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Paulo Tabuada
University of Notre Dame

George J. Pappas
University of Pennsylvania, pappasg@seas.upenn.edu

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Finite Bisimulations of Controllable Linear Systems[1]

Paulo Tabuada
Dept. of Electrical Engineering
University of Notre Dame
Notre Dame, IN 46556
e-mail: ptabuada@nd.edu

George J. Pappas
Dept. of Electrical and Systems Engineering
University of Pennsylvania
Philadelphia, PA 19104
e-mail: pappasg@seas.upenn.edu

Abstract

Finite abstractions of infinite state models have been critical in enabling and applying formal and algorithmic verification methods to continuous and hybrid systems. This has triggered the study and characterization of classes of continuous dynamics which can be abstracted by finite transition systems. In this paper, we focus on synthesis rather than analysis. In this spirit, we show that given any discrete-time, linear *control* system satisfying a generic controllability property, and any finite set of observations restricted to the boolean algebra of *Brunovsky* sets, a finite bisimulation always exists and can be effectively computed.

1 Introduction

Algorithmic approaches to formally verifying continuous and hybrid systems have critically relied on extracting finite-state abstractions while preserving properties expressible in suitable temporal logics. This has been achieved by constructing *finite bisimulations*, which are finite partitions of the infinite state space that preserve a finite set of observations as well as reachability properties. It is well known that bisimilar systems have equivalent properties expressible in various temporal logics. If a hybrid system is bisimilar to a finite state system, then verification of the hybrid system can be equivalently performed on the purely discrete system.

In the hybrid systems community, this approach has originated with the seminal work on timed automata [1], that was subsequent extended to rectangular hybrid automata [9]. Linear differential equations with special eigenstructure for which finite abstractions exist were introduced in [10] by combining tools from geometric model theory and linear systems theory. Nonlinear dynamics were considered in [4] where integrals of motion were used to defined bisimulations. In the linear case, these integrals can be obtained by exploiting the Brunovsky normal form of controllable systems for constant inputs. Although finite bisimulations can be obtained, it is not clear how to patch such bisimulations for different values of the control inputs. The survey paper [2] describes the boundary between hybrid systems with decidable and undecidable model checking problems. Controller synthesis techniques for hybrid systems include, among many others, supervisory control based on approximate finite abstractions [7], logic based synthesis [12], invariants for the continuous dynamics [13], convexity properties of affine systems [8], and mixed integer linear programming [3]. Closer to our approach is the work described in [16] where sufficient conditions are provided to finitely compute controlled invariant sets. Although we essentially use the same controllability conditions, this paper focuses on finite bisimulations rather than controlled invariant sets.

In this paper, we take a novel yet algorithmic approach to hybrid system *synthesis* rather than hybrid system *analysis*. The focus on synthesis differentiates our approach from all previously mentioned verification approaches. Furthermore, we focus on the essence of computability which is the continuous dynamics of hybrid systems. Since we take a synthesis approach, we assume that we are given *continuous control systems* rather than *continuous dynamical systems*. In particular, in this paper, we identify critical properties of discrete time control systems ensuring the existence of finite bisimilar quotients : system controllability, and compatibility between finite observations and the controlled dynamics. We discuss in detail discrete-time, linear control systems and the boolean algebra of Brunovsky sets, showing that this class satisfies the required assumptions. In addition, we also show that finite bisimilar quotients of systems in this class not only exist, but they are also effectively computable.

The contributions presented in this paper extend our previous work described in [15] where we have only considered language equivalent finite abstractions. We thus strengthen language equivalence to bisimulation while enlarging the class of observations for which finite bisimulations can be constructed. In the dual pa-

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per [14], we show how finite controllers for the finite, bisimilar systems can be refined to controllers for the original model. Therefore, this paper in conjunction with [14] and known temporal logic synthesis procedures results in the algorithmic synthesis of controllers for discrete-time, controllable linear systems with respect to temporal logic specifications.

