2013

Understanding the TIPS Beta

Matthew Cocci

University of Pennsylvania

Follow this and additional works at: http://repository.upenn.edu/wharton_research_scholars

Part of the Finance and Financial Management Commons

Cocci, Matthew, "Understanding the TIPS Beta" (2013). Wharton Research Scholars. 101.
http://repository.upenn.edu/wharton_research_scholars/101

This paper is posted at ScholarlyCommons. http://repository.upenn.edu/wharton_research_scholars/101
For more information, please contact libraryrepository@pobox.upenn.edu.
Understanding the TIPS Beta

Abstract
This paper examines the TIPS Beta, a simple regression-based measure of the link between yields on Nominal U.S. Treasury bonds and Treasury Inflation Protected Securities (TIPS) using data beginning in January 2003. In the time period until late-2008 and early-2009, the Beta measure exhibited minor variations about a relatively stable level before subsequently displaying increased variability about a new, lower level. The change marks a deteriorating or less-persistent link between yields in the two markets, which this paper examines in more detail. In particular, a basic model of yields and interest rates movements is given, which is then used to simulate interest rates. By adding uncorrelated, idiosyncratic errors to the two markets, increased uncertainty in inflation expectations, and a shock to the TIPS liquidity premium, the behavior of the Beta statistic over the observed time period can be replicated.

Keywords
TIPS, US Treasury, Beta measure

Disciplines
Business | Finance and Financial Management

This other is available at ScholarlyCommons: http://repository.upenn.edu/wharton_research_scholars/101
Understanding the TIPS Beta
Measuring the Link Between
Nominal and Inflation-Linked Bond Markets

Matthew Cocci
Advisor: Professor Gregory Nini

Abstract This paper examines, the TIPS Beta, a simple regression-based measure of the link between yields on Nominal U.S. Treasury bonds and Treasury Inflation Protected Securities (TIPS) using data beginning in January 2003. In the time period until late-2008 and early-2009, the Beta measure exhibited minor variations about a relatively stable level before subsequently displaying increased variability about a new, lower level. The change marks a deteriorating or less-persistent link between yields in the two markets, which this paper examines in more detail. In particular, a basic model of yields and interest rates movements is given, which is then used to simulate interest rates. By adding uncorrelated, idiosyncratic errors to the two markets, increased uncertainty in inflation expectations, and a shock to the TIPS liquidity premium, the behavior of the Beta statistic over the observed time period can be replicated.
# Contents

1. Background 3

2. The TIPS Beta 5
   2.1. Definition .......................................... 5
   2.2. Applications and Relevance .......................... 9

3. Data 10
   3.1. Yield Data ........................................... 11
   3.2. Inflation Expectations ............................... 13
   3.3. Evolution of Beta Over Time ........................ 15

4. Basic Yield Model 16

5. Simulating TIPS and Nominal Yields 19
   5.1. Specifying a Process for the Nominal Yield .......... 19
   5.2. Scenario 1: Idiosyncratic Market Factors .............. 20
   5.3. Scenario 2: Adding Inflation Expectations .......... 24
   5.4. Scenario 3: Liquidity Premium ....................... 26

6. Conclusion 28

A. Calibrating the CIR Model to Nominal Yields 33
1. Background

First introduced in sufficient quantities by the United Kingdom, inflation-linked bonds have now grown into an important asset class, despite only recently becoming a regular feature in the centuries-old government bond market. For the United States, the inflation-linked bond market began in 1997 with the Treasury Inflation Protected Securities (TIPS) program. It has grown into a fairly substantial market in its own right, at over $700 billion in November of 2011 (Fleming and Krishnan, 2012).

The continued existence of TIPS serves two main purposes, while assisting both investors and governments with financial planning and public policy goals. In particular, TIPS offer investors a simple way to guarantee real, inflation-adjusted returns—provided the bonds are held to maturity and “real” returns are measured relative to the conventional inflation measure. For governments and their central banks, inflation-linked bonds provide a market-based measure of inflation expectations. With these two groups in mind, this paper will examine the link between the TIPS and nominal bond markets, studying one metric, the TIPS Beta, in greater detail.

Before proceeding directly to the Beta metric, a short summary of the TIPS market will provide the necessary background for further discussion. To begin, TIPS differ from traditional nominal bonds by compensating bondholders for inflation that occurs prior to maturity. Although the specified real coupon rate remains fixed over the life of an inflation-linked bond, the principal will periodically adjust upwards (downwards) following increases (decreases) in a conventional price level index. In the U.S., that index is the non-seasonally adjusted Consumer Price Index (CPI). When this fixed coupon rate is applied to the variable principal, such adjustments produce nominal coupon payments that rise and fall with the price level, though remaining fixed in real terms. In this way, TIPS represent an exact mirror image to traditional nominal bonds, which pay a constant nominal coupon (applied to a fixed principal), while the real value of any given coupon
payment is left free to vary inversely with the price level. Clearly, the presence or absence of inflation adjustment will have direct implications for the determination of prices and yields in the two markets.¹

Just as in the nominal bond case, coupon-paying TIPS are easily priced by building from simpler zero-coupon bonds. To begin, consider an inflation-protected zero-coupon bond that pays $1 real dollar at maturity, \( T \), though the nominal repayment value will depend upon the level of some specified price index. This zero-coupon bond has a price today \((t)\) of \( P_{t,T}^{\text{tips}} \), and this price implies a specific TIPS yield just as a nominal zero-coupon bond price implies a unique nominal yield, also included for comparison:

\[
\begin{align*}
y_{t,T}^{\text{tips}} & = -\frac{\ln P_{t,T}^{\text{tips}}}{T - t} \\
y_{t,T}^{\text{nom}} & = -\frac{\ln P_{t,T}^{\text{nom}}}{T - t}
\end{align*}
\]

where the yields are continuously compounded. As with nominal bonds, coupon-paying TIPS can be priced as a portfolio of zero-coupon TIPS, while the yield-to-maturity is a weighted average of zero-coupon TIPS yields. For a more detailed treatment of how yields and various related rates are computed in the two markets, see Gurkaynak et al. (2007) and Gurkaynak et al. (2010).

