



10-6-2009

Mach's Holographic Principle

Justin Khoury

Perimeter Institute for Theoretical Physics, University of Pennsylvania, jkhoury@sas.upenn.edu

Maulik Parikh

Columbia University, Inter-University Centre for Astronomy and Astrophysics

Suggested Citation:

Khoury, J. and Parikh, M. (2009). "Mach's holographic principle." *Physical Review D*. **80**, 084004.

© 2009 The American Physical Society

<http://dx.doi.org/10.1103/PhysRevD.80.084004>.

This paper is posted at Scholarly Commons. http://repository.upenn.edu/physics_papers/52

For more information, please contact repository@pobox.upenn.edu.

Mach's Holographic Principle

Abstract

Mach's principle is the proposition that inertial frames are determined by matter. We put forth and implement a precise correspondence between matter and geometry that realizes Mach's principle. Einstein's equations are not modified and no selection principle is applied to their solutions; Mach's principle is realized wholly within Einstein's general theory of relativity. The key insight is the observation that, in addition to bulk matter, one can also add boundary matter. Given a space-time, and thus the inertial frames, we can read off both boundary and bulk stress tensors, thereby relating matter and geometry. We consider some global conditions that are necessary for the space-time to be reconstructible, in principle, from bulk and boundary matter. Our framework is similar to that of the black hole membrane paradigm and, in asymptotically anti-de Sitter space-times, is consistent with holographic duality.

Disciplines

Physical Sciences and Mathematics | Physics

Comments

Suggested Citation:

Khoury, J. and Parikh, M. (2009). "Mach's holographic principle." *Physical Review D*. **80**, 084004.

© 2009 The American Physical Society

<http://dx.doi.org/10.1103/PhysRevD.80.084004>.

Mach's holographic principleJustin Khoury^{1,2} and Maulik Parikh^{3,4}¹*Perimeter Institute for Theoretical Physics, 31 Caroline St. N., Waterloo, Ontario, Canada N2L 2Y5*²*Center for Particle Cosmology, University of Pennsylvania, Philadelphia, Pennsylvania 19104, USA*³*Institute for Strings, Cosmology, and Astroparticle Physics, Columbia University, New York, New York 10027, USA*⁴*Inter-University Centre for Astronomy and Astrophysics, Post Bag 4, Pune 411007, India*

(Received 16 February 2007; revised manuscript received 9 March 2009; published 6 October 2009)

Mach's principle is the proposition that inertial frames are determined by matter. We put forth and implement a precise correspondence between matter and geometry that realizes Mach's principle. Einstein's equations are not modified and no selection principle is applied to their solutions; Mach's principle is realized wholly within Einstein's general theory of relativity. The key insight is the observation that, in addition to bulk matter, one can also add boundary matter. Given a space-time, and thus the inertial frames, we can read off both boundary and bulk stress tensors, thereby relating matter and geometry. We consider some global conditions that are necessary for the space-time to be reconstructible, in principle, from bulk and boundary matter. Our framework is similar to that of the black hole membrane paradigm and, in asymptotically anti-de Sitter space-times, is consistent with holographic duality.

DOI: [10.1103/PhysRevD.80.084004](https://doi.org/10.1103/PhysRevD.80.084004)

PACS numbers: 04.20.Cv, 11.25.Tq, 98.80.Jk

I. MACH'S PRINCIPLE

Acceleration appears absolute. A snapshot of a rotating bucket of water reveals, through the gentle curve in the water's surface, that the bucket was rotating. Two rocks tied with a rope and set spinning about an axis perpendicular to the rope are measurably distinct from the same two rocks undergoing linear motion: the rope becomes tense. A passenger in an elevator or a windowless spaceship is aware of starts and stops even though the vehicle is a closed system.

With his principle of equivalence, Einstein recognized that gravity was simply acceleration in disguise. Moreover, Einstein's equations, like Newton's law of gravitation, indicate that matter is the source for gravity. But if acceleration and gravity are linked, and if gravity depends on matter, then can acceleration be attributed to matter?

This imprecise notion is the essence of Mach's principle, which asserts that whether the motion of a given observer is inertial or not is determined by "the distant stars," Mach's memorable phrase for the matter distribution in the universe [1]. If this profound claim were true, all motion, not just inertial motion, would be relative. Acceleration would not be absolute, for to accelerate without matter would be meaningless: there would be nothing to accelerate with respect to. The water in a bucket "rotating" all alone in the universe would not rise up at the sides, as there would be no sense in which a solitary bucket could be said to be rotating. As Weinberg poetically points out [2], to appreciate the significance of Mach's principle, one need only perform a pirouette underneath a starry sky. Is it mere coincidence that the frame in which one's arms fly outwards is the same as the frame in which the distant stars appear to spin overhead, or is there a deeper dynamical explanation?

In his landmark paper on the foundations of general relativity [3], Einstein sought to make the relativity of all motion one of the cornerstones of his new theory. But ironically, general relativity did not in the end seem to support Mach's idea. In general relativity whether a given worldline is inertial or accelerating depends on whether or not it satisfies the geodesic equation. This in turn depends on the metric which, indeed, is related through Einstein's equations to the matter distribution, encoded in the stress-energy tensor. However, the point is that ultimately the metric exists whether or not there is matter present. The existence of Minkowski space most emphatically underlines this point: geodesics and inertial frames exist even in the total absence of all matter. Although there are several distinct versions of what is meant by Mach's principle, the example of Minkowski space establishes that one common interpretation—that inertial frames here and now are determined by some kind of averaging over matter elsewhere—can immediately be ruled out.

In 1918, Einstein proposed a different definition of Mach's principle.

"Mach's Principle: The G field is without remainder determined by the masses of bodies. Since mass and energy are, according to results of the special theory of relativity, the same, and since energy is formally described by the symmetric energy tensor ($T_{\mu\nu}$), this therefore entails that the G field be conditioned and determined by the energy tensor [4]."

That is, Mach's principle holds if the metric (the " G field") is, up to diffeomorphisms, uniquely specified by the stress tensor. We interpret Einstein's 1918 formulation to posit a one-to-one correspondence between matter and geometry. This formulation would accept that in Minkowski space, inertial frames exist without matter,

but it would require that Minkowski space be the unique empty space-time. If Minkowski space were the unique space-time devoid of matter, one might still be able to claim that specifying the matter distribution somehow specifies the inertial frames. But of course there exist, besides Minkowski space, a host of perfectly fine solutions to the vacuum Einstein equations, among which are several well-known exact solutions like the Schwarzschild and Kerr black holes. Evidently, this version of Mach's principle is also in trouble.

Indeed, that matter and geometry are, contrary to what Einstein's 1918 proposal suggests, not entirely in one-to-one correspondence can be seen in a variety of ways:

- (i) The Weyl tensor is not determined by matter. Einstein's equations determine the Ricci tensor in terms of the stress tensor:

$$R_{\alpha\beta} = 8\pi G_D \left(T_{\alpha\beta} - \frac{1}{D-2} T g_{\alpha\beta} \right). \quad (1)$$

But the complete geometry is encoded in the Riemann tensor which, in four or more dimensions, includes not only the Ricci tensor but also the Weyl tensor. And, unlike the Ricci tensor, the Weyl tensor is independent of matter (give or take a Bianchi identity). Consequently, matter— $T_{\alpha\beta}$ —does not fully determine the Riemann tensor, and hence the geometry.

- (ii) General relativity permits gravitational waves. A more physical way of stating the problem is to note that general relativity permits gravitational waves. But these can exist as independent fluctuations even in empty space-time; there exist gauge-invariant solutions to the homogeneous wave equation.
- (iii) Einstein's equations need boundary conditions. Since Einstein's equations are second-order partial differential equations, to obtain a unique solution one needs to supplement them by boundary/initial conditions for the metric. These are usually in the form of an induced metric $h_{\alpha\beta}$ and extrinsic curvature $K_{\alpha\beta}$ for some appropriate hypersurface. The boundary conditions are arbitrary and are also apparently independent of matter.

For all of these reasons, Einstein's 1918 version of Mach's principle does not seem to hold. A related embarrassment from a Machian perspective is the existence of solutions with nonzero global angular momentum; Mach's ideas on the relativity of all motion imply that a closed system cannot be rotating—rotating with respect to what?—but in the Kerr black hole, as well as in other examples, general relativity permits solutions that have nonvanishing total angular momentum.

In attempts [5,6] to save Mach's principle, two separate lines of attack have been pursued. According to one,

avored initially by Einstein himself, general relativity is preserved intact but a selection rule is imposed on the space of solutions. Einstein, for example, demanded that cosmologies have compact spatial topology. There are several drawbacks to this approach, not the least its *ad hoc* nature. For instance, demanding compact spatial topology rules out Minkowski space, while $\mathbb{R} \times T^3$ with an arbitrarily large torus is allowed. Furthermore, it fails to eliminate the problem of boundary conditions or of gravitational waves. The second line of attack consists of modifying general relativity. This approach has also not worked. Indeed, so long as the dynamics of gravity are governed by a differential equation, arbitrary and apparently matter-independent boundary conditions are needed. Nor have other variants of Mach's principle met with great success. Thus it would seem that Mach's principle is one of those tantalizingly beautiful ideas that, sadly, are not realized in nature.

Nevertheless, it is the purpose of this paper to argue that Mach's ideas can be realized within Einstein's general theory of relativity. The version of Mach's principle we will implement is strongly inspired by Einstein's 1918 formulation: we will seek to find an equivalence between matter and geometry. The equations are not modified in any way. Instead, we shall show that there exists a recasting of the theory that suggests consistency with Mach's principle.

To understand how this is possible, let us note that all of the aforementioned objections—the matter independence of the Weyl tensor, the existence of gravitational waves, solutions with global rotation—can be related to one thing: the need for boundary conditions. To reiterate, Einstein's equations are second-order differential equations for the metric; to determine their solution they need to be supplemented by boundary conditions. This, then, is the crux of the problem. Boundary conditions are needed. The stress tensor is not enough.

Yet this way of stating the problem also points to an unexplored loophole. In recent years, there has been renewed interest in boundary matter. Boundary matter had largely been neglected in earlier approaches to Mach's principle, perhaps because it seemed too exotic at the time. But boundary matter is part of the bread and butter of theoretical physics today, and it arises in a variety of contexts. It appears in brane-world scenarios, in which our Universe is embedded in a higher-dimensional space. It appears in Hořava-Witten constructions as end-of-the-world branes [7]. It appears again in AdS/CFT as holographically dual matter [8]. It appears in the Brown-York (BY) construction of a boundary stress tensor for the gravitational field [9]. And, perhaps most relevantly, boundary matter appears in the black hole membrane paradigm [10], where the matter lives on the black hole horizon, an internal boundary of space-time for an external observer.

Adding a stress tensor at a boundary does not affect the bulk Einstein equations. Moreover, even space-times that are empty in the bulk, such as Minkowski space, may admit stress tensors living on their boundaries. We will see that, by allowing for two separate boundary stress tensors, every gauge-equivalent class of metrics can be related to a matter distribution, where matter now consists of both bulk and boundary stress tensors. In the process, Einstein's equations are retained and no solutions are sacrificed.

