Towards a factored analysis of legged locomotion models

Richard Altendorfer
University of Michigan

Daniel E. Koditschek
University of Pennsylvania, kod@seas.upenn.edu

Philip Holmes
Princeton University

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Richard Altendorfer, Daniel E. Koditschek
Dept. of Electrical Engineering and Computer
Science,
The University of Michigan, Ann Arbor, MI
48109, USA

Philip Holmes
Dept. of Mechanical and Aerospace
Engineering,
Princeton University, Princeton, NJ 08544, USA

Abstract—In this paper, we report on a new stability analysis for hybrid legged locomotion systems based on factorization of return maps. We apply this analysis to a family of models of the Spring Loaded Inverted Pendulum (SLIP) with different leg recirculation strategies. We obtain a necessary condition for the asymptotic stability of those models, which is formulated as an exact algebraic expression despite the non-integrability of the SLIP dynamics. We outline the application of this analysis to other models of legged locomotion and its importance for the stability of legged robots and animals.

I. INTRODUCTION

This paper introduces a new formalism for studying the stability of legged locomotion gaits and other periodic dynamically dexterous robotic tasks. We are motivated by the need to explain and control the remarkable performance of RHex, an autonomous hexapedal robot runner with unparalleled mobility [1]. Powered by only six actuators, located at the "hips" to drive each of its six passively compliant legs, RHex's locomotion is excited by a single periodic "clock" signal split into phase and anti-phase copies for coordinating its alternating tripod gait. A simple PD controller at each hip motor in a given tripod forces its leg to track the alternately fast and slow clock reference signal corresponding to putative stance and swing phases. RHex's stable gaits in varied terrain strongly motivate the development of an analytical understanding of the relationship between clock signal and steady state gait properties in this "simple" open loop case.

A complete account of this relationship in even the simple case would entail insight into the steady state properties of an under-actuated high degree of freedom hybrid mechanical system whose Lagrangian dynamics switches between a set of 2^6 possible holonomically constrained models depending upon which toes are in contact with the ground. Fortunately, a growing body of simulation study and empirical evidence [2] suggests that RHex, when properly tuned, exhibits sagittal plane stance behavior well approximated by the two degree of freedom SLIP. Thus, in the short term, we seek to understand how adjustments to a coordinating clock signal will determine the steady state performance of the bipedal SLIP underlying the alternating tripod gait. In this paper, we develop the mathematical foundations of a new formalism for distinguishing volume-preserving from non-

volume-preserving hybrid Lagrangian systems. We apply this formalism to the hybrid SLIP model, the results of which suggest new insight into the relationship between clock excitation and steady state gait.

A. Leg Swing Policies and Self Stability in the SLIP Template

The SLIP model provides a ubiquitous description of biological runners in the sagittal plane [3] and, as mentioned above, a broadly useful prescription for legged robot runners such as RHex [4], [1], [2] as well. The closely related three degree of freedom Lateral Leg Spring (LLS), has been recently identified as a candidate template for cockroach running in the horizontal plane [5], [6] and seems likely to be relevant for RHex as well [1]. For present purposes, the most important insight from these models has been to provide a mathematical explanation for their unexpected "self-stability" properties (asymptotically stable equilibrium gaits in the absence of any external sensor based feedback inputs).

The originally discovered self-stability of SLIP [7], [8] arises from a simple leg swing policy that specifies a constant leg touchdown angle at the end of each aerial phase. A similar strategy was earlier shown to yield stable gaits in the LLS model [6]. Recently, a different time-dependent leg retraction policy has been shown to inherit the stability properties of the fixed touchdown angle policy while increasing the robustness of the SLIP system [9]. On the other hand, numerical simulations of a recirculation policy where the SLIP's leg starts recirculating after leg liftoff at a constant angular velocity until leg touchdown suggest not asymptotic but neutral stability (see also [10]). Hence the leg swing policy seems to play a central role in the stability of those low-dimensional models, as is also suggested by high dimensional systems such as RHex [1] or, as established in an independently conceived model inspired by animal locomotion strategies, for a quadrupedal trotter [11].

The stability of such systems is established by reasoning about the local properties of the full-stride return map – the function relating body state at one stride to body state at next – which summarizes all properties relevant to the goal of translating the body center of mass. Unfortunately, even the simplest 2dof SLIP system is non-integrable [12],

37

which precludes a closed form representation of the return map that might illuminate the stability mechanisms. In this paper we will show how the stability of hybrid systems possessing certain symmetries can be analyzed in terms of their non-hybrid components, e.g. flight phase and stance phase in the case of SLIP, thus decomposing the hybrid return map into "partial" return map factors that might be analyzed more easily.

Before doing so, we first introduce certain terminology and notation used throughout the paper by way of reviewing Liouville's theorem (see e.g. [13]), pointing out that the classical result should not be expected to apply in the present case.

B. Liouville's theorem and stability

Liouville's theorem states that volume in phase space of a holonomically constrained conservative dynamical system described by a single Hamiltonian flow is preserved, i.e. a set of initial conditions at $t = t_0$ in phase space will be mapped to a set with identical symplectic volume for any $t \geq t_0$.

Non-holonomically constrained conservative systems are in general not Hamiltonian and Liouville's theorem cannot be applied (see e.g. the asymptotic stability of the Chaplygin sleigh in [14] and references therein).