2 Transition systems and boolean algebras

2.1 Transition systems

Definition 2.1 *A transition system with observations is a tuple $T = (Q, \longrightarrow, O, H)$, where:*

- *Q is a (possibly infinite) set of states,*
- *$\longrightarrow \subseteq Q \times Q$ is a transition relation,*
- *O is a (possibly infinite) set of observations,*
- *$H : Q \to O$ is a map assigning to each $q \in Q$ an observation $H(q) \in O$.*

We say that T is finite when Q, O are finite, and infinite otherwise. We will usually denote by $q \longrightarrow q'$ a pair (q, q') belonging to $\longrightarrow$. As we will only consider transition systems with observations, we shall refer to them simply as transition systems. Given a state $q \in Q$, we denote by $\mathrm{Pre}(q)$ the set of states in Q that can reach q in one step, that is:

$$\mathrm{Pre}(q) = \{q' \in Q \ : \ q' \longrightarrow q\}$$

We extend Pre to sets $Q' \subseteq Q$ in the usual way:

$$\mathrm{Pre}(Q') = \bigcup_{q' \in Q'} \mathrm{Pre}(q')$$

Finally, we recursively define $\mathrm{Pre}^i(Q')$ by:

$$\mathrm{Pre}^1(Q') = \mathrm{Pre}(Q') \qquad \mathrm{Pre}^i(Q') = \mathrm{Pre}(\mathrm{Pre}^{i-1}(Q'))$$

The main objective of this paper is to partition the states of a transition system while preserving various system properties. We start by defining partitions.

Definition 2.2 *A collection of sets $\Pi = \cup_{i \in I}\{S_i\}$ is called a partition of S if $\cup_{i \in I} S_i = S$ and $S_i \cap S_j = \varnothing$ for $i \neq j$. A partition is called finite if I is finite. Given two partitions Π_1 and Π_2 of S, Π_2 is a refinement of partition Π_1 if for every $S_2 \in \Pi_2$ there exists a $S_1 \in \Pi_1$ such that $S_2 \subseteq S_1$.*

Bisimulations are partitions which preserve both observations and transitions.

Definition 2.3 (Bisimulation) *Let $T = (Q, \longrightarrow, O, H)$ be a transition system and $\Pi = \cup_{i \in I}\{S_i\}$ a partition of Q. Partition Π is called a bisimulation when $q_1, q_2 \in S_i$ implies:*

- *$H(q_1) = H(q_2)$,*
- *if $q_1 \longrightarrow q_1'$ there exists a q_2' such that $q_2 \longrightarrow q_2'$ and $q_1', q_2' \in S_{i'}$ for some $i' \in I$.*

Bisimulations are also equivalence relations[1] on Q, and they induce a well defined transition system on the quotient space Q_Π. For a given $q \in Q$, we denote by $S(q)$ the equivalence class $S \in \Pi$ containing q, and denote by Q_Π the set Π. The quotient transition system induced by Π, $T_\Pi = (Q_\Pi, \longrightarrow_\Pi, O, H_\Pi)$, is then defined by:

- $Q_\Pi = \Pi$,
- $\longrightarrow_\Pi \subseteq Q_\Pi \times Q_\Pi$, defined by $S_i \longrightarrow_\Pi S_{i'}$ iff there exist $q \in S_i$ and $q' \in S_{i'}$ such that $q \longrightarrow q'$,
- $H_\Pi : Q/\Pi \to O$ defined by $H_\Pi(S(q)) = H(q)$.

We note that H_Π is well defined since Π respects observations, that is, $S(q) = S(q') \Rightarrow H(q) = H(q')$. As the states of T_Π are given by $Q_\Pi = \Pi$, a finite quotient is obtained when the partition Π is also finite. When Π is a bisimulation, it is straightforward that T is bisimilar [11] to T_Π by the relation $R \subseteq Q \times Q_\Pi$ defined by $(q, S) \in R$ iff $q \in S$. Bisimulations are important as they preserve properties expressible in several temporal logics such as LTL, CTL, CTL* or μ-calculus [5].

2.2 Boolean algebras and stable partitions

We now obtain some conditions at the level of boolean algebras that imply the existence of finite bisimulations. The conditions obtained in this section will be more natural and directly applicable to the main goals of this paper.

