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Multi-vehicle path planning in dynamically changing environments

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Abstract
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Keywords
approximation theory, asymptotic stability, collision avoidance, computational geometry, concave programming, mobile robots, robot dynamics, asymptotic stability, collision free, mobile robot, multivehicle path planning, nonconvex feasibility optimization problem, nonholonomic multivehicle system, nonsmooth dynamical systems, obstacle avoidance, polygonal curve approximation, trajectory control

Comments

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Multi-Vehicle Path Planning in Dynamically Changing Environments

Ali Ahmadzadeh, Nader Motee, Ali Jadbabaie and George Pappas

***Abstract*— In this paper, we propose a path planning method for nonholonomic multi-vehicle system in presence of moving obstacles. The objective is to find multiple fixed length paths for multiple vehicles with the following properties: (i) bounded curvature (ii) obstacle avoidant (iii) collision free. Our approach is based on polygonal approximation of a continuous curve. Using this idea, we formulate an arbitrarily fine relaxation of the path planning problem as a nonconvex feasibility optimization problem. Then, we propound a nonsmooth dynamical systems approach to find feasible solutions of this optimization problem. It is shown that the trajectories of the nonsmooth dynamical system always converge to some equilibria that correspond to the set of feasible solutions of the relaxed problem. The proposed framework can handle more complex mission scenarios for multi-vehicle systems such as rendezvous and area coverage.**

I. Introduction

The problem of path planning for a vehicle in a dynamically changing environment has been an active research area in robotics and control communities [1], [2]. The major trends have been focused on holonomic and non-holonomic kinematic path planning problems. Perhaps Dubins' seminal work [3] is one of the first ones in this area that characterizes shortest bounded-curvature paths for a vehicle in absence of obstacles. It is well-known that finding a shortest bounded-curvature path amidst polygonal obstacles in the plane is NP-hard [4]. Also, researchers have shown that the general feasibility algorithm is exponential in time and space [5]. These results imply that struggling to find an efficient and exact algorithm to solve curvature-constrained path planning problem is hopeless. This partially resulted in developing various approximate methods to solve the path planning problem [6]- [7]. Nevertheless, the existing algorithms are incomplete in the sense that they may not provide a solution even if one exists.

Among different approaches to the path planning problem, the navigation function method is the closest one to our methodology [8]. In this method, the vehicles are steered by some artificially generated forces, defined as the negative gradient of a navigation function. The navigation function is defined so that it can generate attractive forces toward the goal and repulsive forces in the neighborhood of an obstacle. The main disadvantage of the navigation function methods is that they can not handle nonholonomic constraints such as bounded-curvature constraint.

In this paper, our goal is to propose a near-optimal and scalable method for solving the bounded-curvature path planning problem in presence of moving obstacles. We assume that each obstacle can be represented as union of non-overlapping disks and their motion trajectories are known. First, we consider the path planning problem for a single vehicle. Then we extend our results to handle multi-vehicle path planning problems. Our approach is based on polygonal approximation of a continuous curve in the plane. A path connecting the initial and final positions of a vehicle can be approximated by finitely many waypoints. This approximation can be arbitrarily improved by increasing the number of waypoints. In this setting, we can relax the bounded-curvature and collision-free constraints by verifying the constraints only at these waypoints. This relaxation results in a finite-dimensional formulation of the path planning problem as a nonconvex feasibility optimization problem. Every feasible solution to the relaxed problem is an approximate bounded-curvature and collision-free path for the vehicle.

Furthermore, we propose a nonsmooth dynamical systems approach to find feasible solutions of the optimization problem. In this method, each waypoint is treated as a moving particle in the plane. We define interaction forces between the particles such that: (i) the set of equilibria of the system contains all feasible solutions of the optimization problem, and (ii) the corresponding multi-particle system is asymptotically stable. In an equilibrium point the net force on each particle is equal to zero. It is shown that by applying some specific type of nonsmooth interaction forces, the net force on each particle is equal to zero if and only if these particles are representing a feasible path. In other words, for every initial condition the trajectory of the system always converges to a feasible path for the vehicle. Since we are using discontinuous dynamical system, we need nonsmooth analysis and stability of nonsmooth systems to analyze the dynamical system with discontinuous right-hand sides. When studying a discontinuous vector field the classical notion of solution for dynamical system is too restrictive and may not even exist. There are several solution notions for discontinuous systems such as Caratheodory solutions, Krasovskii solutions and Filipov notion of solutions [13]. In this paper, we employ the notion of Filipov's solutions. Filipov in his seminal contribution [10] developed a solution concept for differential equations whose right-hand sides were only required to be Lebesgue measurable in the state and time variables. For our analysis, we will apply Shevitz and Paden's results [14] on nonsmooth Lyapunov stability theory and LaSalles invariance principle for a class of nonsmooth Lipschitz continuous Lyapunov functions.

This paper is organized as follows. In Section II, we formulate the path planning problem for a single vehicle as a feasibility optimization problem. A dynamical system approach to the path planning problem is discussed in Section III. In Section IV, it is shown that by using discontinuous interaction forces we can always guarantee the convergence of the trajectories of the system to feasible paths. The single-vehicle path planning in presence of moving obstacles is presented in Section V. In Section VI, we show that our methodology can be directly applied to the multi-vehicle path planning problem in presence of moving obstacles.

II. Problem Formulation

The goal of this paper is to find a fixed-length bounded curvature trajectory for a vehicle with given initial and final configurations in a dynamically changing environment. We assume that the dubins vehicle is traveling with a constant speed V. Suppose that there are M moving obstacle with known motion patterns in the environments. At any time instant t, each obstacle is assumed to be represented by a disk $\mathbf{D}(c_j(t), r_j(t)) = \{x \mid \|x - c_j(t)\| \leq r_j(t)\}$. We also assume that these disks are not overlapping for all time.

Path Planning with Moving Obstacles: *Let $\kappa_{\max} > 0$ be the maximum allowable curvature and $P, Q \in \mathbb{R}^2$ the initial and final points. Then the problem consists of finding a curve $\gamma : [0, T] \to \mathbb{R}^2$ (parameterized by time where $T > 0$ is a fixed number) such that*

(i) $\gamma(0) = P$ *and* $\gamma(T) = Q$.

(ii) $\kappa(t) \leq \kappa_{\max}$ *for all* $t \in [0, T]$.