In the usual environment of positive inflation, the difference in these two yields offers an approximate measure of inflation expectations, known as the “breakeven inflation (BEI) rate,” defined

\[
\text{(BEI)}_{t,T} = y_{t,T}^{\text{nom}} - y_{t,T}^{\text{tips}}
\]

The breakeven inflation rate approximates the average rate of inflation that, if realized, would make an investor indifferent between holding nominal bonds or TIPS. If average realized inflation exceeds the BEI rate, then TIPS will outperform nominal bonds (and vice versa, should realized average inflation fall short of the BEI rate). Of course,

¹Good discussions of how the TIPS market functions and how TIPS are priced can be found in Fleming and Krishnan (2012), Sack and Elsasser (2002), and Roll (2004).
everything mentioned still applies if expected inflation and the BEI rate are negative, suggesting expected deflation. In this way, the yields on TIPS and nominal bonds give policy makers a market-based measure of inflation expectations, although there are important qualifications that will be discussed in later sections.

The exposition and basic definitions in Equations 1 and 2 above provide enough background to define the TIPS Beta metric and examine the data that will factor into its computation.

2. The TIPS Beta

Now that yields have been defined in the two markets, investors and policy makers might naturally ask about the relationship between them. This section will define the Beta metric to handle that task. Later sections will then examine the instability of this measure, particularly following the financial crisis, before developing simulation methods to better model its behavior.

2.1. Definition

The most obvious method to link yields in the two markets involves regression: predicting yields in one market from yields in the other. Figure 1, which plots nominal zero-coupon yields against TIPS zero-coupon yields (more on the data below), suggests a connection between the two markets. The high correlation, although likely to be biased upwards, still indicates that regression might give a suitable method for bridging the gap between markets.

A first attempt might use the level of the \( n \)-year nominal rate to predict the \( n \)-year TIPS rate:

\[
y_{t,t+n}^{\text{tips}} = \alpha + \beta y_{t,t+n}^{\text{nom}} + \varepsilon_t
\]
However, this approach presents a problem because it strongly violates simple regression assumptions, detailed further below. A much better approach uses the differenced series:

$$\Delta y_{t, t+n}^{\text{tips}} = \alpha + \beta \left( \Delta y_{t, t+n}^{\text{nom}} \right) + \varepsilon_t$$ (3)

Where

$$\Delta y_{t, t+n} = y_{t, t+n} - y_{t-1, t+n-1}$$

To highlight the improvement from the specification of Equation 3, consider Figure 2, which plots the residuals for differenced and undifferenced example regressions. Both use the same 10-year zero-coupon yield series beginning in January 2003, although the basic message remains the same for other maturities. As demonstrated, the Equation 3 specification, which uses differenced yields, leads to roughly normally distributed errors that do not exhibit serial correlation, unlike the undifferenced regression whose residuals are in the bottom row.²

²Of course, this differencing approach does not completely solve the problem. In fact, given the
As a result, the $\beta$ regression coefficient in Equation 3 will then represent the TIPS Beta discussed throughout the remainder of the paper. Figure 3 gives one example of what a beta regression according to Equation 3 might look like.

Finally, before proceeding further, one modeling choice in Equation 3 deserves mention. Namely, some might object that Equation 3 has the regression backwards: nominal yields belong on the left hand side as the dependent variable, since TIPS approximate the real interest rate—a factor that influences nominal yield. However, several considerations suggest Equation 3 instead. First, real rates of returns have historically been an ex-post measure, only available after inflation and nominal rates of return have been realized. A priori, investors might not have well-defined expectations or suitable indi-

large number of observations that fail to display a strictly normal distribution, common tests of normality such as the Jarque-Bera test will strongly reject the normality assumption. Instead, some inspection will show that the daily changes more closely resemble $t$-distributions with under 10 degrees of freedom for the different rates. However, since the Beta metric typically uses simple regression, this paper will follow the same simplifying convention.
cators for the real interest rate. Instead, they may consider nominal rates as dictated by some Taylor Rule, then subtract off expected inflation to recover a reasonable TIPS yield. Using the specification in Equation 3, investors could easily model TIPS yields as functions of more familiar macroeconomic variables like unemployment, the output gap, producer prices, and so forth. This approach seems more appropriate given a central bank that targets nominal interest rates roughly in line with some Taylor Rule and that sets an explicit inflation target, as in many modern economies.

Lastly, the TIPS market, while large in nominal terms, still represents a relatively small fraction of total U.S. Treasury debt outstanding—about 10% in 2008 (Campbell et al., 2009). In addition, dealers and banks do not typically quote rates in terms of real yields or expected real yields. For better or worse, nominal yields—such as LIBOR, the 10 year US Treasury rate, average 30-year mortgage rates—represent the standard units of account in fixed income. Considering these issues, nominal interest rates appear as the more relevant benchmark in financial markets, influencing other variables and suggesting the Equation 3 specification.
2.2. Applications and Relevance

With the definition in hand, focus can now turn to the uses of the metric and the value in properly understanding its shortcomings.

Immediately, the TIPS Beta suggests a practical, heuristic application. In particular, it tells investors and policy makers that, given a 1% change in nominal yields, TIPS yields can be expected to change roughly $\beta\%$. Therefore, if a central bank uses inflation-protected yields as a proxy for real interest rates, it gains a rough sense of how real interest rates will move if it changes nominal yields to hit certain target levels.