Definition 2.4 *A Boolean algebra of subsets of a set S, denoted by $\mathcal{B}(S)$, is a nonempty collection of subsets of S that is closed under union and complementation, that is $S_1 \cup S_2 \in \mathcal{B}(S)$ and $\overline{S_1} = S \backslash S_1 \in \mathcal{B}(S)$ for every $S_1, S_2 \in \mathcal{B}(S)$.*

Note that the above definition implies that $\varnothing, S \in \mathcal{B}(S)$ and also that $A \cap B \in \mathcal{B}(S)$. Boolean algebra endomorphisms are defined as follows:

Definition 2.5 *A map $F : \mathcal{B}(S) \to \mathcal{B}(S)$ is called a Boolean algebra endomorphism if $F(A \cup B) = F(A) \cup F(B)$ and $F(\overline{A}) = \overline{F(A)}$ for every $A, B \in \mathcal{B}(S)$.*

[1] Each set S_i is regarded as an equivalence class.

A Boolean algebra endomorphism is called eventually idempotent if there exists a $k \in \mathbb{N}$ such that $F^{k+1} = F^k$, where F^k denotes k-th iterate of F.

Finally, we define stable partitions under a Boolean algebra endomorphism.

Definition 2.6 *Let $F : \mathcal{B}(S) \to \mathcal{B}(S)$ be a Boolean algebra endomorphism and Π a partition of S. Partition Π is called stable under F if for any $S_i \in \Pi$, $F(S_i) = \cup_{j \in J} S_j$.*

Using the above definition, we can define a bisimulation as a stable (under Pre) partition of Q that, in addition, preserves observations. The following theorem gives us conditions for the existence of finite, stable partitions.

Theorem 2.7 *Let $F : \mathcal{B}(S) \to \mathcal{B}(S)$ be a Boolean algebra endomorphism. If F is eventually idempotent and $\Pi \subseteq \mathcal{B}(\mathbb{R}^n)$ is a finite partition of S, then a finite and stable (under F) refinement of Π exists.*

Proof: Assume that $|\Pi| = p$, let $\alpha : \{0,1\} \times \mathcal{B}(S) \to \mathcal{B}(S)$ be the function defined by $\alpha(0, S_i) = \overline{S_i}$ and $\alpha(1, S_i) = S_i$, and let V be the set $V = \{0,1\}^{p \times k}$. Each element $v \in V$ is a $p \times k$ matrix of zeros and ones. The element (a, b) of such matrix is denoted by v_{ab}. We now consider the refinement of Π defined by slicing each set $S_i \in \Pi$ as $S_i = \cup_{v \in V} S_i^v$, where S_i^v is defined by:

$$S_i^v = S_i \cap \bigcap_{a=1}^{p} \bigcap_{b=1}^{k} \alpha(v_{ab}, F^b(S_a))$$

Intuitively, the sets S_i^v represent the subsets of S_i defined by the points that can reach S_a in b steps when $v_{ab} = 1$ and cannot reach S_a in b steps when $v_{ab} = 0$. We now show that $\Pi^V = \bigcup_{v \in V} \bigcup_{i \in \{1,2,\ldots,p\}} S_i^v$ is stable under F. Consider any $S_i^v, S_{i'}^{v'} \in \Pi^V$ and assume that $S_{i'}^{v'} \cap F(S_i^v) \neq \varnothing$. We have that $F(S_i^v)$ is given by:

$$\begin{aligned}
F(S_i^v) &= F\Big(S_i \cap \bigcap_{a=1}^{p} \bigcap_{b=1}^{k} \alpha(v_{ab}, F^b(S_a))\Big) \\
&= F(S_i) \cap \bigcap_{a=1}^{p} \bigcap_{b=1}^{k} F\big(\alpha(v_{ab}, F^b(S_a))\big) \\
&= F(S_i) \cap \bigcap_{a=1}^{p} \bigcap_{b=1}^{k} \alpha(v_{ab}, F^{b+1}(S_a)) \\
&= F(S_i) \cap \bigcap_{a=1}^{p} \bigcap_{b=2}^{k+1} \alpha(v_{ab-1}, F^b(S_a)) \\
&= F(S_i) \cap \bigcap_{a=1}^{p} \bigcap_{b=2}^{k} \alpha(v_{ab-1}, F^b(S_a)) \\
&= \alpha(1, F(S_i)) \cap \bigcap_{a=1}^{p} \bigcap_{b=2}^{k} \alpha(v_{ab-1}, F^b(S_a)) \quad (2.1)
\end{aligned}$$

where we have used the fact that F is a Boolean algebra endomorphism and the fact that F is eventually idempotent. Equation (2.1) shows that:

$$v'_{ab} = v_{ab-1} \text{ for } 2 \leq b \leq k \quad (2.2)$$
$$v'_{i1} = 1 \quad (2.3)$$