(iii) $\gamma(t) \cap \mathbf{D}(c_j(t), r_j(t)) = \emptyset$ *holds for all* $t \in [0, T]$ *and* $j = 1, \ldots, M$.

Note that $\kappa(t)$ is the curve curvature at time t. One can see that γ is a fixed length curve of length $l = VT$. We refer to the second condition as the obstacle avoidance constraint. The third condition guarantees a bounded curvature curve.

In the sequel, we will tackle this problem in several steps and propose an arbitrarily fine approximation of the optimal solution. In Section II-A, we review polygonal approximation of a continuous curve with equidistant waypoints in $\mathbb{R}^2$. In Section II-B and II-C, we show that conditions (ii) and (iii) can be relaxed by verifying the constraints only at waypoints. In Section II-D, we will see that the path planning problem reduces to a feasibility optimization problem.

A. Polygonal Curve Approximation

Our approach is based on discrete approximation of a continuous curve using finite number of vertices. Consider a polygonal curve $\gamma_p = \overline{p_0 p_1 ... p_n}$ represented by its ordered vertices $p_0, p_1, ..., p_n \in \mathbb{R}^2$ where $p_0 = P$, $p_n = Q$ and $\overline{p_i p_{i+1}}$ is the line segment connecting p_i to p_{i+1}. Under some mild assumptions, for a given error bound $\epsilon > 0$, one can always find points $\{p_0, p_1, \ldots, p_n\}$, for a large number $n > 0$, such that

$$|L(\gamma_p) - l| < \epsilon, \tag{1}$$

where

$$L(\gamma_p) = \sum_{i=1}^{n} \|p_i - p_{i-1}\|.$$

Without loss of generality, we may assume that all points p_i are equidistant. Therefore, it follows that

$$d = \|p_i - p_{i-1}\| \simeq \frac{l}{n}. \tag{2}$$

for all $i = 1, ..., n$.

B. Discrete Curvature

If we assume that $d \ll \frac{1}{\kappa_{\max}}$, then we can use C_i the circle passing through the points (p_{i-1}, p_i, p_{i+1}) (if not all of these three points lie on a line), as an approximation to the osculating circle to the curve at point p_i to calculate the curve curvature at that point. As both p_{i-1} and p_{i+1} move toward p_i, circle C_i approaches a limiting circle with radius r_i which is the same as the osculating circle at point p_i. More importantly, $\frac{1}{r_i}$ is the curvature at p_i. Therefore, we can employ circle C_i to calculate an approximation of the curvature at point p_i. Let A denotes the area of the triangle formed by nodes (p_{i-1}, p_i, p_{i+1}) and $d_{ij} = \|p_i - p_j\|$. The discrete curvature κ_i at point p_i is defined by

$$\kappa_i = \frac{1}{R_i} = \frac{4A}{d_{(i-1)i} d_{i(i+1)} d_{(i-1)(i+1)}} \tag{3}$$

By applying assumption (2) and the fact that the area of the triangle is $A = \sqrt{s(s-a)(s-b)(s-c)}$ where $s = \frac{a+b+c}{2}$, we have

$$\kappa_i = \frac{2\sqrt{d^2 - \frac{d_{(i-1)(i+1)}^2}{4}}}{d^2}. \tag{4}$$

By imposing the following constraint on discrete curvature

$$\kappa_i \leq \kappa_{\max},$$

it follows that

$$\|p_{i-1} - p_{i+1}\| = d_{(i-1)(i+1)} \geq \frac{l}{n}\sqrt{4 - \frac{\kappa_{\max}^2 l^2}{n^2}} = \eta \tag{5}$$

where $i = 1, ..., n-1$.

C. Moving Obstacles

Our goal is to find a path for the Dubins vehicle in presence of moving obstacles with known motion patterns. Suppose that t_i is the time instant at which the vehicle is at waypoint p_i. Therefore, the obstacle avoidance condition (iii) can be written as follow

$$\|p_i - c_j(t_i)\| \geq r_j(t_i) \tag{6}$$

for all $i = 0, \ldots, n$ and $j = 1, \ldots, M$. We assume that

$$\|P - c_j(0)\| \geq r_j(0) \quad \text{and} \quad \|Q - c_j(T)\| \geq r_j(T)$$

for all $j = 1, \ldots, M$.

D. Relaxed Path Planning Problem

A relaxation of the path planning problem can be posed as the following problem.

Relaxed Path Planning Problem as a Feasibility Problem: *There exists a polygonal curve $\gamma_p = \overline{p_0 p_1 ... p_n}$ that satisfies conditions (i)-(iii) if and only if the following optimization problem is feasible*

$$\min_{\{p_1,\dots,p_{n-1}\}\in\mathbb{R}^2} \; 0 \tag{7}$$

$$\begin{aligned}
\text{subject to:} \quad & p_0 = P \quad \text{and} \quad p_n = Q, \\
& \|p_i - p_{i-1}\| = d, \qquad i = 1,...,n \\
& \|p_{i-1} - p_{i+1}\| \geq \eta, \quad i = 1,...,n-1 \\
& \|p_i - c_j(t_i)\| \;\geq\; r_j(t_i), \quad i = 1,\dots,n-1 \\
& \qquad\qquad\qquad\qquad\qquad j = 1,\dots,M.
\end{aligned}$$

where η is defined in (5). The optimization problem (7) is a nonconvex problem. In the following section, we we propose a multi-particle dynamical system approach to solve the feasibility problem (7). First, we consider the path planning problem without obstacles.

III. Path Planning Using Stable Multi-Particle Systems

In this section, we propose a method to find a feasible solution of problem (7) for a single vehicle in the absence of obstacles in the environment. Consider the waypoints $p_0, ..., p_n \in \mathbb{R}^2$. These points can be viewed as point mass particles moving on the plane with some initial random positions. Let m_i be the mass of particle i with position p_i. A force vector $\mathbf{F}_i$ can be associated to point mass particle p_i. Therefore, we have

$$m_i \ddot{p}_i = \mathbf{F}_i \tag{8}$$

where $i = 0, ..., n$. Let $p = [p_0^T, p_1^T, \dots, p_n^T]^T$ denote the state of the overall system. One can impose the following constraints

$$p_0 = P \quad \text{and} \quad p_n = Q$$

on particles 0 and n by assuming that $m_0, m_n > M$ for any large number $M > 0$. In other words, two heavy masses are concentrated at points P and Q and that their positions are fixed.

