These act purely on the boundary value of the gauge field. In that case, the gauge invariant form of the regularity condition is

$$A_{(0)i}(p) + \frac{1}{|p|} A_{(1)i} = i p_i \varphi_{(0)}. \tag{B33}$$

Combined with self-duality thus gives

$$A_{(0)i} + \frac{i\varepsilon}{|p|}\epsilon_{ijk} p_j A_{(0)k} = i p_i \varphi_{(0)}. \tag{B34}$$

4. Some used identities

$$\begin{aligned} \Pi^2 &= \Pi \\ dF &= *d*dA = -t\Box\Pi A \\ \epsilon_{ikm}\epsilon_{jlm}\partial_k\partial_l &= t\Box\Pi_{ij} \\ \Pi * d &= *d\Pi = *d \\ (*d*d)_{ij} &= -M^2\Pi_{ij} = -t\Box\Pi A \\ d(M\Pi)^{-1} * d(M\Pi)^{-1} * d &= -*d \\ \left(\Pi - \frac{\theta}{M} * d\right)^{-1} &= \frac{\Pi + \frac{\theta}{M} * d}{1 + \theta^2}. \end{aligned} \tag{B35}$$

Using regularity, we get

$$\Pi_{ij} v_j = \frac{it}{M^2}\epsilon_{ijk} p_j f_k \tag{B36}$$

so

$$v_i f_i = t A_i F_i. \tag{B37}$$

APPENDIX C: SOLUTION OF THE BOUNDARY SELF-DUAL EQUATION

1. General regular solution

Let us consider the general action (4.7). We rewrite its equations of motion (4.8), combined with the regularity condition (2.33):

$$f = -M\Pi A, \tag{C1}$$

and obtain equations for regular solutions

$$\begin{aligned}(\alpha - \mu M)F + (1-\gamma)M\Pi\cdot A + \delta * dF &= \tilde{J}\\ -\beta M A + (\gamma - \epsilon M)F + \mu * dF &= J.\end{aligned} \tag{C2}$$

We now compute the on-shell action using regularity:

$$\begin{aligned}S_{\text{on-shell}} = \int d^3x\Big(&\Big(\frac{1}{2}\alpha - \mu M\Big)AF - \frac{1}{2}\beta M v \Pi A\\ &+ \Big(\frac{1}{2}\epsilon M^2 - M\gamma + \frac{1}{2}M\Big)A\Pi A + \frac{1}{2}\delta F^2\\ &- A(\tilde{J} - M\Pi\mathcal{J})\Big).\end{aligned} \tag{C3}$$

We can partial integrate to find

$$-\frac{1}{2}\beta\int d^3x v_i M\Pi_{ij}A_j = \frac{t}{2}\beta\int d^3x A_i F_i. \tag{C4}$$

and also $\int F^2 = t\int M^2 A\Pi A$. The effective action is finally

$$\begin{aligned}S_{\text{on-shell}} = \int d^3x\Big(&\Big(\frac{1}{2}\alpha - \mu M + \frac{1}{2}\beta\Big)AF\\ &+ \Big(\frac{1}{2}\epsilon M^2 - M\gamma + \frac{1}{2}M + \frac{1}{2}t\delta M^2\Big)A\Pi A\\ &- A(\tilde{J} - M\Pi\mathcal{J})\Big).\end{aligned} \tag{C5}$$

This expression, although not yet entirely explicit because it still depends on A, and was used in the main text.

