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Locating and Capturing an Evader in a Polygonal Environment

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Volkan Isler, Sampath Kannan, and Sanjeev Khanna

Abstract

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Locating and Capturing an Evader in a Polygonal Environment

I. Introduction

Pursuit-evasion games are among the fundamental problems studied by robotics researchers. In a pursuit-evasion game, one or more pursuers try to capture an evader who, in turn, tries to avoid capture indefinitely. A typical example is the homicidal chauffeur game where a driver wants to collide with a pedestrian and the goal is to determine conditions under which he can (not) do so. Among the numerous applications of this game are collision avoidance and air traffic control.

Recently, there has been increasing interest in developing strategies to capture intelligent evaders with certain sensing capabilities who are contaminating a complex environment [2], [3], [4]. The main ingredient of a pursuit-evasion game is the presence of an adversarial evader who actively avoids capture. Due to this aspect, typically a pursuit strategy is usually different than a search strategy where the target's motion is independent from the pursuer's (e.g. [5], [6]). Obtaining such pursuit-strategies is important for surveillance applications where we would like to locate, and perhaps, capture intruders who may be adversarial. Another application is a search-and-rescue operation where we would like to save a victim. In this setting, even though the victim is not adversarial, a pursuit strategy is still desirable as it guarantees a rescue regardless of the victim's actions.

To capture the adversarial nature of the game, pursuit-evasion games are usually studied in a game theoretic framework [7], [8]. The conditions under which the pursuer can capture the evader are obtained by studying a Hamilton-Jacobi-Isaacs equation which brings together the system equations of the pursuer and the evader. This approach has the advantage of yielding a closed-form solution of the game. Unfortunately, as the environments get complicated, solving Hamilton-Jacobi equations become intractable. Therefore solutions of pursuit-evasion games in complex environments are usually algorithmic.

Perhaps the most well-understood game in this context is the *visibility-based pursuit-evasion game* where one or more pursuers try to locate an evader in a polygonal environment [9], [10], [11]. In this game, the evader is very powerful: it has unbounded speed and global visibility, meaning that it knows the location of the pursuers at all times. In [3], the authors study a similar game in a probabilistic framework (where the evader performs a random walk) and propose a greedy algorithm.

In our present work, we propose randomized pursuer strategies for the visibility based pursuit-evasion problem. Randomization is a powerful technique which allows us to solve many problems that are not solvable by deterministic algorithms and has found wide-spread applications in many areas ranging from computational geometry to cryptography.

As we show in the following sections, it turns out that randomization provides a drastic increase in the power of the pursuers. For example, it is known that there are simply-connected environments where $\Theta(\log n)$ pursuers are required [11] in order to locate the evader with deterministic strategies. In contrast, we show that a single pursuer can locate the evader in any simply-connected environment with high probability, even if the evader knows the pursuer's location at all times and has unbounded speed (Theorem 2). The power of randomized strategies comes from the fact that the evader has no prior knowledge of the random decisions inherent in such strategies. It is worth noting that the randomized strategies work against any evader strategy and require no prior information about the strategy of the evader.

We also address the harder task of capturing the evader. For this problem we present a strategy for two pursuers, one of which is at least as fast as the evader. The strategy is based on the randomized strategy to locate the evader and the solution of a known lion and man problem [1] which is reviewed in Section III-A. The same strategy can be used to capture the evader while protecting a door. This problem was introduced in [12] to model scenarios where the goal is to locate the evader which may leave the polygonal area through a door and win the game.

The two-pursuer strategy can be modified so that a single pursuer can also capture the evader. However, the expected time to capture in this case, though finite, may be significantly longer than the expected time to capture with two pursuers.

Organization of the paper

We start the paper with a motivating example for randomized strategies (Section I-A). We present preliminary concepts and definitions in Section I-B. In Section II, we address the problem of locating a fast, unpredictable evader with global visibility.

Next, in Section III, we address the task of capturing the evader in a simply-connected environment. For this problem we present a randomized strategy for two pursuers, who can communicate at all times, to quickly capture the evader. We show how this strategy can be modified for a single pursuer at the expense of increasing the capture time in Section IV-A. We also present extensions of the basic two-pursuer strategy for the case where the pursuers have limited communication (Section IV-B) and for a scenario where the polygonal room has a door through which the evader can escape (Section IV-C).

A. Randomized strategies

The power of randomization in the context of pursuit-evasion games is nicely illustrated by the example in Figure 1. A similar example can be found in [13].

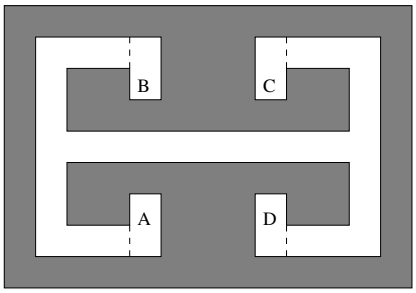

Fig. 1. A single pursuer can not capture an evader using deterministic strategies.

In this example, a single pursuer can never locate the evader using a deterministic strategy: Let us distinguish four regions A, B, C and D as shown in the figure. Now suppose the pursuer has a deterministic strategy of visiting these regions in the order A, B, C, D. In this case, the evader can first hide at B and escape to D while the pursuer is visiting A. Afterwards, it can repeat the same strategy and escape to B when the pursuer is at C. If the pursuer visits the regions in a different order, it is easy to see that the evader can find a similar strategy and avoid the pursuer. Therefore, in this polygon one pursuer can never locate the evader.

