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Abstract

We study the existence and stability of static kinklike configurations of a five-dimensional scalar field, with Dirichlet boundary conditions, along the extra dimension of a warped braneworld. In the presence of gravity such configurations fail to stabilize the size of the extra dimension, leading us to consider additional scalar fields with the role of stabilization. We numerically identify multiple nontrivial solutions for a given five-dimensional action, made possible by the nonlinear nature of the background equations, which we find is enhanced in the presence of gravity. Finally, we take a first step toward addressing the question of the stability of such configurations by deriving the full perturbative equations for the gravitationally coupled multifield system.

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Scalar kinks in warped extra dimensions

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We study the existence and stability of static kinklike configurations of a five-dimensional scalar field, with Dirichlet boundary conditions, along the extra dimension of a warped braneworld. In the presence of gravity such configurations fail to stabilize the size of the extra dimension, leading us to consider additional scalar fields with the role of stabilization. We numerically identify multiple nontrivial solutions for a given five-dimensional action, made possible by the nonlinear nature of the background equations, which we find is enhanced in the presence of gravity. Finally, we take a first step toward addressing the question of the stability of such configurations by deriving the full perturbative equations for the gravitationally coupled multifield system.

I. INTRODUCTION

The possibility of extra spatial dimensions [1,2], hidden from our current experiments and observations through compactification or warping, has opened up a wealth of options for particle physics model building [3–17] and has allowed entirely new approaches for addressing cosmological problems [18–33]. In many implementations, standard model (SM) fields can be confined to a submanifold, or brane, while in others they populate the entire extra-dimensional space. Common to both approaches, however, is the inclusion of bulk fields beyond pure gravity, either because they are demanded by a more complete theory, such as string theory, or because they are necessary to stabilize the extra-dimensional manifold. Thus, a complete understanding of the predictions and allowed phenomenology of extra-dimension models necessarily includes a comprehensive consideration of the configurations of these bulk fields, the simplest of which are real scalars. Indeed, four-dimensional (4D) Poincaré invariance allows for these new bulk fields to acquire nontrivial static configurations along the extra dimensions.

Bulk scalar fields may also prove useful for model building. Static 1D scalar configurations with a node (where the field vanishes) are known to localize wave functions of other fields near that node. In the context of extra dimensions, these kinklike scalar backgrounds can be used, for example, to localize bulk fermions near either boundary [34–37], allowing for interesting constructions of flavor models. They can also affect the localization of other scalar or vector fields leading to a field theoretic description of fat branes (see, for example, the constructions in [38–41]). More recently, multiple bulk scalar fields have

been used to construct so-called soft-wall models, in which the profiles of the scalar fields, coupled to the gravitational background, play the role of a dynamical IR boundary (as opposed to a hard-wall IR boundary or brane, located at a specific location in the IR). Phenomenological model building in these scenarios [42,43] has drawn attention to this possibility, specially because it may alleviate some of the problems of the hard-wall approach, such as the generic tension with precision electroweak measurements [44].

The above examples all make use of static solutions of bulk scalar fields, assuming that such configurations can be dynamically realized. This latter aspect, however, has received little attention, in part because of the difficulty of analyzing a coupled multifield system in the presence of gravity. Kinklike scalar configurations are a particularly interesting case to consider because the boundary conditions make it possible to obtain nontrivial general results regarding both the existence and the stability of such configurations, at least in the case of one flat extra dimension without gravity [45,46]. In this paper we build on these previous results and extend them as far as possible to the case with a gravitating (warped) extra dimension. In the presence of gravity the kinklike configuration cannot fix and stabilize the interbrane distance [47]. It is therefore necessary to assume the existence of at least one additional stabilizing field, coupling either directly or gravitationally to the kink field. We will opt for the latter and introduce additional noninteracting scalar fields. At least one of these additional fields must be given a monotonic profile in order to stabilize the size of the extra dimension (i.e., interbrane modulus) [48]. The analysis of the resulting dynamical multifield system in the presence of gravity is a first step toward understanding the dynamics of more general multifield configurations, such as soft-wall models.

The plan of the paper is as follows. In Sec. II we review the results for kinklike backgrounds in a flat extra dimen-

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sion with no gravity. We then generalize these to include a warped gravitational background, but with no gravitational backreaction from the kink scalar itself. In Sec. IV we consider the coupled system of multiple scalar fields in the presence of gravity, and as a special case consider a kink field with Dirichlet boundary conditions and another scalar whose purpose is to stabilize the whole configuration. We write the background equations for such a system and show graphically how nonlinearities allow a given action to have multiple static solutions.

In the final section we take the first steps toward studying the question of the stability of these static configurations by deriving the complete set of equations for scalar and gravitational perturbations around a given static background. The general procedure is quite complex and involves extended theorems of oscillation theory appropriate to the type of eigenvalue problem we are led to, namely, a matrix Sturm-Liouville problem. We therefore reserve a complete study of the stability of the general system for future work.