Of course, such direct cause-and-effect reasoning presumes that $\beta$ is stable, or at least confined within certain bounds. If the metric proves unstable, then changes to the nominal yield might not necessarily move real yields as much as usual, or even in the correct direction. For central banks trying to pull down the yield curve in the face of recession, this suggests that achieving lower real yields might require more effort, more aggressive reduction in nominal yields, or different policy altogether. In fact, the data described below will demonstrate that beta indeed proved unstable following the financial crisis, while the yield model and simulations in later sections offer one way to understand what can drive this instability.

Second, Beta decomposes changes in nominal yields in a potentially useful way. Suppose nominal yields increase purely because inflation expectations increase. This change should have zero effect on TIPS. Therefore, if shifts in nominal interest rates were always the result of shifts in inflation expectations, a $\beta$ of zero would necessarily follow. At the other extreme, imagine changes in the nominal interest rate always reflected changes in the real interest rate. Then this should affect both TIPS and nominal yields equally, resulting in a $\beta$ of 1. Therefore, a $\beta$ less than unity might convey how often, on average, a change in nominal yields corresponds to a change in real yields. But of course, changes in inflation expectations and real yields rarely occur in such an exclusive, binary manner,
as the two components often exhibit correlation. In this case, $\beta$ roughly signifies the fraction of the change in nominal yield that owes itself to a change in real yields.

Finally, like nominal bonds, TIPS have a well-defined duration; however, that duration represents the sensitivity to changes in TIPS yields. For an investor holding a portfolio of both nominal and inflation-linked bonds, Beta helps arbitrate between the two asset classes. Since there does not exist a closed form expression for the nominal duration of TIPS, the sensitivity to changes in nominal yields must be estimated empirically as in the Beta regression (Roll, 2004). In this case, the effective nominal duration equals the product of Beta and the duration with respect to the real interest rate. So with a reliable estimate in hand, an investor can approximately delta-hedge a portfolio of both TIPS and nominal bonds. Although imperfect relative to a pure delta-hedging strategy transacting only in nominal bonds, such a mixed-asset strategy could be attractive if, say, an inflation-linked bond appears undervalued relative to nominal bonds. For longer-term trading strategies that require holding assets and rebalancing, the basic concepts and intuition behind Beta can be extended to suit the purpose.\(^3\)

Given the above uses, a proper understanding of Beta could prove useful to both market participants and policymakers. Therefore, the remainder of this paper will examine historical data before modeling the driving forces behind the metric. By specifying how those forces might behave, useful information and intuition can be garnered for the purposes listed above.

3. Data

This section describes the data used to gain some intuition and calibrate models for later simulations. The final subsection calculates and shows rolling estimates of Beta, noting

\(^3\)For example, Roll (2004) discusses the concept of effective nominal duration fairly extensively. In his article, he also estimates the sensitivity of real yields to changes in the shape of the nominal yield curve, not just changes in yields.
how it has changed over time (particularly following the financial crisis). In fact, this instability served as a primary motivation for exploring the metric in more detail. It will drive efforts to find out how and why Beta behaves as it does and whether efforts to predict the level of Beta would prove useful.

3.1. Yield Data

As mentioned above, TIPS and US nominal treasury bonds typically pay coupons, but by viewing these securities as a portfolio of zero-coupon bonds, estimating the zero-coupon yield curve reduces to an optimization problem. In particular, a functional form is specified for the zero-coupon yield curve, then parameters are estimated so that the prices of coupon bonds implied by the curve most closely match market prices. Repeating this process daily will produce zero-coupon yields—along with forward rates and par yields—across a range of maturities. In fact, this approach has the attractive feature of estimating a continuous set of rates given only a finite number of bonds. In this way, there can exist a five-year nominal yield for each trading day even if the US government does not issue a five-year zero coupon bond every day (or at all, for that matter).

Two papers by economists at the Federal Reserve, Gurkaynak et al. (2007) and Gurkaynak et al. (2010), implemented this approach for both nominal bonds and TIPS, generating the data for this paper. In particular, they chose the Nelson-Siegel-Svensson curve as their representation of the zero-coupon yield curve. They then found the parameters that best induce this curve to match off-the-run issues of coupon bonds in the two markets.\textsuperscript{4} Not only do their data reach back to the very first issuance of TIPS (and

\textsuperscript{4}“Off-the-run” refers to those bonds which are not members of the most recent issuance of a particular maturity, while “on-the-run” bonds are members. For example, if the US government issues a new round of 5-year nominal bonds today, those securities are “on-the-run” until the government issues new 5-year bonds perhaps a month, several months, or years later. On-the-run securities typically trade at a premium relative to off-the-run assets owing to the greater liquidity of on-the-run securities. Since this features implies that they will typically not representative, off-the-run
much longer for nominal bonds), but their estimates of zero-coupon yields have been updated each trading day since publication, for a wide range of maturities. Because of the consistency in the fitting method and the availability of the data, this paper will use the daily estimates of Gurkaynak et al. (2007) and Gurkaynak et al. (2010) throughout. Figures 4 and 5 plot this data, beginning in 2003 and continuing until late 2012.

Figure 4: Nominal ZCB Yields: 2003 - End Date

![Time Series of Nominal Interest Rates, Various Maturities](image)

Before moving on, some choices regarding the data deserve further attention. First, the data for all but the 2-year TIPS yield begin in January 2003, although TIPS have been issued since 1997. As mentioned by Sack and Elsasser (2002) and Campbell et al. (2009), the TIPS market initially showed poor liquidity until the Treasury resolved doubts later on about their commitment to the program. Around 2003, this issue became less important than at the time of first issuance in 1997; therefore, this paper will focus on yields after 2003 to avoid picking up factors irrelevant to the discussion and unlikely to appear in the future.