Now consider the following randomized strategy: Instead of committing to a deterministic strategy, the pursuer selects one of the regions $\{A, B, C, D\}$ uniformly at random and visits it. It is easy to see that if the pursuer guesses the region where the evader is located correctly, then the evader can not escape and the probability of this desired event is $\frac{1}{4}$. The crucial observation is that since the evader does not know which region the pursuer will visit, it can not choose a strategy based on the order of points visited by the pursuer.

One might suspect that the desired event of locating the evader is not guaranteed since the pursuer can keep guessing a wrong region forever. However, this probability can be made arbitrarily small by repeating the same strategy a few times. This is because, if k is the number of trials, the probability of missing in all k trials is only $(\frac{3}{4})^k$ in this example and this probability decreases exponentially in k. In general, if the probability of capture is p, the expected number of rounds to capture is $\frac{1}{p}$. Note that each round is independent. We can obtain the expected time to locate the evader as follows: Since the length of a round is bounded by the time to travel between two furthest points in the polygon (say T), the expected time to capture is $\frac{T}{p}$. We can convert the expected time to a high probability argument by repeating the experiment roughly $\frac{1}{p}\log\frac{1}{p}$ times using the Chernoff bound. For details of this technique the reader is referred to [14].

B. Preliminaries

Let P be the input polygon including its interior and V be the set of vertices of P. Unless stated otherwise, n denotes the number of vertices of the polygon. Two points $u, v \in P$ can *see* each other if the line segment uv lies entirely in P.

We use $d(u, v)$ to denote the length of the shortest path from u to v that remains inside P. The shortest path has the following property.

Property 1: The shortest path between any two points u and v inside a polygon P is a polygonal path whose inner vertices are vertices of P.

The *shortest path tree from a point x in P* is defined as $\cup_{v\in V} d(x, v)$. A polygon is *simply-connected* if any simple closed curve inside the polygon can be shrunk to a point. All the polygons considered in this paper are simply-connected.

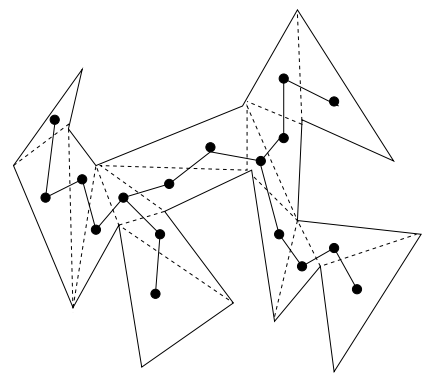

Fig. 2. Triangulation of a polygon and its dual tree.

The *triangulation* of a polygon is a decomposition of the polygon into triangles by a maximal set of non-intersecting diagonals (see Figure 2). The dual of a triangulation is a graph whose vertices correspond to the triangles. There is an edge between two vertices if the corresponding triangles share a side. It is well known that the triangulation of a simply-connected polygon has exactly $n - 2$ triangles. In addition, the dual of the triangulation is a tree [15].

Game formulations

In this paper, we study two pursuit-evasion games with different objectives. Both games take place in a simply-connected polygon P which is known to all players.

The first game, which we call the *locating game*[1] is defined as follows.

It is played between an evader and a single pursuer. An *evader strategy* is a continuous function $e : [0, \infty) \to P$ such that $e(t)$ denotes the evader's position at time t. The *pursuer strategy*, $p(t)$, is defined similarly. We assume that in both games, the evader knows the strategy of the pursuer(s) before the game starts. However, it does not have access to the outcome of the random coin tosses during the execution of the pursuer's strategy. The pursuer, on the other hand, knows nothing about the evader's strategy.

The pursuer moves with unit speed so that $\dot{p} = 1$. The *diameter* of the polygon P, denoted $diam(P)$, is defined as $\max_{u,v\in P} d(u, v)$. Since the pursuer moves with unit speed, the diameter is also equal to the maximum amount of time

[1]The general version of this game is known as the visibility-based pursuit evasion game [11].

it takes the pursuer to travel between two points in P. The evader can be arbitrarily faster than the pursuer but it must move continuously.

After the game starts, the players can observe their surroundings continuously. Further, at any given time t, the pursuer's location $p(t)$ is revealed to the evader. The pursuer wins the game if, in finite time t^*, he can reach a position such that $p(t^*)$ sees $e(t^*)$. The evader wins the game otherwise, i.e. if, for any given pursuer strategy p, there exists a strategy for the evader to avoid being seen by a pursuer who executes p.

The second game is called the *capture game* and is defined as follows.

Let $e(t)$ denote the evader's and $p_i(t)$ denote the ith pursuer's strategy as before. Instead of finding the evader, the pursuers win the capture game if in finite time t^*, they can reach a position such that there exists an i with $p_i(t^*) = e(t^*)$.

In this game, one of the pursuer's is as fast as the evader. Without loss of generality, we assume that $\dot{e} = \dot{p_1} = 1$. Similar to the finding game, the players observe their surroundings continuously and at any given time t, the pursuers' location $p_i(t)$ is revealed to the evader.

We assume that the players move in discrete time intervals and in turns: the evader first, followed by the pursuers. It is easy to see that a pursuit strategy that captures the evader in this formulation can be modified to a pursuit strategy which guarantees that p_1 can reach a point within unit distance from the evader in a game where the players move continuously and simultaneously. It is known that it is not possible to capture the evader in the continuous formulation [16] (see also [17]).