Our goal is to design force vectors $\mathbf{F}_i$ for each particle such that the set of stable equilibria of the dynamical systems (8) is equal to the set of all feasible solutions of the optimization problem (7).

Definition 1: We refer to a real-valued function f_{ij} as *elasticity function* if it satisfies the following conditions:

(i) $f_{ij} = f_{ji}$ for all i and j.
(ii) Functions f_{ij} are nondecreasing.
(iii) The vector $(p_0, ..., p_n)$ is a feasible solution of problem (7) if and only if $f_{ij}(\|p_i - p_j\|) = 0$ for all $i, j = 0, ..., n$.

Throughout the paper, we will also refer to the elasticity functions as spring-like forces.

Theorem 1: All feasible solutions of problem (7) are stable equilibria of the multi-particle system (8) with

$$\mathbf{F}_i = \sum_{\substack{j=0 \\ j\neq i}}^{n} f_{ij}(\|p_i - p_j\|)\, \mathbf{e}_{ij} - \upsilon \dot{p}_i \tag{9}$$

where f_{ij}'s are continuous spring forces, $\mathbf{e}_{ij} = \frac{p_i - p_j}{\|p_i - p_j\|}$, and $\upsilon > 0$ is a constant.

Proof: Let $p = (p_0, ..., p_n)$, for the dynamical system (8) we define Lyapunov function $\mathbf{E}(p, \dot{p})$ as follows:

$$\mathbf{E}(p,\dot{p}) = \sum_{i=0}^{n} \sum_{\substack{j=0 \\ j\neq i}}^{n} \mathbf{W}_{ij}(\|p_i - p_j\|) + \frac{1}{2} \sum_{i=0}^{n} m_i \|\dot{p}_i\|^2, \tag{10}$$

where

$$\mathbf{W}_{ij}(\alpha) = \int_{\alpha_0}^{\alpha} f_{ij}(\sigma) d\sigma$$

and α_0 is a root of function f_{ij}. According to property (ii), f_{ij} is nondecreasing. This implies that $\mathbf{W}_{ij}(x) \geq 0$ for all $x \geq 0$ and $i, j = 0, ..., n$. Therefore, it follows that

$$\mathbf{E}(p,\dot{p}) \geq 0. \tag{11}$$

In addition,

$$\frac{\partial \mathbf{E}}{\partial t} = \sum_{i=0}^{n} \sum_{\substack{j=0 \\ j\neq i}}^{n} \dot{p_i}^T \frac{\partial \mathbf{W}_{ij}(\|p_i - p_j\|)}{\partial p_i} + \sum_{i=0}^{n} m_i \; \dot{p}_i^T \ddot{p}_i. \tag{12}$$

We have that

$$\begin{aligned}
\dot{p_i}^T \frac{\partial \mathbf{W}_{ij}(\|p_i - p_j\|)}{\partial p_i} &= \dot{p_i}^T f_{ij}(\|p_i - p_j\|) \; \frac{\partial \|p_i - p_j\|}{\partial p_i} \\
&= f_{ij}(\|p_i - p_j\|) \; \dot{p_i}^T \mathbf{e}_{ji} \\
&= -f_{ij}(\|p_i - p_j\|) \; \dot{p_i}^T \mathbf{e}_{ij}.
\end{aligned}$$

By applying (8) and (9) to substitute for $\ddot{p}_i$, we get

$$\frac{\partial \mathbf{E}}{\partial t} = -\upsilon \sum_{i=0}^{n} \|\dot{p}_i\|^2 \leq 0. \tag{13}$$

From the basic Lyapunov theorem [9], we can conclude that favorable equilibria of system (8) with force vectors (9) are stable. This establishes the stability but not the asymptotic stability of favorable equilibria. In fact, favorable equilibria are actually locally asymptotically stable. We use LaSalle's invariance principle to prove local asymptotical stability of the favorable equilibria. Let point $(p, \dot{p}) = (p_0, 0)$ be a favorable equilibrium of the dynamical system. Consider $N((p_0, 0), \epsilon)$ a ϵ-neighborhood of the favorable equilibrium in which there is no unfavorable equilibrium in $N((p_0, 0), \epsilon)$. Also let $\mathcal{C} = \{(p, \dot{p}) \mid \mathbf{E}(p, \dot{p}) \leq 1\}$. Also, we define $\Omega = \mathcal{C} \bigcap N((p_0, 0), \epsilon)$. Note that $\mathcal{C}$ is a closed set; therefore, Ω is compact. In addition $\dot{\mathbf{E}} \leq 0$, so every solution to the autonomous dynamical system (8) that starts in Ω remains in Ω. As a consequence of LaSalle's invariance principle, the trajectory enters the largest invariant set of

$$\Omega \bigcap \{(p,\dot{p}) \mid 0 \in \dot{\mathbf{E}}\} \subseteq \Omega \bigcap \{(p,\dot{p}) \mid \dot{p} = 0\}.$$

To obtain the largest invariant set in this region, note that

$\dot{p} = 0$ and p are constant. Thus, the trajectory converges to an equilibrium. But in Ω there is no unfavorable equilibrium, consequently the manifold of favorable equilibria is attractive. ∎

Remark 1: One should note that dynamical system (8) with continuous vector forces (9) may have some additional unfavorable equilibria. A simple analysis shows that in equilibrium the net force on each particle p_i can be zero while some of the force components are not zero (see Fig. IV). In fact, nonzero spring-like forces in equilibrium imply infeasibility of the corresponding solution (path). This verifies the possibility of converging to infeasible solutions (paths). In Section IV, we will show that by employing discontinuous forces such (unfavorable) possibilities can be withdrawn. We will show that all unfavorable equilibria (corresponding to infeasible paths) are unstable.

Remark 2: Some additional restrictions on the initial and final orientations of the vehicle can be imposed. This can be done by fixing the positions of particles p_1 and p_{n-1} additional to p_0 and p_n by imposing the constraints $m_2, m_{n-1} > M$ for some large enough $M > 0$.

IV. Stability Analysis of Multi-Particle System with Discontinuous Forces

In this section, we use discontinuous elasticity functions and by means of net force analysis in the equilibrium, we show that an equilibrium is stable if and only if all of the forces are equal to zero. We should emphasize that forces are zero if and only if constraints are satisfied. We consider the following class of discontinuous elasticity functions

$$f(z) = \begin{cases} 0 & \text{if} \quad z \geq \eta \\ -w & \text{if} \quad z < \eta \end{cases}, \tag{14}$$

where $w, \eta \geq 0$ are constant.