We now solve the equations of motion in general, for regular Euclidean bulk solutions. Using regularity in the equations of motion (4.8), after some algebra we get:

$$\begin{aligned}-M(\beta + \mu t M)\Pi A + (\gamma - \epsilon M)F &= *dJ\\ -M(\gamma - 1 + \delta t M)\Pi A + (\alpha - \mu M)F &= *d\tilde{J}.\end{aligned} \tag{C6}$$

This has now become purely a system of equations for A. Since there is but one independent equation for A, both equations have to be consistent. Without loss of generality, let us take this to be the first equation above, which we rewrite as

$$-a\Pi A + bF = J, \tag{C7}$$

After some algebra, we can rewrite this as a massive equation for A, as we did before:

$$(a^2 - tb^2M^2)\Pi A = j, \tag{C8}$$

where we defined a new conserved current

$$j = -b * dJ - aJ. \tag{C9}$$

The on-shell effective action can now be written entirely in terms of j.

2. General topologically massive solution

In this appendix we explicitly solve the equations of motion of the massive topological boundary theory.

We showed in Appendix B that the self-duality solution combined with regularity give the following equation in momentum space:

$$A_i^{\mathrm{T}}(p) = -\frac{i\varepsilon}{|p|}\epsilon_{ijk}p_j A_k^{\mathrm{T}}(p). \tag{C10}$$

The solution is expanded in a basis of polarization vectors p_i, p_i^*, k_i, where $k\cdot p = k\cdot p^* = 0$. Choosing coordinates $p_i = (p_0, p_1, p_2)$, we have $p_i^* = (-p_0, p_1, p_2)$, and we can choose for example $k_i = (0, -p_1, p_2)$. Expanding

$$A_i^{\mathrm{T}}(p) = \frac{p_i^*}{|p|}f(p) + \frac{k_i}{|p|}g(p), \tag{C11}$$

we find the constraints

$$p\cdot p^* = 0 \qquad g(p) = -\frac{4i\varepsilon p_1 p_2}{\sqrt{2}p_0^2}\mathrm{sgn}(p_0)f(p). \tag{C12}$$

Imposing the massive boundary condition (2.50) simply adds to this the constraint $p^2 = m^2$ to the above solution.

It will be useful to have the general solution of the self-dual theory

$$A_i = \frac{1}{m}\epsilon_{ijk}\partial_j A_k. \tag{C13}$$

in position space. First of all, (C13) automatically implies the gauge-fixing condition

$$\partial^i A_i = 0. \tag{C14}$$

We can rewrite (C13) as follows:

$$(\Box + m^2)A_i = 0, \tag{C15}$$

where $\Box$ is with respect to the flat Euclidean 3-dimensional metric. The general solution of this is best written in terms of Bessel functions of the first kind:

$$\begin{aligned}A_i(R,\Omega_2) = \frac{1}{\sqrt{R}}\sum_{lm}(&a_{i(1)lm}J_{l+(1/2)}(mr)\\ &+ a_{i(2)lm}J_{-l-(1/2)}(mr))Y_{lm}(\Omega_2)\end{aligned} \tag{C16}$$

in polar coordinates (R, Ω_2), for some constants $a_{i(1)}$ and $a_{i(2)}$. These constants are however not all arbitrary. Indeed,

it is easy to show that the theory has a single massive degree of freedom, described by a *single* pair of polarizations $a_{i(1)}$, $a_{i(2)}$ One of the three A_i's is solved for from (C13). The remaining one is solved from (C14).

Solving the equation in momentum space as in (C11), we find

$$p \cdot p^* = 0 \qquad g(p) = \frac{4\sqrt{2}i}{m}\mathrm{sgn}(p_0)p_1 p_2 f(p). \tag{C17}$$

3. Boundary terms for the self-dual problem

As remarked in the main text, because of the invariant of the self-dual solutions, self-dual boundary conditions do not lead to unique boundary terms. We give here two other actions that lead to the same boundary problem:

$$\begin{aligned} S_{\mathrm{bdy}}[A, f] &= \int d^3x\left(Af + \frac{\varepsilon}{2m}F^2 - \frac{1}{m}fF\right) \\ S_{\mathrm{bdy}}[A, f] &= \int d^3x\left(Af + \frac{1}{2}\varepsilon AF - \frac{1}{m}fF\right). \end{aligned} \tag{C18}$$

It would be interesting to check that these are connected by an $SO(1, 1)$ transformation.