In the case of piecewise-defined holonomically constrained conservative Hamiltonian systems with different flows f_α^t (with α in an indexing set I) the transition to a new Hamiltonian flow is triggered by so-called threshold functions (for a general definition of hybrid systems see [15]). In almost all settings within robotics, these threshold functions between different flows f_α^t and f_β^t depend upon state, and often have no explicit time dependence at all. Examples include a discrete version of the Chaplygin sleigh [14], [16] and low-dimensional models of legged locomotion in the horizontal and sagittal plane [6], [7], [8], which all exhibit partial asymptotic stability for certain parameter settings. Here, the (local) asymptotic stability of those hybrid system at a fixed point means that the eigenvalues of the components of their linearized return maps defined by Poincaré sections lie within the unit circle (excluding those corresponding to conserved quantities).

In those cases, the return map $\mathscr{R}$ defined by a Poincaré section is composed of several "factor" maps r_α^β that relate the state variables directly after one transition to those directly before the next. Additional transition mappings $\mathscr{T}_\alpha^\beta$ are used at transitions from flows f_α^t to f_β^t. They arise in practice from the fact that different flows are most easily handled analytically in different coordinate systems. Thus the return map can be decomposed into factors $\mathscr{R} = \mathscr{T}_\gamma^\alpha \circ r_\gamma^\alpha \circ \cdots \circ \mathscr{T}_\alpha^\beta \circ r_\alpha^\beta$. Liouville's theorem is not applicable to return maps where the Poincaré section is defined by the vanishing of a threshold function that is not exclusively time-dependent, since the underlying

Hamiltonian flow will map an open set of initial values to an open set on the Poincaré section with *different* evolution times.

C. Contribution of this paper

In this paper, we focus on the role of volume preservation in flows and transition maps of models of legged locomotion as an indicator of local stability. Although we cannot invoke Liouville's theorem for reasons just reviewed, we nevertheless deduce the necessity for volume preservation at a given fixed point in these flows, manifesting itself as the condition of unity determinant in the associated linearized return map. In general, in order to check whether the return map of a hybrid system $\mathscr{R}$ is volume-preserving at a fixed point, the map must be computed explicitly [16], [6], [7], [8]. However, we will show that if all vector fields f_α, $\alpha \in I$ possess time reversing symmetries $\mathscr{S}_\alpha$ and if the periodic orbit giving rise to a fixed point of $\mathscr{R}$ is composed of pseudosymmetric orbits (to be defined below) on each flow domain V_α, then volume-preservation of the whole system can be determined by volume-preservation on individual flow domains. Moreover, if the transition function, h_α^β, from a flow domain, V_α, to a next flow domain, V_β, enjoys a certain symmetry related to the vector field's time reversing symmetry $\mathscr{S}_\alpha$, then volume preservation on V_α can be determined in certain cases without an explicit expression for the flow map on that domain.

This paper introduces the rigorous foundations for the formalism just outlined, and provides an example of its value for robotics by application to the LLS [6] and SLIP [7], [8] models. As explained above, the question of whether a hybrid system is volume preserving or not has immediate consequences for gait stability. Since our conditions for volume preservation of the full stride return map may be checked with respect to a simple "factor map," we are able to study the effects of a broad range of leg swing policies that are implementable in the analytically tractable "flight" phase of the leg. Specifically, we show how the volume preserving properties of SLIP under different leg recirculation strategies can be determined by simple differentiation. Notably, in Sec.III.A.3 we give for the first time necessary conditions for asymptotic and neutral stability of a RHex-like leg recirculation scheme applied to the SLIP model, without explicit computation of the stance phase flow map.

II. FACTORING RETURN MAPS WITH TIME REVERSING SYMMETRIES

Assume a hybrid mechanical system whose time evolution is described by holonomically constrained autonomous conservative vector fields f_α, $\alpha \in I$ with configuration space variables q_α: $\dot{x}_\alpha = f_\alpha(x_\alpha)$ with $x_\alpha = (q_\alpha \ \dot{q}_\alpha)^\top \in V_\alpha$. The open flow domains V_α are called

charts [15]. Transitions between vector fields are governed by threshold functions h^β_α which can depend on the initial condition $x_{\alpha_0} = x_\alpha(t = 0) \in V_\alpha$, time t, and the current state $f^t_\alpha(x_{\alpha_0})$.[1] We require that for each chart there is only one threshold function h^β_α. Transitions to the vector field f_β are uniquely defined by $t^\beta_{\alpha_0}(x_{\alpha_0}) = \min_{t>0}\{t : h^\beta_\alpha(f^t_\alpha(x_{\alpha_0}),x_{\alpha_0},t) = 0\}$. In addition, all transition mappings $\mathscr{T}^\beta_\alpha$ are assumed to be volume preserving. The *flow map* r^β_α for the αth vectorfield is implicitly defined via $t^\beta_{\alpha_0}$

by $r^\beta_\alpha : x_{\alpha_0} \mapsto f^{t^\beta_{\alpha_0}}_\alpha(x_{\alpha_0})$, where x_{α_0} is assumed to be the result of a preceding chart transition.

We will define a partial return map for evolutions on individual charts V_α. Under the assumption that a periodic orbit of the whole dynamical system is composed of pseudo-symmetric orbits (to be defined below) on individual charts, volume preservation of the full return map $\mathscr{R}$ can be determined in some cases without explicitly computing r^β_α. This will be the case if the vector field f_α has additional structure, namely a time reversal symmetry.