II. Locating the evader

In this section, we study the finding game and show that for any simple polygon P, the pursuer can locate the evader in $O(n \cdot diam(P))$ expected time.

A. The pursuer strategy

Given polygon P, the pursuer first triangulates the polygon. The pursuer's strategy is divided into rounds of length at most $diam(P)$. Let T be the triangulation tree (see Figure 2) rooted at the triangle that contains the pursuer's initial location at the beginning of a round. For any triangle t let $t_1, .., t_k$, $k \leq 3$ be the children of t. We use the notation $T(t)$ to denote the subtree of T rooted at the triangle t. Figure 3 is provided for quick reference to the notation used in this section.

The pursuer's strategy relies on the following observation: Suppose the pursuer is inside triangle t and the evader is located inside a triangle contained in $T(t_j)$ for some j. Then, while the pursuer is located at t, the evader can not enter any triangle contained in $T(t_i)$, $i \neq j$ without being seen by the pursuer. This is because the triangle t is a separator for the subtrees $T(t_i)$. Moreover, this property is preserved if the pursuer moves to the triangle t_j.

Therefore, had the pursuer known the subtree that contains the evader, he could gradually move towards it while preventing the evader to move from one subtree to another. This process guarantees that the pursuer can enter the triangle which contains the evader and this clearly implies that the evader would be located. Of course, the pursuer does not know where the evader is. This is where we will utilize randomization.

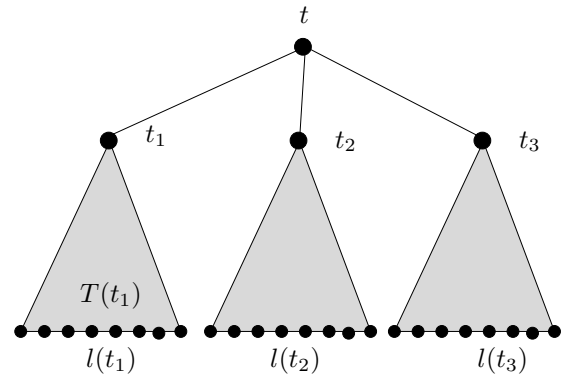

Fig. 3. Notation used for Lemma 1. Each vertex of the tree corresponds to a triangle in the triangulation tree.

The pursuer will guess the subtree that contains the evader according to the following rule:

Let $l(t)$ denote the number of leaves of the subtree $T(t)$. Suppose the pursuer is located in triangle t and let $t_1, .., t_k$ be the children of t (see Figure 3). Let $L = \sum_{i=1}^{k} l(t_i)$. With probability $\frac{l(t_i)}{L}$, the pursuer picks the child t_i and moves there. The round is over whenever the pursuer arrives at a leaf of T.

Next, we show that using this guessing strategy, the pursuer efficiently locates the evader.

Lemma 1: Let T be the triangulation tree rooted at the triangle that contains the pursuer's initial location in the beginning of the round. At each round, if the pursuer follows the guessing strategy described above, he can locate the evader with probability at least $\frac{1}{L}$.

Proof: The lemma is proven by induction on the height of T. The basis, where the height of t is 0, corresponds to the case where the input polygon is a triangle. The pursuer trivially locates the evader with probability 1 in this case.

Let $p(t)$ be the probability that the evader is located within a round, starting from triangle t and inductively assume that the lemma is true for all trees of height less than or equal to j.

Given a triangulation tree of height $j+1$, the probability of success starting from the root t is:

$$p(t) \geq \min\left\{\frac{l(t_1)}{L}p(t_1), \ldots, \frac{l(t_k)}{L}p(t_k)\right\} \tag{1}$$

Note that all the subtrees $T(t_i)$ have height at most j, therefore by the inductive hypothesis we have $p(t_i) \geq \frac{1}{l(t_i)}$ for all i and the lemma follows. ■

Clearly, the number of leaves of any triangulation tree is less than the number of vertices of the polygon, therefore at each round the evader is located with probability at least $\frac{1}{n}$. Moreover, since the length of a round is $diam(P)$, we have the main result of this section:

Theorem 2: In any simply connected environment P, against any evader strategy, the expected time to locate the evader with a single pursuer is at most $n \cdot diam(P)$ where n is the number of vertices and $diam(P)$ is the diameter of the polygon.

Remark 3: Any simply-connected polygon can be partitioned into a minimum number of disjoint convex polygons in polynomial time [18], [19]. The dual of such a partition will also be a tree. Therefore, instead of using a triangulation dual, the pursuer can execute the strategy described above using the dual of the convex partition. However, in general this does not improve the expected capture time. For example, for the polygon shown in Figure 4, the number of leaves of the triangulation dual is equal to the number of leaves of the dual of a minimum convex partition.

B. Lower bounds

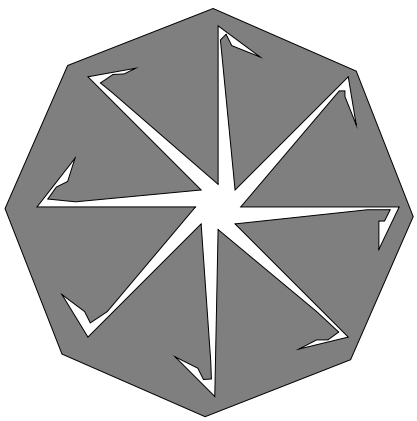

Fig. 4. For any randomized pursuer strategy, the expected time to capture the evader in this star with hooks is $O(n \cdot diam(P))$.

