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WORKSPACE DELINEATION OF CABLE-ACTUATED PARALLEL MANIPULATORS

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ABSTRACT

While there is extensive literature available on parallel manipulators in general, there has been much less attention given to cable-driven parallel manipulators. In this paper, we address the problem of analyzing the reachable workspace using the tools of semi-definite programming. We build on earlier work [1,2] done using similar techniques by deriving limiting conditions that allow us to compute analytic expressions for the boundary of the reachable workspace. We illustrate this computation for a planar parallel manipulator with four actuators.

INTRODUCTION

There is extensive literature on parallel manipulators going back to the early work on the Cauchy-Gough-Stewart platform [3, 4]. The book by Merlet [5] provides an excellent review. Much of this literature addresses the kinematic geometry of parallel manipulators and the kinematics of closed chain mechanisms. However, the design of cable driven platforms has received considerably less attention despite their attractive features such as scalability.

Albus *et al* developed a six degree-of-freedom robot crane design based around a parallel platform driven by cables [6]. Roberts *et al* addressed the kinematic analysis of such systems by presenting a numerical approach for testing whether a given configuration lies within the reachable workspace [1]. Oh and Agrawal used a similar numerical approach to develop and test a numerically optimized controller for planning and executing movements [2]. Takeda and Funabashi addressed the question of synthesizing such mechanisms to optimize force transmission characteristics [7].

In this paper, we build on this work by explicitly delineating the reachable workspace for a cable-driven parallel platform. Unlike [1, 2], in which the workspace is numerically computed, we derive limiting conditions that allow us to compute analytical expressions for the boundary of the reachable workspace, similar in intent to the general approach used by Husty [8] when solving the forward kinematics problem for the Cauchy-Gough-Stewart platform. We illustrate this computation for a planar parallel manipulator with four actuators.

It is worth noting that the kinematic analysis of platforms driven by cables is similar to the analysis of multifingered manipulation in the robotics literature [9,10]. The equilibrium equations with inequalities on cable tensions are similar to the equations of equilibrium for the grasped object with constraints on finger forces. The static and kinematic analysis of such systems can be reduced to semi-definite programs [11]. A survey of some of the recent work in semi-definite programming is available in [12].

In this paper, we first describe the planar parallel manipulator geometry used in our example application. In the next section we approach the analysis of cable-driven parallel platforms using the tools of semi-definite programming and give an algorithmic

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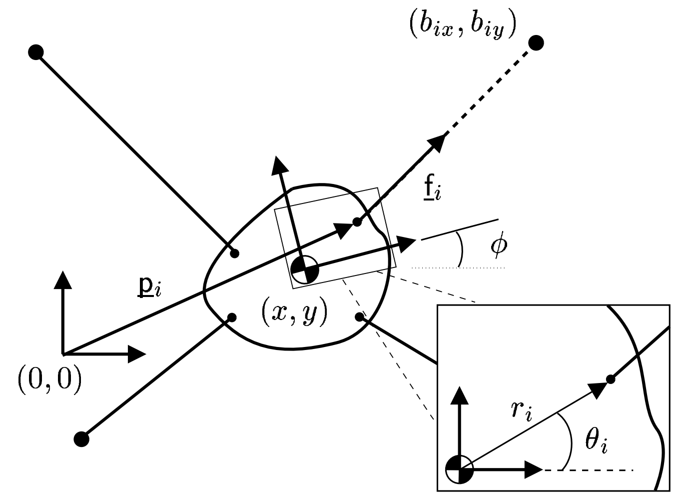

Figure 1. GENERAL PLANAR PLATFORM GEOMETRY USED AS AN EXAMPLE.

procedure for defining the boundary of the reachable workspace. Finally, we conclude by applying the technique to a four-cable planar parallel manipulator and demonstrating the flexibility of our analytic approach.

MODELING

Consider a planar platform held in place by four cables, as shown in Fig. 1.

The equations of motion for this system can be written as one matrix equation:

$$\begin{bmatrix} \hat{w}_1 & \hat{w}_2 & \hat{w}_3 & \hat{w}_4 \end{bmatrix}_{3x4} \underline{T}_{4x1} = \underline{b}_{3x1} \tag{1}$$

where the $\hat{w}_i$ are unit-length wrench vectors describing the effect of the forces and calculated about the spatial origin ($(0,0)$ in Fig. 1). The vector $\underline{b}$ represents dynamic terms or external forces.