II. KINKED SCALARS IN FLAT EXTRA DIMENSIONS

In [45,46] a 5D flat scenario including one real scalar field with an arbitrary scalar potential was studied and the general conditions for the existence and perturbative stability of static, nontrivial, background scalar field configurations were presented. In this section we briefly review the main results and slightly extend the discussion of the energy densities of different kink configurations.

Consider a real scalar field in five dimensions (labeled by indices $M, N, \ldots = 0, 1, 2, 3, 5$) with a flat background metric, and defined by the action

$$ S = \int d^5x \left[\frac{1}{2} \eta^{MN} (\partial_M \phi) \partial_N \phi - V(\phi) \right]. \quad (1) $$

The extra dimension is compactified on an S_1/Z_2 orbifold with the scalar field $\phi(x, y)$ being odd under Z_2 reflections along the extra coordinate $x^5 \equiv y$ [i.e., $\phi(x, y) = -\phi(x, -y)$]. Here the orbifold interval is defined as $[0, \pi R]$, with its size πR assumed to be fixed. The potential $V(\phi)$ must then be invariant under the discrete symmetry $\phi \to -\phi$, and is chosen to have at least two degenerate global minima at $\phi = \pm v$, with $v \neq 0$. To simplify notation, we will also choose the potential to vanish at $\phi = 0$.

Under these conditions, it was shown in [45] that there will always be static solutions, nontrivial along the extra coordinate y, satisfying the (static) field equation

$$ \phi'' - \frac{\partial V}{\partial \phi} = 0, \quad (2) $$

where a prime denotes a derivative with respect to y. The profiles of these solutions, satisfying Dirichlet boundary conditions, resemble that of a kink solution patched to an antikink in the middle of the interval. The possible solu-
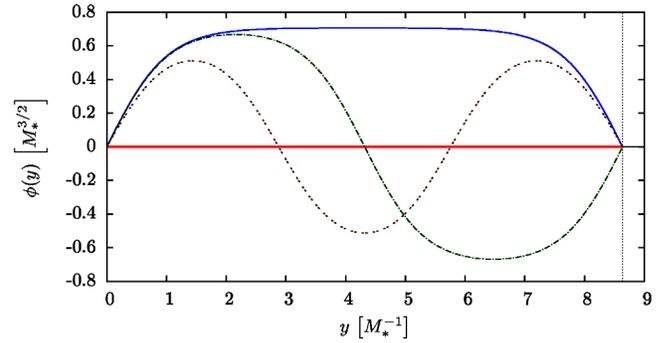

FIG. 1 (color online).   Profiles in the extra-dimension interval $[0, \pi R]$ of different static configurations of the Dirichlet scalar field ϕ, defined by the scalar potential $V(\phi) = -\frac{1}{2}|\mu^2|\phi^2 + |\lambda|\phi^4$ ($\mu^2 = 2M_*^2$, $\lambda = 1M_*^{-1}$, $\pi R = 8.6375M_*^{-1}$). The solutions with nodes in the interval (dashed curves) are unstable, while the stability of the nodeless and trivial solutions (solid curves) depend on the parameters of the model. The vertical dashed line indicates the location of the second brane.

tions were classified in two groups, namely, those with nodes in the interval (multiple kink-antikink solutions patched together) and those with no nodes, vanishing only at the end points of the orbifold (see Fig. 1). It was shown that all static kink solutions with nodes are perturbatively unstable, whereas the stability of nodeless solutions depends on the parameters of the model in a particularly simple way.

The Dirichlet solutions of Eq. (2) with no nodes in the interval form a continuous one-parameter family of functions. A simple choice for the parameter is the amplitude A of the solution, i.e., the maximum value of the nontrivial solution $\phi_A(y)$. Solutions with different amplitudes A generally vanish at different points along the extra dimension, which correspond to different possible orbifold radii R (see Fig. 2). However, in order to obtain the stability condition
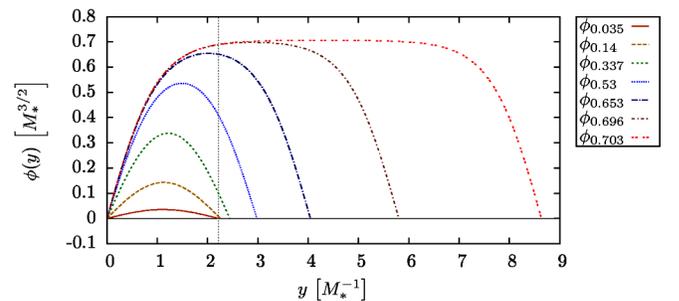
FIG. 2 (color online).   Nodeless static configurations of the kink-scalar field ϕ, defined by the scalar potential $V(\phi) = -\frac{1}{2}|\mu^2|\phi^2 + |\lambda|\phi^4$ ($\mu^2 = 2M_*^2$, $\lambda = 1M_*^{-1}$). Configurations with different amplitudes are solutions to different physical problems, corresponding to different stabilization radii of the extra dimension. The vertical dashed line indicates the minimal radius R_c, below which nodeless solutions do not exist with this potential.

for these solutions it is extremely useful to consider the full family of solutions.