One might suspect that the expected time to locate an evader can be improved using a more sophisticated strategy. Unfortunately, this is not possible: The polygon in Figure 4 is a $k-$star with hooks attached at the end of each spike (in the figure $k = 8$). The evader's strategy is to choose a hook at random and hide there until the end of the game. In order to locate the evader, the expected number of spikes searched by the pursuer is $\frac{k}{2}$ and it takes $diam(P)$ steps to travel from one spike to another. Since the number of vertices is a constant multiple of k, the time it takes to locate the evader is $\Omega(n \cdot diam(P))$. In fact, using the well-known technique due to Yao, this argument can be extended to show that the expected time to capture the evader for *any* randomized pursuer strategy is $\Omega(n \cdot diam(P))$ (see [14] for details).

We present the results of a simulation of the pursuer and the evader strategy for such an environment in Figure 5. For the simulation, the visibility-based pursuit evasion game was played 1000 times. The average number of rounds for locating the evader is 8.9. This is in agreement with Lemma 1 since the number of the leaves of the triangulation dual is 9.

On the other hand, the randomized strategy may not be optimal for some environments. The simplest example of such an environment is a star-shaped polygon such as the one shown in Figure 6. In this environment, the optimal strategy is to go to a point (e.g. x in the figure) from where the entire polygon (hence, the evader) will be visible.

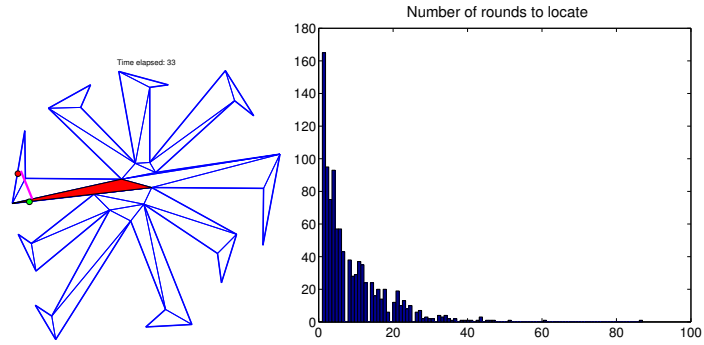

Fig. 5. **Left:** An instance of the simulator showing the triangulation of the environment as well as the hiding location of the evader. **Right:** The histogram of the number of rounds required to locate evader in 1000 simulations. The mean μ and the standard deviation σ of the number of rounds was $\mu = 8.8960$ and $\sigma = 9.0479$.

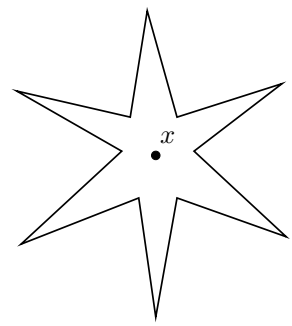

Fig. 6. A star-shaped polygon. The optimal strategy is to go to a point (e.g. x in the figure) from where the entire polygon is visible.

III. Capturing the evader with two pursuers

In this section, we move on to the more challenging task of capturing the evader, defined as moving to the same point as the evader. We start by presenting a pursuit strategy for two pursuers (who can communicate at all times) to capture the evader. Later, we will show how to modify this strategy to obtain a strategy for i) two pursuers who have only line-of-sight communication ii) a single pursuer (at the expense of increasing the expected capture time).

The strategy of one of the pursuers is based on the solution to a problem known as the *lion and man* problem [1]. We present an extension of this strategy in the case of a (possibly non-convex) polygonal environment. One of the major difficulties for our pursuers is that the evader may not be visible at all times, in which case the lion's strategy is not well-defined. The second pursuer will use the strategy presented in the previous section to tackle this difficulty.

We start with a review of the lion's strategy.

A. The lion and man problem

The *lion and man* problem with discrete time in the non-negative quadrant of the plane is attributed to David Gale [20]. Let the initial positions of the lion and man be $L_0 = (x_0, y_0)$ and $M_0 = (x'_0, y'_0)$, respectively. In each round, first the man moves to any point in the quadrant at distance at most 1 from his current position, and then the lion does the same. The lion wins if he moves to the current position of the man. The man

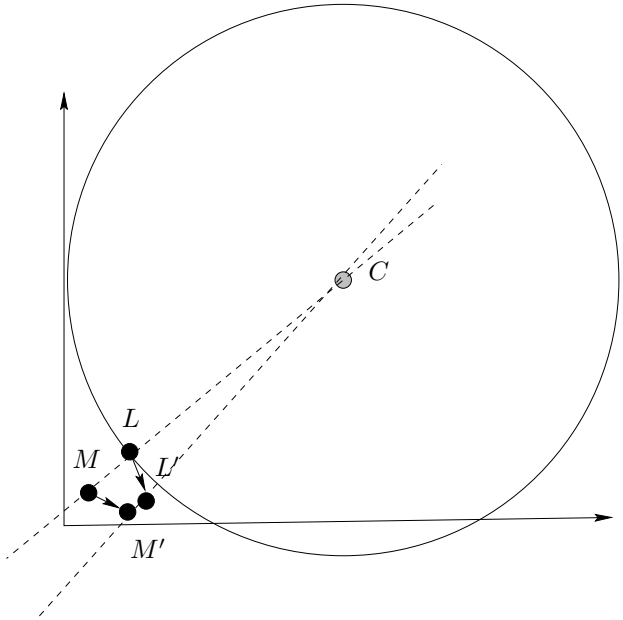

Fig. 7. Lion's strategy

wins if he can keep escaping for infinitely many rounds. In [1], Sgall proves that, when both $x_0' < x_0$ and $y_0' < y_0$, the lion always catches the man in a finite number of rounds. The number of moves required is bounded by a quadratic function in x_0, y_0 and the slope (or its inverse) of the line segment L_0M_0.