To be more precise, we start with a solution to Einstein's equations. The Ricci tensor can already be read off from the bulk stress tensor everywhere. But the boundary conditions on the metric remain to be represented in terms of matter. We therefore identify two stress tensors on a timelike boundary, $T^{\text{BY-in}}$ and $T^{\text{BY-out}}$, which determine the induced metric $h_{\alpha\beta}$ and extrinsic curvature $K_{\alpha\beta}$ of the boundary through Israel-like junction conditions and boundary Einstein equations. Note that these stress tensors are not independent of $h_{\alpha\beta}$ but are instead understood as functions of the latter; as usual in general relativity, matter stress energy cannot be meaningfully defined without specifying a metric. The key point is that we obtain a precise relation between $(T^{\text{BY-in}}, T^{\text{BY-out}})$ and $(h_{\alpha\beta}, K_{\alpha\beta})$. In turn, the latter serve as boundary data to evolve the bulk equations of motion. Of course, since a timelike boundary lies within its own causal future, $T^{\text{BY-in}}$ and $T^{\text{BY-out}}$ must satisfy subtle, nonlocal consistency conditions. For generic boundary stress tensors, no solution for the bulk metric would exist. We cannot therefore attempt to specify $T^{\text{BY-in}}$ and $T^{\text{BY-out}}$ *a priori*. Instead, we always proceed from a bona fide bulk solution to Einstein's equations, from which we read off the boundary stress tensors. By construction, the $T^{\text{BY-in}}$ and $T^{\text{BY-out}}$ so obtained satisfy all of the necessary causality conditions. Our Machian proposal is then to interpret these boundary stress tensors, along with any bulk stress tensor throughout space-time, as encoding the bulk geometry. And, in this precise sense, we have a map between matter and geometry.

This interpretation ideally requires that the bulk metric uniquely follows from these boundary (and bulk) stress tensors. While this is certainly plausible intuitively, at least for a certain class of space-times, a rigorous proof of uniqueness lies beyond the scope of this work. Later we will give physical arguments and provide necessary conditions for uniqueness to hold. These conditions are essentially the timelike counterparts to the conditions that a spacelike hypersurface be a Cauchy surface.

Trading boundary conditions for appropriate sources is an oft-employed technique in physics. An example that immediately springs to mind is the method of image charges in electrostatics in which conducting boundary conditions are replaced by fictitious charges. A closer analogue to our proposal is the membrane description of black hole horizons [10,11]. The membrane paradigm is the remarkable notion that, from the perspective of an

outside observer, a black hole behaves precisely as if it were cloaked in a fluid membrane living at the event horizon. That is, the equations of motion of fields in the background of a black hole, with regular boundary conditions at the horizon, can be rewritten so that the same equations describe the fields interacting with a source at the horizon—a membrane. Yet, the membrane approach is more than a mathematical trick. To an observer hovering outside the horizon, the membrane appears to behave like a real, dynamical fluid. It conducts electricity according to Ohm's law, it generates heat through Joule's law, and it flows following the Navier-Stokes equation. Only by jumping into the black hole can the observer realize the illusory nature of the membrane.

In some sense our proposal can be viewed as an “inside-out” version of the membrane paradigm, a parallel most apt for space-times with causal horizons. Take de Sitter space. The natural location for the boundary stress energy in this case is on the “stretched” horizon, a timelike surface hovering just inside the causal horizon. Much like the black hole case, this membrane is dynamical and satisfies a host of classical equations. Intriguingly its surface energy density inferred from Israel-like junction conditions has the equation of state of dust. To a bulk observer this boundary dust plays the role of the distant stars, relative to which accelerated motion in the bulk can meaningfully be defined.

Moreover, our boundary stress tensors $T^{\text{BY-in}}$ and $T^{\text{BY-out}}$ have a compelling interpretation in terms of the Brown-York stress energy [9] of the gravitational field in the interior and exterior regions of space-time, respectively. It had long been conjectured that general relativity could be proven to be Machian by somehow taking into account gravitational stress energy. But of course a local notion of stress energy for gravity is meaningless. Stress tensors are usually constructed from fields and first derivatives of fields but, for the metric, both of these can be made trivial at any point by a suitable choice of coordinates. Instead, as argued by Brown and York using Hamilton-Jacobi theory, the natural location for gravitational stress energy is at the boundary.

Finally, in asymptotically anti-de Sitter (AdS) space-times, our proposal is consistent with the holographic correspondence. (Intriguingly, a connection between Mach's principle and holography has also been made in Hořava's Chern-Simons M theory [12].) The Brown-York stress tensors mentioned above are now understood as the holographic stress tensors for the dual field theory in one lower dimension. Indeed, the correspondence identifies the radial direction (say in Arnowitt-Deser-Misner coordinates) in an asymptotically AdS space-time with the renormalization group (RG) scale of the dual theory [13–15]. Placing an effective boundary at some radial location therefore represents an ultraviolet cutoff in the dual theory where RG initial conditions can be specified. Furthermore,

replacing the exterior region with matter on the boundary corresponds in the dual language to integrating out high-energy degrees of freedom, whose quantum stress-energy tensor is just the Brown-York tensor for the exterior gravitational field. Similarly, the Brown-York tensor for the interior is the stress tensor for the low-energy degrees of freedom. This interpretation also makes it clear that there is nothing special about the location of the boundary. In the Wilsonian sense, shifting the boundary along the radial direction simply corresponds to choosing a different RG cutoff.

To summarize, our proposal makes contact with the membrane paradigm, the Brown-York notion of stress energy for the gravitational field, and the holographic correspondence. It amounts to a rewriting of Einstein's theory, combined with a rule for obtaining the boundary stress tensors, that together make a compelling case for the implementation of Mach's principle in general relativity. This reformulation does not affect Einstein's equations; indeed, any formulation of Mach's principle that was *not* consistent with those equations would already be in trouble.

Often in physics the recasting of an existing theory has deepened our understanding of nature. Consider again the black hole membrane paradigm. While superficially just a rewriting of classical equations, the key insight that the event horizon behaves as a dynamical fluid led to a host of conceptual breakthroughs: black hole entropy was understood as a local property; astrophysical phenomena like the Blandford-Znajek process [16] were clarified; the no-hair theorem became intuitive; and the complementarity principle was motivated as an approach to the information puzzle [17,18]. The rewritten equations are deservedly termed a new paradigm. We hope that our framework may similarly help to shed new light on old problems, such as the origin of noninertial forces and the relativity of all motion.

A brief outline of this work is as follows. In Sec. II we argue that the various obstacles in the way of a realization of Mach's principle all boil down to the need for boundary conditions. Section III presents an electromagnetic counterpart to Mach's principle; the boundary conditions are encoded in charges and currents living on a kind of Faraday cage. We then show that the boundary conditions for gravity too can be regarded as originating in boundary sources, and we propose a particular kind of boundary stress tensor that encodes the boundary conditions. In Sec. IV we show that our particular prescription for obtaining the boundary stress tensor also takes care of the problem of net global angular momentum by precisely canceling any global rotation of the space-time. In Sec. V we apply our proposal for boundary stress tensors to a variety of well-known space-times and we read off the form of the boundary matter. Remarkably the boundary matter typically turns out to be simply pressureless dust. In

Sec. VI we discuss existence and uniqueness issues in relation to the initial-value problem in general relativity. Section VII contrasts our proposal with earlier attempts to reconcile Mach's principle with gravity. We conclude in Sec. VIII with a brief summary and some directions for future work.

II. BOUNDARY CONDITIONS

Three reasons for the apparent failure of Mach's principle is that general relativity admits gravity waves, the geometry is encoded partly in the Weyl tensor, and Einstein's equations are subject to boundary conditions. In this section we argue that these three objections are intimately related. This will motivate us to cast the problem entirely in terms of boundary conditions, in preparation for the next section wherein we will capture those boundary conditions through the addition of boundary matter.

We begin by reviewing the relation between gravitational wave solutions and the Weyl tensor in $D > 3$ space-time dimensions. Given a metric, the space-time geometry can be characterized by calculating the Riemann curvature tensor in the usual way. Now the part of the Riemann tensor that does not figure in the Einstein equations is the Weyl tensor, which, again in $D > 3$ space-time dimensions, is related to the Riemann and Ricci tensors via

$$C_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} + \frac{2}{D-2}(g_{\alpha[\delta}R_{\gamma]\beta} + g_{\beta[\gamma}R_{\delta]\alpha}) + \frac{2}{(D-1)(D-2)}Rg_{\alpha[\gamma}g_{\delta]\beta}, \quad (2)$$

where the commutator is normalized according to $x_{[a,b]} \equiv (x_{ab} - x_{ba})/2$. Since Weyl is unspecified by the local stress tensor, it follows that homogeneous solutions to Einstein's equations—gravity waves—are encoded in Weyl. Indeed, one can make this more explicit by noting that the Bianchi identity $\nabla_{[\alpha}R_{\beta\gamma]\delta}{}^{\epsilon} = 0$, combined with Einstein's equations, implies a constraint on Weyl:

$$\nabla^{\delta}C_{\alpha\beta\gamma\delta} = 16\pi G_D \left(\frac{D-3}{D-2}\right) \left(\nabla_{[\beta}T_{\alpha]\gamma} - \frac{1}{D-1}g_{\gamma[\alpha}\nabla_{\beta]}T\right). \quad (3)$$

Then, after some work [19], it is possible to show that the linearized version of this equation can be repackaged as the wave equation for a massless spin-2 field, establishing the anticipated relation between the Weyl tensor and gravitational waves. Incidentally, (3) implies that the Weyl tensor is not entirely independent of bulk matter. Since the stress tensor only appears acted upon with derivatives, however, the Weyl tensor at a point qualitatively encodes the part of the curvature due to matter elsewhere [20].

Having motivated the equivalence between gravity waves and Weyl, we turn to Weyl versus boundary con-

ditions. Solving the Einstein equations requires boundary conditions for the metric, for instance the induced metric $h_{\alpha\beta}$ and extrinsic curvature $K_{\alpha\beta}$ of a codimension one hypersurface. (For the purpose of this introductory section, we postpone a discussion of consistency and well posedness to Sec. VI.) For a timelike boundary with unit normal n^α (with mostly positive metric signature), these are given by

$$h_{\alpha\beta} = g_{\alpha\beta} - n_\alpha n_\beta; \quad K_{\alpha\beta} = h_\alpha^\gamma \nabla_\gamma n_\beta. \quad (4)$$

Meanwhile, (3) is a first-order differential equation for Weyl. This means that, given some bulk matter, a solution for the metric requires specifying one boundary condition: the boundary value of the Weyl tensor. Thus it remains to establish a correspondence between the boundary value of the Weyl tensor and the choice of $h_{\alpha\beta}$ and $K_{\alpha\beta}$.

To do so, we assume for simplicity an empty bulk, $T_{\alpha\beta} = 0$. The bulk Weyl tensor evaluated at the boundary can be decomposed into its electric and magnetic parts:

$$\begin{aligned} E_{\mu\nu} &\equiv C_{\alpha\beta\gamma\delta} n^\alpha n^\gamma h_\mu^\beta h_\nu^\delta; \\ B_{\mu\nu\gamma} &\equiv C_{\alpha\beta\gamma\delta} h_\mu^\alpha h_\nu^\beta n^\delta. \end{aligned} \quad (5)$$

Both are traceless, $E^\mu{}_\mu = 0$ and $B^\alpha{}_{\nu\alpha} = 0$, and they have the following symmetries: $E_{\mu\nu} = E_{\nu\mu}$; $B_{\alpha\beta\gamma} = -B_{\beta\alpha\gamma}$; $B_{[\alpha\beta\gamma]} = 0$.

To derive a relation between E and (h, K) , we take the trace, over α and γ of the Gauss relation ${}^{(D-1)}R_{\alpha\beta\gamma\delta} = h_\alpha^\kappa h_\beta^\mu h_\gamma^\nu h_\delta^\sigma R_{\kappa\mu\nu\sigma} + K_{\alpha\gamma} K_{\beta\delta} - K_{\beta\gamma} K_{\alpha\delta}$ and use $R_{\alpha\beta} = 0$ to obtain

$$E_{\mu\nu} = -{}^{(D-1)}R_{\mu\nu} + K K_{\mu\nu} - K_\mu^\alpha K_{\nu\alpha}. \quad (6)$$

Next, take the identity $R_{\alpha\beta\gamma\delta} n^\delta = (\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha) n_\gamma$. Contracting with $h_\mu^\alpha h_\nu^\beta$ and substituting the expression for the Weyl tensor and the extrinsic curvature, we find

$$B_{\mu\nu\gamma} = D_\mu K_{\nu\gamma} - D_\nu K_{\mu\gamma}, \quad (7)$$

where D_μ is the covariant derivative associated with h . Since E and B are traceless, taking the trace of the above expressions incidentally gives the usual initial-value constraints of general relativity. Equations (6) and (7) display the explicit map between the boundary data h and K and the boundary value of the Weyl tensor. By (3), the components of the Weyl tensor at some point in the bulk are then also implicitly functions of the boundary data, which is what we wanted to show.