Consider now any point $x \in S_{i'}^{v'}$. Such point satisfies the reachability properties expressed by (2.2) and (2.3) and consequently $x \in F(S_i^v)$. This shows that $S_{i'}^{v'} \cap F(S_i^v) \neq \varnothing \Rightarrow S_{i'}^{v'} \subseteq F(S_i^v)$. Let now Z be the subset of $I \times V$ defined by $(i', v') \in Z$ iff $S_{i'}^{v'} \cap F(S_i^v) \neq \varnothing$. It then follows that:

$$F(S_i^v) = \bigcup_{(i',v') \in Z} S_{i'}^{v'}$$

To conclude the proof we only need to show that Π^V is finite, however this follows immediately from the fact that V is finite. ■

Theorem 2.7 will be our main tool in order to show that finite bisimulations exist for linear control systems and the Boolean algebra of Brunovsky sets.

3 Finite bisimulations of controllable systems

In this section we show that finite bisimulations of discrete time linear systems exist provided that a controllability assumption is satisfied, and that we work with a carefully chosen Boolean algebra of sets.

3.1 Controllable linear systems

A discrete time linear system:

$$\Sigma : \quad x(t+1) = Ax(t) + Bu(t)$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $A \in \mathbb{Q}^{n \times n}$ and $B \in \mathbb{Q}^{n \times m}$ defines an infinite transition system:

$$T_\Sigma = (\mathbb{R}^n, \longrightarrow_\Sigma, O_\Sigma, H_\Sigma)$$

where $\longrightarrow_\Sigma \subseteq \mathbb{R}^n \times \mathbb{R}^n$ is defined by $x \longrightarrow_\Sigma x'$ iff there exists a $u \in \mathbb{R}^m$ such that $x' = Ax + Bu$. To complete the definition of T_Σ we must also provide a *finite* observation set O_Σ and the observation map H_Σ. Since we are interested in finite observations we naturally obtain the partition associated with observational equivalence.

Definition 3.1 *Let $T_\Sigma = (\mathbb{R}^n, \longrightarrow_\Sigma, O_\Sigma, H_\Sigma)$ be a transition system associated with a discrete time control system Σ. The partition of $\mathbb{R}^n$ defined by:*

$$\bigcup_{o \in O_f} \{H_\Sigma^{-1}(o)\}$$

where H_Σ^{-1} denotes the set-valued inverse of H_Σ, that is, $H_\Sigma^{-1}(o) = \{x \in \mathbb{R}^n \ : \ H_\Sigma(x) = o\}$, is called the observation partition.

The nature of the observation map H_Σ will be carefully chosen in the next section depending on the structure of the control system. For now we will assume that the observation partition is contained in some Boolean algebra of sets $\mathcal{B}(\mathbb{R}^n)$. Controllability of the control system will be crucial in obtaining a finite, stable refinement of the observation partition.

Proposition 3.2 *Let Σ be a discrete time linear control system of dimension n and T_Σ its associated transition system. If Σ is controllable, then for any $x \in \mathbb{R}^n$:*

$$\mathrm{Pre}^n(x) = \mathbb{R}^n$$

In particular $\mathrm{Pre}^{n+1}(x) = \mathrm{Pre}^n(x)$, hence Pre is eventually idempotent.

The previous result immediately suggest that one should regard the Pre operator as our Boolean algebra endomorphism.