B. Lion's strategy

Let the initial positions of the lion and man be $L_0 = (x_0, y_0)$ and $M_0 = (x_0', y_0')$, respectively. In the beginning of the game, the lion finds a point C on the line M_0L_0 such that L_0 is inside the segment M_0C and the circle with center C, radius $|CL_0|$ and passing through L_0 intersects both axes. Among all possible such circles, it chooses the one whose center is closest to the origin. C remains fixed throughout the game.

Let L and M denote the current positions of the lion and the man respectively (see Figure 7). Let M' denote the point the man moves to, $|MM'| \leq 1$. If $|LM'| \leq 1$, the lion catches the man. Otherwise, it moves to a point L' on the line $M'C$ such that $|L'L| = 1$. There are two such points, it chooses the one closer to the man.

Definition 4: We will refer to this move as the *lion's move from L with respect to C and M′*.

The lion's move maintains the following:

Lemma 5 ([1]): If the lion does not catch the man in the current move then

(i) M' has both coordinates strictly smaller than C,

(ii) L' is inside the segment $M'C$, and

(iii) $|L'C|^2 \geq 1 + |LC|^2$.

Proof: See [1]. ■

C. The strategy to capture the evader

Let $p_1(t), p_2(t)$ and $e(t)$ denote the locations of the pursuers and the evader, respectively, at time t. In the beginning of the game the two pursuers move together and search for the evader using the strategy described in the previous section. Without loss of generality, we assume that the game starts at $t = 0$ where $p_1(0) = p_2(0) = o$ and $e(0)$ is visible from o. We will sometimes refer to point o as the *origin*. The origin will be fixed until the evader is captured. Let $d_1(t) = d(p_1(t), o)$, $d_2(t) = d(p_2(t), o)$, and $d_e(t) = d(e(t), o)$.

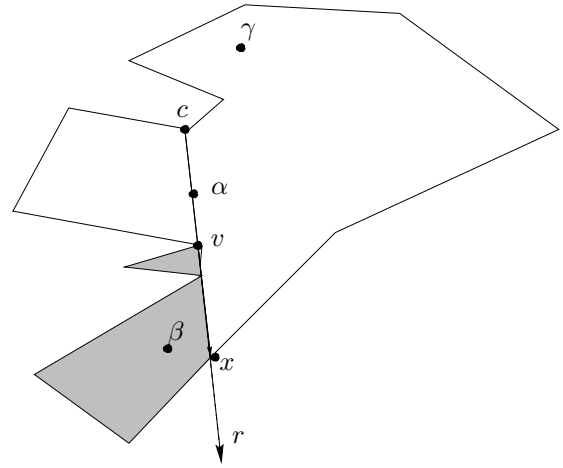

Fig. 8. Pocket with respect to c and v

Definition 6: Suppose $e(t)$ is visible from $p_1(t)$ but $e(t+1)$ is not visible from $p_1(t)$. This means that the shortest path P from $p_1(t)$ to $e(t+1)$ is composed of at least two line segments (Property 1). The first vertex on the path from $p_1(t)$ to $e(t+1)$ is called a *pseudo-blocking vertex*.

Let r be the ray starting from a vertex c and passing from another vertex v that is not adjacent to c. In the sequel, c will be the center of the circle for the lion's move and v will be the pseudo-blocking vertex. Consider the first time the ray r leaves the polygon P after it passes through v and let x be the point on $r \cap P$ just before this happens (see Figure 8). The line segment vx splits the boundary of the polygon into two chains. The chain which does not contain the point c, together with the line segment vx defines a polygon. We will refer to this polygon as *the pocket with respect to c and v*. The line segment vx is referred to as the *entrance of the pocket*.

We will utilize the following properties of pockets:

Property 2: Let α be a point on the line segment cv and β be a point in the pocket with respect to c and v. The line segment αv is contained in the shortest path from α to β (Figure 8).

Property 3: Let R be a pocket with respect to c and v inside a polygon P. Any path from $\beta \in R$ to $\gamma \in P - R$ crosses the entrance of the pocket (Figure 8).

Looking ahead, let us describe how we will utilize these properties: Suppose pursuer p_1 is moving towards the evader and the evader disappears. Let v be the current pseudo-blocking vertex. If p_1 moves towards v, Property 2 implies that it is still moving on the shortest path from the evader to the origin. If the evader becomes visible before p_1 reaches v, Property 3 implies that it must cross the entrance of the pocket and p_1 can continue its strategy (described in the next section) as if the evader has not disappeared.