The exact form of a wrench vector for the system in Fig. 1 is found as:

$$\hat{w}_i = \frac{1}{N_i}\underline{w}_i = \frac{1}{N_i}\begin{pmatrix} b_{ix} - x - r_i\cos(\phi+\theta_i) \\ b_{iy} - y - r_i\sin(\phi+\theta_i) \\ b_{ix}\left(r_i\cos(\phi+\theta_i)+x\right) - b_{iy}\left(r_i\sin(\phi+\theta_i)+y\right) \end{pmatrix} \tag{2}$$

where the normalizing term is:

$$N_i = \sqrt{\begin{matrix} (b_{ix}-x)^2 + (b_{iy}-y)^2 + r_i^2 \ldots \\ -2r_i\left[(b_{ix}-x)\cos(\phi+\theta_i) + (b_{iy}-y)\sin(\phi+\theta_i)\right] \end{matrix}} \tag{3}$$

An equivalent problem setup can be performed for a spatial platform as well; spatial wrench vectors will have three translational and three rotational components.

If cables are added to the system, additional columns will be added to the wrench matrix and the tension vector will scale accordingly.

PROBLEM SETUP/THEORY

Unfortunately, the solution to Eqn. 1 is complicated by one important facet of this problem: only positive tensions are possible. The updated problem statement is:

$$W\underline{T} = \underline{b},\ \underline{T} \succeq 0 \tag{4}$$

where the symbol $\succeq$ denotes that $t_i \geq 0$ for each component t_i of $\underline{T}$. (Likewise, the symbol $\succ$ would denote that $t_i > 0$ for each component of $\underline{T}$.)

The possibility exists that there is no solution to this problem, but guidance is available in the form of Farkas's Lemma [13]:

Farkas's Lemma: *The system $W\underline{T} = \underline{b}$, $\underline{T} \succeq 0$ has no solution if and only if the system $W^T\underline{q} \succeq 0$, $\underline{b}^T\underline{q} < 0$ has a solution.*

The geometric interpretation of this lemma is to consider $\underline{q}$ as the normal vector of a separating hyperplane. This hyperplane separates the convex hull formed by positive combinations of the wrench vectors that make up W from the point defined by $\underline{b}$. Intuitively, this condition makes sense; the convex hull formed by the wrench vectors is the space of all possible vectors that could be formed by $W\underline{T}$, $\underline{T} \succeq 0$. If $\underline{b}$ does not belong to this set, as demonstrated by the separating hyperplane, then no solution can exist.

Note that this separation is strict; if $\underline{b}$ were to lie on the hyperplane, then $\underline{b}^T\underline{q} = 0$ and a solution to the first system exists according to the lemma.

In order to find a solution to Eqn. 4, we must be able to demonstrate that no separating hyperplane exists according to Farkas's Lemma. This can be accomplished using a procedure taken from Grassmann geometry [14] as applied to cable/platform systems by Roberts *et al* [1]. It goes as follows:

1. Extract from W the set of m wrench vectors in $\mathbb{R}^n$. A candidate hyperplane can be formed from $n-1$ linearly independent vectors. For vectors in $\mathbb{R}^3$, the normal is given by

$\underline{q} = \underline{w}_i \times \underline{w}_j$; in higher dimensions this normal is given by:

$$\underline{q} = \det \begin{bmatrix} \underline{e}_1 & \cdots & \underline{e}_n \\ \underline{w}_{1,1} & \cdots & \underline{w}_{1,n} \\ \vdots & \ddots & \vdots \\ \underline{w}_{(n-1),1} & \cdots & \underline{w}_{(n-1),n} \end{bmatrix} \quad (5)$$

where $\underline{e}_i$ are the canonical basis vectors, and $\underline{w}_{ij}$ is the i^{th} component of the j^{th} wrench vector.

2. Verify that the remaining wrench vectors lie on one side of the hyperplane while the forcing vector $\underline{b}$ lies on the other. This is accomplished by verifying that $sign\left(\underline{w}_i^T \underline{q}\right) \neq sign\left(\underline{b}^T \underline{q}\right)$ for all remaining $\underline{w}_i$ not used in forming the hyperplane. For vectors in $\mathbb{R}^3$ this procedure involves checking that $sign\left(\left(\underline{w}_i \times \underline{w}_j\right) \cdot \underline{w}_k\right) \neq sign\left(\left(\underline{w}_i \times \underline{w}_j\right) \cdot \underline{b}\right)$ for all $k \neq i, j$, or, equivalently, that $sign\left(\det\left[\underline{w}_i \ \underline{w}_j \ \underline{w}_k\right]\right) \neq sign\left(\det\left[\underline{w}_i \ \underline{w}_j \ \underline{w}_k\right]\right)$. Similarly for higher dimensions, this test becomes $sign\left(\det\left[\underline{w}_1 \cdots \underline{w}_{n-1} \ \underline{w}_k\right]\right) \neq sign\left(\det\left[\underline{w}_1 \cdots \underline{w}_{n-1} \ \underline{b}\right]\right)$.
3. Assuming that $\underline{q}$ was not a separating hyperplane, choose another set of wrench vectors and repeat step 2. There will be $\binom{m}{n-1}$ sets to form planes from and then $m-n+2$ determinants to test ($m-n+1$ other wrench vectors plus the forcing vector $\underline{b}$).