Assume that the vector field f_α admits a time reversing symmetry (For a survey see [17].) $\mathscr{S}_\alpha : V_\alpha \to V_\alpha$, i.e.

$$\frac{d}{dt}(\mathscr{S}_\alpha \circ x_\alpha) = f_\alpha(\mathscr{S}_\alpha \circ x_\alpha) \Leftrightarrow \dot{x}_\alpha = f_\alpha(x_\alpha) \quad (1)$$

that is also an involution: $\mathscr{S}_\alpha \circ \mathscr{S}_\alpha = id$. This implies $\mathscr{S}_\alpha \circ f^t_\alpha \circ \mathscr{S}_\alpha = f^{-t}_\alpha$. We now investigate a composition of two *partial return maps*, $x_{\alpha_1} = \mathscr{S}_\alpha \circ r^\beta_\alpha(x_{\alpha_0})$, followed by $x_{\alpha_2} = \mathscr{S}_\alpha \circ r^\beta_\alpha(x_{\alpha_1})$. If the functional identity $t^\beta_{\alpha_1} = t^\beta_{\alpha_0}$ holds, then

$$\mathscr{S}_\alpha \circ r^\beta_\alpha \circ \mathscr{S}_\alpha \circ r^\beta_\alpha = \mathscr{S}_\alpha \circ f^{t^\beta_{\alpha_0}}_\alpha \circ \mathscr{S}_\alpha \circ f^{t^\beta_{\alpha_0}}_\alpha = \quad (2)$$
$$f^{-t^\beta_{\alpha_0}}_\alpha \circ f^{t^\beta_{\alpha_0}}_\alpha = id ,$$

i.e. $\mathscr{S}_\alpha \circ r^\beta_\alpha$ is also an involution. A sufficient condition for $t^\beta_{\alpha_1} = t^\beta_{\alpha_0}$ is given by

$$\left(h^\beta_\alpha(f^{t^\beta_{\alpha_0}}_\alpha(x_{\alpha_1}),x_{\alpha_1},t^\beta_{\alpha_0}) = 0 \Leftrightarrow \right.$$
$$\left. h^\beta_\alpha(f^{t^\beta_{\alpha_0}}_\alpha(x_{\alpha_0}),x_{\alpha_0},t^\beta_{\alpha_0}) = 0\right) \Rightarrow t^\beta_{\alpha_1} = t^\beta_{\alpha_0} , \quad (3)$$

or more explicitly

$$\left(h^\beta_\alpha(f^{t^\beta_{\alpha_0}}_\alpha \circ S_\alpha \circ f^{t^\beta_{\alpha_0}}_\alpha(x_{\alpha_0}),S_\alpha \circ f^{t^\beta_{\alpha_0}}_\alpha(x_{\alpha_0}),t^\beta_{\alpha_0}) = \right.$$
$$\left. h^\beta_\alpha(S_\alpha(x_{\alpha_0}),S_\alpha \circ f^{t^\beta_{\alpha_0}}_\alpha(x_{\alpha_0}),t^\beta_{\alpha_0}) = 0\right)$$
$$\Leftrightarrow \left(h^\beta_\alpha(f^{t^\beta_{\alpha_0}}_\alpha(x_{\alpha_0}),x_{\alpha_0},t^\beta_{\alpha_0}) = 0\right) \quad (4)$$

which can be considered an invariance of the threshold function under the partial return map $\mathscr{S}_\alpha \circ r^\beta_\alpha : x_{\alpha_0} \mapsto S_\alpha \circ$

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[1] Note that this definition generalizes [15], where h^β_α only depends on $f^t_\alpha(x_{\alpha_0})$.

$r^\beta_\alpha(x_{\alpha_0})$. This essentially checks that the threshold function h^β_α "preserves" the time reversal symmetry of f_α.

Given that $\mathscr{S}_\alpha \circ r^\beta_\alpha \circ \mathscr{S}_\alpha \circ r^\beta_\alpha = id$ we obtain

$$D(\mathscr{S}_\alpha \circ r^\beta_\alpha \circ \mathscr{S}_\alpha \circ r^\beta_\alpha) = \mathbf{1}_{n \times n} \quad (5)$$
$$= D\mathscr{S}_\alpha(r^\beta_\alpha \circ \mathscr{S}_\alpha \circ r^\beta_\alpha) \cdot Dr^\beta_\alpha(\mathscr{S}_\alpha \circ r^\beta_\alpha) \cdot D\mathscr{S}_\alpha(r^\beta_\alpha) \cdot Dr^\beta_\alpha$$

Next, we call a trajectory on a chart V_α *pseudosymmetric* if its initial condition, $\bar{x}_\alpha$, is a fixed point of the partial return map $\mathscr{S}_\alpha \circ r^\beta_\alpha$, i.e. $\mathscr{S}_\alpha \circ r^\beta_\alpha(\bar{x}_\alpha) = \bar{x}_\alpha$. Evaluation of expression (5) at such a fixed point then allows us to determine the square of the determinant of the Jacobian of r^β_α (the monodromy matrix):

$$\left(D\mathscr{S}_\alpha(r^\beta_\alpha(\bar{x}_\alpha)) \cdot Dr^\beta_\alpha(\bar{x}_\alpha)\right)^2 = \mathbf{1}_{n \times n}$$
$$\det^2\left(D\mathscr{S}_\alpha(r^\beta_\alpha(\bar{x}_\alpha)) \cdot Dr^\beta_\alpha(\bar{x}_\alpha)\right) = 1$$

If $\mathscr{S}_\alpha$ is linear, as is the case in all of the examples in this paper, then $\det^2(D\mathscr{S}_\alpha) = 1$ and therefore

$$\det^2(Dr^\beta_\alpha(\bar{x}_\alpha)) = 1 .$$

Hence if a periodic orbit described by a fixed point of the full return map $\mathscr{R}$ is composed of pseudosymmetric orbits on V_α and conditions (3) hold on each chart, the full return map $\mathscr{R}$ is volume-preserving at this fixed point. If conditions (3) do not hold or if periodic orbits of the full return map are not composed of pseudosymmetric orbits, no conclusions can be drawn from this argument. Note that here "volume" is defined with respect to the state variables chosen, and in general is not a phase space volume, and that the computation (at the fixed point) is local.