As a check on the number of degrees of freedom, note that h and K are symmetric tensors in $D - 1$ dimensions, for a total of $D(D - 1)$ components; each is covariantly conserved: $D^\mu h_{\mu\nu} = 0$ and $D^\mu K_{\mu\nu} - D_\nu K = 0$, bringing the total down to $D(D - 3) + 2$; the trace of (6) gives the Hamiltonian constraint ${}^{(D-1)}R = K^2 - K_{\alpha\beta} K^{\alpha\beta}$; and the final condition corresponds roughly speaking to the usual freedom in specifying the codimension one hypersurface,

which requires one constraint. This leaves us with $D(D - 3)$ arbitrary gauge-invariant degrees of freedom at the boundary which, to complete the circle, is indeed twice the number of graviton polarizations in D dimensions.

III. THE DISTANT STARS: MATTER AT THE BOUNDARY OF SPACE

In this section we show how the boundary conditions for the metric can be related to stress tensors on the boundary. Barring a few caveats that are discussed later, this realizes Einstein's statement of Mach's principle—that the metric field be encoded in matter stress energy—as long as “matter” refers to both bulk and boundary stress tensors. To illustrate our procedure we first consider a related problem in electrodynamics.

A. Electromagnetic analogy

There exists an electromagnetic counterpart to the version of Mach's principle presented here. Asking whether the metric—the gravitational field—is determined entirely by matter sources is akin to asking whether the electromagnetic field is entirely determined by electromagnetic sources—charges and currents. The answer is no because, once again, there are boundary conditions. Alternatively, we note that Maxwell's equations in vacuum allow for electromagnetic waves, much as the vacuum Einstein equations support gravitational waves. Indeed, specification of the $D - 1$ form electromagnetic current j does not yield the gauge-invariant electromagnetic tensor F , because even though

$$d * F = j, \quad (8)$$

we are free to add a term F' to F satisfying $d * F' = 0$. Writing $F' = dA'$, the term undetermined by the sources satisfies

$$d * dA' = 0. \quad (9)$$

In Lorentz gauge, $d * A' = 0$, we obtain the wave equation

$$(*d * d + d * d*)A' = \Delta A' = 0, \quad (10)$$

where Δ is the d'Alembertian. So we see explicitly that sources determine fields up to electromagnetic waves. Then, since there are $D - 2$ propagating degrees of freedom, a total of $2(D - 2)$ functions on the boundary must be specified as boundary conditions.

But the understanding that electromagnetic waves are responsible for the failure of the sources to determine the field also points to a way out. We know that electromagnetic waves are blocked by a Faraday cage. Hence, if we could place a Faraday cage around the region of interest, all electromagnetic waves coming from the interior would be blocked, and the boundary electromagnetic fields could be attributed to charges and currents living on the cage. Indeed, this is more or less what we will show. We will see that our proposal amounts to interpreting the boundary

surrounding some region of space as some hypothetical material—call it a dual Faraday cage—which eliminates the tangential magnetic field. This surface absorbs all incident electromagnetic waves, since the latter cannot propagate through without a magnetic field component tangential to the surface. In that sense our approach brings to mind the old Wheeler-Feynman absorber theory, proposed for different reasons, in which electromagnetic waves in the bulk are emitted or absorbed by distant boundary sources [21,22].

Our procedure for implementing Machian electromagnetism begins by splitting the Maxwell action,

$$S = \int d^D x \left(-\frac{1}{4g^2} F^2 + J^\alpha A_\alpha \right), \quad (11)$$

as integrals over two regions of space, \mathcal{M}_{in} and \mathcal{M}_{out} , separated by a boundary Σ . In principle Σ can be null, spacelike or timelike, although to make contact with common experience, let us take it to be timelike. Call S_{in} and S_{out} the actions restricted to an integration over \mathcal{M}_{in} and \mathcal{M}_{out} , respectively. Now, neither the stationarity of S_{in} nor the stationarity of S_{out} is sufficient on its own to yield the classical equations of motion, because there remains a variation of the field at Σ , after integration by parts. Following the action formulation of the membrane paradigm [11], one deals with this leftover variation by adding and subtracting a boundary action on Σ ,

$$S = S_{\text{in}} + S_\Sigma + S_{\text{out}} - S_\Sigma, \quad (12)$$

with S_Σ chosen so that $\delta(S_{\text{in}} + S_\Sigma) = 0$ classically. Thus, from the point of view of an observer in \mathcal{M}_{in} , S_Σ encodes all physical effects of the exterior region.

Varying the relevant action for this observer yields the bulk equations of motion for the gauge field, $\partial_\alpha F^{\alpha\beta} = -g^2 J^\beta$, as well as a boundary term

$$\delta S = \int_\Sigma d^{D-1} x \left(-\frac{1}{g^2} n_\alpha F^{\alpha\beta} + \frac{\delta S_\Sigma[A]}{\delta A_\beta} \right) \delta A_\beta. \quad (13)$$

Such a boundary term is usually set to zero by imposing Dirichlet boundary conditions on the variation: $\delta A = 0$ on Σ . However, we can also choose not to fix any definite boundary conditions at Σ . In the membrane paradigm, one does not impose boundary conditions at the horizon, because the horizon is regarded as dynamical. Here, we do not fix boundary conditions, because we want to capture them in terms of matter. Instead, we choose the variation of the boundary action, S_Σ , to cancel the residual variation from the bulk action:

$$\frac{1}{g^2} n_\alpha F^{\alpha\beta} = \frac{\delta S_\Sigma[A]}{\delta A_\beta}. \quad (14)$$

This condition implies that the normal component of the electric field E_\perp as well as the tangential component of the magnetic field B_\parallel are canceled by the boundary action. Thus, as advertised, the boundary acts as a dual Faraday

cage preventing incident electromagnetic waves from going through. (We say ‘‘dual,’’ because an ordinary Faraday cage would eliminate E_\parallel and B_\perp ; the blocking effect on electromagnetic radiation is identical.)

A general choice for S_Σ , consistent with all symmetries and with at most two derivatives, is

$$S_\Sigma = \int_\Sigma d^{D-1} x \left(-\frac{1}{4g_b^2} F^2 + j_{(1)}^\alpha A_\alpha \right), \quad (15)$$

where $j_{(1)}$ is a surface current, and where we have allowed for a different coupling constant g_b on the boundary. In this case the matching condition (14) gives Maxwell’s equations on the boundary,

$$\partial_\alpha F^{\alpha\beta} = -g_b^2 \left(j_{(1)}^\beta - \frac{1}{g^2} n_\alpha F^{\alpha\beta} \right). \quad (16)$$

From the point of view of a fiducial observer on Σ , the last term acts as a current, which we therefore denote by

$$j_{(2)}^\beta \equiv -\frac{1}{g^2} n_\alpha F^{\alpha\beta}. \quad (17)$$

In the absence of charge transfer between bulk and boundary, $j_{(2)}$ is conserved on the boundary, since $\partial_\alpha j_{(2)}^\alpha \sim n_\alpha J^\alpha = 0$. It then follows from (16) that $j_{(1)}$ is separately conserved.

Specifying the boundary currents $j_{(1)}$ and $j_{(2)}$ completely determines, through (16) and (17) respectively, the gauge field on the boundary and its normal derivative, and therefore encodes the boundary conditions necessary to solve for the electromagnetic field in the bulk.

As a check on the number of degrees of freedom in general D dimensions, each current is a vector in $D - 1$ dimensions for a total of $2(D - 1)$ components, but each satisfies a continuity equation which brings the total down to $2(D - 2)$. Sure enough, this is the requisite number of boundary conditions as a spin-1 massless particle has $D - 2$ propagating degrees of freedom.

Let us sketch the method with a simple example. Consider a region free of charge but permeated by a divergence-free electrostatic field, $F^{0i} = -\partial_i A^0 = -\partial_i \phi$, where ϕ is the Coulomb potential. This region is taken to be enclosed by a fictitious timelike surface Σ . We would like to determine the required surface charge densities $j_{(i)}^0 \equiv \sigma_{(i)}$, $i = 1, 2$, on Σ that would reproduce the electrostatic field inside. Consulting (17), we find that the normal component of the electric field is accounted for by $\sigma_{(2)}$:

$$\sigma_{(2)} = \frac{1}{g^2} \partial_n \phi. \quad (18)$$

Using the usual equation for the jump in the normal component of E due to surface charge, $E_\perp^{\text{out}} - E_\perp^{\text{in}} = g^2 \sigma$, we find incidentally that $E_\perp^{\text{out}} = 0$. Similarly, $\sigma_{(1)}$ is determined by the tangential component through (16):

$$\sigma_{(1)} = -\frac{1}{g_b^2} \nabla_{\parallel}^2 \phi - \sigma_{(2)}. \quad (19)$$

In particular, if the field lines happen to hit Σ everywhere normal to the surface, then the boundary can be interpreted physically as a thin conductor.

The straightforward simplicity of this derivation should not suggest that it is somehow tautological. For example, had we relied only on the equations of motion—as opposed to on an action formulation—we would have been faced with the following problem. We know that surface charges cause a discontinuity in the normal electric field component: $E_{\perp}^{\text{out}} - E_{\perp}^{\text{in}} \sim \sigma$. Now, in order to trade boundary conditions for charges— E_{\perp}^{in} for σ —one has to impose some other condition, as otherwise we have only one equation relating three variables. But *any* independent equation would have done the trick. However, as we shall see, all but one of these conditions fail to possess such desirable properties as the vanishing of global angular momentum, the extinction of incident waves, and consistency with holography and the membrane paradigm. That one condition, which in Machian electromagnetism corresponds to $E_{\perp}^{\text{out}} = 0$, follows naturally, as shown above, from an action principle.

B. Gravity

We turn now to Mach's principle in general relativity. For simplicity we focus here on a vanishing bulk stress tensor, save for the usual cosmological term. The prescription is much the same as for the electromagnetic case studied above, expect for one important subtlety. Unlike electromagnetism for which boundary sources can be specified independently of the gauge field, in gravity it makes no sense to define a stress tensor without a metric. For example, even in a scalar field theory, a metric is involved in contracting the derivatives in the kinetic term. More generally, diffeomorphism invariance requires that the stress tensor be covariantly conserved, and this entails a compatibility check between the stress tensor and the metric, because the covariant derivative itself must be metric compatible. Thus we see that, unlike in the electromagnetic case where we could specify charges at the boundary independently of the field, here, as is usual in general relativity, the boundary stress tensors are understood as functions of the induced metric. Nevertheless, we will obtain a relation between these stress tensors and the induced metric and extrinsic curvature of the boundary, which can then be used (in principle) as boundary data to integrate the bulk Einstein equations. (The important issue of whether the bulk metric can be uniquely “reconstructed” from such boundary data is postponed until Sec. VI.) We interpret the relation between boundary stress tensors and the vacuum bulk metric as the manifestation of Mach's principle: matter encodes geometry.

Formally, our construction applies equally to spacelike and timelike boundaries. However, we want to identify some physical matter living on Σ , with which a bulk observer can potentially interact. The boundary conditions on a timelike surface are more readily interpreted as matter than boundary conditions on a spacelike surface. (The null case is also physical but requires separate analysis, as the boundary data for the characteristic problem is different [23]. We leave this to future work.) Furthermore, a timelike boundary makes contact with the Brown-York definition of the bulk gravitational stress tensor, as well as with the membrane paradigm. As we will see explicitly in Sec. V, the latter is particularly relevant for de Sitter space where Σ is a surface hovering inside some observer's event horizon. Henceforth, we will assume the boundary to be timelike. Of course to get a bulk solution from timelike sources requires that we specify boundary conditions for all times. This will be the case here, just as the membrane description of a black hole horizon is valid for all times for an external observer.