Theorem 2: Consider the multi-particle dynamical system (8) with

$$\mathbf{F}_i = \sum_{\substack{j=0 \\ j \neq i}}^{n} f_{ij}(\|p_i - p_j\|)\, \mathbf{e}_{ij} - \upsilon \dot{p}_i \tag{15}$$

where the elasticity functions f_{ij} are either continuous as in definition 1 or discontinuous as in (14). Then for almost all initial conditions, the trajectories of the multi-particle dynamical system (8) asymptotically converge to an equilibrium. Furthermore, a feasible solution of problem (7) is a locally asymptotically stable equilibrium of the multi-particle dynamical system (8) if all the corresponding spring-like forces are equal to zero.

Proof: See the appendix for a proof. ∎

Theorem 2 shows that for properly chosen continuous elasticity functions the set of all stable equilibria of the multi-particle dynamical system (8) contains all feasible solutions of (7). In the following theorem, it is shown that by means of discontinuous elasticity functions one can actually prove that all unfavorable equilibria of the multi-particle dynamical system (8) are unstable. In other words, theorem (3) shows that for almost all initial conditions (except for those where

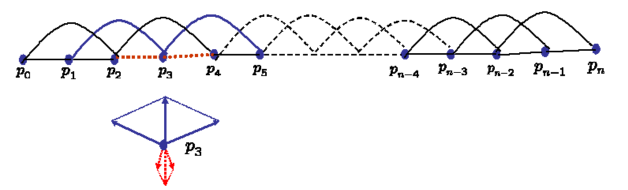

Fig. 1. Analysis of the net forces in the equilibrium which shows that the net forces in p_3 could be zero even though the forces are not zero)

the particles lie on a straight line passing through p_0 and p_n which constitute a set of measure zero) the trajectories of the multi-particle dynamical system asymptotically converges to an equilibrium which is a feasible solution of problem (7). In the sequel, we consider the following class of discontinuous elasticity functions

$$f_{i(i+1)}(z) = \begin{cases} w_1 & \text{if} \quad (z - \frac{l}{n}) \geq \frac{w_1}{k_f} \\ k_f(z - \frac{l}{n}) & \text{if} \quad -\frac{w_1}{k_f} \leq z \leq \frac{w_1}{k_f} \\ -w_1 & \text{if} \quad (z - \frac{l}{n}) \leq -\frac{w_1}{k_f} \end{cases} \tag{16}$$

and

$$f_{(i-1)(i+1)}(z) = \begin{cases} 0 & \text{if} \quad z \geq \eta \\ -w_2 & \text{if} \quad z < \eta \end{cases}, \tag{17}$$

where $\eta = \frac{l}{n}\sqrt{4 - \frac{\kappa_{\max}^2 l^2}{n^2}}$ and $w_1, w_2, k_f > 0$ are some constant numbers.

Theorem 3: Consider the multi-particle dynamical system described by (8) with $2k$ particles (i.e., $n = 2k - 1$) and discontinuous elasticity functions defined as (16) and (17). If $w_2 > 2w_1$, then all of the infeasible equilibria are either unstable or saddle with measure zero region of attraction.

Proof: Without loss of generality, we may assume that $\|p_0 - p_n\| > l$. If $\|p_0 - p_n\| < l$, then problem (7) is infeasible. When $\|p_0 - p_n\| = l$, there is only one stable equilibrium that corresponds to the case where all particles lie on a straight line connecting p_0 to p_n.

In Theorem 2, we showed that the multi-particle dynamical system is stable. This means that the trajectories of the multi-particle system converge to stable equilibrium. In an equilibrium point, the net force on a given particle is equal to zero. This does not necessarily means that all elasticity functions acting on that particle are zero. There are two types of springs vector forces: (i) To enforce particles to be equidistant:

$$f_{i(i+1)}(\|p_i - p_{i+1}\|)\, \mathbf{e}_{i(i+1)},$$

(ii) To satisfy curvature constraints:

$$f_{(i-1)(i+1)}(\|p_{i-1} - p_{i+1}\|)\, \mathbf{e}_{(i-1)(i+1)}.$$

It is easy to see that

$$|f_{i(i+1)}| \leq w_1 < \frac{w_2}{2}, \tag{18}$$

and

$$|f_{(i-1)(i+1)}| = 0 \quad \text{or} \quad w_2. \tag{19}$$

We can associate a graph to the multi-particle system with nodes representing the mass particles. There is an edge between a pair of nodes if there is a nonzero spring force between the two particles. From Fig. IV, one can see that each particle p_i for $i = 2, ..., 2k-3$ is connected to four other particles (the positions of the particles p_0 and p_{2k-1} are fixed). In an equilibrium point, there are three spring-like forces with magnitudes f_{01}, f_{12} and f_{13} associated with node p_1. Since the net force is zero at this node, we have

$$f_{01}\ \mathbf{e}_{01} + f_{12}\ \mathbf{e}_{12} + f_{13}\ \mathbf{e}_{13} = \mathbf{0}. \tag{20}$$

It follows that

$$\|f_{01}\ \mathbf{e}_{01} + f_{12}\ \mathbf{e}_{12}\| = |f_{13}|. \tag{21}$$

Assume that $f_{13} = w_2$, then from the above equation we get

$$\|f_{01}\ \mathbf{e}_{01} + f_{12}\ \mathbf{e}_{12}\| = |f_{13}| = w_2. \tag{22}$$

On the other hand, we have

$$\|f_{01}\ \mathbf{e}_{01} + f_{12}\ \mathbf{e}_{12}\| \leq |f_{01}| + |f_{12}| \leq 2w_1 < w_2. \tag{23}$$

This is a contradiction, because from our assumptions we know that $w_2 > 2w_1$. Therefore, we conclude that $f_{13} = 0$. In other words, there are only three spring-like forces acting on particle p_3, i.e., f_{23}, f_{34} and f_{35}. Using a similar argument, we can also show that $f_{35} = 0$. By repeating the same procedure on the other nodes, it follows that

$$f_{35} = f_{57} = \ldots = f_{(2k-3)(2k-1)} = 0. \tag{24}$$

Therefore, at nodes with odd indices all spring-like forces resulting from curvature constraints are zero and that can be eliminated from the graph. Similarly, we can argue that at node p_{2k-2} we have $f_{(2k-2)(2k-4)} = 0$. By performing a similar analysis, we can show that

$$f_{(2k-2)(2k-4)} = f_{(2k-4)(2k-6)} = \ldots = f_{20} = 0. \tag{25}$$

From (24) and (25), we conclude that all spring-like forces corresponding to curvature constraints are equal to zero. Thus, the only possibility in order to have $f_{i(i+1)} \neq 0$ (for all $i = 0, \ldots, 2k-1$) in an equilibrium is that all particles to lie on a straight line passing through p_0 to p_{2k-1}. This formation of particles is clearly saddle because we assumed that $\|p_0 - p_n\| > l$ and all particles will have expansion forces acting on them and infinitesimal deviation from the line push the particles further away from line which makes the formation unstable.