III. APPLICATION TO HYBRID MODELS OF LEGGED LOCOMOTION

A. SLIP with pitching

The SLIP model consists of a rigid body of mass m and moment of inertia I with a massless springy leg attached to an unactuated hip joint which is a distance d away from the center of mass (for details see [8]). A full stride consists of a stance phase with the foothold fixed and the leg under compression, and a flight phase in which the body describes a ballistic trajectory. Hence there are two vector fields f_1 (for stance) and f_2 (for flight) and the return map can be written as $\mathscr{R} = \mathscr{T}^1_2 \circ r^1_2 \circ \mathscr{T}^2_1 \circ r^2_1$. We assume that a periodic orbit of period one is composed of pseudosymmetric stance and flight phases.[2]

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[2] This was proven in [8] for SLIP without pitching dynamics and without gravity in stance.
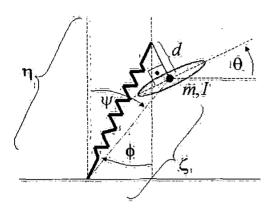
39

Fig. 1. Parametrization of SLIP with pitching dynamics.

1) Stance phase of SLIP with pitching: The equations of motion that describe the stance phase of SLIP with mass m and moment of inertia I read in the conventions of Fig. 1 (see also [8]):

$$\ddot{\zeta} = \zeta \dot{\psi}^2 - g\cos\psi - \frac{\partial_\eta V(\eta)}{m\eta}(\zeta + d\cos(\psi + \theta))$$

$$\zeta\ddot{\psi} = -2\dot{\zeta}\dot{\psi} + g\sin\psi + d\frac{\partial_\eta V(\eta)}{m\eta}(\sin(\psi + \theta))$$

$$\ddot{\theta} = d\zeta\frac{\partial_\eta V(\eta)}{\eta I}\sin(\psi + \theta) \qquad (6)$$

with spring potential $V(\eta)$ where the compressed spring length $\eta = \sqrt{d^2 + \zeta^2 + 2d\zeta\cos(\psi + \theta)} \leq \eta_0$, the relaxed spring length being denoted by η_0. The distance between the 'hip' pivot and the mass center is d and all joints, including the toe-ground pivot, are assumed frictionless and moment-free.

The involutive time reversing symmetry $\mathscr{S}_1$ of (6) acting on $x_1 = (\zeta \ \psi \ \theta \ \dot{\zeta} \ \dot{\psi} \ \dot{\theta})^\top$ is given by the linear map

$$\mathscr{S}_1 = \text{diag}(1, -1, -1, -1, 1, 1) . \qquad (7)$$

Under $\mathscr{S}_1$, the spring length η remains invariant: $\mathscr{S}_1(\eta) = \eta$. Transitions to flight occur when the spring length has reached its rest length η_0 which is also the initial spring length at the beginning of stance. Hence the threshold function can be written as $h_1^2(x_1(t), x_{1_0}, t) = \eta^2(t) - \eta_0^2$. Then the partial return map $\mathscr{S}_1 \circ r_1^2$ is volume-preserving at a fixed point, because (4) holds:

$$h_1^2(x_1(t_{1_0}^2), x_{1_0}, t_{1_0}^2) = \eta^2(t_{1_0}^2) - \eta_0^2 = 0$$

$$\Leftrightarrow h_1^2(\mathscr{S}_1 \cdot x_{1_0}, \mathscr{S}_1 \cdot x_1(t_{1_0}^2), t_{1_0}^2) = \eta_0^2 - \eta^2(t_{1_0}^2) = 0 .$$

This result is independent of the specific form of the spring potential $V(\eta)$. For $d = 0$, the two-dimensional SLIP system describing a point mass on a massless spring is recovered.

2) Flight phase of SLIP with pitching: The equation of motion of the center of mass that describe the flight phase of SLIP read

$$\ddot{y} = 0, \qquad \ddot{z} = -g, \qquad \ddot{\theta} = 0 \qquad (8)$$

where the z-axis points vertically upwards, the y-axis points in a horizontal direction, and θ denotes the SLIP body's pitching angle with respect to the horizontal. The linear involutive time reversing symmetry $\mathscr{S}_2$ of (8) acting on $x_2 = (y \ z \ \theta \ \dot{y} \ \dot{z} \ \dot{\theta})^\top$ is given by

$$\mathscr{S}_2 = \text{diag}(\pm 1, 1, -1, \mp 1, -1, 1) . \qquad (9)$$

Here, the sign ambiguity in θ was resolved by matching θ's transformation to that of the stance phase (7). The transformation law of y under $\mathscr{S}_2$ is not needed in subsequent calculations[3] and is left unresolved. Hence from now on, we work with a reduced state vector $x_2 \rightarrow (z \ \theta \ \dot{z} \ \dot{\theta})^\top$.

The simplicity of the equations of motion (8) allows us to explicitly compute the determinant of the monodromy matrix of $\mathscr{S}_2 \circ r_2^1$ for a given leg recirculation scheme. Therefore the application of the formalism of Sec. 2, which only provides a sufficient condition for volume-preservation, but not for non-preservation, is relegated to the appendix.