Consider then the Einstein-Hilbert action, including a cosmological term:

$$S = \frac{1}{16\pi G_D} \int_{\mathcal{M}} d^D x \sqrt{-g} (R - 2\Lambda), \quad (20)$$

which again splits as integrals over two space-time regions \mathcal{M}_{in} and \mathcal{M}_{out} separated by Σ . Once again we add and subtract a boundary action to get $\delta(S_{\text{in}} + S_{\Sigma} + S_{\text{GH}}) = 0$, where the Gibbons-Hawking term

$$S_{\text{GH}} = \frac{1}{8\pi G_D} \int_{\Sigma} d^{D-1} x \sqrt{-h} K \quad (21)$$

is necessary to obtain a well-defined variational principle on \mathcal{M}_{in} . (Our convention is that the normal vector to Σ is inward pointing, hence the sign of Gibbons-Hawking.) Thus, from the point of view of an observer in \mathcal{M}_{in} , S_{Σ} encodes the physical effects of the exterior region. We take S_{Σ} to be a general two-derivative action in $D - 1$ dimensions,

$$S_{\Sigma} = \frac{1}{16\pi G_{D-1}} \int_{\Sigma} d^{D-1} x \sqrt{-h} ((D-1)R - 2\lambda) + S_{\Sigma}^{\text{matter}}[h], \quad (22)$$

describing intrinsic gravity coupled to a cosmological constant λ and boundary matter. This action is a functional of intrinsic boundary quantities only and so it leaves the bulk equations unaffected.

Performing the variation $\delta(S_{\text{in}} + S_{\Sigma} + S_{\text{GH}}) = 0$ then yields the bulk Einstein equations $G_{\alpha\beta} = -\Lambda g_{\alpha\beta}$ as well as a surface term:

$$\delta S = \frac{1}{2} \int_{\Sigma} d^{D-1}x \sqrt{-h} \left\{ \frac{1}{8\pi G_D} (K_{\alpha\beta} - Kh_{\alpha\beta}) + \frac{2}{\sqrt{-h}} \frac{\delta S_{\Sigma}[h]}{\delta h^{\alpha\beta}} \right\} \delta h^{\alpha\beta}. \quad (23)$$

Usually, this would be set to zero by imposing Dirichlet boundary conditions: $\delta h|_{\Sigma} = 0$. But alternatively, one can choose $S_{\Sigma}[h]$ to cancel this term. This gives an Israel matching condition, with the only difference being that the extrinsic curvature term for the “exterior” region vanishes in this case—the physics of the exterior region is encoded in the boundary action. A similar Israel condition arises in the action description of the membrane paradigm for black holes, this time with the region interior to the black hole horizon replaced by boundary matter [10,11]. The upshot is that (23) fixes the canonical momentum $K_{\alpha\beta} - Kh_{\alpha\beta}$ and hence half of the boundary conditions for the gravitational field. The vanishing of (23) is thus the gravitational analogue of (14) for electromagnetism—our boundary acts as a *gravitational dual Faraday cage*. (As in electromagnetism, junction conditions by themselves do not identify the precise relation between boundary condition and boundary sources, because an arbitrary choice for the field value beyond the boundary can be made; the action formulation fixes that choice.)

For the choice of S_{Σ} given in (22), the vanishing of the above surface term gives an Einstein equation on the boundary,

$$\begin{aligned} (D-1)G_{\alpha\beta} + \lambda h_{\alpha\beta} &= \frac{G_{D-1}}{G_D} (Kh_{\alpha\beta} - K_{\alpha\beta}) \\ &+ 8\pi G_{D-1} T_{\alpha\beta}^{\text{BY-out}}, \end{aligned} \quad (24)$$

where

$$T_{\alpha\beta}^{\text{BY-out}} \equiv -\frac{2}{\sqrt{-h}} \frac{\delta S_{\Sigma}^{\text{matter}}}{\delta h^{\alpha\beta}} \quad (25)$$

is the stress tensor for the boundary matter. This matter stress tensor is recognized as the Brown-York stress tensor for the exterior space-time region \mathcal{M}_{out} , hence the superscript. In analogy with classical mechanics where the energy of a system can be expressed through the Hamilton-Jacobi action as $E = -\partial S/\partial t$, a quasilocal notion of stress energy for a space-time region can be defined on its boundary as [9]

$$\begin{aligned} T_{\alpha\beta}^{\text{BY}} &\equiv \frac{2}{\sqrt{-h}} \frac{\delta S_{\text{class}}}{\delta h^{\alpha\beta}} \\ &= \frac{1}{8\pi G_D} (K_{\alpha\beta} - Kh_{\alpha\beta}) + \frac{2}{\sqrt{-h}} \frac{\delta S_{\text{reg}}}{\delta h^{\alpha\beta}}. \end{aligned} \quad (26)$$

The regulating term S_{reg} is a local action on the boundary required to cancel potential infrared divergences as the boundary is taken to infinity or to some horizon, whatever the case may be. For asymptotically locally AdS space-

times, such divergences correspond through the AdS/CFT duality to the usual UV divergences in field theory. The cancellation of divergences will end up fixing G_{D-1} and λ , which at this stage might appear arbitrary. For a 3 + 1-dimensional bulk, the case of interest, we will find in Sec. V the following counter-term action:

$$S_{\text{reg}} = \frac{\ell}{16\pi G_4} \int_{\Sigma} d^3x \sqrt{-h} \left({}^{(3)}R + \frac{4}{\ell^2} \right), \quad (27)$$

which is of the form (22) with $\lambda = -2/\ell^2$ and $G_3 = G_4/\ell$, where ℓ is the AdS radius [24]. Substituting this in (24) and comparing the result with (26), we indeed see that from the point of view of an observer in \mathcal{M}_{in} , the effects of the exterior region can be encoded with matter on Σ , whose corresponding stress tensor $T_{\alpha\beta}^{\text{BY-out}}$ is the Brown-York stress tensor for the gravitational field in \mathcal{M}_{out} .

The form of (24) as Einstein’s equations on the boundary suggests that the extrinsic curvature term be interpreted by a boundary observer as a second stress tensor

$$Kh_{\alpha\beta} - K_{\alpha\beta} = 8\pi G_D T_{\alpha\beta}^{\text{BY-in}}. \quad (28)$$

From the discussion above, $T_{\alpha\beta}^{\text{BY-in}}$ is just the Brown-York stress tensor for the interior space-time region \mathcal{M}_{in} . The boundary matter behaves very much like real matter. For example, when there is no matter exchange between boundary and bulk, $T_{\alpha\beta}^{\text{BY-in}}$ is conserved on the boundary

$$D^{\alpha} T_{\alpha\beta}^{\text{BY-in}} = 0. \quad (29)$$

It follows from (24) and the Bianchi identity on the boundary, $D^{\alpha} G_{\alpha\beta} = 0$, that $T_{\alpha\beta}^{\text{BY-out}}$ is covariantly conserved as well. Even more compellingly, when matter does extend to the boundary, the contracted Gauss-Codazzi equation yields

$$D^{\alpha} T_{\alpha\beta}^{\text{BY-in}} = -T_{\gamma\delta}^{\text{bulk}} n^{\gamma} h^{\delta}_{\beta}. \quad (30)$$

This has a very satisfying interpretation: it is a continuity equation between Machian boundary matter and bulk matter. This is the gravitational counterpart of charge conservation on the boundary, obtained by taking the divergence of (17). Thus we see from (29) and (30) that the boundary matter behaves as matter should; it participates in a continuity equation with “real” bulk matter when such matter interacts with the boundary, and is conserved otherwise.

Now, although (24) is itself a differential equation for h , in 2 + 1 dimensions it has a unique solution up to global identifications—the Weyl tensor vanishes identically in 2 + 1 dimensions, thus curvature is entirely determined by the Ricci tensor. Similarly, (28) can be solved uniquely. Therefore, we have established a map between $T^{\text{BY-in}}$ and $T^{\text{BY-out}}$ and h and K . The latter constitute boundary data that in principle would allow the reconstruction of the bulk gravitational field. An important caveat is whether this procedure would yield a unique bulk metric, an issue we explore in more detail in Sec. VI.

[In more than four dimensions, the situation is more complicated. The boundary Einstein equations (24) themselves require boundary data in order for a unique solution to obtain. In general, the boundary will have a nonvanishing Weyl tensor, corresponding to gravitational waves circulating within the boundary. Then the procedure described here would have to be iterated to lower dimensional surfaces in order to encode the Weyl tensor of the boundary. However, in many situations the Weyl tensor of the boundary vanishes, something that is automatically true when the bulk space-time is four dimensional. In these cases it might be possible to choose a surface within the boundary—of codimension two from the bulk point of view—on which there is no matter. A further complication in higher dimensions is that the counter-term action in (27) generically includes higher curvature terms. This introduces higher derivative modifications to (24), which therefore requires still more boundary data. In this respect, it would be interesting to look at Lanczos-Lovelock gravity, which, being quasilinear in second derivatives, might have a more tractable description in terms of Machian boundary matter. In any case, the two-derivative action presented here suffices for the 2 + 1-dimensional boundary of a 3 + 1-dimensional space-time.]

The stress tensors $T^{\text{BY-in}}$ and $T^{\text{BY-out}}$ have the natural interpretation of describing the stress energy in the gravitational field for the exterior and interior space-time regions delimited by Σ . Thus our proposal supports the old suspicion that general relativity might be reconciled with Mach's ideas by taking into account not only the matter stress tensor, but also that of the gravitational field. The natural location for this stress tensor, as Brown and York realized, is on the boundary.

As a check on the counting of degrees of freedom in D dimensions, each stress tensor is a symmetric tensor in $D - 1$ dimensions which is covariantly conserved, for a total of $D(D - 3) + 2$ degrees of freedom. As argued at the end of Sec. II, the Hamiltonian constraint and the choice of Σ give two more conditions, bringing the total of freely specifiable functions down to $D(D - 3)$, in agreement with the required boundary data for the $D(D - 3)/2$ graviton degrees of freedom.

IV. VANISHING TOTAL ANGULAR MOMENTUM

In this brief section we show that our setup implies that the total angular momentum of an isolated system is zero. This is in harmony with the Machian precept that motion is only defined in a relative sense. In fact, the vanishing of all global charges is an immediate consequence of the definition of our boundary term: charges are calculated using the Brown-York stress tensor, $T_{\alpha\beta}^{\text{BY-in}}$, but since we are adding precisely $-T_{\alpha\beta}^{\text{BY-in}}$ on the boundary, the two contributions cancel exactly.

Concretely, corresponding to a given Killing vector ξ^α of the boundary geometry, one can define a conserved

current

$$j_\alpha = T_{\alpha\beta}^{\text{BY-in}} \xi^\beta \quad (31)$$

with associated conserved charge

$$Q^{\text{bulk}} = \oint_B d^{D-2}x \sqrt{\sigma} u^\alpha j_\alpha. \quad (32)$$

Here, u^α is a timelike vector on Σ , which is normal to a $D - 2$ -dimensional closed surface B with induced metric $\sigma_{\alpha\beta} = h_{\alpha\beta} + u_\alpha u_\beta$. From (23) and (28), it is clear that the addition of S_Σ gives a second boundary current which precisely cancels the bulk contribution:

$$Q^{\text{bulk}} + Q^{\text{bdy}} = 0. \quad (33)$$

For example, consider the angular momentum for the Kerr geometry in 3 + 1 dimensions. Here, we choose Σ to be at some large radius where the metric is approximately

$$ds^2 \approx -\left(1 - \frac{2M}{r}\right) dt^2 - \frac{4J}{r} \sin^2\theta dt d\phi + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2. \quad (34)$$

The Arnowitt-Deser-Misner angular momentum, corresponding to the Killing vector $\xi^\alpha = (\partial/\partial\phi)^\alpha$, is then

$$J^{\text{bulk}} = \oint_B d^2x \sqrt{\sigma} u^\alpha T_{\alpha\phi}^{\text{BY-in}} = \frac{3J}{8\pi} \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin^3\theta = J, \quad (35)$$

which is just the statement that the parameter J really is the angular momentum of the space-time. We see that it is indeed given by the Brown-York stress tensor for the interior. It then follows from (33) that *the boundary rotates in the opposite direction* with angular momentum $-J$, such that the combined bulk plus boundary sources have precisely zero net angular momentum. Note that a cancellation would not have occurred had one picked a different mapping of the boundary conditions to boundary sources; it is only in our prescription, originating in the action formulation, that conserved charges vanish.