Remark 3: In the proof of Theorem 3, we assumed that net force is equal to zero in an equilibrium. In continuous systems, this is always the case. However, the vector field could be nonzero in an equilibrium of a nonsmooth dynamical system. We should note that we proved in Theorem 2 that for almost all initial conditions the trajectory of the multi-particle dynamical system converges to a stable equilibrium. Therefore, we only need to show that if the net force is not zero in an equilibrium, then the equilibrium is either unstable or is saddle with measure zero region of attraction. We refer to Appendix VIII-B for a formal proof and further discussion on this issue.

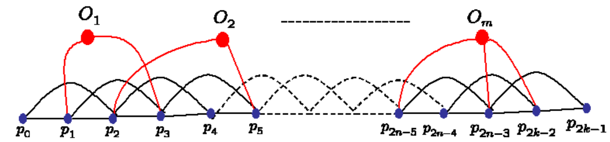

Fig. 2. Graph representation of the forces in the presence of obstacles.

V. Single-Vehicle Path Planning in Presence of Moving Obstacles

In this section, our goal is to find a path for Dubins' vehicle in presence of moving obstacles with known motion trajectories. We assume that each obstacle j can be represented by a disk $\mathbf{O_j}(c_j(t), r_j(t))$ for all $j = 1, \ldots, M$. Furthermore, we assume that at any time instant these disks are not overlapping. Suppose that t_i is the time instant at which the vehicle is at waypoint p_i. Therefore, the obstacle avoidance condition can be written as follow:

$$\|p_i - c_j(t_i)\| \ \geq \ r_j(t_i), \tag{26}$$

for all $i = 0, \ldots, n$ and $j = 1, \ldots, M$. Similar to the static obstacles, in order to enforce obstacle avoidance constraints, we define a new spring-like force between obstacle $\mathbf{O}_j(c_j(t), r_j(t))$ and particle p_i as $f_{ij}(\|p_i - c_j(t_i)\|)\mathbf{e}_{ij}$ with the following elasticity function

$$f_{ij}(z) = \begin{cases} 0 & \text{if} \quad z \ \geq \ r_j(t_i) \\ -w_3 & \quad \text{otherwise} \end{cases}. \tag{27}$$

■

Similar to the static obstacle case, the elasticity functions defined by (27) belong to the class of elasticity functions defined by (14). Therefore, the stability conditions of Theorem (2) hold and the trajectories of the multi-particle dynamical system (8) with new obstacle-avoidance forces asymptotically converge to equilibrium.

Theorem 4: Consider the multi-particle dynamical system (8) with $2k$ particles and M moving obstacles with known motion trajectories. The obstacles are represented by non-overlapping disks $\mathbf{O_j}(c_j(t), r_j(t))$ for $j = 1, \ldots, M$. If the spring-like forces and obstacle avoidance forces are defined as (16), (17), and (27) with the following constraints

$$\begin{aligned} 2w_1 &< w_2, \\ 2(w_1 + w_2) &< w_3, \end{aligned} \tag{28}$$

then all stable equilibria of the multi-particle dynamical system (8) are feasible.

Proof: We only need to show that in an equilibrium all the particles lie outside the obstacles. Then from Theorem 3 it follows that all stable equilibria are feasible. For this purpose, we assume that particle p_i is inside the obstacle with center $c_j(t_i)$. This means that there are five force components acting on particle p_i with zero net force,

$$\underbrace{f_{(i-1)i}\mathbf{e}_{(i-1)i} + f_{i(i+1)}\mathbf{e}_{i(i+1)}}_{\text{spring forces to enforce particles to be equidistant}} + \underbrace{f_{(i-2)i}\mathbf{e}_{(i-2)i} + f_{i(i+2)}\mathbf{e}_{i(i+2)}}_{\text{spring forces to impose curvature constraints}} - \underbrace{w_3\mathbf{e}_{ij}}_{\text{obstacle avoidance force}} = \mathbf{0}$$

It follows that

$$\|f_{(i-1)i}\mathbf{e}_{(i-1)i} + f_{i(i+1)}\mathbf{e}_{i(i+1)} + f_{(i-2)i}\mathbf{e}_{(i-2)i} + f_{i(i+2)}\mathbf{e}_{i(i+2)}\| = \|w_3\mathbf{e}_{ij}\| = w_3.$$

On the other hand,

$$\begin{aligned}\|f_{(i-1)i}\mathbf{e}_{(i-1)i} &+ f_{i(i+1)}\mathbf{e}_{i(i+1)} \\ &+f_{(i-2)i}\mathbf{e}_{(i-2)i} + f_{i(i+2)}\mathbf{e}_{i(i+2)}\| \\ &\leq \|f_{(i-1)i}\mathbf{e}_{(i-1)i}\| + \|f_{i(i+1)}\mathbf{e}_{i(i+1)}\| + \\ &\quad \|f_{(i-2)i}\mathbf{e}_{(i-2)i}\| + \|f_{i(i+2)}\mathbf{e}_{i(i+2)}\|\end{aligned}$$

$$\leq w_1 + w_1 + w_2 + w_2 = 2(w_1 + w_2) < w_3$$

Thus, it follows that $w_3 < w_3$ which is a contradiction. Therefore, in an equilibrium there are no force components between the particles and the obstacles. Therefore, all the particles lie outside the obstacles. ■

VI. Multi-Vehicle Path Planning in Presence of Moving Obstacles

The developed framework in previous sections can be employed to handle multi-vehicle path planning problem in presence of moving obstacles. We associate a multi-particle dynamical system (representing a path) with each vehicle. For example, for N vehicles we need to have N different multi-particle dynamical systems. In order to guarantee a collision-free path for each vehicle, we need to introduce a new force component so called *collision-avoidance* force.