3.2 Brunovsky sets

We must now provide a Boolean algebra of sets for which the Pre operator will be a Boolean algebra endomorphism. When considering linear control systems it is natural to consider the Boolean algebra of semi-linear sets (boolean combinations of affine (in)equalities), since this class of sets is closed under the Pre operator. However, semi-linear sets do not satisfy, in general, $\mathrm{Pre}(\overline{S_1}) = \overline{\mathrm{Pre}(S_1)}$, hence the Pre operator is not an endomorphism for the Boolean algebra of semi-linear sets. This motivates the study of a subclass of semi-linear sets which we now introduce, called Brunovsky sets, which satisfy this property. To introduce Brunovsky sets, we start by reviewing the Brunovsky normal form for controllable linear systems.

Definition 3.3 (Brunovsky normal form) *Consider a linear control system of dimension n with m inputs defined by the pair of matrices (A, B) and let $k = (k_1, k_2, \ldots, k_r)$ be a sequence of integers satisfying:*

$$k_1 \geq k_2 \geq \ldots \geq k_r \ \wedge \ k_1 + k_2 + \ldots + k_r = n \tag{3.1}$$

We say that the pair (A, B) is in Brunovsky normal form if matrices A and B are of the following form:

$$A = \begin{bmatrix} A_{k_1} & 0 & \ldots & 0 \\ 0 & A_{k_2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & A_{k_r} \end{bmatrix}$$

$$B = \begin{bmatrix} b_{k_1} & 0 & \ldots & 0 & 0 & \ldots & 0 \\ 0 & b_{k_2} & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & b_{k_r} & 0 & \ldots & 0 \end{bmatrix}$$

where matrix A is partitioned in r^2 blocks while matrix B is partitioned in mr blocks. Each block A_{k_i} and b_{k_i} are of the form:

$$A_{k_i} = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix} \qquad b_{k_i} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{3.2}$$

Any controllable linear system can be effectively transformed to Brunovsky normal form by feedback and a change of state and input coordinates as asserted in the next result.

Theorem 3.4 ([6]) *For every controllable linear system, there exists a unique sequence of integers $k = (k_1, k_2, \ldots, k_r)$ satisfying (3.1), a linear transformation $G \in \mathbb{Q}^{m \times n}$ and invertible linear transformations $F \in \mathbb{Q}^{n \times n}$ and $H \in \mathbb{Q}^{m \times m}$ such that the pair $(A', B') = (F(A - BH^{-1}G)F^{-1}, FBH^{-1})$ is in Brunovsky normal form.*

Note that the transformed system $\Sigma' = (A', B')$ is related to system $\Sigma = (A, B)$ by an invertible state/input transformation $U : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^m$ with rational entries:

$$\begin{bmatrix} x' \\ u' \end{bmatrix} = U \begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} F & \mathbf{0}_{n \times m} \\ G & H \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \tag{3.3}$$

where $\mathbf{0}_{n \times m}$ is the $n \times m$ matrix of zeros. For systems already in Brunovsky form, we consider the Boolean algebra of sets that is generated by (and are hence finite unions and complements of) the following elementary sets:

$$\{y \in \mathbb{R}^n \mid \theta_1(y) \sim_1 0 \wedge \ldots \wedge \theta_a(y) \sim_a 0\} \tag{3.4}$$

where $\sim_i \in \{=, >\}$, θ_i are affine functions of the form $\pm y_j + c_j$ with $c_j \in \mathbb{Q}$, $j \in \{1, 2, \ldots, n\}$ and y_j denotes the jth component of vector y. Such sets satisfy the following properties:

Proposition 3.5 *Let Σ be a discrete time controllable linear system in Brunovsky normal form. Then, for any elementary set E of the form (3.4),* Pre(E) *is given by:*

$$\{y \in \mathbb{R}^n \mid \widehat{\theta}_1(y) \sim_1 0 \wedge \ldots \wedge \widehat{\theta}_c(y) \sim_c 0\} \qquad (3.5)$$

with $\widehat{\theta}(y) = \pm y_{i+1} + c_i$ for some $\theta(y) = \pm y_i + c_i$ defining E and $i \notin \{k_1, k_1 + k_2, \ldots, k_1 + k_2 + \ldots + k_r\}$. Hence, Pre *transforms elementary sets into elementary sets.*

Proof: From the Brunovsky normal form of Σ we see that there is a transition from y to y' iff $y_{i+1} = y'_i$ for $i \notin \{k_1, k_1 + k_2, \ldots, k_1 + k_2 + \ldots + k_r\}$. Consequently, y' satisfies $\pm y'_i + c \sim 0$, iff y satisfies $\pm y_{i+1} + c \sim 0$ which leads to (3.5). ∎