The equations of motion for the z and θ coordinates can be explicitly solved and read in dimensionless variables:

$$\tilde{z}(\tilde{t}) = \tilde{z}_0 + \dot{\tilde{z}}_0 \tilde{t} - \frac{\tilde{t}^2}{2}$$

$$\dot{\tilde{z}}(\tilde{t}) = \dot{\tilde{z}}_0 - \tilde{t}$$

$$\tilde{\theta}(\tilde{t}) = \tilde{\theta}_0 + \dot{\tilde{\theta}}_0 \tilde{t}$$

$$\dot{\tilde{\theta}}(\tilde{t}) = \dot{\tilde{\theta}}_0 \qquad (10)$$

with $\tilde{t} = t\sqrt{\frac{g}{\eta_0}}$, $\tilde{z} = \frac{z}{\eta_0}$, $\dot{\tilde{z}} = \frac{\dot{z}}{\sqrt{\eta_0 g}}$, $\tilde{\theta}_0 = \theta_0$, and $\dot{\tilde{\theta}} = \dot{\theta}\sqrt{\frac{\eta_0}{g}}$. We now want to explore different strategies to position the leg during flight. Since the leg is assumed massless, any leg angle trajectory $\phi(\tilde{t}, \tilde{x}_{2_0})$ where ϕ is defined in Fig. 1 can be commanded.

The threshold function h_2^1 for a recirculating leg reads in dimensionless variables

$$h_2^1(\tilde{x}_{2_0}(\tilde{t}), \tilde{x}_{2_0}, \tilde{t}) = \tilde{z}(\tilde{t}) + \tilde{d}\cos(\tilde{\theta}(\tilde{t})) - \cos(\phi(\tilde{t}, \tilde{x}_{2_0})) \qquad (11)$$

where $\tilde{d} = \frac{d}{\eta_0}$. Setting (11) to zero determines the time from leg liftoff ($\tilde{t}_{LO} = 0$) to leg touchdown $\tilde{t}_{TD} := \tilde{t}_{2_0}^1$, for which in general a closed form solution does not exist. Then the flow map r_2^1 takes the dimensionless state vector $\tilde{x}_{2_0} = (\tilde{z} \ \dot{\tilde{z}} \ \tilde{\theta} \ \dot{\tilde{\theta}})^\top$ from its value at leg liftoff to that at

[3]y is determined by conservation of energy and y is not a periodic variable.

40

touchdown: $r_2^1(\tilde{x}_{2_0}) = \tilde{x}_2(\tilde{t}_{TD})$. A fixed point of a pseu-dosymmetric flight trajectory satisfies $\tilde{x}_{2_0} = \mathscr{S}_2 \circ r_2^1(\tilde{x}_{2_0})$.

The determinant of the monodromy matrix of r_2^1 can easily be computed from the flight trajectory expressions (10), bearing in mind that the flight time $\tilde{t}_{TD}$ also depends on the initial conditions:

$$\det(Dr_2^1) = 1 - \partial_{\tilde{z}_0}\tilde{t}_{TD} + \dot{\tilde{z}}_0\partial_{\tilde{z}_0}\tilde{t}_{TD} + \dot{\tilde{\theta}}_0\partial_{\tilde{\theta}_0}\tilde{t}_{TD} \qquad (12)$$

In this expression, the leading term 1 is a consequence of Liouville's theorem, because $\tilde{z}$ and $\dot{\tilde{z}}$ and $\tilde{\theta}$ and $\dot{\tilde{\theta}}$ are canonically conjugate up to a trivial rescaling, whereas the remaining terms make the non-applicability of Liouville's theorem to this hybrid system with a state-dependent threshold function (11) explicit. Hence using implicit differentiation of (11) the determinant can be written in terms of partial derivatives of $\phi(\tilde{t}, \tilde{x}_{2_0})$:

$$\det\left(Dr_2^1(\tilde{x}_{2_0})\right) = 1 + \left.\frac{\Delta_2^{1\,num}}{\Delta_2^{1\,den}}\right|_{\tilde{t}=\tilde{t}_{TD}} \qquad (13)$$

with

$$\Delta_2^{1\,num} = \sin(\phi(\tilde{t}, \tilde{x}_{2_0})) \cdot$$

$$\left(\partial_{\tilde{z}_0}\phi(\tilde{t}, \tilde{x}_{2_0}) - \dot{\tilde{z}}_0\partial_{\tilde{z}_0}\phi(\tilde{t}, \tilde{x}_{2_0}) - \dot{\tilde{\theta}}_0\partial_{\tilde{\theta}_0}\phi(\tilde{t}, \tilde{x}_{2_0})\right)$$

$$+\tilde{t} - \dot{\tilde{z}}_0 + \tilde{d}\sin(\tilde{\theta}_0)\dot{\tilde{\theta}}_0$$

$$\Delta_2^{1\,den} = \sin(\phi(\tilde{t}, \tilde{x}_{2_0}))\partial_{\tilde{t}}\phi(\tilde{t}, \tilde{x}_{2_0}) - \tilde{t} + \dot{\tilde{z}}_0 - \tilde{d}\sin(\tilde{\theta}_0)\dot{\tilde{\theta}}_0 \ .$$

Albeit $\tilde{t}_{TD}$ cannot be computed explicitly in general, $\tilde{t}_{TD} = 2\dot{\tilde{z}}_0$ at a symmetric (period-1) fixed point of $\mathscr{S}_2 \circ r_2^1$, $\sin(\phi(\tilde{t}_{TD}, \tilde{x}_{2_0})) = -\sqrt{1 - (\tilde{z}_0 + \tilde{d}\cos(\tilde{\theta}_0))^2}$ and $\theta(\tilde{t}_{TD}) = -\tilde{\theta}_0$. The eigenvalues of the partial return map $\mathscr{S}_2 \circ r_2^1$ at such a fixed point are $\lambda_1 = 1$ (vertical energy), $\lambda_2 = 1$ (rotational energy), $\lambda_3 = -1$, and $\lambda_4 = -\det(Dr_2^1(\tilde{x}_{2_0}))$. It must be emphasized, however, that the eigenvalues of this partial return map are not equal to the eigenvalues of the total return map $\mathscr{R}$ at the fixed point. In particular, the eigenvalues of $\mathscr{R}$ can be complex, as in Fig. 2, below.