We finally also give the values of the constants in (4.8) and (4.11):

$$\begin{aligned} a_1 &= \beta\delta - (\gamma - 1)\mu \qquad a_2 = \gamma\delta - \alpha\mu \\ a_3 &= \delta\epsilon - \mu^2 \qquad b_1 = \beta\mu - (\gamma - 1)\epsilon \\ b_2 &= \gamma\mu - \alpha\epsilon, \end{aligned} \tag{C19}$$

and

$$a = \frac{a_3^2}{a_1 b_2 - a_2 b_1} \qquad b = \frac{a_3(a_1 - b_2)}{a_1 b_2 - a_2 b_1}, \tag{C20}$$

both in Euclidean signature.

4. Near-extremal solutions

A particularly interesting boundary condition is the generalization of (2.48) to nonextremal solutions where the electric and magnetic field are still aligned:

$$f = \lambda F. \tag{C21}$$

This boundary condition preserves conformal invariance. Combining it with the massive boundary condition (2.50), we get the following action:

$$S_{\mathrm{bulk}}^{\mathrm{on\text{-}shell}} = \int d^3x\left(\gamma Af + \frac{1}{2}\varepsilon AF + \frac{\epsilon}{2}f^2\right), \tag{C22}$$

The parameter λ and the mass are given by:

$$\lambda = \frac{1}{1 - \gamma} \qquad m = -\frac{\gamma}{\epsilon}. \tag{C23}$$

Near extremality, $\lambda \simeq \varepsilon$, we get that γ is small. m is kept finite by making ϵ small as well.

APPENDIX D: HOLOGRAPHIC RENORMALIZATION OF ABELIAN GAUGE FIELDS

The total Euclidean action is:

$$\begin{aligned} S = &\int d^4x\sqrt{g}\left(-\frac{R - 2\Lambda}{16\pi G_N} + \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right) \\ &- \frac{1}{8\pi G_N}\int d^3x\sqrt{\gamma}K + S_{\mathrm{bdy}}[A] \end{aligned} \tag{D1}$$

where $\Lambda = -3/\ell^2$, $K_{ij} = \frac{1}{2}\partial_r\gamma_{ij}$, and $\gamma = \frac{\ell^2}{r^2}g$ where g is the metric (2.2). The gravity equations of motion are

$$\begin{aligned} G_{\mu\nu} + \Lambda g_{\mu\nu} &= 8\pi G_N T_{\mu\nu} \\ T_{\mu\nu} &= F_{\mu\lambda}F_\nu{}^\lambda - \tfrac{1}{4}g_{\mu\nu}F_{\lambda\sigma}F^{\lambda\sigma}. \end{aligned} \tag{D2}$$

For conformally invariant matter, the stress-energy tensor is traceless. Therefore, it is most convenient to write Einstein's equations as follows:

$$R_{\mu\nu} + \frac{d}{\ell^2}g_{\mu\nu} = 8\pi G_N T_{\mu\nu}. \tag{D3}$$

The Ricci tensor is:

$$\begin{aligned} R_{rr} &= -\frac{1}{2}\mathrm{Tr}(g^{-1}g'') + \frac{1}{4}\mathrm{Tr}(g^{-1}g')^2 + \frac{1}{2r}\mathrm{Tr}(g^{-1}g') - \frac{d}{r^2} \\ R_{ri} &= \frac{1}{2}[\nabla^j g'_{ij} - \nabla_i \mathrm{Tr}(g^{-1}g')] \\ R_{ij} &= R_{ij}[g] - \frac{1}{2}g''_{ij} + \frac{1}{2}(g'g^{-1}g')_{ij} - \frac{1}{4}g'_{ij}\mathrm{Tr}(g^{-1}g') \\ &\quad + \frac{1}{2r}(g_{ij}\mathrm{Tr}(g^{-1}g') + (d-1)g'_{ij}) - \frac{d}{r^2}g_{ij}, \end{aligned} \tag{D4}$$