Proposition 3.6 *Let Σ be a discrete time controllable linear system in Brunovsky normal form and $\mathcal{B}(\mathbb{R}^n)$ the Boolean algebra generated by elementary sets of the form (3.4). Then the following equality holds:*

$$\mathrm{Pre}(\overline{S}) = \overline{\mathrm{Pre}(S)} \qquad \forall S \in \mathcal{B}(\mathbb{R}^n) \qquad (3.6)$$

Proof: We shall make use of the well known facts that elements of a boolean algebra can be written as unions of elementary sets and that $\mathrm{Pre}(S_1 \cup S_2) = \mathrm{Pre}(S_1) \cup \mathrm{Pre}(S_2)$ for any sets S_1 and S_2 in order to prove equality (3.6). We shall only consider the case where $S = E_i \cup E_j$ with $E_i = \{y \in \mathbb{R}^n \mid y_i + c = 0\}$ and $E_j = \{y \in \mathbb{R}^n \mid y_j + d > 0\}$ since the general case follows the same arguments.

The complement of E_i can be written as the union of the following elementary sets:

$$E_i^1 = \{y \in \mathbb{R}^n | y_i + c > 0\} \qquad E_i^2 = \{y \in \mathbb{R}^n | -y_i - c > 0\}$$

while the complement of E_j is given by the union of:

$$E_j^1 = \{y \in \mathbb{R}^n | y_j + d = 0\} \qquad E_j^2 = \{y \in \mathbb{R}^n | -y_j - d > 0\}$$

It then follows that:

$$\begin{aligned}
\mathrm{Pre}(\overline{S}) &= \mathrm{Pre}(\overline{E_i \cup E_j}) \\
&= \mathrm{Pre}\big((E_i^1 \cup E_i^2) \cap (E_j^1 \cup E_j^2)\big) \\
&= \big(\mathrm{Pre}(E_i^1) \cup \mathrm{Pre}(E_i^2)\big) \cap \big(\mathrm{Pre}(E_j^1) \cup \mathrm{Pre}(E_j^2)\big) \\
&= (E_{i+1}^1 \cup E_{i+1}^2) \cap (E_{j+1}^1 \cup E_{j+1}^2) \qquad (3.7)
\end{aligned}$$

Consider now $\overline{\mathrm{Pre}(S)}$:

$$\begin{aligned}
\overline{\mathrm{Pre}(S)} &= \overline{\mathrm{Pre}(E_i \cup E_j)} \\
&= \overline{\mathrm{Pre}(E_i) \cup \mathrm{Pre}(E_j)} \\
&= \overline{\mathrm{Pre}(E_i)} \cap \overline{\mathrm{Pre}(E_j)} \\
&= \overline{E_{i+1}} \cap \overline{E_{j+1}} \\
&= (E_{i+1}^1 \cup E_{i+1}^2) \cap (E_{j+1}^1 \cup E_{j+1}^2) \qquad (3.8)
\end{aligned}$$

Since (3.7) equals (3.8), the desired identity is proved. ∎

Propositions 3.5 and 3.6 are now used to show that Pre is a Boolean algebra endomorphism:

Proposition 3.7 *Let Σ be a discrete time controllable linear system in Brunovsky normal form. Then,* Pre *is a Boolean algebra endomorphism for $\mathcal{B}(\mathbb{R}^n)$, the Boolean algebra generated by elementary sets of the form (3.4).*

Proof: Let $S \in \mathcal{B}(\mathbb{R}^n)$, then $\mathrm{Pre}(S) = \mathrm{Pre}(\cup_{i \in I} E_i)$ since any $S \in \mathcal{B}(\mathbb{R}^n)$ can be written as the union of elementary sets E_i of the form (3.4). As the equality $\mathrm{Pre}(S_1 \cup S_2) = \mathrm{Pre}(S_1) \cup \mathrm{Pre}(S_2)$ is satisfied for any sets S_1 and S_2, it follows that $\mathrm{Pre}(S) = \cup_{i \in I} \mathrm{Pre}(E_i)$ which by Proposition 3.5 is a union of elementary sets and therefore an element of $\mathcal{B}(\mathbb{R}^n)$. This shows that Pre transforms elements in the Boolean algebra into elements of the Boolean algebra.