Consider two vehicles with constant speeds V_1 and V_2 traveling distances l_1 and l_2. The corresponding waypoints for these two vehicles are represented by $\{p^1_0, p^1_1, ..., p^1_n\}$ and $\{p^2_0, p^2_1, ..., p^2_m\}$. In order to avoid collision between the two vehicles, we must guarantee that both vehicles are not going to arrive at a waypoint simultaneously. In other words, if there exist waypoints p^1_i and p^2_j for which

$$\left| \frac{l_1}{nV_1} i - \frac{l_2}{mV_2} j \right| \leq \epsilon \tag{29}$$

for some time-error $\epsilon > 0$, then the following constraint has to be imposed on the corresponding waypoints

$$\|p^1_i - p^2_j\| \geq \epsilon' \tag{30}$$

for some position-error $\epsilon' > 0$. Therefore, we can introduce new elasticity functions for all pairs of points p^1_i and p^2_j satisfying condition (29) as follows

$$f_{ij}(z) = \begin{cases} 0 & \text{if} \quad z \geq \epsilon' \\ -w_4 & \text{otherwise} \end{cases} \tag{31}$$

We emphasize that the interaction force (31) is defined between waypoints of two different trajectories. According to Theorem 2, it is straightforward to show that the resulting multi-particle dynamical systems under collision-avoidance constraints (forces) are stable. Furthermore, the following theorem shows that all unfavorable equilibria of the overall system are unstable.

Theorem 5: Suppose that N is the number of vehicles, M the number of moving obstacles, and $2k_i$ the number of particles representing a trajectory for vehicle i for $i = 1, \ldots, N$. Each obstacle is represented by a disk $\mathbf{D}(c_j(t), r_j(t))$ for $j = 1, \ldots, M$. At any given time, assume that no more than two vehicles can possibly collide. Then all stable equilibria of the resulting multi-particle dynamical system with elasticity functions defined by (16), (17), (27) and (31) in which

$$\begin{aligned} 2w_1 &< w_2 \\ 2(w_1 + w_2) &< w_3 \\ 2(w_1 + w_2) + w_3 &< w_4 \end{aligned}$$

are feasible.

Proof: We only need to prove that in an equilibrium paths are collision-free. Then according to Theorem 4 one can conclude that all stable equilibria are feasible. Let assume that vehicles 1 and 2 collide, i.e., there exist waypoints p^1_i and p^2_j that satisfy (29) and $\|p^1_i - p^2_j\| \leq \epsilon'$. In this case, there are (at most) six force components acting on particle p^1_i with zero net force (in the following equations for the sake of simplicity superscript 1 is dropped)

$$\underbrace{f_{(i-1)i}\mathbf{e}_{(i-1)i} + f_{i(i+1)}\mathbf{e}_{i(i+1)}}_{\text{spring forces to enforce particles to be equidistant}} + \underbrace{f_{i(i+2)}\mathbf{e}_{i(i+2)} + f_{(i-2)i}\mathbf{e}_{(i-2)i}}_{\text{spring forces to impose curvature constraints}} - \underbrace{f_{ik}\mathbf{e}_{ik}}_{\text{obstacle avoidance force}} - \underbrace{w_4\mathbf{e}_{ij}}_{\text{collision avoidance force}} = \mathbf{0}$$

for some obstacle with index k. It follows that

$$\|f_{(i-1)i}\mathbf{e}_{(i-1)i} + f_{i(i+1)}\mathbf{e}_{i(i+1)} + f_{i(i+2)}\mathbf{e}_{i(i+2)} + f_{(i-2)i}\mathbf{e}_{(i-2)i} - f_{ik}\mathbf{e}_{ik}\| = \|w_4\mathbf{e}_{ij}\| = w_4$$

Therefore, we have

$$\begin{aligned}\|w_4\mathbf{e}_{ij}\| \leq\ & \|f_{(i-1)i}\mathbf{e}_{(i-1)i}\| + \|f_{i(i+1)}\mathbf{e}_{i(i+1)}\| + \\ & \|f_{i(i+2)}\mathbf{e}_{i(i+2)}\| + \|f_{(i-2)i}\mathbf{e}_{(i-2)i}\| + \|f_{ik}\mathbf{e}_{ik}\|\end{aligned}$$

$$\leq w_1 + w_1 + w_2 + w_2 + w_3 = 2(w_1 + w_2) + w_3 < w_4$$

This is a contradiction. Therefore, the collision-avoidance force f_{ij} must be equal to zero. This means that in equilibrium all paths are collision free. ■

VII. Conclusion

We formulated an arbitrarily fine relaxation of the path planning problem for nonholonomic vehicles as a nonconvex feasibility optimization problem. Then, we proposed a nonsmooth dynamical systems approach to find feasible solutions of the nonconvex optimization problem. We showed that the set of equilibria of the nonsmooth dynamical systems contains all feasible solutions of the optimization problem and that the dynamical system is asymptotically stable. This method can be applied to compute feasible paths for multi vehicles in presence of moving obstacles.

VIII. Appendix

A. Proof of Theorem 1

For the multi-particle dynamical system (8), we define the following Lyapunov function

$$\mathbf{E}(p,\dot{p}) = \sum_{i=0}^{n-1}\sum_{j=i+1}^{n} \mathbf{W}_{ij}(\|p_i - p_j\|) + \frac{1}{2}\sum_{i=0}^{n} m_i\ \|\dot{p}_i\|^2. \tag{32}$$