Assuming that none of the hyperplanes tested were separating hyperplanes, then Farkas's Lemma tells us that Eqn. 4 has a solution.

This procedure can be carried out using analytic forms of the wrench vectors, such as those given in Eqn. 2, or carried out directly using a numeric linear programming solver such as *linprog* in MATLAB.

The more useful analysis is to consider what configurations lie on the boundary between solvable and unsolvable systems. As mentioned before, this will occur when $\underline{b}$ lies on the hyperplane given by $\underline{q}$ since the system has just barely passed the criterion for solvability. In the context of this hyperplane test, the hyperplane is formed using vectors from W, and so $\underline{b}$ must be coplanar with these vectors. This corresponds to two things happening: all of the wrench vectors not used in forming the hyperplane lie *on one side of* the hyperplane and $\underline{b}$ lies *on* the hyperplane (causing the matrix determinant involving $\underline{b}$ to evaluate to zero). Since all of the wrench vectors must non-negatively combine to form $\underline{b}$, all of the wrench vectors not in the hyperplane must have zero magnitude. Therefore, the boundaries of the valid configuration space will correspond to the case when one or more of the tensions in the cables becomes zero and $\underline{b}$ is coplanar with the remaining wrench vectors.

Now consider the case when $\underline{b} = \underline{0}$. This could arise in the planar platform case when gravity is orthogonal to the motion of the platform. The problem now looks like:

$$W\underline{T} = 0, \underline{T} \succeq 0 \quad (6)$$

Now, the second system in Farkas's Lemma will never have a solution because $\underline{b}^T \underline{q} = \underline{0} \not< 0$. Therefore Eqn. 6 always has a solution. Unfortunately, this can always be the trivial solution $\underline{T} = \underline{0}$. Practically speaking, this is unacceptable since it is impossible to move a platform from one point to another with zero tension. Consider a restatement of the problem:

$$W\underline{T} = \underline{0}, \underline{T} \succ 0 \quad (7)$$

To handle this case, we use Stiemke's theorem [15]:

Stiemke's Theorem: *The system $W\underline{T} = 0, \underline{T} \succ 0$ has no solution if and only if the system $W^T \underline{q} \succeq 0, W^T \underline{q} \neq \underline{0}$ has a solution.*

Again, the geometric interpretation is to consider $\underline{q}$ as the normal vector of a hyperplane, but in this case, the hyperplane is a supporting hyperplane. This means that the convex hull lies entirely on one side of the plane. The same procedure given previously will work, but now the goal is to see if all of the vectors not used to form the candidate hyperplane lie on one side of the hyperplane. If this is the case, then the hyperplane is supporting and Eqn. 7 has no solution.

Similarly to the conditions that defined the boundary of solvability for Eqn. 4, the (open) boundary in this problem will occur when the system barely fails the test of solveability. If several wrenches lie on the supporting hyperplane but one or more lie together on one side, then setting the magnitudes of those wrenches to zero might still yield a non-negative tension solution where the tension is not identically zero. Such a solution will not satisfy Eqn. 7, but still provides an acceptable solution to Eqn. 6 and thus could be considered as the boundary. Just as before, this boundary will correspond to one or more tensions being set to zero and then solving the reduced system.

This strict-solution result is important because it represents a sufficient condition for the existence of a solution to the general problem represented by Eqn. 4. To see this, consider the form of a general solution to $W\underline{T} = \underline{b}$:

$$\underline{T} = \underline{T}_p + N\underline{c} \quad (8)$$

where $\underline{T}_p$ is the particular (least-squares) solution and N is the matrix of null vectors of W. If Eqn. 7 has a solution, then this solution represents a vector in the null space of and can be expressed by $N\underline{c}$ using a multiple of some unit $\underline{c}$. Now, since this vector is strictly positive, it can be arbitrarily scaled so that any negative components of $\underline{T}_p$ are canceled out by the addition. The desired solution $\underline{T} \succeq 0$ is then obtained.

Since the existence of a solution to Eqn. 7 guarantees that Eqn. 4 has a solution, the forcing term $\underline{b}$ is free to vary and the platform can resist arbitrary wrenches.