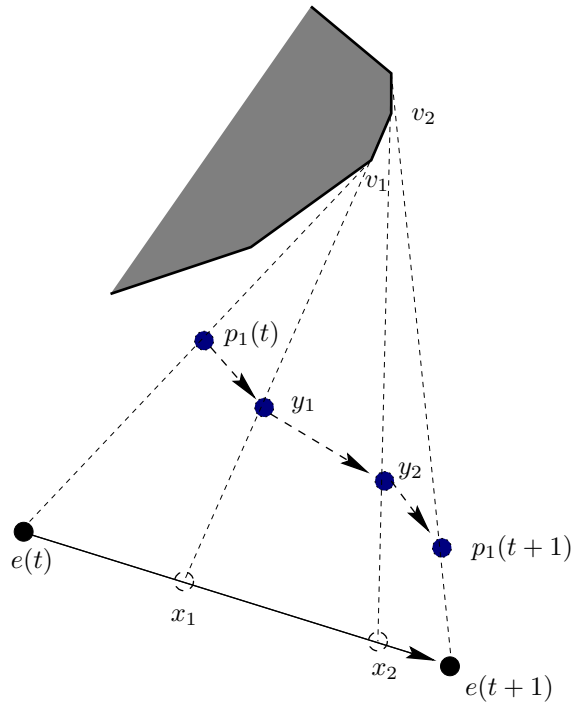

Fig. 9. The extended lion's move

If the evader is not visible when p_1 arrives at v, then v becomes a *blocking vertex*. At this point, the second pursuer will enter the game.

Next, we present the details of the strategies of p_1 and p_2.

D. Strategy of Pursuer p_1

As stated earlier, we assume that pursuer p_1 is at least as fast as the evader. At time step t, p_1 moves according to the following strategy:

If the evader is visible, he performs a *extended lion's move* which is defined as follows: Let τ be the shortest path from $e(t)$ to $e(t+1)$. Without loss of generality, p_1 will pretend that the evader followed τ. As a point x moves from $e(t)$ to $e(t+1)$ along τ, the vertices on the shortest path from x to the origin o may change. However, the number of changes is at most n: The first vertex on the shortest path from x to o must be one of the vertices of the polygon. Since τ is the shortest path from $e(t)$ to $e(t+1)$, each vertex of the polygon can be this first vertex for at most one contiguous sub-path in τ. Let $x_1, \ldots, x_{k-1}$ correspond to the points on τ where such changes occur, we define $x_0 = e_t$ and $x_k = e(t+1)$. Let v_i be the first vertex on the shortest path from x_i to o. The extended lion's move consists of $k-1$ phases. During Phase i, $i = 1, \ldots, k$, pursuer p_1 performs the lion's move with respect to v_i and x_i (see Figure 9). Note that the time spent by the pursuer in Phase i is equal to the time spent by the evader in traveling from x_{i-1} from x_i.

If the evader was visible in the previous time step, but is not visible any more, let v be the pseudo-blocking vertex. Pursuer p_1 moves towards v until he reaches it. If the evader becomes visible before p_1 arrives at v, he continues with the lion's move. Otherwise, v becomes a blocking vertex.

If the evader is still not visible after p_1 reaches the blocking vertex, he waits for p_2 to report the location of the evader. Let R be the current pocket defined with respect to the blocking vertex and the current center. There are two possibilities.

1) The evader reveals itself to p_1. Then, by Property 3, this must happen before the evader crosses the entrance of R. In this case p_1 continues the game with the lion's move.

2) Pursuer p_2 finds the evader located at e. Let v' be the first vertex on the shortest path from v to e and R' be the pocket with respect to v and v'. In this case, v' becomes a pseudo-blocking vertex, R' becomes the new pocket and p_1 continues his strategy by moving towards v'.

E. Strategy of Pursuer p_2

The task of pursuer p_2 is to search for the evader when it is not visible to p_1. When the evader disappears from the sight of p_1, pursuer p_2 waits until p_1 reaches the blocking vertex. Afterwards, p_2 locates the evader using the strategy described in the previous section and reports the location of the evader to p_1.

F. Properties of Pursuer p_1's strategy

Lemma 7: For all times t, pursuer p_1 maintains the following invariants until the evader is caught:

(I1) $p_1(t)$ is on the shortest path from o to $e(t)$.

(I2) $d_1(t+1)^2 \geq d_1(t)^2 + \frac{1}{n}$ if $p_1(t) \neq p_1(t+1)$.

Proof of Invariant I1: We prove the invariant by induction. Assume that it holds at time t.

First consider the case where p_1 can see e at time t. Let the first vertex on the shortest path from $e(t)$ to o be u. It follows that $p(t)$ is in the line segment joining u to $e(t)$, since if $p(t)$ is between o and u on the shortest path, he would not be able to see $e(t)$.

Let x denote the evader's position at an arbitrary time in the time interval $[t, t+1)$. Suppose when the evader is at x, the first vertex on the shortest path from x to o changes from u to v. Note that p_1 can see the evader until this point. Then, the shortest path from x to o passing through u and the shortest path from x to o passing through v have the same length. This implies that u, v, and x have to be collinear. For otherwise, a shorter path from x to o can be found in the interior of the polygon formed by these two presumed shortest paths from x to o, which is a contradiction.

This implies that either u is an ancestor or a descendant of v in the shortest path tree rooted at o. If u is an ancestor, at the point x where the switch occurs, p_1 could either be on the segment vx in which case it can continue the lion's move in the next phase or p_1 is on the segment uv, in which case e will become invisible to p_1 after x. In this case, p_1 must be either moving towards a pseudo-blocking vertex or waiting at a blocking vertex. In both cases, the invariant is maintained by Property 2. If u is a descendant of v, then p_1 is already on the segment ux and hence on the segment vx. Hence it can continue the lion's move in the next phase. The invariant is therefore maintained as a corollary of Lemma 5.