where d is the dimension of the boundary. From the fact that the curvature scalar is always constant we get

$$\begin{aligned} &\mathrm{Tr}(g^{-1}g'') - \frac{d}{r}\mathrm{Tr}(g^{-1}g') - \frac{3}{4}\mathrm{Tr}(g^{-1}g')^2 \\ &\qquad + \frac{1}{4}(\mathrm{Tr}(g^{-1}g'))^2 - R[g] = 0. \end{aligned} \tag{D5}$$

As usual in holographic renormalization, we solve Einstein's equations perturbatively in the distance to the boundary:

$$g(r, x) = g_{(0)}(x) + r^2 g_{(2)}(x) + r^3 g_{(3)} + r^4 g_{(4)} + \cdots \tag{D6}$$

We find [19]

$$g_{(2)} = -\frac{1}{d-2}\left(\mathrm{Ric}[g_{(0)}] - \frac{1}{2(d-1)}g_{(0)}R[g_{(0)}]\right) = 0, \tag{D7}$$

where we used the fact that in our applications we have the boundary condition $g_{(0)ij} = \delta_{ij}$. At the next order, from the first and third of (D4) and (D3) we find the equation

$$\mathrm{Tr}\, g_{(3)} = 0 \qquad (d-3) g_{(3)} = 0, \tag{D8}$$

which leaves the traceless part of $g_{(3)}$ undetermined, as it should, since at this order we find the holographic stress-energy tensor [19]:

$$g_{(3)} = \frac{16\pi G_N}{3\ell^2}\langle T_{ij}\rangle. \tag{D9}$$

The second equation then gives

$$\nabla^j g_{(3)ij} = \frac{16\pi G_N}{3\ell^2}\tau_{ri}|_{r=0}, \tag{D10}$$

therefore

$$\nabla^j \langle T_{ij}\rangle = f^j F_{ij}. \tag{D11}$$

Since $f^i = -\langle O^i_{\Delta=2}\rangle$, where $O^i_{\Delta=2}$ is an operator of dimension 2, this is in exact agreement with the Ward identity derived in field theory (see formula (4.19) of [33]). Indeed, one considers the effective action

$$\delta S_{\mathrm{eff}} = \int d^3x \sqrt{g_{(0)}}\left(\frac{1}{2}\delta g^{ij}_{(0)}\langle T_{ij}\rangle + \langle O^i_{\Delta=2}\rangle \delta A_i\right). \tag{D12}$$

Invariance of the action under diffeomorphism then implies (D11). As remarked earlier, in this paper, instead of regarding $O^i_{\Delta=2}$ as an operator, we will regard it as a conserved current.

At the next order, we find

$$\begin{aligned} g_{(4)} &= -\frac{4\pi G_N}{\ell^2}\left(\tau_{ij} - \frac{1}{2}\delta_{ij}\,\mathrm{Tr}\tau\right)\bigg|_{r=0} \\ &= -\frac{4\pi G_N}{\ell^2}\left(f_i f_j - F_i F_j - \frac{1}{4} g_{ij}(f^2 - F^2)\right). \end{aligned} \tag{D13}$$

Regularity of the matter solution amounted to a relation between f and A. When we include gravity, we have to demand regularity of the coupled gravity-Maxwell system. This will, in particular, involve a special choice of the boundary stress-energy tensor, which is left undetermined by (D11). We will not pursue this further here, but simply assume that such a choice indeed exists and is enough to make the solutions well behaved at $r = \infty$.