Second, subsequent analysis will focus on the four maturities depicted in Figures 4 and 5. The 2- and 20-year rates represent the shortest and longest yield curve estimates issues are used instead.
available in both markets, while the 5- and 10-year rates fill out the medium-term. Yields with maturities shorter than two years would exhibit substantially variability due to carry considerations, particularly in the TIPS market; therefore, they are excluded (Gurkaynak et al., 2010). Finally, Figure 5 shows that the 2-year rate begins later than the other series. Since the earliest issues of TIPS had the shortest maturity at five years, no bonds with remaining maturity of two years were available until about 2004. To avoid extrapolating the yield curve beyond observed maturities, the 2-year series begins later (Gurkaynak et al., 2010).

### 3.2. Inflation Expectations

Prior to Breakeven Inflation Rates, surveys of economists and market participants typically represented the only way to collect a consensus view on inflation. The most well-known, the Survey of Professional Forecasters, has been conducted since 1968 until the present, with quarterly updates now released by the Philadelphia Federal Reserve. The 5- and 10-year inflation forecasts offer the most information since introduction of
TIPS, citing the mean, median, and dispersion of forecasts (which other maturities often lack). The dispersion equals the difference between the 75th and 25th percentiles and, therefore, acts as measure of uncertainty about expected inflation.

Now since breakeven inflation rates roughly estimate the market’s view of average inflation, BEI rates should not wildly diverge from forecasts. However, a glance at Figure 6, which plots BEI rates versus median forecasts, tells a different story. Although the survey only offers quarterly forecasts (so that a connected line through survey dates may overstate the stability of those forecasts), a sharp divergence nonetheless existed around the time of the financial crisis in late 2008, early 2009. In fact, a few survey dates occurred during the collapse in BEI rates. Yet oddly, even accounting for the minor increase in forecast dispersion and uncertainty, BEI rates fall well outside middle 50% range of forecasts.

This observation implies either that professional forecasters have very different views than the majority of TIPS investors or that some other factors influence BEI rates beyond merely inflation expectations. This latter influence will be explored more fully and explicitly below when constructing a model of yields in the two markets.
3.3. Evolution of Beta Over Time

A common approach to estimating Beta relies on periodically conducting the regression specified in Equation 3 over a window of the past few weeks or months.

As an example, take Figure 7, which plots the Beta coefficient resulting from rolling regressions over the past 120 days at each data point. Immediately obvious, the coefficient estimate varies, and in some cases, it does so considerably. The variability in the Beta estimate would feature even more prominently if the number of days used in each regression were decreased to, say, 30 or 60 days.

In addition, the estimates and the correlation between yields in the two markets (displayed in the bottom panel) noticeably drop following the financial crisis, and this dip has important consequences. First, the lower correlation implies a weaker link between yields in the two markets: changes in the nominal yield offer substantially less information about changes in TIPS yields. For investors who need to calculate the effective nominal duration of inflation-protected bonds, this might require a change of approach. For policymakers using TIPS yields as an approximation of real yields, the behavior of Beta (post-crisis) implies that lower nominal yield might not necessarily lower real yields as much as usual.
Finally, note that the observed break in the beta coefficient and the corresponding drop in correlation persists across different regression windows and maturities, although only one maturity is shown above. Therefore, in order to understand the driving forces behind this general result, the next section will model some fundamental factors at work in the two markets.

4. Basic Yield Model

The discussion above has made reference to several concepts including expected inflation, the real interest rate, and liquidity differentials, all of which influence TIPS and nominal yields in very specific ways. In order to clarify and lay the groundwork for simulation, this section will make those concepts explicit.

Take the following simple specification of yields in the two markets, which captures the primary determinants of yield:

\begin{align}
    y_{t,T}^{\text{tips}} &= r_{t,T} + \ell_{t,T} + \varepsilon_{t}^{\text{tips}} \\
    y_{t,T}^{\text{nom}} &= r_{t,T} + E_{t}[\pi_{t,T}] + p_{t,T} + \varepsilon_{t}^{\text{nom}}
\end{align}

TIPS zero-coupon yields represent the sum of some real interest interest rate \((r_{t,T})\), a liquidity premium \((\ell_{t,T})\) owing to the relative illiquidity of TIPS versus nominal bonds, as well as an error term \(\varepsilon_{t}^{\text{tips}}\) accounting for various idiosyncratic factors that might influence the market.\(^5\) Note that the real interest rate and the liquidity premium can vary as a function of the bond’s maturity.

Next, nominal yields include that same real interest rate \((r_{t,T})\), average expected inflation over the life of the bond given current information \(E_{t}[\pi_{t,T}]\), an inflation risk premium \((p_{t,T})\), and an error term \(\varepsilon_{t}^{\text{nom}}\) accounting for various idiosyncratic factors

\(^5\)See Campbell et al. (2009) on the existence of a TIPS risk or liquidity premium.
that might influence the market. The inflation risk premium compensates for the risk that realized inflation might differ from expected inflation. Going forward, combining the two yields will now offer some more insight into the discussions above.