In order to give a physical significance to all of this theory, consider the consequences of Farkas's Lemma and Stiemke's Theorem with the interpretation of the vector $\underline{q}$ as a twist. This is entirely appropriate because all of this work has dealt with wrenches, and wrenches and twists are complementary [9]. For a wrench $\underline{w}$ and twist $\underline{q}$, the quantity $\underline{w}^T\underline{q}$ represents the work done by the wrench for the motion described by the twist. Therefore the condition $W^T\underline{q} \succeq 0$ says that there is a motion described by a twist, $\underline{q}$, such that the work done by all of the wrenches is non-negative. Likewise, $\underline{b}^T\underline{q} < 0$ says that the work done by the forcing wrench for this same motion is strictly negative. The work done cannot be both positive and negative and so the original system has no solution. For the Stiemke's Theorem conditions, if $W^T\underline{q} \succeq 0$ and at least one $\underline{w}_i^T\underline{q} \neq 0$, then there is a motion such that the system of wrenches does positive work. But since there is no forcing wrench, the total work done must be zero in order for a solution to exist.

APPLICATION TO THE PLANAR PARALLEL MANIPULATOR

In order to illustrate the use of Farkas's Lemma and Stiemke's Theorem, we now present an example using the planar platform geometry introduced earlier. Beginning with an analytic form for the wrenches, as given by Eqn. 2, we apply the separating hyperplane search procedure to form a set of analytic conditions that a platform configuration must satisfy in order for a positive tension solution to exist. Using these, we generate plots to visualize the valid configuration space.

Our example platform has four cables and therefore four wrenches to deal with. If we can use Stiemke's Theorem to prove that a solution exists to Eqn. 7, then we know that a solution exists to the general problem given by Eqn. 4. This proof requires that no supporting hyperplane can be found or, equivalently, that every hyperplane is separating.

For the planar platform geometry, each wrench looks like a vector in $\mathbb{R}^3$, and we can form planes using pairs of vectors. We can form $\binom{4}{2} = 6$ different planes and then test $4-3+1=2$ other wrench vectors against each plane to determine if the plane is separating or not. For each plane, two determinants can be formed, and the signs of the determinants must be opposite if the plane is separating. This leads to six comparisons, but some of these are redundant. Now, we define four determinants, one for each combination of three wrench vectors:

$$\begin{array}{l} D_4 = \det\left[\underline{w}_1\ \underline{w}_2\ \underline{w}_3\right] \\ D_3 = \det\left[\underline{w}_1\ \underline{w}_2\ \underline{w}_4\right] \\ D_2 = \det\left[\underline{w}_1\ \underline{w}_3\ \underline{w}_4\right] \\ D_1 = \det\left[\underline{w}_2\ \underline{w}_3\ \underline{w}_4\right] \end{array} \tag{9}$$

There are two things to note: first, the subscript of D tells which wrench has been dropped to form this submatrix of W; second, these wrench vectors are not normalized. Since the important information we are looking for is the relative directions of the vectors, all we care about is the sign of the determinants; the magnitude is irrelevant and can be ignored. This greatly simplifies the analytic form of each determinant.

Now recall that swapping columns inside the determinants has the effect of changing the sign of the determinant. Using this fact, the full enumeration of the six tests is summarized in Tab. 1 and then rewritten using the defined determinants. It is easy to verify that these six tests are self-consistent.

Table 1. SUMMARY OF THE DETERMINANT SIGN TEST AND EQUIVALENT REPRESENTATION

$sign(\det[A]) \neq sign(\det[B])$		$\Rightarrow$ $sign(A) \neq sign(B)$	
$\underline{w}_1, \underline{w}_2, \underline{w}_3$	$\underline{w}_1, \underline{w}_2, \underline{w}_4$	D_4	D_3
$\underline{w}_1, \underline{w}_3, \underline{w}_2$	$\underline{w}_1, \underline{w}_3, \underline{w}_4$	$-D_4$	D_2
$\underline{w}_1, \underline{w}_4, \underline{w}_2$	$\underline{w}_1, \underline{w}_4, \underline{w}_3$	$-D_3$	$-D_2$
$\underline{w}_2, \underline{w}_3, \underline{w}_1$	$\underline{w}_2, \underline{w}_3, \underline{w}_4$	D_4	D_1
$\underline{w}_2, \underline{w}_4, \underline{w}_1$	$\underline{w}_2, \underline{w}_4, \underline{w}_3$	D_3	$-D_1$
$\underline{w}_3, \underline{w}_4, \underline{w}_1$	$\underline{w}_3, \underline{w}_4, \underline{w}_2$	D_2	D_1

These six constraints really simplify to three: D_1 and D_2 must have opposite signs; D_1 and D_3 must have the same sign; and D_1 and D_4 must have opposite signs. If all three constraints are satisfied, then a solution exists to Eqn. 7; therefore, a solution exists to Eqn. 4.