V. EXAMPLES

In this section we illustrate our construction with several explicit examples. Little has been said thus far about the location of the boundary matter and there is indeed ample flexibility in this choice. However, space-times with horizons offer natural candidate surfaces: the Rindler horizon of an accelerated observer, the cosmological horizon in de Sitter space, etc. In these cases the close connection to the membrane paradigm for causal horizons [25] is manifest. For concreteness, we work in 3 + 1 dimensions throughout this section.

A. Minkowski space

Minkowski space-time has long been considered the quintessential anti-Machian solution to the Einstein field equations. Flat space has no matter, yet there is a well-defined notion of inertia everywhere. Indeed, the same holds true for any of the other vacuum solutions to Einstein's equations. In all of these space-times, inertial frames are unambiguously defined and yet there is no (bulk) matter whatsoever. Thus it would seem that acceleration is defined absolutely and not relatively. Were it not for boundary matter, one could even say that while Newtonian mechanics postulates the existence separately of absolute space and absolute time, Einstein's theory of gravity allows for absolute space-time.

Our proposal draws a different conclusion: it is boundary matter that plays the role of the distant stars. Consider spherically-symmetric coordinates with a fiducial observer sitting at the origin. A natural location for the boundary is the world volume of a two-sphere at some fixed large radius r_0 , with topology $\mathbb{R} \times S^2$. See Fig. 1(a). The boundary geometry is just the Einstein static universe, with the only nonvanishing component of the Einstein tensor given by $G^t_t = -1/r_0^2$. Meanwhile, the extrinsic curvature tensor is given by $K^t_t = 0$; $K^A_B = -\delta^A_B/r_0$, where A, B denote angular variables on the two-sphere. Substituting in (28), we obtain

$$(T^{\text{BY-in}})^t_t = -\frac{1}{8\pi G_4} \frac{2}{r_0}; \quad (T^{\text{BY-in}})^A_B = -\frac{\delta^A_B}{8\pi G_4 r_0}. \quad (36)$$

The $1/r$ falloff leads to conserved charges for the space-time which diverge in the limit $r_0 \rightarrow \infty$, as seen from (32). From the Brown-York perspective, these can be canceled with appropriate G_3 and λ to yield a finite $T^{\text{BY-out}}$. Indeed, substituting $T^{\text{BY-in}}$ and the boundary Einstein tensor in (24) gives

$$(T^{\text{BY-out}})^t_t = \frac{1}{8\pi G_3} \left(-\frac{1}{r_0^2} + \lambda \right) + \frac{1}{8\pi G_4} \frac{2}{r_0}; \quad (T^{\text{BY-out}})^A_B = \delta^A_B \left(\frac{\lambda}{8\pi G_3} + \frac{1}{8\pi G_4 r_0} \right). \quad (37)$$

Canceling the diverging terms fixes the cosmological term

$$\lambda = -\frac{1}{r_0^2}, \quad (38)$$

as well as the intrinsic Newton's constant on the boundary

$$\frac{G_3}{G_4} = \frac{1}{r_0}, \quad (39)$$

in agreement with brane-world calculations [26]. In particular, we see that gravity decouples in the limit that the boundary is sent to infinity.

Unlike its counterpart in $3+1$ dimensions, the Einstein static universe in $2+1$ does not require a nonvanishing cosmological term, only dust. This can be seen directly from the components of the boundary Einstein tensor. Interpreting (24) as Einstein's equations on the boundary, it follows that the cosmological term (38) and the two boundary stress tensors, $T^{\text{BY-in}}$ and $T^{\text{BY-out}}$, behave collectively as *pressureless dust*, with energy density

$$\rho_{\text{dust}} = \frac{1}{8\pi G_4 r_0}. \quad (40)$$

We may interpret this dust sprinkled on the boundary as determining a cosmic rest frame for Minkowski space with respect to which accelerated motion in the bulk is defined. The question "What does Newton's bucket spin with respect to in empty space?" finally has a Machian answer: "The bucket rotates with respect to dust on the surrounding Einstein static universe."

B. de Sitter space

Another space-time of historical significance for Mach's principle is de Sitter space. It was de Sitter's example of a closed, matter-free cosmological model that eventually led Einstein to abandon Mach's principle. Today de Sitter space is of obvious relevance for early-universe inflation and late-time cosmic acceleration.

Any pair of observers can have two-way communication only within a finite portion of the space-time, the so-called

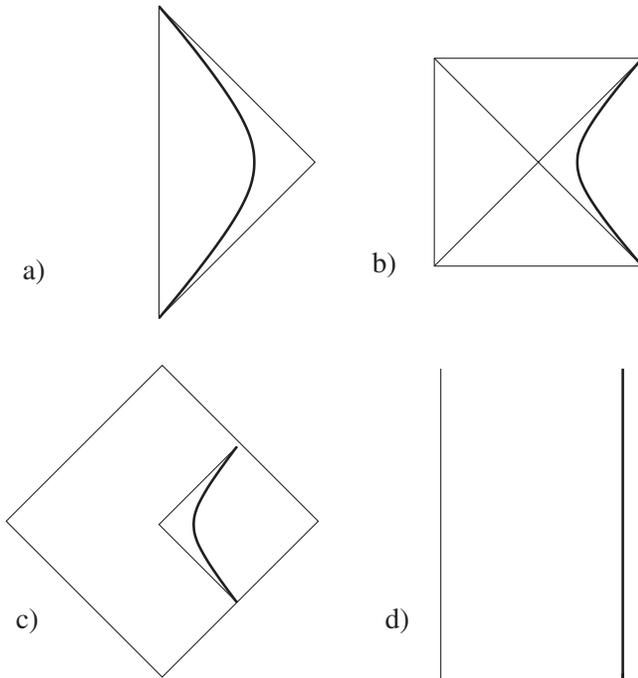


FIG. 1. Penrose diagrams illustrating our construction for a) Minkowski, b) de Sitter, c) Rindler, d) AdS spaces. The thick curve in each case represents the location of the boundary stress energy.

causal diamond. A natural location for our stress energy in this case is the boundary of this region, which is the causal horizon for this observer. (Proponents of the conjectured dS/CFT correspondence have instead focused on I^\pm [27]. While our construction is certainly formally applicable to that case as well, the physical interpretation of matter on a spacelike surface is a little unclear.)

Choosing a null boundary leads to some well-known complications, for example, the vector normal to the surface is also on the surface, and the volume element vanishes. Fortunately, the technical subtleties of dealing with a null surface can be circumvented by choosing a timelike surface hovering just inside the horizon, a *stretched horizon*, and then taking the limit in which it approaches the true null horizon. See Fig. 1(b). This is in precise analogy with the membrane paradigm for black holes [11]. (We continue to refer to this timelike surrogate horizon as the stretched horizon even though in de Sitter space, it should probably be called a shrunken horizon.) Unlike the black hole membrane paradigm, for which only those observers who remain outside the black hole see a membrane, here every observer has its own stretched horizon. The physical interpretation of the membrane in such a scenario probably involves some form of observer complementarity [28].

The metric in the causal diamond is given by

$$ds^2 = -(1 - \Lambda r^2)dt^2 + \frac{dr^2}{1 - \Lambda r^2} + r^2 d\Omega^2, \quad (41)$$

with our fiducial observer sitting at $r = 0$, and his causal horizon located at $r = \Lambda^{-1/2}$. The stretched horizon, on which boundary matter will be pasted, is defined as the timelike surface $r = r_0$ such that the lapse function satisfies

$$\alpha \equiv \sqrt{1 - \Lambda r_0^2} \ll 1. \quad (42)$$

At the end of the day we will be interested in the limit $\alpha \rightarrow 0$, in which the stretched horizon merges with the true horizon.

Substituting the extrinsic curvature components $K^t_t = \Lambda r_0/\alpha$ and $K^A_B = -\delta^A_B \alpha/r_0$ in (28) yields the following ‘‘in’’ stress tensor:

$$\begin{aligned} (T^{\text{BY-in}})^t_t &= -\frac{1}{8\pi G_4} \frac{2\alpha}{r_0} \equiv -\frac{\theta}{8\pi G_4}; \\ (T^{\text{BY-in}})^A_B &= \frac{\delta^A_B}{8\pi G_4} \left[\frac{\alpha}{r_0} + \frac{1}{\alpha r_0} \right] \equiv \frac{\delta^A_B}{8\pi G_4} \left[\frac{\theta}{2} + g \right], \end{aligned} \quad (43)$$

where θ is the expansion parameter for a congruence of radial null geodesics, while $g \equiv r_0^{-1} \alpha^{-1}$ is the proper surface gravity. Thus, exactly as in the membrane paradigm for black holes, we recognize the stress tensor of a Newtonian fluid with energy density $\rho = \theta/8\pi G_4$, pressure $P = g/8\pi G_4$, and bulk viscosity $\zeta = -1/16\pi G_4$. Moreover, this fluid satisfies a host of nonrelativistic equations, such as the Navier-Stokes equation, Ohm's law, and

Joule's law [10]. We see therefore that de Sitter horizons also possess a membrane interpretation.

The $1/\alpha$ divergence in the pressure can be canceled by counterterms with suitable choices of G_3 and λ in (24) to yield a finite $T^{\text{BY-out}}$. Substituting in (24) the above $T^{\text{BY-in}}$, as well as the boundary Einstein tensor with components $G^t_t = -1/r_0^2$ and $G^A_B = 0$, we find

$$\begin{aligned} (T^{\text{BY-out}})^t_t &= \frac{1}{8\pi G_3} \left(-\frac{1}{r_0^2} + \lambda \right) + \frac{1}{8\pi G_4} \frac{2\alpha}{r_0}; \\ (T^{\text{BY-out}})^A_B &= \delta^A_B \left[\frac{\lambda}{8\pi G_3} - \frac{1}{8\pi G_4} \left(\frac{\alpha}{r_0} + \frac{1}{\alpha r_0} \right) \right]. \end{aligned} \quad (44)$$

It is easily seen that the required cosmological term and gravitational constant are

$$\lambda = \frac{1}{r_0^2}; \quad \frac{G_3}{G_4} = \frac{\alpha}{r_0}. \quad (45)$$

Gravity therefore decouples in the limit that the stretched and true horizons merge, analogous to our Minkowski analysis.

As in Minkowski space, the boundary geometry is once again Einstein's static universe in $2 + 1$ dimensions. All sources on the boundary, including the cosmological term, therefore add up to an effective dust fluid with surface energy density

$$\rho_{\text{dust}} = \frac{1}{8\pi G_4 r_0 \alpha}. \quad (46)$$

This diverges as $\alpha \rightarrow 0$, since the dust fluid has infinite proper acceleration in this limit. The divergence is familiar from the black hole membrane paradigm for which the energy density on the membrane also diverges in the limit that the timelike stretched horizon approaches the null event horizon.

C. Rindler space

Our next example is Rindler space, describing a uniformly accelerated observer in Minkowski space. The Rindler trajectory starts at I^- and ends at I^+ , thereby defining a causal horizon. A natural location for our stress tensor is the stretched horizon hovering over this Rindler horizon, as shown in Fig. 1(c).