In particular, Equations 4 and 5 hint at how the breakeven inflation rate approximates expected inflation. Using the formula in Equation 2, the BEI rate becomes

\[
(\text{BEI})_{t,T} = y_{t,T}^{\text{nom}} - y_{t,T}^{\text{tips}} = (r_{t,T} + E_t [\pi_{t,T}] + p_{t,T} + \varepsilon_{t}^{\text{nom}}) - (r_{t,T} + \ell_{t,T} + \varepsilon_{t}^{\text{tips}})
\]

\[
= E_t [\pi_{t,T}] + (p_{t,T} - \ell_{t,T}) + \left(\varepsilon_{t}^{\text{nom}} - \varepsilon_{t}^{\text{tips}}\right) \quad (6)
\]

Clearly, expected inflation plays a major role in determining the level of the breakeven rate, but the other factors might disallow such a direct interpretation, as noted by Gurkaynak et al. (2010). In particular, different idiosyncratic shocks, such as large trades, a security trading “special,” or atypical activity in a particular maturity, might influence the BEI rate on any given day, as denoted by the error terms. However, those effects should average out over time so that the errors matter much less than the net premium \((p_{t,T} - \ell_{t,T})\) in expectation (although they will certainly increase the variance of the BEI rate). This net premium factor includes two terms that will likely (and justifiably) differ from zero, biasing the observed BEI rate if they fail to cancel. This feature seems to influence the BEI rates versus professional forecasts for 2 year rates, plotted in Figure 8. Over most of the data, forecasts lie above BEI rates.

Perhaps more instructive than the pure BEI rate, the change in the BEI rate might offer a cleaner interpretation:

\[
\Delta(\text{BEI})_{t,T} = \Delta E_t [\pi_{t,T}] + \Delta (p_{t,T} - \ell_{t,T}) + \Delta \left(\varepsilon_{t}^{\text{nom}} - \varepsilon_{t}^{\text{tips}}\right) \quad (7)
\]

Again, the error terms should balance out in expectation. In addition, a stable inflation risk premium coupled with an unchanging liquidity differential between the two markets
implies a change in the net premium, \( \Delta (p_{t,T} - \ell_{t,T}) \), equal to zero. In this way, changes in the BEI rate might more closely approximate changes in inflation expectations even if the level of the BEI rate is a biased estimator of the expected inflation.

Lastly, Equations 4 and 5 suggest how to simulate the yields in the two markets. Specifically, the simulations below will take the nominal yield as the benchmark, for reasons argued above in Section 2.1. Then, given a simulated time series for the \( n \) year rate, \( y_{t,T+n}^{nom} \), Equations 4 and 5 directly imply a way to recover the TIPS yield:

\[
y_{t,T+n}^{tips} = y_{t,T+n}^{nom} - E_t [\pi_{t,T+n}] + (\ell_{t,T+n} - p_{t,T+n}) + (\varepsilon_{t,T+n}^{tips} - \varepsilon_{t,T+n}^{nom})
\]

The coming sections will simplify this equation and modify the simulations somewhat; however, this provides a good starting point for describing the general method.
5. Simulating TIPS and Nominal Yields

As mentioned, TIPS yields will follow from a nominal yield, less a net premium and expected inflation. The following subsections will build up simulations of increasing complexity, incorporating the different factors from the yield model above, each one at a time. Every additional step will build upon the previous, providing some extra descriptive power and intuition on the way towards a full replication of the observed behavior of the Beta metric.

Throughout, the simulations will focus on 10-year yields, although observations about the behavior of 10-year yields also apply to yields of different maturities. Only the scale differs, while the general patterns remain the same. In addition, each subsection will typically present a single realization of the specified processes; however, the effects and conclusions hold for repeated simulations of those conducted in each subsection.

5.1. Specifying a Process for the Nominal Yield

Since nominal interest rates will vary randomly over time, a stochastic differential equation will best model nominal yield movements, but with the slight modification of a small additional error term after moving to discrete time (justification below). In particular, a separate CIR model can be calibrated for each $n$-year yield using historical data. It will have the form

$$
y_{t, t+n}^{\text{nom}} = x_n(t) + \epsilon_{t}^{\text{nom}}$

$$
dx_n(t) = \kappa \left[ \theta - x_n(t) \right] dt + \sigma \sqrt{x_n(t)} dW(t) \tag{9}$$

where $\epsilon_t^{\text{nom}}$ represents some normally distributed random error term with small variance.

For now, discussion of the $\epsilon_t$ term will be postponed to the next subsection in order to focus more fully on the stochastic component, $x_n(t)$. The Appendix describes how to
calibrate the parameters $\kappa$, $\theta$, and $\sigma$ in Equation 9 using discrete-time data. Then, once calibrated, this particular interest rate model will exhibit several desirable properties—properties that recommend this interest rate model over many competing alternatives:

- **Mean Reversion**: The stochastic factor, $x_n(t)$, reverts to $\theta$ at rate $\kappa$, a reasonable property for interest rates.

- **Volatility Scaling**: As yields decline, the volatility declines as well.

- **Closed Form Solution**: Given a starting point, future values of $x_n(t)$ can be drawn from a Non-Central Chi-Square distribution without requiring discretization. This avoids discretization error and potentially negative rates.

- **Non-Negativity**: Since nominal yields remain positive, the interest rate process should respect that property. The Equation 9 specification accomplishes that, unlike simpler models. In fact, this feature served as the primary property recommending this model over other simpler models, such as the Vasicek model.

Figure 9 displays one realization of the process for $x_n(t)$ (calibrated to 10 year rates). To compare this to nominal yields, the actual path of 10-year TIPS yields have been included in the Figure as well.

### 5.2. Scenario 1: Idiosyncratic Market Factors

This subsection will demonstrate that the addition of any idiosyncratic market factors will lead to a variable $\beta$. This property, while easy to demonstrate and grasp, provides important intuition. Namely, if independent stochastic shocks influence yields in the two markets substantially, Beta estimates using the recent past—say, the previous 90 days—might function no better than a longer run estimate. More fundamental macroeconomic factors might not drive the variability in Beta estimates at all. Rather, short term
historical estimates might be largely influenced random, market-specific factors that have no implication for future estimates of Beta.