These three constraints can be rewritten as three inequalities:

$$\begin{array}{l} D_1D_2 < 0 \\ D_1D_3 > 0 \\ D_1D_4 < 0 \end{array} \tag{10}$$

Once again, the magnitudes of the determinants are unimportant.

It is important to realize that, for a fixed geometry, this is really an analytic set of inequalities whose functions involve powers of $x, y, \cos(\phi)$, and $\sin(\phi)$. In order to create polynomial equations, a change of variable can be performed:

$$\begin{aligned} u &= \tan(\tfrac{\phi}{2}) \\ \cos(\phi) &= \tfrac{1-u^2}{1+u^2} \\ \sin(\phi) &= \tfrac{2u}{1+u^2} \end{aligned} \tag{11}$$

This change will make the functions polynomials of order 12; the highest power of x and y and is 4, and the highest power of u is 8. This can be seen from the analytic form of the quantities D_1, D_2, D_3, D_4 given in the appendix. (Each is a polynomial of x, y, and u, with the highest power of x and y as 2 and the highest power of u as 4–the leading terms drop out when plugged into Eqn. 10).

So the workspace boundaries are given by an analytic system of inequalities:

$$\begin{aligned} f_1(x,y,u) &< 0 \\ f_2(x,y,u) &> 0 \\ f_3(x,y,u) &< 0 \end{aligned} \tag{12}$$

The region enclosed by these inequalities is the configuration space of valid configurations. The boundary of this region is given by: $D_1D_2 = 0$, $D_1D_3 = 0$, and $D_1D_4 = 0$. Clearly, at a boundary, at least one of the determinants is zero, i.e. $D_1 = 0$, $D_2 = 0$, $D_3 = 0$, or $D_4 = 0$. We can use this to validate the intuition developed earlier. Consider Eqn. 7 with one of the tensions set to zero (e.g. cable four):

$$\begin{aligned} &\left[\hat{w}_1\ \hat{w}_2\ \hat{w}_3\ \hat{w}_4\right]\left[t_1\ t_2\ t_3\ 0\right]^T = \underline{0} \\ \Rightarrow &\left[\hat{w}_1\ \hat{w}_2\ \hat{w}_3\right]\left[t_1\ t_2\ t_3\right]^T = \underline{0} \end{aligned} \tag{13}$$

We still require that t_1, t_2, t_3 are not zero, and so the determinant of the wrench submatrix must be identically zero. Since a zero determinant speaks only of the relative directions of the component vectors and not the magnitudes, this determinant being zero is equivalent to $D_4 = 0$. Therefore the boundary is given by setting one of the tensions to zero and solving the reduced system, validating the earlier intuition.

These analytic functions were computed for a square platform with side lengths of 1 meter and cable connections at the corners. The locations of the cable anchors were chosen as $(0, 0)$, $(0, 5)$, $(6, 5)$, and $(6, 0)$, all in meters. The forms of the determinants in Eqn. 9 were found using these dimensions and are given in the appendix.

By evaluating the inequalities given by Eqn. 12, one can quickly determine if a given position and angle are valid. Furthermore, these inequalities can be used to visualize the space of valid configurations. Figures 2–6 are all examples of analysis based on plotting these inequalities. Please note that although the jagged lines seem to suggest that these plots have been created using a numeric routine, they are in fact based on closed-form equations obtained by applying the theory given earlier. The jagged lines are a plotting artifact introduced by the patching procedure of Maple's implicit plotting function.

Visualizing Eqn. 12 in two dimensions requires fixing one of the parameters x, y, u and then plotting the others. Each inequality will produce a valid region; the overlap of the three regions corresponding to the three inequalities will produce the complete valid region. For example, Fig. 2 shows the three regions corresponding to $u = 0.015$. The overlap of these three regions is shown in Fig. 3 and corresponds to the valid (x,y) configurations given this angle.

Figure 4 shows the valid region with $u = 0$. Notice that if the platform is not rotated, any configuration where the platform corners do not cross the anchor boundaries is valid.

As the platform orientation is changed slightly, there is a large change in the valid configuration space. To see this more clearly, Fig. 5 shows a plot of Eqn. 12 with the position fixed at $(2, 2.5)$ and only varying the platform angle. The angles which lead to solutions are shown by the highlighted portion of the u-axis.

Parametric studies can be done as well. Suppose that one of the cable anchors can be moved around during operation in order to ensure that all tensions are positive. For a given platform position and angle, the inequalities will become functions of b_{ix} and b_{iy}, where i denotes which cable anchor is moveable. Figure 6 shows two examples of valid anchor placements for fixed platform position and angle. Given the form of the wrench vectors in Eqn. 2 and their linear dependence on anchor position, the roughly linear boundaries for the valid anchor positions are to be expected.