where

$$\mathbf{W}_{ij}(\alpha) = \int_{\alpha_0}^{\alpha} f_{ij}(\xi)d\xi \tag{33}$$

and α_0 is a root of function f_{ij}, i.e. $f_{ij}(\alpha_0) = 0$. Since function f_{ij} is nondecreasing , $\mathbf{W}_{ij}(\alpha) \geq 0$, therefore, $\mathbf{E}(p,\dot{p}) \geq 0$. In order to be able to apply nonsmooth Lyapunov theorem, first we need to show that $\mathbf{E}(p,\dot{p})$ is a regular and locally Lipschitz function. Therefore, we need to show that $h_{ij}(p) = \mathbf{W}_{ij}(\|p_i - p_j\|)$ is regular and locally Lipschitz for all $i,j = 1,...,n$. According to the definition of f_{ij}, $h_{ij}(p)$ can be written as the pointwise maximum of a set of smooth functions[1]. This means that $h_{ij}(p)$ is regular. It can be shown that $\mathbf{W}_{ij}$ is locally Lipschitz [2]. Function $h_{ij}(p)$ which is composition of two locally Lipschitz functions is also locally Lipschitz. Now, we can use nonsmooth Lyapunov theorem. For simplicity we drop $(p,\dot{p})$ from $\mathbf{E}(p,\dot{p})$. Also we use $f_{ij}(.)$ in replacement of $f_{ij}(\|p_i - p_j\|)$. We apply the Chain Rule [14] to compute $\dot{\tilde{\mathbf{E}}}$. Therefore, we have [3]

$$\dot{\tilde{\mathbf{E}}} = \bigcap_{\xi\in\partial\mathbf{E}} \xi^T \mathbf{K} \begin{bmatrix} \dot{p} \\ \sum_{j\neq 1} f_{1j}(.)\mathbf{e}_{1j} - \upsilon\dot{p_1} \\ \vdots \\ \sum_{j\neq n} f_{nj}(.)\mathbf{e}_{nj} - \upsilon\dot{p_n} \end{bmatrix}. \tag{34}$$

[1]Please note that in the definition of dynamical system there is an implicit assumption that $p_i \neq p_j$, because $\mathbf{e}_{ij} = (p_j - p_i)/\|p_i - p_j\|$ and $\|.\|$ is smooth everywhere except at the origin

[2]It can be proven that the restriction of a locally Lipschitz continuous function $f : [a,b] \rightarrow R$ to the intervals $[a = c_0, c_1], [c_1,c_2], \cdots, [c_{n-1}, c_n = b]$ is also locally Lipschitz on $[a,b]$ where $c_0 < c_1 < \ldots < c_n$.

[3]In this section, our notations are standard and are consistent with that of [14]. For example, we have the following definition from [14]

$$\mathbf{K}[f](x) = \bigcap_{\delta>0}\bigcap_{\mu N=0} \bar{co} f(B(x,\delta) - N)$$

and $\bigcap_{\mu N=0}$ denotes the intersection over all sets N of Lebesgue measure zero. In this formula, $\bar{co}$ denotes convex closure, and μ denotes the Lebesgue measure.

Suppose that at p some of the $f_{ij}(\|p_i - p_j\|)$ are discontinuous and that S denote the set of indices of such discontinuous functions at p. We define $\underline{a}_{ij}$ and $\overline{a}_{ij}$ as follows

$$\underline{a}_{ij} = \lim_{\epsilon\rightarrow 0^-} f_{ij}(\|p_i - p_j\| + \epsilon),\ \ \overline{a}_{ij} = \lim_{\epsilon\rightarrow 0^+} f_{ij}(\|p_i - p_j\| + \epsilon)$$

for all $i,j \in S$. Since functions f_{ij} are nondecreasing, it follows that $\underline{a}_{ij} < \overline{a}_{ij}$. Thus

$$\partial\mathbf{E} = -\begin{bmatrix} \sum_{\{1,j\}\in S} [\underline{a}_{1j}, \overline{a}_{1j}]\mathbf{e}_{1j} + \sum_{\{1,j\}\notin S} f_{1j}(.)\mathbf{e}_{1j} \\ \vdots \\ \sum_{\{n,j\}\in S} [\underline{a}_{nj}, \overline{a}_{nj}]\mathbf{e}_{nj} + \sum_{\{n,j\}\notin S} f_{nj}(.)\mathbf{e}_{nj} \\ \dot{p} \end{bmatrix}. \tag{35}$$

Also, we have

$$\mathbf{K}\begin{bmatrix} \dot{p} \\ \sum_{j\neq 1} f_{1j}(.)\mathbf{e}_{1j} - \upsilon\dot{p_1} \\ \vdots \\ \sum_{j\neq n} f_{nj}(.)\mathbf{e}_{nj} - \upsilon\dot{p_n} \end{bmatrix}$$

$$\subseteq \begin{bmatrix} \dot{p} \\ \sum_{\{1,j\}\in S} [\underline{a}_{1j}, \overline{a}_{1j}]\mathbf{e}_{1j} + \sum_{\{1,j\}\notin S} f_{1j}(.))\mathbf{e}_{1j} - \upsilon\dot{p_1} \\ \vdots \\ \sum_{\{n,j\}\in S} [\underline{a}_{nj}, \overline{a}_{nj}]\mathbf{e}_{nj} + \sum_{\{n,j\}\notin S} f_{nj}(.))\mathbf{e}_{nj} - \upsilon\dot{p_n} \end{bmatrix} \tag{36}$$

Assume that $\xi \in \partial\mathbf{E}(p,\dot{p})$. Therefore, from equation (35) it follows that

$$\xi = \begin{bmatrix} -\sum_{\{1,j\}\in S} \xi_{1j}\mathbf{e}_{1j} - \sum_{\{1,j\}\notin S} f_{1j}(.)\mathbf{e}_{1j} \\ \vdots \\ -\sum_{\{n,j\}\in S} \xi_{nj}\mathbf{e}_{nj} - \sum_{\{n,j\}\notin S} f_{nj}(.)\mathbf{e}_{nj} \\ \dot{p} \end{bmatrix}. \tag{37}$$

where $\xi_{ij} \in [\underline{a}_{ij}, \overline{a}_{ij}]$. By substituting (36) and (37) into (34), we get

$$\dot{\tilde{\mathbf{E}}} \subseteq \bigcap_{\xi_{ij}\in[\underline{a}_{ij},\overline{a}_{ij}]} \sum_{\{i,j\}\in S} [\underline{a}_{ij} - \xi_{ij}, \overline{a}_{ij} - \xi_{ij}]\dot{p}_i \cdot \mathbf{e}_{ij} - \upsilon\sum_{i=1}^{n}\|\dot{p_i}\|^2$$