Otherwise, if p_1 does not see the evader at time t, he must be either waiting at a blocking vertex or moving towards

a pseudo-blocking vertex. In both cases, the invariant is maintained by Property 2. ■

Proof of Invariant I2:

If p_1 is moving towards a pseudo-blocking vertex, his distance to the origin is increasing by 1 and the invariant is maintained.

Next, we show that the extended lion's move maintains the invariant: Suppose the lion's move has $k \leq n$ phases and consider phase i of the extended lion's move where the evader moves from the point x_{i-1} to x_i. Suppose, during this phase the pursuer p_1 moved from point y_{i-1} to y_i (see Figure 9) and let v_i be the center of the circle for the lion's move during this phase.

Let $\alpha_i = d(o, y_i) - d(o, y_{i-1})$.

As a corollary of Lemma 5 we have

$d(y_i, o)^2 \geq d(y_{i-1}, o)^2 + \alpha_i^2$.

Summing up over all phases we get the total progress to be $\sum_{i=1}^{k} \alpha_i^2$.

This expression when subject to $\sum_{i=1}^{k} \alpha_i = 1$ is minimized when all $\alpha_1 = \ldots = \alpha_k = \frac{1}{k}$. Therefore we have $d_1(t+1)^2 \geq d_1(t)^2 + \frac{1}{k}$ which implies the invariant I2. ■

The combined strategy of the two pursuers can be viewed as follows: Pursuer p_1 moves only when it knows the shortest path from the evader to the origin o. Performing the lion's move is equivalent to growing a disk inside the polygon whose center is at the origin o and passes through the current location of p_1. By invariant I1, the evader can never enter the disk. Further, the disk is still protected if p_1 does not move. Invariant I2 implies that, whenever p_1 moves, the disk monotonically grows and the evader is eventually squeezed between p_1 and the polygon boundary.

Pursuer p_2 moves only when p_1 does not know the evader's path to the origin. It locates the evader using the randomized strategy given in the previous section and reports its location to p_1 so that p_1, in turn, can keep growing the disk and eventually capture the evader.

G. Expected time to capture

Let $T_1 = diam(P)$ be the time it takes pursuer 1 (who performs the lion's move) to travel the diameter of the polygon. By Invariant I2 (Lemma 7), this pursuer will capture the evader in nT_1^2 steps. However, in the meantime, pursuer 2 may have to search for the evader. Each such search ends in time $T_2 \cdot n \cdot \log n$ with high probability, where T_2 is the time for pursuer p_2 to travel the diameter of the polygon. Moreover, once a vertex becomes a blocking vertex, it will never become a blocking vertex again (it will be included in the ball defined by the origin and p_1), therefore at most n such searches will be necessary. In conclusion, the expected time to capture the evader is $O(nT_1^2 + T_2 \cdot (n^2 \log n))$ with high probability.

IV. Extensions of the two-pursuer strategy

In this section, we present three extensions of the two-pursuer strategy presented in the previous section. In Section IV-A, we show how a single pursuer can implement the same strategy at the expense of increased capture time. In Section IV-B, we show how the global communication requirement can be relaxed. Finally, in Section IV-C, we show that two pursuers can capture the evader even if the polygon has a door through which the evader can escape and win the game.

A. Capturing the evader with a single pursuer

Suppose we have only pursuer p_1. In this case, instead of waiting for p_2 to find the evader, p_1 can guess the first vertex on the shortest path from the evader to his current location and move there.

Consider the shortest path tree T from the origin o to the vertices of the polygon. For each vertex v, let $l(v)$ be the number of leaves of the subtree $T(v)$ of T rooted at the vertex v. Then the probability that the pursuer's guess will be successful if he is located at v is at least $\frac{1}{l(v)}$. If the guess is correct and the evader is visible, the pursuer continues with the lion's move. However, in case of a wrong guess the evader may end up in an advantageous location and move towards the origin o, in which case the pursuer must restart the game. Further, if all the guesses are correct, no vertex can be a blocking vertex more than once. Continuing this way we can obtain a worst-case lowerbound on the probability of success. Unfortunately, this bound can be possibly exponentially small in the number of reflex vertices in the environment. However, the expected time to capture the evader is still finite for any simply connected environment and this strategy may still be practical for simple settings.

One might suspect that an analysis similar to the one in Section II can be applied to prove that the expected time to capture is polynomial. The reason such an analysis does not apply directly is that even if the pursuer and the evader are co-located in a leaf triangle, the capture game still continues and the evader can move to another triangle in the tree. Therefore the number of guesses may exceed the depth of the tree, resulting in a possibly exponential capture time. This poses an interesting trade-off between the pursuer's visibility and the capture time. If the pursuer can somehow track the evader at all times (perhaps using a satellite), then Lemma 7 implies that he could capture the evader in time $O(n \cdot diam(P)^2)$. If this is not possible though, he can either use a second pursuer for locating the evader and still capture it in polynomial time or simultaneously search and capture which results in a much longer capture time.

B. Relaxing the global communication requirement

Consider the following sequence of events. Pursuer 1 arrives at the blocking vertex v. Suppose at that time the evader is not visible from v. Therefore pursuer 2 starts searching for the evader and finds it at time t.