It remains to show that Pre respects complement since $\mathrm{Pre}(S_1 \cup S_2) = \mathrm{Pre}(S_1) \cup \mathrm{Pre}(S_2)$ is satisfied for any sets S_1 and S_2. However, this is ensured by Proposition 3.6. ∎

Proposition 3.7 shows that for the Boolean algebra $\mathcal{B}(\mathbb{R}^n)$, the Pre operator is a Boolean algebra endomorphism. Even though the previous result holds for systems already in Brunovsky normal form, we can transfer the previous results to all controllable linear systems not necessarily in Brunovsky form. This can be easily achieved using the isomorphism (3.3) which relates the original form and the Brunovsky normal form of any given controllable linear system.

In Brunovsky coordinates, a constraint of the form $\pm y_i + c_i \sim_i 0$, $\sim_i \in \{=, >\}$ can also be represented as:

$$wy + c \sim 0, \quad w \in \left\{ \begin{bmatrix} \pm 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}^T, \begin{bmatrix} 0 \\ \pm 1 \\ \vdots \\ 0 \end{bmatrix}^T, \ldots, \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \pm 1 \end{bmatrix}^T \right\} \qquad (3.9)$$

where b^T denotes the transpose of vector b. Since the Brunovsky coordinates are related to the original coordinates by an invertible linear transformation $Fx = y$, we can express (3.9) in the original coordinates, using the equality $wy = wFF^{-1}y = wFx$, as:

$$fx + c \sim 0, \quad f \in \pm\{f_1, \ldots, f_n\}, \quad \sim \in \{=, >\} \quad (3.10)$$

where f_i are the rows of matrix F. This motivates the following definition.

Definition 3.8 *Let $\Sigma = (A, B)$ be a discrete-time controllable linear system. We define the Boolean algebra of Brunovsky sets as the Boolean algebra generated by elementary sets of the form:*

$$\{x \in \mathbb{R}^n \mid \theta_1(x) \sim_1 0 \wedge \ldots \wedge \theta_a(x) \sim_a 0\}, \quad \sim_i \in \{=, >\}$$

where the functions θ_i are of the form $\theta_i(x) = fx + c$ with $f \in \pm\{f_1, \ldots, f_n\}$ and $c \in \mathbb{Q}$.

Note that for systems already in Brunovsky normal form, $F = I_n$. Furthermore, the Boolean algebra of Brunovsky sets is system dependent, as the elementary sets critically depend of the matrix F, which in turn depends on system matrices A and B. The presented properties of linear control systems and Brunovsky sets allow us to directly apply Theorem 2.7.

Theorem 3.9 *Let $T_\Sigma = (\mathbb{R}^n, \longrightarrow_\Sigma, O_\Sigma, H_\Sigma)$ be a transition system associated with a discrete-time linear control system Σ. If Σ is controllable and the observation partition is finite and contained in the Boolean algebra of Brunovsky sets, then a finite bisimulation of T_Σ exists and is effectively computable.*

Theorem 3.9 places no conditions on the eigenstructure of the matrix A, which was necessary in the case of finite bisimulations of linear dynamical systems with finite (but semi-algebraic) sets [10]. Theorem 3.9 also motivates the use of observation maps of the form:

$$p(x) = \begin{cases} 1 & \text{if } \theta(x) \sim 0 \\ 0 & \text{otherwise} \end{cases} \qquad H_\Sigma(x) = \begin{bmatrix} p_1(x) \\ p_2(x) \\ \vdots \\ p_l(x) \end{bmatrix}$$

and finite observation sets $O = \{0, 1\}^l$.

4 Discussion

In this paper, we have identified important properties leading to the existence of finite bisimilar quotients for discrete-time control systems. Finite bisimulations can be computed for the class of linear control systems and Brunovsky sets. More general classes of sets and systems satisfying the important properties are the subject of current research. Another direction for research focuses on efficiently approximating more general, semi-linear sets by Brunovsky sets.

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