Setting (13) to 1 yields a partial differential equation for leg recirculation schemes $\phi(\tilde{t}, \tilde{x}_{2_0})$ that are volume preserving. In the following, for simplicity, we apply formula (13) to different leg recirculation schemes for SLIP without pitching dynamics, i.e. $\tilde{d} = 0$ and no $\tilde{\theta}, \dot{\tilde{\theta}}$ dependence. The investigation of leg recirculation strategies of SLIP with pitching dynamics will be detailed in [10]. For SLIP without pitching, the determinant at the fixed point simplifies to

$$\det(Dr_2^1(\tilde{x}_{2_0})) = 1 + \qquad (14)$$

$$\left.\frac{\sin(\phi(\tilde{t}, \tilde{x}_{2_0}))\left(\partial_{\tilde{z}_0}\phi(\tilde{t}, \tilde{x}_{2_0}) - \dot{\tilde{z}}_0\partial_{\tilde{z}_0}\phi(\tilde{t}, \tilde{x}_{2_0})\right) + \tilde{t} - \dot{\tilde{z}}_0}{\sin(\phi(\tilde{t}, \tilde{x}_{2_0}))\partial_{\tilde{t}}\phi(\tilde{t}, \tilde{x}_{2_0}) - \tilde{t} + \dot{\tilde{z}}_0}\right|_{\tilde{t}=\tilde{t}_{TD}}$$

3) Analysis of Recirculation Strategies:
Consider now the following family of leg recirculation schemes

$$\phi(\tilde{t}, \tilde{x}_{2_0}) = k\arccos(\tilde{z}_0) + \alpha(\tilde{t} - l\dot{\tilde{z}}_0) : \quad k, l > 0 \qquad (15)$$

For $k \neq 0$ leg recirculation starts at an angle proportional to the leg liftoff angle; if $l = 1$, a certain angular trajectory is specified starting at apex (see the leg retraction scheme in [9]).

The application of the partial return map $\mathscr{S}_2 \circ r_2^1$ on $\tilde{x}_{2_0}$ in the threshold function h_2^1 (4) in order to determine for which parameters of the leg recirculation schemes (15) volume is preserved is relegated to the appendix. Instead we proceed by explicitly computing $\det(Dr_2^1(\tilde{x}_{2_0}))$. With the angular trajectory (15), the determinant becomes

$$\det(Dr_2^1(\tilde{x}_{2_0})) = 1 + \qquad (16)$$

$$\left.\frac{\sin(\phi(\tilde{t}, \tilde{x}_{2_0}))\left(-l\alpha(\tilde{t} - l\dot{\tilde{z}}_0) - \frac{k\dot{\tilde{z}}_0}{\sqrt{1-\tilde{z}_0^2}}\right) + \tilde{t} - \dot{\tilde{z}}_0}{\sin(\phi(\tilde{t}, \tilde{x}_{2_0}))\alpha(\tilde{t} - l\dot{\tilde{z}}_0) - \tilde{t} + \dot{\tilde{z}}_0}\right|_{\tilde{t}=\tilde{t}_{TD}}$$

Hence for different leg angle protocols we obtain

1) Constant leg touchdown angle protocol: $k = l = 0$, $\alpha = 2\pi - \beta \Rightarrow \det(Dr_2^1(\tilde{x}_{2_0})) = 0$. The two-dimensional monodromy matrix has rank one for all $\tilde{t}_{TD}$, and the return map becomes one-dimensional. In [7] this return map was parametrized by apex height, whereas in [8] the angle of the touchdown velocity was chosen. No information about the behavior of this lower-dimensional return map can be obtained from this argument.

2) Leg retraction [9]: $k = 0$, $l = 1$, and $\alpha(\tilde{t} - \dot{\tilde{z}}_0) = \alpha_A + \bar{\omega}(\tilde{t} - \dot{\tilde{z}}_0)$ where α_A is a constant angle and $\bar{\omega} = \omega\sqrt{\frac{n_0}{g}}$ is a constant dimensionless angular velocity. Then again $\det(Dr_2^1(\tilde{x}_{2_0})) = 0$ and the behavior of the remaining one-dimensional return map cannot be determined from this argument.

3) Leg recirculation (starting at leg liftoff): $l = 0$, $k > 0$, and $\alpha(\tilde{t}) = \alpha_A + \bar{\omega}\tilde{t}$. This exemplifies the fast rotation phase of the open loop policy used by RHex [1], although a full analysis is beyond the scope of the present paper. Then the determinant of the monodromy matrix *at the symmetric fixed point* becomes:

$$\det(Dr_2^1(\tilde{x}_{2_0})) = 1 - \frac{\dot{\tilde{z}}_0(1-k)}{\dot{\tilde{z}}_0 + \bar{\omega}\sqrt{1-\tilde{z}_0^2}} \qquad (17)$$

$$= \begin{cases} <1 & : \quad 0 < k < 1 \\ 1 & : \quad k = 1 \\ >1 & : \quad k > 1 \end{cases}$$

In order to illustrate the predictive power of (17), we numerically approximate the determinant $\det(D\mathscr{R}(\tilde{x}))$ of