$$= \{0\} - \upsilon\sum_{i=1}^{n}\|\dot{p_i}\|^2 \leq 0.$$

The last equality is the result of the fact that for any given interval $[a,b]$ we have $\bigcap_{\xi\in[a,b]}[a-\xi, b-\xi] = 0$. This establishes the stability but not the asymptotic stability of favorable equilibria. We use nonsmooth version of LaSalle's theorem to prove local asymptotical stability of the favorable equilibria. Let point $(p,\dot{p}) = (p_0, 0)$ be a favorable equilibrium of the multi-particle dynamical system. We consider $N((p_0,0),\epsilon)$ a ϵ-neighborhood of the favorable equilibrium in which there is no unfavorable equilibrium in $N((p_0,0),\epsilon)$. We also define $\mathcal{C} = \{(p,\dot{p}) \mid \mathbf{E}(p,\dot{p}) \leq 1\}$ and $\Omega = \mathcal{C} \bigcap N((p_0,0),\epsilon)$. Since $\mathcal{C}$ is a closed set, Ω is a compact set. Moreover, $\dot{\tilde{\mathbf{E}}} \leq 0$. Thus, every Filippov solution to the autonomous dynamical system (8) that starts in Ω remains in Ω. As a consequence of the LaSalle's theorem [14], the trajectory enters the largest invariant set in

$$\Omega \bigcap \{(p,\dot{p}) \mid 0 \in \dot{\tilde{\mathbf{E}}}\} \subseteq \Omega \bigcap \{(p,\dot{p}) \mid \dot{p} = 0\}.$$

One can see that $\dot{p} = 0$ and p is constant. Therefore, the trajectory converges to an equilibrium. On the other hand, in Ω there is no unfavorable equilibrium. It concludes that the manifold of favorable equilibria is attractive. ■

B. Instability of Filipov's Equilibria with Nonzero Net Force

Theorem 6: For the multi-particle dynamical system defined as in Theorem 3, the Filipov's equilibria with nonzero vector fields are unstable.

Proof: Let $p = [p_0^T, p_1^T, ..., p_{2k-1}^T]^T$ be an equilibrium of the multi-particle dynamical system. Assume that the net force on particle p_1, i.e.,

$$F_1(p) = f_{10}(\|p_1 - p_0\|)\mathbf{e}_{10} + f_{12}(\|p_1 - p_2\|)\mathbf{e}_{12} + f_{13}(\|p_1 - p_3\|)\mathbf{e}_{13} \neq 0 \quad (38)$$

is nonzero. This can only happen if $0 \in \mathbf{K}[F_1](p)$. Since functions $f_{10}(.)$ and $f_{12}(.)$ are continuous, we have

$$0 \in \{f_{10}(\|p_1 - p_0\|)\mathbf{e}_{10} + f_{12}(\|p_1 - p_2\|)\mathbf{e}_{12}\} + \mathbf{K}[f_{13}](\|p_1 - p_3\|)\mathbf{e}_{13}. \quad (39)$$

From equations (38) and (39), we can conclude that f_{13} is discontinuous at $\|p_1 - p_3\|$. Therefore, $\|p_1 - p_3\| = \eta$ and it follows that

$$0 \in \{f_{10}(\|p_1 - p_0\|)\mathbf{e}_{10} + f_{12}(\|p_1 - p_2\|)\mathbf{e}_{12}\} + [-w_2, 0]\mathbf{e}_{13}.$$

Assuming

$$v_1 = f_{10}(\|p_1 - p_0\|)\mathbf{e}_{10} + f_{12}(\|p_1 - p_2\|)\mathbf{e}_{12},$$

we get

$$0 \in \{v_1\} + [-w_2, 0]\mathbf{e}_{13}. \quad (40)$$

Since $v_1 \neq 0$, equation (40) can only hold if

$$v_1 = \alpha\, \mathbf{e}_{13} \quad \text{and} \quad 0 < \alpha \leq 2w_1 < w_2.$$

An infinitesimal deviation of particle p_1 from its original position results in an infinitesimal change in vector v_1 (functions $f_{10}(.)$ and $f_{12}(.)$ are continuous), but it causes a substantial change in the value of $f_{13}(\|p_1 - p_3\|)$ to either 0 or $-w_2$. Therefore, around infinitesimal neighborhood of p_1 we have

$$\ddot{p_1} = v_1 + f_{13}(\|p_1 - p_3\|)\mathbf{e}_{13}. \quad (41)$$

Without loss of generality, we may assume that p_3 is at the origin and p_1 is initially located on the y-axis. Let $p_1 = [x_p \;\; y_p]^T$. Therefore,

$$\begin{bmatrix} \ddot{x}_p \\ \ddot{y}_p \end{bmatrix} = \alpha \begin{bmatrix} 0 \\ -1 \end{bmatrix} - \frac{1}{\sqrt{x_p^2 + y_p^2}} f_{13}\left(\sqrt{x_p^2 + y_p^2}\right) \begin{bmatrix} x_p \\ y_p \end{bmatrix}$$

where

$$f_{13}(z) = \begin{cases} 0 & \text{if} \quad z \geq \eta \\ -w_2 & \text{if} \quad z < \eta \end{cases}.$$

We can rewrite the above dynamical system as follows

$$\ddot{x}_p = \begin{cases} 0 & \text{if} \quad \sqrt{x_p^2 + y_p^2} \geq \eta \\ \dfrac{w_2 x_p}{\sqrt{x_p^2 + y_p^2}} & \text{if} \quad \sqrt{x_p^2 + y_p^2} < \eta \end{cases},$$

$$\ddot{y}_p = \begin{cases} -\alpha & \text{if} \quad \sqrt{x_p^2 + y_p^2} \geq \eta \\ \dfrac{(w_2 - \alpha) y_p}{\sqrt{x_p^2 + y_p^2}} & \text{if} \quad \sqrt{x_p^2 + y_p^2} < \eta \end{cases}.$$

Therefore, if $\sqrt{x_p^2 + y_p^2} < \eta$, by looking at x-component of the dynamical system at p_1, we can conclude that x_p moves away from 0. If $\sqrt{x_p^2 + y_p^2} \geq \eta$ then y_p decreases until $\sqrt{x_p^2 + y_p^2} < \eta$ and therefore x_p moves away from 0. Thus, the dynamical system (41) is unstable. Also, one can prove that an equilibrium in which more than two discontinuous elasticity functions are involved is also unstable. A similar analysis can be performed for the other particles. Therefore, all Filipov's equilibria of the multi-particle dynamical system with nonzero vector fields are unstable. ■

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