CONCLUSIONS

We approached the problem of evaluating the reachable workspace for a cable-driven parallel platform by using the tools of semi-definite programming to obtain analytic expressions for the boundaries of this workspace. Starting with a general statement of the problem, we applied Farkas's Lemma to provide the necessary and sufficient condition for a solution to exist. This condition required that there was no hyperplane that separated the convex hull formed by the cable wrench vectors and the forcing wrench. In order to assess if this was the case, we presented an algebraic procedure to find such a hyperplane. We then extended this method to finding the sufficient condition for ensuring that the parallel platform could resist an arbitrary applied wrench.

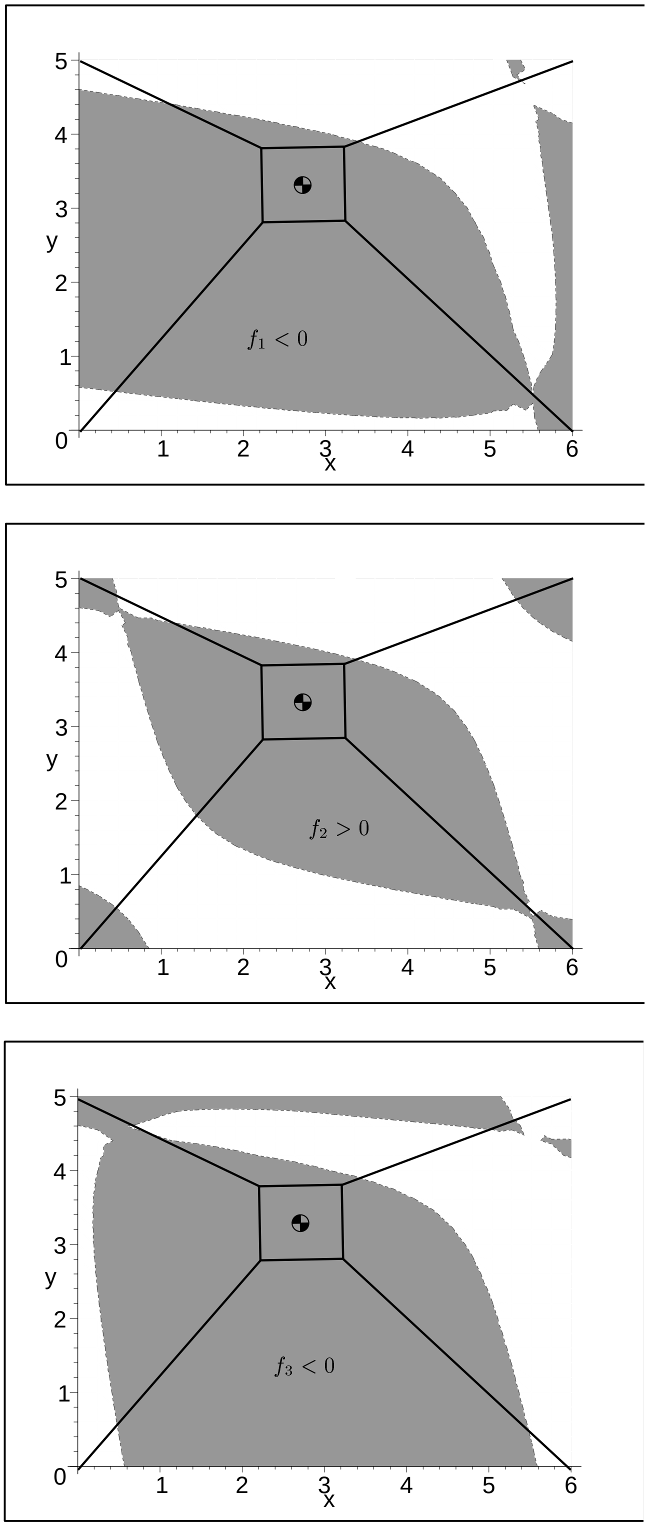

Figure 2. PLOTS SHOWING (x,y) REGIONS THAT SATISFIY THE *INDIVIDUAL* INEQUALITIES IN EQN. 12. PLATFORM ANGLE IS FIXED AT $u = 0.015$.