We now consider the effect of backreaction on the asymptotic expansion of the on-shell action, which determines the divergent part of the action. The matter part of the action has been analyzed in the main text. Including backreaction does not change this, as the on-shell action was shown to be a total derivative. For the gravitational part of the action we can now use the formula in [19]

$$\begin{aligned} S = \int_{\rho\geq\epsilon} d\rho \frac{3}{\rho^{5/2}}\sqrt{\det g(\rho, x)} + \frac{1}{\rho^{3/2}}(-6\sqrt{\det g(\rho, x)} \\ + 4\rho\partial_\rho\sqrt{\det g(\rho, x)}) \end{aligned} \tag{D14}$$

where $\rho = r^2$. This is the regulated part of the action. Now it is easy to check that the contribution from $\rho = \epsilon$ is

$$S = -4\epsilon^{-3/2}. \tag{D15}$$

This is the leading term near the conformal vacuum. As usual, this is canceled by adding a covariant counterterm $2(1-d)/\epsilon^{d/2}$.

From (D14) we also see that $g_{(4)}$ and any higher order pieces only contribute terms that do not contribute to two-point functions. On the other hand, there is a potential contribution from $g_{(3)}$, which contains the stress-energy tensor. This gives the usual coupling to the metric (D11)

$$\frac{1}{2}\int d^3x \sqrt{g_{(0)}} g^{ij}_{(0)} T_{ij}, \tag{D16}$$

but since with our choice of $g_{(0)}$ the boundary stress-energy tensor is traceless, this vanishes. Therefore, only the contribution from the matter part of the action is present in the two-point functions. In fact, there will be additional contributions since we will be considering a problem with generalized boundary conditions.

APPENDIX E: ANALYSIS OF SUPERSYMMETRIC BOUNDARY CONDITIONS

The bosonic fields we discuss here are the bosonic fields of the four-dimensional gauged $\mathcal{N} = 2$ supergravity [50]. The gauge symmetry is $SO(2) = U(1)$ and can be embedded in the $SO(5)^V$ truncation of the gauged maximal $SO(8)$ supergravity [51] where all scalar fields are projected out. The full field contents are the graviton, two real gravitini ψ^i_μ, $i = 1, 2$ and the Abelian gauge field, which form the $\mathcal{N} = 2$ gravity multiplet. The complete action with the two real gravitini combined into $\psi_\mu = \psi^1_\mu + i\psi^2_\mu$ reads (setting $\kappa = 1$) [57]

$$\begin{aligned} S = \int d^4x\sqrt{g}\Big(&-\frac{R}{4} + \frac{\Lambda}{2} + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\bar{\psi}_\mu\gamma^{\mu\nu\rho}D_\nu\psi_\rho \\ &- \frac{1}{2\ell}\bar{\psi}_\mu\gamma^{\mu\nu}\psi_\nu \\ &+ \frac{i}{4}\left(F^{\mu\nu} - \frac{1}{2}\,\mathrm{Im}(\bar{\psi}^\mu\psi^\nu)\right)\bar{\psi}_\rho\gamma_{[\mu}\gamma^{\rho\sigma}\gamma_{\nu]}\psi_\sigma\Big) \end{aligned} \tag{E1}$$

where

$$D_\mu = \nabla_\mu - \frac{i}{\ell}A_\mu \qquad \nabla_\mu = \partial_\mu + \frac{1}{4}\lambda_\mu{}^{ab}\gamma_{ab} \tag{E2}$$

and the spin connection is given by

$$\begin{aligned} \lambda_{\mu ab} &= \Omega_{\mu ab} - \Omega_{\mu ba} - \Omega_{ab\mu} \\ \Omega_{\mu\nu}{}^a &= \partial_{[\mu}e^a_{\nu]} - \frac{1}{2}\,\mathrm{Re}(\bar{\psi}_\mu\gamma^a\psi_\nu) \end{aligned} \tag{E3}$$