41

the full return map for fixed SLIP parameters $\tilde{E} = \frac{E}{mg\eta_0} =$
2.1, $\gamma = \frac{\kappa\eta_0}{mg} = 13$, and fixed recirculation parameters
$\alpha_A = \pi$, $\tilde{\omega} = 14$ for different $k \in \{1/6, 0.5, 1, 2, 3.3\}$.
Here, E is the total energy of the system and the spring
potential is $V(\eta) = (\kappa/2)(\eta - \eta_0)^2$. We then compare
these values to the values of the determinant obtained by
inserting the numerically determined fixed points $\tilde{x}_{2_0} =$
$(\tilde{z}_0 \ \tilde{\dot{z}}_0)^{\top}$ into (17). The determinants obtained in those
two different ways are plotted in Fig. 2a and agree to
a high precision $(|\det(D\mathscr{R}(\bar{x})) - \det(Dr_2^1(\tilde{x}_{2_0}))| < 10^{-7})$.
Barring an improbable numerical cancellation between
stance and flight phase dynamics, this also demonstrates
that the SLIP's stance phase is volume preserving.[4] In
Figs. 2b-d iterations of the return map in $(\tilde{z}_0 \ \tilde{\dot{z}}_0)$-space
are shown for $k \in \{1/6, 1, 3.3\}$ and initial conditions off
the fixed point. The eigenvalues are complex conjugate
pairs in all three cases. For $k = 1/6$ the trajectory spirals
towards the fixed point, as expected from a stable fixed
point (Fig. 2c), for $k = 1$ the trajectory is a deformed circle
around the fixed point,[5] indicating neutral stability (Fig.
2d), and for $k = 3.3$ the trajectory spirals away from the
fixed point, indicating instability (Fig. 2b).

B. Lateral leg-spring model

The lateral leg-spring (LLS) was introduced in [6]. We
focus here on the three-degree-of-freedom version with
pairs of 'virtual' elastic legs in intermittent contact with
the ground. A full stride consists of two stance phases: a
phase where the first elastic leg pivots around a "foothold"
on one side of the rigid body, followed by a phase where
the second elastic leg pivots around a "foothold" on the
opposite side. See [6] for details. The equations of motion
of both stance phases can be cast into the form (6) (with
$g = 0$). Hence the stance phases from leg touchdown
to liftoff are volume preserving. They are related by a
transition mapping $\mathscr{T}_1^2$ which maps the state at liftoff
of the 1. leg to the state at touchdown of the 2. leg,
and an analogous map $\mathscr{T}_2^1$. Thus the return map reads
$\mathscr{R} = \mathscr{T}_2^1 \circ r_2^1 \circ \mathscr{T}_1^2 \circ r_1^2$.

The dynamics of the LLS model can be described by
four state variables $(v, \delta, \theta, \omega)$, where v is the center of
mass speed, δ is the angle between the body axis and the
mass center velocity vector, θ is the angle between the
body axis and an inertial frame and $\omega = \dot{\theta}$. In [6] these
four variables are augmented by two fixed parameters: β,

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[4]This is not true for approximations to the stance phase dynamics
which violate the time reversing symmetry $\mathscr{S}_1$.

[5]Numerical evidence shows that closed curves persist in any neighbor-
hood around the fixed point. Since the determinant away from the fixed
point is not 1, the standard KAM theorem for 2-dimensional maps (see
e.g. [18]) is not applicable here. However, the reverse time symmetry
of the leg recirculation strategy in this example can be shown to entail
the existence of Kolmogorov tori around the fixed point. This is a
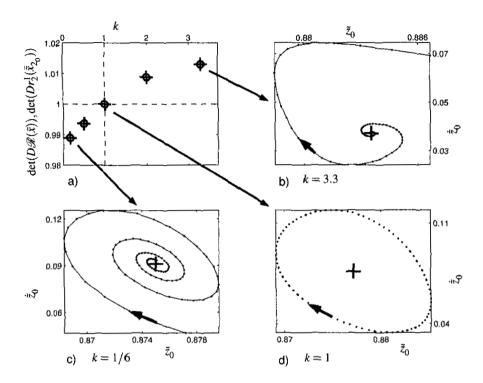consequence of a result on reversible systems [19, Theorem 2.9], for
more details see [10].

the leg touchdown angle with respect to the body axis, and
l_0, the relaxed leg length; and from the values of these six
quantities at liftoff one can find the initial data for the next
stance phase.

In these variables, the transition mapping $\mathscr{T}_1^2$ (omitting
l_0) reads

$$\mathscr{T}_1^2\left((v_n^{LO} \ \delta_n^{LO} \ \theta_n^{LO} \ \omega_n^{LO} \ \beta_n^{LO})\right) = (v_n^{LO} \ \delta_n^{LO} \ \theta_n^{LO} \ \omega_n^{LO} \ \beta)$$

where n stands for the nth stance phase, LO for liftoff, and
β is held constant for all stance phases. If, as implicitly
assumed for the SLIP treated above, β (and/or l_0) are re-
garded as state variables rather than parameters, then since
they are 'reset' to fixed values at touchdown, independent
of their values at liftoff, the transition mapping $\mathscr{T}_1^2$ has
rank four, volume is not preserved, and no deductions
can be made regarding the reduced four-dimensional map.
Restoring a nontrivial dynamical role to the variable β, for
example, via a leg swing feedback strategy similar to (15),
could lead to a non-degenerate mapping.