The construction is very similar to the previous example since the near-horizon geometry of de Sitter space is Rindler space. To see this explicitly, we introduce a new radial coordinate $z \equiv \alpha/\Lambda r$ and a dimensionless time $\tilde{t} \equiv \sqrt{\Lambda} t$, in terms of which the line element (41) takes the form

$$ds^2 = (1 + \Lambda z^2)^{-1} \left\{ -z^2 d\tilde{t}^2 + dz^2 + \frac{1}{\Lambda} d\Omega^2 \right\}. \quad (47)$$

In the limit $\Lambda \rightarrow 0$, the metric on the sphere becomes approximately that of a two-dimensional plane, and we have

$$ds^2 = -z^2 d\tilde{t}^2 + dz^2 + dx_i dx^i, \quad (48)$$

which describes Rindler space. In this coordinate system, the trajectory of the Rindler observer is $z = \text{constant}$, the causal horizon is at $z = 0$, and we will denote by $z_0 \ll 1$ the location of the stretched horizon.

In terms of the original coordinates, the above limit corresponds to taking $\Lambda \rightarrow 0$, $\alpha \rightarrow 0$, such that $\alpha/\sqrt{\Lambda}$ is finite. Thus we can easily obtain all of the desired quantities by taking this limit of our results for de Sitter space. For instance, we deduce that the ‘‘in’’ stress tensor has components $(T^{\text{BY-in}})^t_t = 0$ and $(T^{\text{BY-in}})^i_j = \delta^i_j/8\pi G_4 z_0$. Moreover, $\lambda \rightarrow 0$ and $G_3 \rightarrow 0$, but such that $\lambda/8\pi G_3 \rightarrow 1/8\pi G_4 z_0$. Once again one can think of the total stress energy plus the cosmological term as a boundary dust fluid with energy density

$$\rho_{\text{dust}} = \frac{1}{8\pi G_4 z_0}, \quad (49)$$

which, as before, diverges in the limit $z_0 \rightarrow 0$ where the stretched horizon merges with the true horizon.

D. Anti-de Sitter space

Next, we consider four-dimensional anti-de Sitter space. In global coordinates, the AdS₄ line element can be written as

$$ds^2 = -\left(1 + \frac{r^2}{\ell^2}\right) dt^2 + \left(1 + \frac{r^2}{\ell^2}\right)^{-1} dr^2 + r^2 d\Omega^2, \quad (50)$$

with ℓ as the AdS radius. A natural location for our boundary stress energy in this case is a large sphere at $r = r_0 \gg \ell$. See Fig. 1(d). From (28), the components of the ‘‘in’’ stress tensor are calculated straightforwardly:

$$\begin{aligned} (T^{\text{BY-in}})^t_t &= -\frac{1}{4\pi G_4 \ell} \sqrt{1 + \ell^2/r_0^2} \\ &\approx -\frac{1}{4\pi G_4 \ell} - \frac{\ell}{8\pi G_4 r_0^2}; \\ (T^{\text{BY-in}})^A_B &= -\frac{\delta^A_B}{8\pi G_4 \ell} \frac{2 + \ell^2/r_0^2}{\sqrt{1 + \ell^2/r_0^2}} \approx -\frac{\delta^A_B}{4\pi G_4 \ell}. \end{aligned} \quad (51)$$

The constant and r_0^{-2} terms both result in infrared divergences in global charges such as the AdS energy and must be regulated with appropriate counterterms. Since the boundary is once again $\mathbb{R} \times S^2$, the intrinsic Einstein equations (24) yields

$$\begin{aligned} (T^{\text{BY-out}})^t_t &= \frac{1}{8\pi G_3} \left(-\frac{1}{r_0^2} + \lambda\right) + \frac{1}{4\pi G_4 \ell} + \frac{\ell}{8\pi G_4 r_0^2}; \\ (T^{\text{BY-out}})^A_B &= \delta^A_B \left(\frac{\lambda}{8\pi G_3} + \frac{1}{4\pi G_4 \ell}\right). \end{aligned} \quad (52)$$

The requirement that the divergent terms drop out uniquely fixes the cosmological term and gravitational coupling

constant on the boundary

$$\lambda = -\frac{2}{\ell^2}; \quad G_3 = \frac{G_4}{\ell}, \quad (53)$$

which confirms (27) and is in perfect agreement with earlier calculations in AdS/CFT [24,26,29].

We regard the agreement with AdS/CFT as an important validation of our approach. And yet, at first sight, Mach’s principle seems to fly against one of the great achievements of string theory: the unification of particle physics with gravity. Indeed our proposal treats gravity and matter fields on different footings since, in the final analysis, gravity, unlike matter, is determined also by sources at the boundary. However, in the spirit of AdS/CFT, one can envision a generalization of Mach’s principle in which not just gravity, but all fields corresponding to closed string excitations have matter counterparts at the boundary. Indeed, our boundary stress tensors can be thought of as arising from an effective action obtained by integrating out both UV and IR degrees of freedom [30,31]. While in string theory, such a holographic duality has been realized only in certain backgrounds, in particular, in asymptotically AdS space-times, our construction provides a hint that it could be more generally true.

In the context of string theory, it is worth mentioning an intriguing manifestation of Mach’s principle in Hořava’s Cherns-Simons proposal for M theory [12]. Here, the emergence of macroscopic space-time requires turning on a large number of Wilson lines in the gauge theory. The inertia of a propagating excitation is then understood as arising from interactions with this background ‘‘matter.’’

E. Expanding universe

So far we have considered static vacuum space-times, with or without a cosmological constant. As an example of a more general space-time, we next apply our Machian prescription to a Friedmann-Robertson-Walker universe, a space-time which is both time dependent and has bulk matter extending to the boundary. Consider then a flat homogenous and isotropic universe in four dimensions:

$$ds^2 = -dt^2 + a^2(t)(dr^2 + r^2 d\Omega^2). \quad (54)$$

Instead of taking the Machian boundary to be at constant comoving coordinate r , suppose we take it to be at constant proper distance $R \equiv a(t)r$. Switching from r to R expresses the line element in Painlevé-type coordinates [32]:

$$\begin{aligned} ds^2 &= -(1 - H^2(t)R^2)dt^2 - 2H(t)Rdt dR + dR^2 \\ &\quad + R^2 d\Omega^2, \end{aligned} \quad (55)$$

where $H(t)$ is Hubble’s constant. In these coordinates, the bulk stress tensor is

$$\begin{aligned} T^t_t &= -\rho; & T^r_r &= 0; \\ T^R_R &= -H(t)R(\rho + p); & T^\theta_\theta &= T^\phi_\phi = p, \end{aligned} \quad (56)$$

where ρ and p are the energy density and pressure. A surface of constant R has an inward-pointing normal one-form given by $n_a = -(1 - H^2(t)R^2)^{-1/2} \delta_a^R$. Then one of the nonvanishing components of $T^{\text{BY-in}}$ is

$$(T^{\text{BY-in}})^t = \frac{1}{8\pi G} \frac{2}{R\sqrt{1 - H^2(t)R^2}}. \quad (57)$$

It is similarly straightforward to calculate $(T^{\text{BY-in}})^{AB}$ and $T^{\text{BY-out}}$. Now, because our surface is not fixed in comoving coordinates, bulk matter is continually swept out of the interior of the surface by the expansion of the universe. Energy-momentum is therefore exchanged between the boundary and the bulk matter. Then a nice check that our boundary stress tensors actually behave like stress tensors is found upon taking their divergence. While the ‘‘out’’ stress tensor turns out to be divergenceless, $D_\nu(T^{\text{BY-out}})^{\mu\nu} = 0$, we find, rather more interestingly, that

$$D_\alpha(T^{\text{BY-in}})^{\alpha} = -\frac{1}{8\pi G} \frac{2\dot{H}(t)H(t)R}{(1 - H^2(t)R^2)^{3/2}}. \quad (58)$$

Using (56) and the fact that $\rho + p = -\dot{H}(t)/4\pi G$, we can write this as

$$D_\alpha(T^{\text{BY-in}})^{\alpha} = -h^{t\gamma} n_\beta T_{\gamma\text{bulk}}^\beta = -h^{tt} n_R T_{R\text{bulk}}^R, \quad (59)$$

where $h^{\alpha\beta} = g^{\alpha\beta} - n^\alpha n^\beta$, so that $h^{tt} = -1/(1 - H^2(t)R^2)$. That is, the divergence of the Brown-York boundary stress tensor is related to the flux of outgoing bulk matter onto the boundary, exactly as one would expect if there were actually matter at the boundary; (59) is a continuity equation in which bulk matter feeds into boundary matter. We regard this as further evidence that the boundary stress tensor should be taken seriously as matter.

VI. RELATION TO CAUCHY PROBLEM

A physical theory is said to admit an initial-value formulation if, given appropriate initial data on some appropriate codimension one hypersurface, a solution to the dynamical equations exists and is unique. General relativity, like electromagnetism, is a constrained theory and so the data cannot be completely arbitrary. But given data that satisfy the constraints, general relativity has been shown to solve the initial-value or Cauchy problem provided the data are specified on a spacelike Cauchy surface or on a pair of intersecting null characteristic surfaces. Moreover, the initial-value problem in general relativity is well posed, meaning that it possesses two additional properties: changes in the data propagate causally, affecting only the solution in the Cauchy development of the part of the initial-value surface where the changes were made, and the solutions have a continuous dependence on the data, where continuity is defined with respect to some suitable choices of topology on the space of initial data and on the space of solutions. All of these statements have precise and careful mathematical formulations that are the subject of a

rich body of literature, but a general result is that there is no well-posed ‘‘initial’’ value formulation for data specified on a timelike boundary. This result makes it clear that, were one to specify arbitrary symmetric matrices on a timelike hypersurface as candidates for the boundary metric and extrinsic curvature, one would not in general be assured that a bulk solution exists, or that such a solution would be unique, even aside from issues of constraints, causality, or continuity. Does this fact pose problems for our Machian paradigm?

At issue here is whether our boundary stress tensors can be relied on to yield a unique, physical space-time in the bulk. We stress at the outset that this is a question of principle; in practice, the task of obtaining a bulk solution from boundary data is nontrivial even for spacelike data [33]. But, as a problem of principle, we shall see that concerns stemming from the Cauchy problem do not for the most part affect our Machian formulation. The reason, of course, is that the Machian boundary matter is not freely specified; rather, the boundary matter is read off from a bona fide bulk space-time. That is, since the boundary conditions that the boundary matter encodes are taken from a known solution to Einstein's equations, the existence aspect of the Cauchy problem is, by definition, guaranteed. By the same token, there is no danger of the constraints not being satisfied; since the constraints are satisfied in the bulk, they are also satisfied at the boundary. Nor does the timelike nature of the boundary lead to any difficulties. Whereas for freely-specified timelike data, there would be severe acausal consistency requirements. The Machian boundary matter leads to no acausal behavior unless the bulk space-time it is derived from is itself somehow causally pathological; indeed, in the membrane paradigm, the boundary conditions at the stretched horizon are also expressed as matter at a timelike boundary, but this leads to no acausality in the bulk since of course the boundary matter is derived from a physically well-behaved space-time. Finally, the question of Cauchy stability is also irrelevant for boundary matter unless it is freely specified. Thus we see that most of the issues revolving around the initial-value problem are not germane to our construction.

There does, however, remain one nontrivial issue: uniqueness. Are there multiple solutions of Einstein's equations with the same timelike boundary data? The answer depends in part on the global structure of the space-time. Indeed, this is true for spacelike data as well. More precisely, even for data specified on a spatial hypersurface, it may turn out that the space-time contains Cauchy horizons, in which case there are then multiple possible extensions of the solution in the region beyond the Cauchy horizon, all of which share the same initial data. It is perhaps worth pointing out that we do not discard general relativity as a theory, because it admits such globally nonhyperbolic solutions, nor even do we discard these space-times as unphysical; indeed, many well-known

space-times have Cauchy horizons, among them such familiar examples as Reissner-Nordström black holes, anti-de Sitter space, Taub-NUT space, and the Gödel universe. Note also that the eventual formation of a Cauchy horizon cannot be easily inferred from scrutinizing the initial data on a putative Cauchy surface. Rather, the presence or absence of such horizons requires knowledge of the global structure of the space-time and, in particular, the presence or absence of null geodesics that fail to intersect the surface of boundary data. But a rule of thumb is that manifolds with timelike boundaries or timelike singularities have Cauchy horizons, and we will see that similar limitations to uniqueness arise in the Machian context when “timelike” is replaced with “spacelike.”