In the previous section, we assumed that the two pursuers can communicate all the time. Hence, after finding the evader at time t, pursuer 2 can compute the first vertex u on the shortest path from $p_1(t)$ to $e(t)$ and report u to pursuer 1. Afterwards, u becomes the pseudo-blocking vertex and pursuer 1 starts moving toward u (Figure 10).

The two pursuers can implement this strategy even if they have only line-of-sight communication. That is, the pursuers

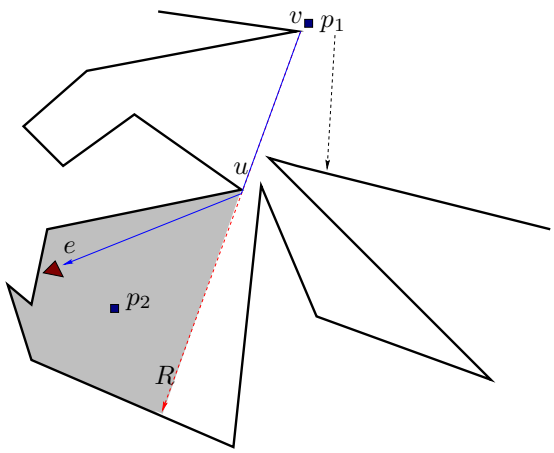

Fig. 10. Pursuer p_1 is waiting at the current blocking vertex v. Pursuer p_2 finds the evader in the pocket with respect to v and u (the shaded region). The vertex u will become the new pseudo-blocking vertex.

can communicate if and only if they can see each other. The only modification required in the strategy is the following.

After finding the evader and computing the pseudo-blocking vertex u at time t, if pursuer 1 and pursuer 2 can not see each other, pursuer 2 can not report the vertex u (see Figure 10). In this case, pursuer 2 starts moving towards pursuer 1 along the shortest path from $p_2(t)$ to $p_1(t)$. Let R be the pocket with respect to the current blocking vertex v and the pseudo-blocking vertex u (the shaded area in Figure 10). Since pursuer 1 will eventually see pursuer 2, there are two possibilities based on whether the evader becomes visible to pursuer 1 in the meantime.

If pursuer 1 sees the evader before seeing pursuer 2, this means that the evader is leaving R (Property 3). Further, at this time, pursuer 1 will be on the evader's shortest to the origin by Property 2. Therefore, in this case, pursuer 1 proceeds with the extended lion's move – maintaining both invariants $I1$ and $I2$.

Otherwise, if pursuer 1 sees pursuer 2 first, the evader must be in the pocket with respect to v and u. In this case, pursuer 2 reports the new pseudo-blocking vertex u and pursuer 1 proceeds as before by moving towards u.

C. Polygonal rooms with a door

In [12], Lee et al. studied the following variant of the pursuit-evasion problem: The input is a pair (P, d) where P is the polygonal room the game is played in and d is a *door*, a point marked on the boundary of P. The goal is to devise a strategy for the pursuer to eventually see the evader, in such a way that the evader can not escape through the door. The authors presented a characterization of polygons where a single pursuer with very narrow visibility (represented by a single ray) can locate the evader before it reaches the door.

In a similar scenario, the two pursuer algorithm presented in Section III can be used to *capture* an evader before it exits through the door. The only modification necessary is the following: Initially, pursuer p_1 is located at the door d and waits until pursuer p_2 locates the evader. Afterwards, he continues with the lion's move with respect to d. This ensures that the evader can never enter the disk whose origin is d and passes through the current location of p_1. Therefore, the door is always protected until the evader is captured.

V. Conclusion and Future Work

In this paper, we studied the visibility-based pursuit evasion game and showed that using a randomized strategy a single pursuer can locate an unpredictable evader in any simply-connected polygonal environment. The evader may be arbitrarily faster than the pursuer and it may know the location of the pursuer at all times.

The randomized strategy has some desirable properties: First, as shown in [11], there are polygonal environments which require an arbitrary number of pursuers if they are restricted to deterministic strategies. Therefore on such environments, a randomized strategy is mandatory for locating the evader with a single pursuer. Moreover, even if the polygon is deterministically searchable by a single pursuer, it is known that some of these polygons require revisiting parts of the polygon $\Omega(n)$ times [11]. Hence, the expected time to capture with a randomized strategy is comparable to the time to capture with a deterministic strategy. However, the randomized strategies may be preferable to the deterministic strategies, as they do not require complicated data structures and costly preprocessing.

Second, the randomized strategy to locate the evader does not require an exact map of the environment: It is based on the dual graph of the triangulation, therefore it is insensitive to errors in the map of the environment. An interesting research direction is to incorporate the navigation strategies in [6] which require a minimal representation of the environment.

Another interesting extension is the case of non-polygonal environments. The randomized strategy can be used to locate the evader in non-polygonal, simply-connected environments. For example, this could be done by replacing the triangulation tree (Lemma 1) with the decomposition studied in [21].

We have also studied the more challenging problem of capturing the evader. For this problem, we presented a strategy for two pursuers (one of which is as fast as the evader) to capture the evader in expected time polynomial in the number of vertices and the diameter of the environment. The strategy can be modified for a single pursuer, however it is not clear whether the expected time to capture remains a polynomial in the number of vertices. We leave this as a future research direction.

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