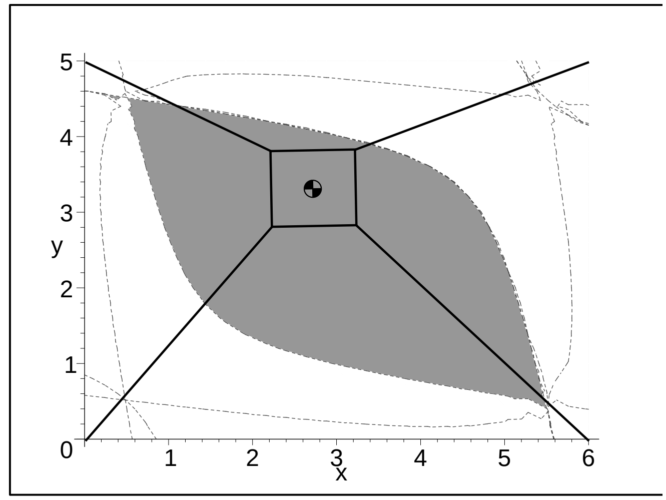

Figure 3. PLOT SHOWING (x,y) REGION THAT SATISFIES *ALL* OF THE INEQUALITIES IN EQN. 12. PLATFORM ANGLE IS FIXED AT $u = 0.015$.

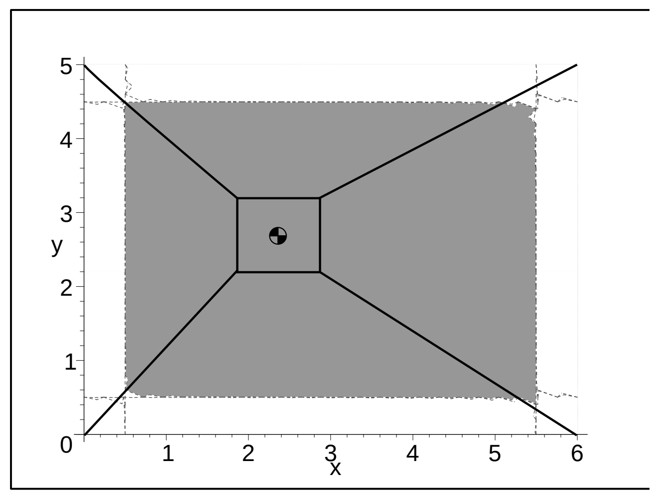

Figure 4. PLOT SHOWING (x,y) REGION THAT SATISFIES *ALL* OF THE INEQUALITIES IN EQN. 12. PLATFORM ANGLE IS FIXED AT $u = 0$.

Throughout this discussion, we used geometric intuition to qualitatively describe the boundaries of the reachable workspace in the context of the semi-definite programming techniques used. Finally, we applied the hyperplane search procedure to a four-cable planar parallel platform and obtained the analytic expressions describing the boundaries of the reachable workspace. This quantitative description of the boundaries matched the intuitive description given earlier.

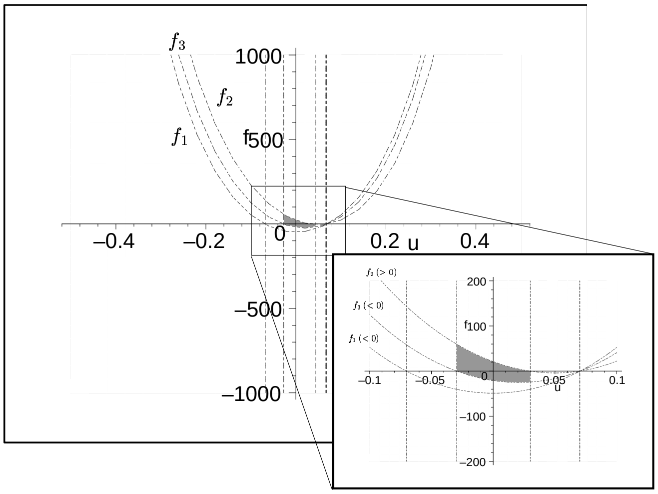

Figure 5. PLOT OF FUNCTIONS IN EQN. 12 WITH POSITION FIXED AT (2, 2.5). GRAY AREA SIGNIFIES REGION WHERE ALL THREE INEQUALITIES ARE SATISFIED; THIS CORRESPONDS APPROXIMATELY TO $u \in [-0.03, 0.03]$.