Let us therefore turn to the question of uniqueness. We emphasize at the outset that we do not have a rigorous proof of uniqueness. Rather, we will identify some global conditions that are necessary, at least at the level of perturbative gravity, for uniqueness to hold. We will also provide some intuitive reasons for why, within perturbative gravity, these necessary conditions may also be sufficient. Consider, to begin with, a Ricci-flat space-time with no horizons and trivial topology, and assume that we have the corresponding boundary matter. We have already seen that the number of components of the Weyl tensor matches the number of degrees of freedom of the boundary matter. However, we can make a somewhat stronger argument. Suppose we perturb the space-time in such a way that linearized gravity is valid everywhere. If there were to exist gauge-independent perturbations that did not affect the boundary, then we would have a counterexample to uniqueness. But, at least at the linearized level, any perturbation can be regarded as a superposition of gravity waves. These gravity waves travel on null trajectories with respect to the background metric. Now, if a gravity wave intersects the timelike boundary, its amplitude and derivatives leave a corresponding perturbation on the boundary matter, via (6) and (7), and then (24) and (28).

Phrased this way, we see that *a necessary condition for uniqueness to hold at the linearized level is that every null geodesic must intersect the timelike boundary at least once*. If we choose our timelike boundary to skirt I^\pm , then it is clear that all null geodesics in Minkowski space that are interior to the boundary will intersect it. Hence Minkowski space, as well as all space-times that are gravitational perturbations of Minkowski space, have a unique correspondence with boundary matter, at least perturbatively. In other words, there is no way to deform Minkowski space by adding gravitational waves without causing a corresponding change in the boundary matter. Parallel arguments apply also to deformations of anti-de Sitter space.

Next, consider Ricci-flat space-times with nontrivial topology. Obviously, if there are multiple asymptotically flat regions, then we need timelike boundaries for each

such region. But suppose we have a black hole. One might have hoped that by choosing two timelike boundaries, one might be able to reconstruct the space-time of a maximally extended black hole. However, this is not so. Not all null geodesics move at 45° on a standard Penrose diagram; those with angular momentum do not. There are such null geodesics interior to the horizon that do not intersect either of the two timelike boundaries. For example, in four dimensions, there is a future-directed null trajectory that winds around the two-sphere up from the white hole singularity to the black hole singularity and which is parameterized by

$$\theta(r) = \arctan\left(\frac{\frac{G_4 M}{r} - 1}{\sqrt{\frac{2G_4 M}{r} - 1}}\right), \quad (60)$$

using standard Schwarzschild coordinates inside the horizon. A gravitational perturbation of the maximally extended black hole could move along this trajectory and fail to be picked up by the boundary matter; thus, uniqueness fails here. As this example indicates, when the boundary matter asymptotes I , the geometry in a region from which some future- or past-directed light rays do not reach I cannot be uniquely determined from the Machian matter. The boundary of such a region of course defines a future or past event horizon. Evidently, event horizons are to our boundary data what Cauchy horizons are to spacelike data.

In fact, the situation is more delicate than that. Consider a Schwarzschild black hole with mass M in D space-time dimensions with or without a cosmological constant Λ . Outside the horizon, a null orbit with angular momentum L is determined by nonrelativistic motion in the effective potential

$$V_{\text{eff}}(r) = \frac{1}{2} \left(1 - \frac{2G_D M}{r^{D-3}} - \frac{2\Lambda}{(D-1)(D-2)} r^2 \right) \frac{L^2}{r^2}. \quad (61)$$

Null geodesics with sufficiently low energy near the horizon are bounded in radius and never reach the boundary matter. Such trajectories emerge from the white hole horizon and reach a finite radius before falling into the black hole horizon. One way to deal with gravitational perturbations that trace such orbits is to introduce additional boundary matter enveloping the horizon. This matter of course is none other than a black hole membrane. We see then that, for black holes, the Machian prescription is not only inspired by, but even requires the membrane paradigm. (Here, the membrane paradigm has to be generalized slightly to include intrinsic gravity on the stretched horizon.)

Even so, there remains one ray that fails to intersect either the stretched horizon or the outside boundary matter. This is the solitary closed orbit at the critical radius

$$r_c = ((D-1)G_D M)^{1/(D-3)}. \quad (62)$$

A δ -function perturbation moving precisely on this un-

stable trajectory would fail to be picked up by either the matter at the stretched horizon or the Machian matter at the boundary. On a Penrose diagram, this null trajectory extends from i^- to i^+ . More generally, one can also consider multi-black hole space-times. A static configuration is given by the Majumdar-Papapetrou metric, which describes an asymptotically flat space-time with a number of extremal black holes whose mutual electrostatic repulsion precisely cancels their mutual gravitational attraction. For this space-time, our prescription to encode the geometry using matter calls for stretched horizons around each black hole as well as Machian matter surrounding the entire configuration. Nevertheless, there are an infinite number of closed orbits that are not imprinted in any of the fictitious matter [34].

As with the solitary circular orbit around a single Schwarzschild black hole, perturbations traveling on such null geodesics would not be picked up by any of the fictitious matter and, to that extent, uniqueness would fail. However, in the space of null trajectories, these trajectories constitute the separatrices between different basins of attraction, namely, for those trajectories that either fall into a black hole or escape to infinity. As phase boundaries, such trajectories are both unstable and of measure zero in the space of trajectories (though generally they may be of fractal dimension [35]). The latter fact means that, if gravitational perturbations are required to be smooth, as seems physically reasonable, then the missing set of measure zero delta-function deformations can be “filled in” by continuity and, in any case, would be unphysical. Similarly, we can perturb the Kerr black hole geometry. The classification of null geodesics in the Kerr metric is again the subject of a vast body of literature [36], but the upshot is that a stretched horizon around the black hole plus boundary matter near infinity captures all null geodesics except for a set of measure zero. That is, any sufficiently smooth deformation of the Kerr metric can be represented in terms of Machian matter.

So far we have identified some necessary conditions for uniqueness to hold. Now we will argue, without pretense of rigor, that these conditions might also be sufficient. In the perturbative regime, expanded around flat space, gravity is a linear theory satisfying the wave equation. The behavior of a wave everywhere is fixed by its amplitude and derivative at the boundary. But these are precisely what our boundary stress tensors encode. So the necessary conditions for the wave to reach the boundary should also be sufficient. Moreover, we know that the Wheeler-Feynman theory, in which electromagnetic waves are uniquely represented as boundary sources, is more or less equivalent to standard Maxwell theory, at least in topologically trivial space-times. But, in the perturbative regime, gravity is not so different from classical electromagnetism. So intuitively we might indeed expect, in this regime at least, that uniqueness holds.

Two further extensions can be made. First, we can introduce matter in the bulk. For example, if there is a star, then the external geometry would still be described by the Schwarzschild solution, leading to bounded null orbits, but there would be no stretched horizon to capture the perturbations on these orbits. However, this presumably creates no difficulties of principle as now there is bulk matter. Since all matter couples gravitationally, any gravitational wave arising out of or falling back on the star would have some effect on the bulk matter. Second, we can go beyond the perturbative discussion here. Here, it is difficult to say anything concrete. However, intuitively we expect that any deformation of a given geometry will have some energy. But in general relativity, energy is determined by a surface integral at infinity. Thus it would seem that any deformation of the geometry would be picked up by surface integrals at infinity [37]. Moreover, unless there is some other symmetry (such as spherical symmetry), the deformations will generically not be degenerate. So any deformation should leave a unique imprint at the boundary. Even if there is additional symmetry, then the charge that generates the symmetry will also generically be expressible as a surface term. It would be interesting to make these statements more rigorous. But, in any case, our perturbative discussion suggests that for regions outside event horizons, all of the commonly studied space-times are uniquely connected to boundary matter, at least locally in the space of solutions.

VII. COMPARISON WITH OTHER MACHIAN PROPOSALS

There have been numerous previous attempts to reconcile Machian ideas with a relativistic theory of gravity. For the purpose of contrasting our framework with existing ideas, we will focus our attention on a few key proposals. In terms of strategy, these have either i) imposed a selection rule to remove unwanted solutions to the Einstein equations, based on some criterion of Machianity, or ii) sought an alternative to Einstein's theory with the hope of fulfilling some Machian expectations.

The classic example of a “selection rule,” proposed by Einstein himself and later pursued by Wheeler [38], is the requirement that the universe have closed spatial topology. This condition removes the need for spatial boundary or asymptotic conditions on the three metric. Moreover, there is no net global charge in a closed universe, thereby fulfilling another Machian expectation. The prime example of such a universe is Einstein's static universe with global topology $\mathbb{R} \times S^3$.

The Einstein-Wheeler universe has been the subject of ample literature, from which we draw three main objections. First, in a technical sense the need for boundary data is not entirely obviated, since a unique solution to the Einstein equations still requires some extra data specified, e.g., on an initial surface. Intuitively this is because, even in

a spatially closed universe, one can still add a generic superposition of standing gravity waves, each satisfying the appropriate closed boundary conditions; the effect of compactifying spatial dimensions is to discretize the wave numbers of gravity waves, not to eliminate them altogether. Second, there is potentially a problem with causality [39,40]. Since the radius of curvature of our Universe is much larger than the horizon size, as indicated by cosmological observations, how can the local notion of inertia be determined by conditions beyond the observable universe? A third objection, which inevitably afflicts any selection rule approach, is that it is nothing more than an *ad hoc* patch. For instance, the Einstein-Wheeler prescription forbids Minkowski space since its spatial slices are noncompact, to the relief of Machian proponents, but permits $\mathbb{R} \times T^3$ with an arbitrarily large torus.

The second approach gives up on the idea that Machianity is somehow realized within the framework of general relativity and proposes instead some modified theory of gravity. The poster-child example in this category, and the theory perhaps best motivated by subsequent developments in particle physics, is the scalar-tensor theory of Brans and Dicke [41]. At its core is the principle that a Machian universe must satisfy

$$\frac{G_4 M}{R} \sim 1, \quad (63)$$

where, heuristically, M and R are the mass and radius of the observable universe. This relation is deduced from inertial induction arguments [42,43]—essentially the requirement that accelerating an observer with respect to the distant stars be physically equivalent to accelerating all of the other matter in the universe. For instance, in the Lense-Thirring effect [44–46], inertial frames inside a rotating shell of mass M and radius R are dragged at the same angular velocity as the shell in the limit where (63) is satisfied.

One can view (63) as a constraint on the matter of the universe, requiring special initial or boundary conditions. From a modern perspective, (63) follows from the near flatness of our Universe, which in turn traces back to early-universe inflation, i.e., to initial conditions. The key insight of Brans and Dicke is that such a relation can be dynamically satisfied if Newton’s constant G_4 is time dependent, continuously adjusting its value according to the matter content of the universe. But while (63) may indeed be an essential property of a Machian universe, at the end of the day Brans-Dicke theory does not escape the need for boundary/initial data. In Einstein frame, the Brans-Dicke scalar field governing the space-time evolution of G_4 is merely an extra matter field coupled to gravity, and gravity still requires boundary data. Indeed, any gravitational theory that asks for solutions to a differential equation cannot avoid the problem of boundary conditions.