Define the supercovariantized field strength

$$\hat{F}_{\mu\nu} = F_{\mu\nu} - \mathrm{Im}(\bar{\psi}_\mu\psi_\nu) \tag{E4}$$

and the supercovariant derivative

$$\mathcal{D}_\mu = D_\mu + \frac{1}{2\ell}\gamma_\mu + \frac{i}{4}\hat{F}_{ab}\gamma^{ab}\gamma_\mu \tag{E5}$$

we can write the supersymmetry transformations as

$$\delta e^a_\mu = \mathrm{Re}(\bar{\epsilon}\gamma^a\psi_\mu) \qquad \delta A_\mu = \mathrm{Im}(\bar{\epsilon}\psi_\mu) \qquad \delta\psi_\mu = \mathcal{D}_\mu\epsilon \tag{E6}$$

where $\epsilon = \epsilon^1 + i\epsilon^2$. The $U(1)$ gauge transformation is

$$\delta A_\mu = \partial_\mu\alpha, \qquad \delta\psi_\mu = -i\alpha\psi_\mu \tag{E7}$$

To determine the supersymmetric boundary conditions, we take the approach recently studied in [52]. First, we look for boundary conditions on the bosonic and fermionic fields which render the symplectic flux through the boundary vanishing. As we showed in Appendix D, we can expand close to the boundary $r = 0$

$$g(r, x) = \delta + r^3 g_{(3)} + r^4 g_{(4)} + \cdots \tag{E8}$$

which gives the boundary condition on metric, esp. $\delta g = O(r^3)$. The gauge field contributes to the symplectic current

$$*\,\omega_r[A] = -\delta_1 A^\mu \delta_2\left(F_{r\mu} + \frac{i}{2}\bar{\psi}_\rho\gamma_{[r}\gamma^{\rho\sigma}\gamma_{\mu]}\psi_\sigma\right) + \delta_2 A^\mu\delta_1\left(F_{r\mu} + \frac{i}{2}\bar{\psi}_\rho\gamma_{[r}\gamma^{\rho\sigma}\gamma_{\mu]}\psi_\sigma\right) \tag{E9}$$

Notice that the pure gauge field contribution vanishes provided the matrices in (4.13) are symmetric. The fermion bilinear terms contribute to the symplectic flux together with those involving only the gravitini, which we now discuss.

Compared to the minimal $\mathcal{N} = 1$ theory, the $\mathcal{N} = 2$ action contains in addition coupling to the gravi-photon field and extra fermion interactions. However, it is straightforward to show that these couplings do not contribute to the gravitini symplectic current. Hence we find the symplectic current from the spin-2 and 3/2 fields [52]

$$*\omega^r[g,\psi] = r^3 P^{ijklmr}[\delta_1 g_{ij}\delta_2\lambda_{klm} - (1\leftrightarrow 2)] + r^{-2}\epsilon^{rijk}\left[\delta_1\tilde{\bar{\psi}}_i\gamma_5\tilde{\gamma}_j\delta_2\tilde{\psi}_k + \frac{1}{2}\delta_1 g_j{}^m\tilde{\bar{\psi}}_i\gamma_5\tilde{\gamma}_m\delta_2\tilde{\psi}_k - (1\leftrightarrow 2)\right] - r\frac{i}{2}g^{rr}[\delta_1 A^m\delta_2(\tilde{\bar{\psi}}_\rho\tilde{\gamma}_{[r}\tilde{\gamma}^{\rho\sigma}\tilde{\gamma}_m\tilde{\psi}_\sigma) - (1\leftrightarrow 2)] \tag{E10}$$

where

$$P^{\alpha\beta\gamma\tau\sigma\mu} = -\tfrac{1}{2}g^{\alpha\beta}g^{\gamma\tau}g^{\mu\sigma} + \tfrac{1}{2}g^{\alpha\mu}g^{\gamma\tau}g^{\beta\sigma} + \tfrac{1}{2}g^{\alpha\gamma}g^{\beta\tau}g^{\mu\sigma}. \tag{E11}$$