Proceeding to the implementation, we assume that nominal and TIPS yields equal

\[ y_{t+n}^{\text{nom}} = x_n(t) + \varepsilon_{t+n}^{\text{nom}} \]
\[ y_{t+n}^{\text{tips}} = x_n(t) - E[\pi] - \varepsilon_{t+n}^{\text{tips}} \]

where \( E[\pi] \) remains constant and where the error terms, \( \varepsilon_{t+n}^{\text{nom}} \) and \( \varepsilon_{t+n}^{\text{tips}} \), are normally and independently distributed with identical variance. To begin, the variance will be set so that \( \sigma = 0.005\% \), in which case roughly 95% of the idiosyncratic errors fall within plus or minus one basis point.

Figure 10 depicts a simulation of 10-year nominal and TIPS yields. Once the yields have been simulated, Figure 11 plots the Beta and correlation estimates over time. This Figure shows that the addition of independent stochastic factors can sufficiently generate the inherent variability displayed in the Beta estimates.

Finally, just a remark about the simulation of nominal yields. The addition of an
Figure 10: Simulated TIPS and Nominal 10-year Yields

Figure 11: Resulting Beta and Correlation Estimates with Idiosyncratic Errors
extra stochastic factor ($\varepsilon_t^{\text{nom}}$) might seem redundant since the primary component of the nominal yield, $x_n(t)$, is already stochastic. However, $\varepsilon_t^{\text{nom}}$ serves a practical purpose in the simulations: to mimic the higher variability that nominal yields display empirically. Because nominal yields respond to inflation unlike TIPS, daily changes in the former will understandably exhibit greater variability. To see this, Figure 12 shows the histograms for the change in yields at a maturity of 10 years, with notably more dispersion in the nominal distribution above (Roll, 2004).

Now consider how the TIPS yield will be simulated as in Equation 8: it will equal the nominal yield less expected inflation, the net premium, and an error term—all of which can vary stochastically. As a result, even if these other stochastic factors have zero correlation with the nominal yield, the variance of TIPS yields will still disagree with empirical evidence, exhibiting greater variance than nominal yields:

$$\text{Var}(y^\text{tips}_{t, t+n}) = \text{Var}(y^\text{nom}_{t, t+n} - E_t[\pi_{t, t+n}] + (\ell_{t, t+n} - p_{t, t+n}) + [\varepsilon^\text{tips}_t - \varepsilon^\text{nom}_t])$$

If Uncorrelated: $= \text{Var}(y^\text{nom}_{t, t+n}) + \text{Var}(E_t[\pi_{t, t+n}]) + \text{Var}(\ell_{t, t+n} - p_{t, t+n}) + \text{Var}(\varepsilon^\text{tips}_t - \varepsilon^\text{nom}_t)$

Figure 12: Histogram of Daily Changes in Yields, Both Markets

<table>
<thead>
<tr>
<th>Days to Day Change, Nominal Yields, 10 Year Maturity</th>
</tr>
</thead>
</table>
| $\mu = -0.0012$  
| $\sigma = 0.065$ |

<table>
<thead>
<tr>
<th>Days to Day Change, TIPS Yields, 10 Year Maturity</th>
</tr>
</thead>
</table>
| $\mu = -0.0014$  
| $\sigma = 0.054$ |
The additional error term, $\varepsilon_{t}^{\text{nom}}$, corrects an empirical inconsistency that would result from the method of simulation.

5.3. Scenario 2: Adding Inflation Expectations

Next, inflation expectations that vary stochastically will add a more realistic element to the simulations. In particular, this subsection will show that changing inflation expectations drive Beta estimates below one, closer to the values typically seen in historical estimates. Additionally, an increase in the variance of the process for inflation expectations—consistent with the greater dispersion observed in professional forecasts—can account for the drop in Beta following the financial crisis.

To begin, expected inflation will follow the process

\[
E_t[\pi] = 2.5 + \eta_t
\]

\[
\eta_{t+1} = \phi \eta_t + \varepsilon_t \quad \eta < 1
\]

\[
\varepsilon_t \sim N(0, \sigma_t)
\]

This process specifies that expected inflation equals the sum of some long-run level and a stationary process. As shown above in Figure 6, the median professional forecasts for average 10-year inflation remained fairly consistent at roughly 2.5%, suggesting the above long-run level. Next, the process, $\eta_t$, reflects the uncertainty regarding expected inflation, demonstrated by the dispersion of professional forecasts and new information reaching markets each day. The parameters $\phi$ and $\sigma_t$ can be specified or estimated from historical breakeven data, while $\sigma_t$ can vary with time to reflect greater uncertainty in inflation expectations.

Figure 13 plots a sample path for average 10-year inflation expectations, with $\phi = 0.99$—roughly the autocorrelation estimate from 2003-2007. The value of $\sigma_t$ remains at
0.02% until halfway through the period (around the beginning of 2008), when it increases to 0.04%. These values roughly approximate the variance of historical breakeven rates, which were used to obtain a sense of the magnitude. Of course, breakeven rates vary not only when inflation expectations change, but this serves as a reasonable starting point.

With that, $n$-year nominal and TIPS yields will be modeled as

$$y_{t,t+n}^{\text{nom}} = x_n(t) + \varepsilon_t^{\text{nom}} + \eta_t$$
$$y_{t,t+n}^{\text{tips}} = x_n(t) - 2.5 - \varepsilon_t^{\text{tips}}$$

Again, the stochastic component of inflation expectations, $\eta_t$, is added to nominal yields rather than subtracted so that TIPS yields do not exhibit greater variance—a concern addressed in the previous subsection. Moreover, nominal yields should absorb the variability due to volatile inflation expectations, as TIPS do not respond to this factor.