At a spiritual level, our proposal is perhaps closest to the boundary matter contemplated by Einstein for some time

in 1916 [47,48]. The idea is to suppose that the metric at large distances from the matter distribution assumes the degenerate form

$$g_{\mu 0} \rightarrow \infty; \quad g_{ij} \rightarrow 0. \quad (64)$$

This form of the metric, being invariant under coordinate transformations to arbitrarily accelerated coordinate frames (keeping t fixed), ensures the asymptotic equivalence of all observers [49]. Einstein reasoned that the matter distribution responsible for the transition from a nearly flat metric in our neighborhood to the above degenerate form cannot be the distant stars, for otherwise the large change in the gravitational field would lead to unacceptably large redshift of their spectra. Instead, it must be attributed to unseen boundary matter beyond the matter distribution. However, the existence of such dark matter seemed unappealing at the time, and Einstein dropped this idea in favor of his closed cosmological model.

VIII. SUMMARY AND OUTLOOK

Any successful implementation of Mach’s principle must hinge on an understanding of boundary conditions, as emphasized by Feynman in his lectures on gravitation.

“Thus it appears to me that we might learn whether Mach’s principle is consistent with our present theory by studying the meaning of the boundary conditions [50].”

Akin to the Wheeler-Feynman theory of electromagnetism, we have argued that Machian ideas may be realized within Einstein’s general theory of relativity by relating the space-time metric (and its inertial frames) to bulk and boundary sources on a timelike surface. Given a space-time, our construction allows us to read off boundary stress tensors that are interpreted as giving rise to the bulk geometry. In concrete terms, this boundary matter plays the role of the distant stars for bulk observers and heuristically selects a frame with respect to which inertial and accelerated motion have meaning.

The reverse problem—specifying boundary stress tensors and integrating the bulk geometry—is in general ill posed since this involves specifying data on a timelike surface. Our proposal circumvents this by determining boundary stress tensors from a bona fide bulk geometry. Whether this well-defined boundary data corresponds to a unique bulk space-time remains an open issue. But we have proposed necessary global conditions that a space-time must satisfy in order for uniqueness to hold.

Our specific proposal is validated by consistency with several other prominent themes in gravity ranging from the membrane paradigm to the idea of a boundary stress tensor for gravitational energy to holography. It satisfies many of the properties that have previously been considered under the rubric of Mach’s principle—distant matter as the source of inertial frames, the absence of global angular momentum, and the relativity of all motion—and it does so

in the most conservative way, without harming either Einstein's equations or their solutions.

Our framework for implementing Mach's principle in general relativity opens many unexplored avenues for further research.

(i) *Total relativity*

Specifying bulk and boundary stress energy singles out a metric and, through it, a set of preferred world-lines, namely, the inertial ones. This is, loosely speaking, reminiscent of spontaneous symmetry breaking in field theory and hints at a larger symmetry group underlying a complete relativization of motion [51]. Specifically, while a particular spacetime is only diffeomorphism invariant, the action might be invariant under additional transformations that connect related spaces, such as rotating geometries with different total angular momenta. Our framework suggests that the boundary is the natural location for such hidden symmetries. There are indeed examples in general relativity of transformations that leave the asymptotic geometry invariant; it is tempting to speculate that the so-called Bondi-Metzner-Sachs and Spi groups for asymptotically flat spaces might be relevant here.

(ii) *Mach's principle in 2 + 1 dimensions*

Something intriguing happens in three dimensions: Mach's principle automatically holds. In three dimensions, there are no gravitational waves, the Weyl tensor vanishes identically, and the Riemann tensor is completely determined by the Ricci tensor. Moreover, there is no freedom to choose gauge-invariant physical boundary conditions with which to solve the Einstein equations. Hence our interpretation of Einstein's 1918 version of Mach's principle already holds in 2 + 1 dimensional gravity. Nevertheless, there exist different types of geometries with the same bulk stress tensor. For example, besides AdS₃, there are also Bañados-Teitelboim-Zanelli black holes of various masses and angular momenta, all obtainable via identifications on the universal covering space of AdS. It may be worthwhile to study these in the context of our realization of Mach's principle.

(iii) *Observational tests*

We described in detail the application of our formalism to de Sitter space. Given the mounting evidence for a small cosmological constant, it is likely that our Universe will asymptote to de Sitter space. It is therefore imperative to study potential

observable consequences of the boundary-matter description; after all, the black hole membrane paradigm was designed to perfectly mimic the observations of a specific kind of observer.

(iv) *Implications for frame dragging*

The dragging of inertial frames due to a rotating source is often hailed as the quintessential Machian signature in general relativity. While an important and testable prediction of general relativity [52], this effect generically comes up short of achieving the complete relativity of motion. For instance, inertial frames inside a rotating mass shell are dragged with angular velocity smaller than that of the shell unless the mass and radius of the shell become of the same order, corresponding roughly speaking to a black hole [44,45]. In our framework, this generally incomplete dragging is to be expected, because the distant boundary matter also influences the geometry within the shell. It would be interesting to study the relative contributions of the shell and the boundary matter in determining inertial frames within the shell, to understand, in particular, why complete dragging is achieved in the black hole limit. Similarly, our framework should also be applied to the cosmological case [39] to understand what has been referred to as Mach0 [53]—why the distant matter distribution in our Universe does not rotate relative to local inertial frames.

To conclude, we are hopeful that, just as the membrane paradigm was valuable in the study of black holes, so our boundary approach to Mach's principle may prove fruitful to better understand some of the foundations of general relativity.

ACKNOWLEDGMENTS

Our gratitude goes to B. Greene, P.J.E. Peebles, L. Smolin, A. Starinets, E. Verlinde, and F. Wilczek for many insightful discussions. We are indebted to the referees for a stimulating exchange and for many helpful suggestions that have improved this manuscript. The research of J. K. at the Perimeter Institute is supported in part by the Government of Canada through NSERC and by the Province of Ontario through MEDT. The research of M.P. at Columbia University was supported in part by DOE Grant No. DE-FG02-92ER40699. The authors also acknowledge support of Foundational Questions Institute (FQXi) Grant No. RFP1-06-20.

- [1] E. Mach, *The Science of Mechanics: A Critical and Historical Account of Its Development (1893)*, translated by T.J. McCormach (Open Court, LaSalle, IL, 1960).
- [2] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity* (John Wiley and Sons, New York, 1972).
- [3] A. Einstein, Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.) **1914**, 1030 (1914).
- [4] A. Einstein, Ann. Phys. (Leipzig) **55**, 241 (1918).
- [5] *Mach's Principle: From Newton's Bucket to Quantum Gravity*, edited by J. Barbour and H. Pfister (Birkhauser, Cambridge, United Kingdom, 1995).
- [6] J.A. Wheeler, *MTW Reprint Supplement* (Princeton University, Princeton, NJ, 1974).
- [7] P. Hořava and E. Witten, Nucl. Phys. **B475**, 94 (1996).
- [8] See, e.g., O. Aharony, S.S. Gubser, J.M. Maldacena, H. Ooguri, and Y. Oz, Phys. Rep. **323**, 183 (2000).
- [9] J.D. Brown and J.W. York, Jr., Phys. Rev. D **47**, 1407 (1993).
- [10] *Black Holes: The Membrane Paradigm*, edited by K.S. Thorne, R.H. Price, and D.A. Macdonald (Yale University Press, New Haven, CT, 1986).
- [11] M.K. Parikh and F. Wilczek, Phys. Rev. D **58**, 064011 (1998).
- [12] P. Hořava, Phys. Rev. D **59**, 046004 (1999).
- [13] M. Porrati and A. Starinets, Phys. Lett. B **454**, 77 (1999).
- [14] V. Balasubramanian and P. Kraus, Phys. Rev. Lett. **83**, 3605 (1999).
- [15] J. de Boer, E.P. Verlinde, and H.L. Verlinde, J. High Energy Phys. **08** (2000) 003.
- [16] R.D. Blandford and R.L. Znajek, Mon. Not. R. Astron. Soc. **179**, 433 (1977).
- [17] L. Susskind, L. Thorlacius, and J. Uglum, Phys. Rev. D **48**, 3743 (1993).
- [18] G. 't Hooft, Nucl. Phys. **B335**, 138 (1990).
- [19] R. Penrose and W. Rindler, *Spinors and Space-time* (Cambridge University Press, Cambridge, United Kingdom, 1984).
- [20] S.W. Hawking and G.F.R. Ellis, *The Large Scale Structure of Space-time* (Cambridge University Press, Cambridge, United Kingdom, 1973).
- [21] J.A. Wheeler and R.P. Feynman, Rev. Mod. Phys. **17**, 157 (1945).
- [22] J.A. Wheeler and R.P. Feynman, Rev. Mod. Phys. **21**, 425 (1949).
- [23] R.K. Sachs, J. Math. Phys. (N.Y.) **3**, 908 (1962).
- [24] S.S. Gubser, Phys. Rev. D **63**, 084017 (2001).
- [25] T. Jacobson and R. Parentani, Found. Phys. **33**, 323 (2003).
- [26] S. de Haro, K. Skenderis, and S.N. Solodukhin, Classical Quantum Gravity **18**, 3171 (2001).
- [27] A. Strominger, J. High Energy Phys. **10** (2001) 034.
- [28] M.K. Parikh, I. Savonije, and E.P. Verlinde, Phys. Rev. D **67**, 064005 (2003).
- [29] V. Balasubramanian and P. Kraus, Commun. Math. Phys. **208**, 413 (1999).
- [30] J. Khoury and H.L. Verlinde, Adv. Theor. Math. Phys. **3**, 1893 (1999).
- [31] E.P. Verlinde and H.L. Verlinde, J. High Energy Phys. **05** (2000) 034.
- [32] M.K. Parikh, Phys. Lett. B **546**, 189 (2002).
- [33] N.O. Murchadha and J.W. York, Jr., Phys. Rev. D **10**, 428 (1974).
- [34] S. Chandrasekhar, Proc. R. Soc. A **421**, 227 (1989).
- [35] C.P. Dettmann, N.E. Frankel, and N.J. Cornish, Fractals **3**, 161 (1995).
- [36] S. Chandrasekhar, *The Mathematical Theory of Black Holes* (Oxford University Press, New York, 1992).
- [37] V. Balasubramanian, D. Marolf, and M. Rozali, Gen. Relativ. Gravit. **38**, 1529 (2006); Int. J. Mod. Phys. D **15**, 2285 (2006).
- [38] J.A. Wheeler, in *Relativity, Groups and Topology*, edited by C. DeWitt and B. DeWitt (Gordon and Breach, New York, 1964).
- [39] C. Schmid, Phys. Rev. D **74**, 044031 (2006).
- [40] J. Bičák, D. Lynden-Bell, and J. Katz, Phys. Rev. D **69**, 064011 (2004).
- [41] C. Brans and R.H. Dicke, Phys. Rev. **124**, 925 (1961).
- [42] D.W. Sciama, Mon. Not. R. Astron. Soc. **113**, 34 (1953).
- [43] R.H. Dicke, Science **129**, 621 (1959).
- [44] J. Lense and H. Thirring, Phys. Z. **19**, 156 (1918).
- [45] D.R. Brill and J.M. Cohen, Phys. Rev. **143**, 1011 (1966).
- [46] L. Lindblom and D.R. Brill, Phys. Rev. D **10**, 3151 (1974).
- [47] W. de Sitter, Proceedings of the Section of Sciences, Koninklijke Akademie van Wetenschappen **19**, 527 (1916).
- [48] A. Einstein, Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.) **1917**, 142 (1917).
- [49] P.J.E. Peebles, *Principles of Physical Cosmology* (Princeton University Press, Princeton, NJ, 1993).
- [50] R.P. Feynman, F.B. Moringo, and W.G. Wagner, *Feynman Lectures on Gravitation* (Addison-Wesley, Reading, MA, 1995).
- [51] F. Wilczek, Phys. Today **57**, No. 04, 10 (2004).
- [52] *Gravity Probe B* experiment, see <http://einstein.stanford.edu>.
- [53] H. Bondi and J. Samuel, arXiv:gr-qc/9607009.