The symplectic from is written in terms of rescaled metric g_{ij} and fermions $\tilde{\psi}_k = r^{1/2}\psi_k$. Now it can be seen that the graviton contribution to the symplectic flux vanishes

$$\int_{r=0}\omega[g] = 0 \tag{E12}$$

where the dual symplectic current

$$\omega[g] = \epsilon_{ijkr} * \omega^r dx^i\wedge dx^j\wedge dx^k = O(r^2). \tag{E13}$$

In the last equation we used $\delta g = r^3\delta_{g(3)}$. We also see that the gravitini contribution

$$\int_{r=0}\omega[\psi] = \int_{r=0}\left(\frac{1}{r^2}\delta_1\tilde{\bar{\psi}}_i\tilde{\gamma}_j\gamma_5\delta_2\tilde{\psi}_k - \frac{i}{2r}\tilde{\epsilon}_{ijk}\delta_1 A^m\delta_2(\tilde{\bar{\psi}}_\rho\tilde{\gamma}_{[r}\tilde{\gamma}^{\rho\sigma}\tilde{\gamma}_{m]}\tilde{\psi}_\sigma)\right)dx^i\wedge dx^j\wedge dx^k \tag{E14}$$

is finite but nonvanishing if $\tilde{\psi}_\mu = O(r)$. In order that the symplectic flux vanishes, we need $\tilde{\psi}_i = O(r^2)$ and $\tilde{\psi}_r = O(r)$. In terms of ψ_μ we impose

$$\psi_r = O(r^{1/2}), \qquad \psi_i = O(r^{3/2}). \tag{E15}$$

Having found the boundary conditions consistent with quantization, it is interesting to see whether supersymmetry is preserved asymptotically. This is equivalent to that supersymmetry transformations keep the symplectic current invariant on the boundary. Plugging (E15) into (E6) leads to

$$\delta_\epsilon A_r = O(r^{1/2}), \qquad \delta_\epsilon A_i = O(r^{3/2}). \tag{E16}$$

From this we see that $\delta_\epsilon A_i$ contributes a vanishing term to the gauge field symplectic current. Evaluating the transformation of $g_{\mu\nu}$ is slightly more complicated, but one can show [52] that up to an infinitesimal diffeomorphism the susy variation takes the form $\delta_\epsilon g = O(r^2)$. The infinitesimal diffeomorphism $\mathcal{L}V_\epsilon$ changes $g_{\mu\nu}$ at order $O(r)$ and simply signals a departure from the Gaussian normal gauge. In practice, one can take the supersymmetry generator to be $s_\epsilon - \mathcal{L}_V$. It now remains to show that the ψ_μ boundary conditions are also preserved by supersymmetry.

For convenience of notation we define the operator $\mathcal{D}_0 = \mathcal{D}|_{A=0}$. Using the result of Appendix D the stress tensor $T_{\mu\nu} = O(r^2)$, one can show following [52] that for η a AdS_4 killing spinor

$$\delta^0_\eta \psi_\mu = \mathcal{D}_{0\mu}\eta = O(r^{3/2}). \tag{E17}$$

What remains is to find the behavior of

$$\delta^A_\eta \psi_\mu = (\mathcal{D}_\mu - \mathcal{D}_{0\mu})\eta = -\left(\frac{i}{\ell}A_\mu - \frac{i}{4}\gamma^{\alpha\beta}\gamma_\mu F_{\alpha\beta}\right)\eta. \tag{E18}$$

For generic values of A and F, the first term is order $O(r^0)$ while the second is $O(r)$. Moreover since A_μ is a $U(1)$ gauge field, both A_μ and $F_{\alpha\beta}$ are just numbers and the two terms are linearly independent due to the γ-matrix. Recall $\tilde{\psi}_\mu = r^{1/2}\psi_\mu$ and (E14) we find $A_\mu = O(r)$ in order that the symplectic current is finite at the boundary. This suggests the only supersymmetric boundary condition for the gauge field is the Dirichlet boundary condition with

$$A_\mu = 0. \tag{E19}$$

Neglecting the radial component this is then $A_i = 0$.

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