Figure 14, which plots the 10-year yields, shows that adding to the simulation stochastic inflation expectations (on top of idiosyncratic market differences) does not have a very noticeable effect upon yields, although it does have a substantial effect upon Beta.
Estimates fall from around $\beta = 1$ in the previous section, and the greater inflation uncertainty in the second half pushes Beta and correlation even lower on average, while increasing overall variability the estimates. Comparing this result to the historical behavior of Beta (Figure 7), allowing for stochastic inflation expectations clearly provides a more realistic simulation.

5.4. Scenario 3: Liquidity Premium

To frame the final addition, note that nothing in the simulations so far can successfully account for the sharp drop in BEI rates first depicted in Figure 6. Recall that the sharp downturn did not correspond to drastic changes in professional forecasts, even accounting for the increased uncertainty in those forecasts. As the only element remaining in the yield model that could explain such a phenomenon, this section considers a sharp increase in the net premium, defined as $(\ell - p)$ above. Such an increase agrees with work done by Campbell et al. (2009), and it could be explained by the technical factors outlined by Hu and Worah (2009) that were associated with Lehman’s bankruptcy.
Therefore, suppose that the net premium takes the form

$$(\ell - p)_t = k + \lambda_t + z_t,$$

$$\lambda_t = ae^{-b(t-c)}$$

$$z_t \sim N(0, \lambda_t \cdot \sigma^2)$$

This specification implies that the net premium has some long-run level $k$, spikes at time $c$ to level $a$ at speed $b$, and has a stochastic component that scales with the level of the spike.

Figure 15 plots one realization of the net premium process. As demonstrated, such characteristics can mimic a sudden liquidity scare, where investors flock to the larger nominal bond market, where they have a greater chance of finding willing counterparties for trades. This follows from the larger size of the nominal bond market and the liquidity problems TIPS had previously experienced, noted in prior sections.
Figure 16: Simulated 10-Year Yields with Net Premium Spike. Note the spike around 2008.

Incorporating this effect into the simulations of 10-year yields, Figure 16 plots a realization of those yields and the resulting Beta and correlation estimates, where

\[ y_{t,t+n}^{\text{nom}} = x_n(t) + \varepsilon_{t}^{\text{nom}} + \eta_t \]
\[ y_{t,t+n}^{\text{tips}} = x_n(t) - 2.5 + (\ell - p)t + \varepsilon_{t}^{\text{tips}} \]

and every other component carries over from previous subsections.

The Figure clearly shows that an increase in the net premium drives down the correlation substantially, consistent with what occurred in the market around the time of the financial crisis. Moreover, this also generates the activity observed in BEI rates, the sharp downturn in particular (Figure 17).

6. Conclusion

This paper began by introducing the inflation-protected bond market—specifically focusing on bonds issued by the U.S. Treasury. After discussing the principal differences between nominal bonds and TIPS regarding inflation compensation, the introduction
then discussed how yields followed from pricing in the two markets.

A suitable definition of yields then led to a definition of the TIPS Beta, a regression based measure linking yields in the two markets. This definition required a consideration of how best to specify the regression, with changes in yields as opposed to levels as the variables of interest. The discussion also made the case for choosing the nominal yield—the product of a larger, more liquid, and more familiar market—as the benchmark and, therefore, explanatory variable.

This definition then gave way to a discussion of the uses of the metric. The first use included heuristic value, allowing investors and policy makers to understand how changes in the nominal yield might translate into changes in TIPS yields and the broader concept of the “real” interest rate. Second, Beta provides a decomposition of nominal yield changes into real yield and inflation components. In this way, Beta can provide a measure of the uncertainty of inflation expectations. Finally, Beta enables the calculation of effective nominal duration. This quantity allows TIPS investors to gauge the sensitivity of their portfolios to nominal interest rate movements, facilitating sensitivity analysis.
and delta hedging.

With this basic framework in place, focus then turned to historical data ranging from 2003 until late 2012. This section noted the substantial drop in breakeven inflation rates—so substantial that BEI rates were an extreme deviation from professional forecasts, suggesting liquidity issues or other factors may have been at work in the TIPS market. Also noted, the Beta metric—calculated each trading day over rolling windows of constant length—displays a notable and sustained decrease in its average level along with an increase in variability.

In order to interpret these observations with greater precision, this paper then turned to a basic model of yields in the two markets which explicitly defined the factors influencing yields and, therefore, Beta. By specifying these factors, stochastic simulations of each could then illustrate how the factors affected estimates of Beta, successfully replicating the observed characteristics of the metric over time.

The simulations also provided some insight for those using the measure for the purposes suggested above. In particular, idiosyncratic market factors and the stochastic nature of the determinants of yield could generate variability in Beta that has no macroeconomic content. This suggests that longer-term averages or an entire simulated distribution of beta estimates might prove more useful than historical estimates using the most recent past (as in a regression over the past 90 days). This would avoid picking up simple noise that has no fundamental information content. Of course, if large changes occur (such as liquidity concerns or greater uncertainty about inflation) on the scale of late 2008 and early 2009, historical estimates might over- or understate the level of Beta, recommending the use of simulation. Second, increasing uncertainty regarding inflation expectations will tend to drive the estimate down, weakening the link between nominal and TIPS yields. Finally, a spike in the net premium best explains the sharp downturn in BEI rates within the simple yield model underlying the simulations.
Together, these issues highlight some of the shortcomings of using the Beta measure. However, by specifying a framework for those factors affecting yields, the analysis above offers one way to simulate and, therefore, put a distribution on the metric. This process allows for a better understanding of the link between the two markets, and it suggests a way to derive intuition from the Beta measure while understanding the caveats that should inform intuition.
References


Hu, G. and Worah, M. (2009). Why tips real yields moved significantly higher after the lehman bankruptcy. PIMCO, Newport Beach, California.