IV. Conclusions

In this paper we used the example of the SLIP loco-
motion model to show how factored analysis of the return
map may be a useful new tool in the stability analysis
of hybrid Lagrangian systems. Specifically, we obtained a
necessary condition for the asymptotic stability of SLIP
in the presence of a leg recirculation strategy relevant to
the operation of the robot RHex [1]. This condition is
formulated in Sec. III.A.3 for a particular family of leg
recirculation strategies as an exact algebraic expression
despite the non-integrability of the SLIP system. Hence leg
recirculation strategies that violate the above condition can
be discarded without recourse to cumbersome numerical
simulations. Application of this formalism to the robot
RHex requires a more elaborate parametrization of leg
recirculation schemes modeled after RHex's open loop
controller.

This analysis can provide for the first time a partial ex-
planation for the surprising self-stable behavior observed
empirically in RHex. It also paves the way for a more
principled investigation of detailed, biologically motivated
leg placement strategies in the LLS model [6] which
captures many aspects of cockroach locomotion [20].

V. Acknowledgements

VI. Appendix: Invariance of the threshold equation for SLIP without pitching

In this appendix we show that the invariance of the
threshold equation (11) under $S_2 \circ r_2^1 : \tilde{x}_{2_0} \mapsto S_2 \circ f_2^{irD}(\tilde{x}_{2_0})$

Fig. 2. a) Comparison of the numerically computed determinant $\det(D\mathscr{R}(\bar{x}))$ (+) of the return map monodromy matrix to to the determinant $\det(Dr_2^1(\bar{\bar{x}}_{2_0}))$ (o) obtained by using the numerically determined fixed points in (17). b)-d) Trajectories around a fixed point. Because of slow convergence, only every 9th iteration in plot b) and every 5th iteration in plot c) is shown.

with $\bar{x}_{2_0} = (\bar{z}_0 \ \dot{\bar{z}}_0)^\top$ corresponds to $|\det(Dr_2^1)| = 1$ for the leg recirculation family (15). First we observe that

$$f_2^{\bar{t}}(\bar{x}_{2_0}) = \left(\bar{z}_0 + \dot{\bar{z}}_0\bar{t} - \frac{\bar{t}^2}{2} \quad \dot{\bar{z}}_0 - \bar{t}\right)^\top$$

$$\mathscr{S}_2 \circ f_2^{\bar{t}}(\bar{x}_{2_0}) = \left(\bar{z}_0 + \dot{\bar{z}}_0\bar{t} - \frac{\bar{t}^2}{2} \quad -(\dot{\bar{z}}_0 - \bar{t})\right)^\top$$

$$f_2^{\bar{t}} \circ \mathscr{S}_2 \circ f_2^{\bar{t}}(\bar{x}_{2_0}) = \left(\bar{z}_0 \quad -\dot{\bar{z}}_0\right)^\top$$

The threshold function $h_2^1(f_2^{\bar{t}}(\bar{x}_{2_0}), \bar{x}_{2_0}, \bar{t})$ becomes

$$h_2^1(f_2^{\bar{t}}(\bar{x}_{2_0}), \bar{x}_{2_0}, \bar{t}) = \bar{z}(\bar{t}) - \cos(\phi(\bar{z}_0, \dot{\bar{z}}_0, \bar{t})) \quad (18)$$

with $\phi(\bar{z}_0, \dot{\bar{z}}_0, \bar{t}) = k \arccos(\bar{z}_0) + \alpha(\bar{t} - l\dot{\bar{z}}_0)$.

Then

$$h_2^1(f_2^{\bar{t}_{TD}} \circ \mathscr{S}_2 \circ f_2^{\bar{t}_{TD}}(\bar{x}_{2_0}), \mathscr{S}_2 \circ f_2^{\bar{t}_{TD}}(\bar{x}_{2_0}), \bar{t}_{TD}) =$$

$$\bar{z}_0 - \cos\left(k \arccos(\bar{z}_0 + \dot{\bar{z}}_0\bar{t}_{TD} - \frac{\bar{t}_{TD}^2}{2})\right.$$
$$\left. + \alpha\left(\bar{t}_{TD} + l(\dot{\bar{z}}_0 - \bar{t}_{TD})\right)\right) = 0$$

For a solution of this equation with the leg recirculating only once during flight $\phi(\bar{z}_0, \dot{\bar{z}}_0, \bar{t}_{TD}) \in (\frac{3}{2}\pi, 2\pi)$. This must

be taken into account when inverting the cosine:

$$\arccos(\bar{z}_0) = -\left(k \arccos(\bar{z}_0 + \dot{\bar{z}}_0\bar{t}_{TD} - \frac{\bar{t}_{TD}^2}{2})\right.$$
$$\left. + \alpha\left(\bar{t}_{TD} + l(\dot{\bar{z}}_0 - \bar{t}_{TD})\right)\right) + 2\pi$$

$$\Leftrightarrow \quad \cos\left(k \arccos(\bar{z}(\bar{t}_{TD}))\right) - \cos\left(\arccos(\bar{z}_0)\right.$$
$$\left. + \alpha\left(\bar{t}_{TD} + l(\dot{\bar{z}}_0 - \bar{t}_{TD})\right)\right) = 0$$

$$\overset{k=1,l=0}{\Leftrightarrow} \quad \bar{z}(\bar{t}_{TD}) - \cos\left(\arccos(\bar{z}_0) + \alpha(\bar{t}_{TD})\right) = 0$$

For $k = 1$ and $l = 0$ this does reduce to the original threshold function (18) and we conclude that $|\det(Dr_2^1)| = 1$, as was explicitly derived in (17). For other values of k and l this does not in general reduce to (18), although we have not ruled out that for specific values of k and l and a specific form of α the original threshold function (18) is recovered.

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43

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