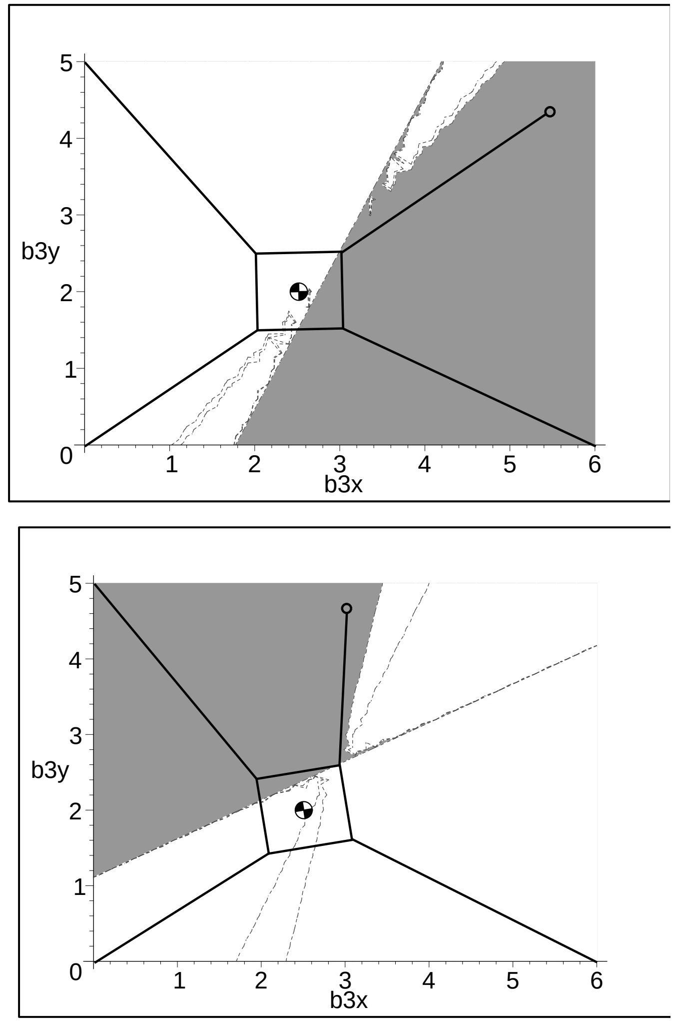

Figure 6. PLOT SHOWING (b_{3x}, b_{3y}) REGION (I.E. CABLE ANCHOR LOCATION) THAT SATISFIES THE INEQUALITIES IN EQN. 12. PLATFORM IS FIXED AT $(2.5, 2)$; THE ANGLE IS $u = 0.01$ FOR THE FIRST PLOT AND $u = 0.10$ FOR THE SECOND PLOT.

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Appendix: Analytic Form of Non-Normalized Determinants

The determinants given in Eqn. 9 are quite large if left in fully symbolic form. To reduce their size, we present them evaluated for an anchor placement of $b_{1x}=0$, $b_{1y}=0$, $b_{2x}=0$, $b_{2y}=5$, $b_{3x}=6$, $b_{3y}=5$, $b_{4x}=6$, $b_{4y}=0$ and a platform shape of $r_1=r_2=r_3=r_4=1/\sqrt{2}$, $\theta_1=5\pi/4$, $\theta_2=3\pi/4$, $\theta_3=\pi/4$, $\theta_4=7\pi/4$. (See Fig. 1 as reference.)

$$D_1=\left(\frac{1}{1+u^2}\right)^2\left[\left(-\frac{13}{2}y-\frac{11}{2}x+yx+\frac{143}{4}\right)u^4 + \left(\frac{133}{2}+59x-10x^2+61y-12y^2\right)u^3+(-11+x+y)\,u^2 + \left(-10x^2+61x+59y-12y^2-\frac{111}{2}\right)u+\left(\frac{11}{2}y-yx-\frac{99}{4}+\frac{9}{2}x\right)\right] \tag{14}$$

$$D_2=\left(\frac{1}{1+u^2}\right)^2\left[\left(yx+\frac{1}{2}x-\frac{13}{2}y-\frac{13}{4}\right)u^4 + \left(-10x^2+59x+\frac{143}{2}-12y^2+59y\right)u^3+(6+y-x)\,u^2 + \left(-\frac{121}{2}-10x^2+61y+61x-12y^2\right)u+\left(\frac{11}{2}y-yx+\frac{1}{2}x-\frac{11}{4}\right)\right] \tag{15}$$

$$D_3=\left(\frac{1}{1+u^2}\right)^2\left[\left(\frac{1}{2}y+\frac{1}{4}+\frac{1}{2}x+yx\right)u^4 + \left(\frac{131}{2}+59y+61x-10x^2-12y^2\right)u^3+(-x-y)\,u^2 + \left(-10x^2+61y-12y^2+59x-\frac{109}{2}\right)u-\left(\frac{1}{4}-yx+\frac{1}{2}x+\frac{1}{2}y\right)\right] \tag{16}$$

$$D_4=\left(\frac{1}{1+u^2}\right)^2\left[\left(yx+\frac{1}{2}y-\frac{11}{2}x-\frac{11}{4}\right)u^4 + \left(\frac{121}{2}+61x-10x^2-12y^2+61y\right)u^3+(-y+x+5)\,u^2 + \left(59x-12y^2-10x^2-\frac{99}{2}+59y\right)u+\left(\frac{1}{2}y-yx+\frac{9}{2}x-\frac{9}{4}\right)\right] \tag{17}$$

where $u=\tan\left(\frac{\phi}{2}\right)$ and (x,y,ϕ) are the position and orientation of the planar platform.