A. Calibrating the CIR Model to Nominal Yields

This appendix derives those estimates detailed in Backus et al. (1998) using much of the same notation. In addition to the CIR model, the paper also discusses several other popular interest rate models, along with their discrete-time counterparts.

So first, recall the specification of the governing process for the $n$-year nominal yield:

$$y_{t,n}^{nom} = x_n(t) + \varepsilon_t^{nom}$$

$$dx_n(t) = \kappa [\theta - x_n(t)] \, dt + \sigma \sqrt{x_n(t)} \, dW(t)$$  \hspace{1cm} (12)

Often, interest rate models involve pricing under a risk-neutral or martingale measure, particularly when pricing interest rate derivatives. However, the analysis above requires calibration under the historical measure, using time series data. To do so, Equation 12 must be discretized, taking $\Delta t = 1$ for one trading day.

$$x_n(t+1) - x_n(t) = \kappa [\theta - x_n(t)] + \sigma \sqrt{x_n(t)} \varepsilon_t$$  \hspace{1cm} (13)

$$\varepsilon_t \sim N(0,1)$$

Calibration will require us to find the mean, variance, and first autocorrelation coefficient of the process, then express $\kappa$, $\theta$, and $\sigma$ as function of those statistics, then finally plug in the values from the data.

First, let’s rewrite Equation 13 more suggestively:

$$x_n(t+1) = x_n(t) + \kappa [\theta - x_n(t)] + \sigma \sqrt{x_n(t)} \varepsilon_t$$

$$= \kappa \theta + x_n(t)[1 - \kappa] + \sigma \sqrt{x_n(t)} \varepsilon_t$$

$$= (1 - \varphi) \theta + \varphi x_n(t) + \sigma \sqrt{x_n(t)} \varepsilon_t$$  \hspace{1cm} (14)

where $\varphi = 1 - \kappa$
We can obtain an estimate for $\varphi$ by regressing $x_n(t+1)$ on $x_n(t)$, an order 1 autoregression.

Second, let’s get the mean of the process, which we will obtain by iterated expectations:

\[
E[x_n(t+1)|x_n(t)] = E \left[ (1 - \varphi)\theta + \varphi x_n(t) + \sigma \sqrt{x_n(t)} \, \varepsilon_t \mid x_n(t) \right]
\]

\[
= (1 - \varphi)\theta + \varphi x_n(t) + \sigma \sqrt{x_n(t)} \, E[\varepsilon_t|x_n(t)]
\]

\[
= (1 - \varphi)\theta + \varphi x_n(t) + \sigma \sqrt{x_n(t)} \cdot 0
\]

\[\Rightarrow \quad E[x_n(t+1)|x_n(t)] = (1 - \varphi)\theta + \varphi x_n(t) \quad (15)\]

\[
E[E[x_n(t+1)|x_n(t)]] = E [(1 - \varphi)\theta + \varphi x_n(t)]
\]

\[
E[x_n(t+1)] = (1 - \varphi)\theta + \varphi E[x_n(t)]
\]

\[\Leftrightarrow \quad E[x_n(t)] = (1 - \varphi)\theta + \varphi E[x_n(t)] \quad (16)\]

Now for the variance, using a familiar decomposition and Equation 15:

\[
\text{Var}(x_n(t+1)) = \text{Var}(E[x_n(t+1) \mid x_n(t)]) + E[\text{Var}(x_n(t+1) \mid x_n(t))]
\]

\[
= \text{Var}((1 - \varphi)\theta + \varphi x_n(t))
\]

\[
+ E\left[ \text{Var} \left( (1 - \varphi)\theta + \varphi x_n(t) + \sigma \sqrt{x_n(t)} \, \varepsilon_t \mid x_n(t) \right) \right]
\]

\[
= \varphi^2 \text{Var}(x_n(t)) + E \left[ \sigma^2 x_n(t) \, \text{Var}(\varepsilon_t \mid x_n(t)) \right]
\]

\[
= \varphi^2 \text{Var}(x_n(t+1)) + E \left[ \sigma^2 x_n(t) \, \text{Var}(\varepsilon_t \mid x_n(t)) \right]
\]

\[
= \varphi^2 \text{Var}(x_n(t+1)) + \sigma^2 E \left[ x_n(t) \cdot 1 \right]
\]

\[
= \varphi^2 \text{Var}(x_n(t+1)) + \sigma^2 \theta
\]

\[\Rightarrow \quad \text{Var}(x_n(t+1)) = \frac{\theta \sigma^2}{1 - \varphi^2} \quad (18)\]
Finally, we can rearrange Equations 16 and 18 to get the parameter estimates:

\[ \kappa = 1 - \varphi, \quad \theta = E[x_n(t)], \quad \sigma^2 = \frac{1 - \varphi^2}{\theta} \text{Var}(x_n(t)) \]  

(19)

The above process using Equations 19 was repeated for each of the four maturities referenced in this paper, yielding the following parameter estimates:

<table>
<thead>
<tr>
<th>Maturity</th>
<th>( \theta )</th>
<th>( \kappa )</th>
<th>( \sigma )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2-year</td>
<td>2.17</td>
<td>0.000936</td>
<td>0.0474</td>
</tr>
<tr>
<td>5-year</td>
<td>2.94</td>
<td>0.002148</td>
<td>0.0469</td>
</tr>
<tr>
<td>10-year</td>
<td>3.94</td>
<td>0.003991</td>
<td>0.0405</td>
</tr>
<tr>
<td>20-year</td>
<td>4.61</td>
<td>0.004919</td>
<td>0.0339</td>
</tr>
</tbody>
</table>