The value of the 4D effective energy density of a given static solution $\phi_A(y)$ is

$$E(A) = \int_0^{T(A)} \left(\frac{1}{2} \phi_A'^2 + V(\phi_A) \right) dy, \qquad (3)$$

where $T(A)$ is the length of the solution in the extra dimension. This can be conveniently rewritten as an integral over ϕ using properties of Eq. (2) and its solutions $\phi_A(y)$

$$E(A) = 2\sqrt{2} \int_0^A \frac{V(A) - 2V(\phi)}{\sqrt{V(\phi) - V(A)}} d\phi. \qquad (4)$$

We are now equipped to state the general results of [45,46] in a slightly modified, although more revealing, version:

Proposition 1.—A static solution to Eq. (2), with $\delta > 0$ nodes inside the orbifold interval, is always unstable.

Proposition 2.—A static, nodeless solution $\phi_{A_*}(y)$ to Eq. (2), with amplitude A_* and associated energy density $E(A_*)$, is stable if

$$\left. \frac{dE}{dA} \right|_{A=A_*} < 0. \qquad (5)$$

This is a powerful result since it means that given any scalar potential $V(\phi)$ we immediately know which of the nontrivial nodeless solutions ϕ_A will be stable or unstable, without the need to actually know explicitly their analytic form.

With this result it is possible to understand the vacuum structure of any single scalar field theory with Dirichlet boundary conditions in five dimensions when the metric along the extra dimension is flat. Possible static solutions consist of the trivial solution $\langle \phi \rangle = 0$ (which may or may not be stable), kinklike solutions with nodes in the interval (which are always unstable), and kinklike solutions without nodes in the interval [some stable and some unstable, depending on condition (5)]. As remarked in [45,46], the trivial solution may be the true vacuum solution even in the case of a negative mass term $-|\mu^2|\phi^2$ in the 5D potential, as long as the inequality $|\mu^2| < |1/R^2|$ is preserved. Therefore, for a given orbifold radius R, many different perturbatively stable vacuum solutions are possible, and it is necessary to identify which one is the true vacuum of the theory.

The true vacuum of the theory will depend on the size of the radius R. This can be seen as follows: Without loss of generality, one may define the energy density of the trivial solution to be zero by choosing the 5D potential $V(\phi)$ to vanish at $\phi = 0$. It was shown in [45,46] that there is a critical radius R_c below which nontrivial nodeless solutions do not exist (see Fig. 2). The energy density associated with the critical nontrivial nodeless solution will be either positive or exactly zero, so that the transition from

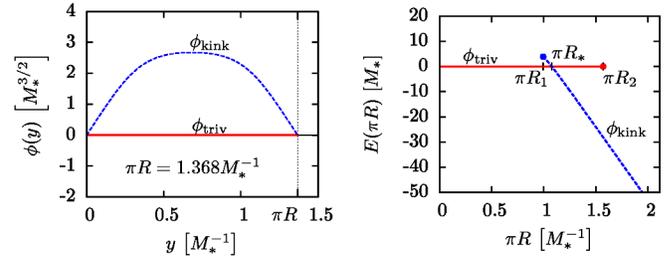FIG. 3 (color online). Profiles (left panel) in the extra-dimension interval $[0, \pi R]$ of the two possible stable static configurations of the Dirichlet scalar field ϕ, defined by the scalar potential $V(\phi) = -\frac{1}{2}|\mu^2|\phi^2 - \frac{1}{4}|\lambda|\phi^4 + \frac{1}{6}|\xi|\phi^6$ (with $\mu^2 = 4M_*^2$, $\lambda = 4M_*^{-1}$, $\xi = 0.6M_*^{-4}$, and $\pi R = 1.368 M_*^{-1}$). In the right panel, we show the energy of the two stable solutions as a function of πR, and it is seen how the absolute stability of coexisting static configurations is determined by the size of the radius R. The dots show critical points where the scalar perturbations contain a massless mode.

one vacuum to another can be either second order or first order, as one varies the radius R.

In Fig. 3 we show an example of a simple setup defined by the scalar field potential $V(\phi) = -\frac{1}{2}|\mu^2|\phi^2 - \frac{1}{4}|\lambda|\phi^4 + \frac{1}{6}|\xi|\phi^6$, with $\mu^2 = 4M_*^2$, $\lambda = 4M_*^{-1}$, and $\xi = 0.6M_*^{-4}$. In the right panel, the energy density of two static solutions is plotted as a function of R, showing clearly that below a critical radius R_1 only the trivial solution is possible and above a critical radius R_2 only the kink solution is possible. For $R_1 < R < R_2$, both solutions are perturbatively stable. At the radius R_* the two solutions are degenerate, marking the transition from one true vacuum to another ($\phi_{\rm triv}$ for $R < R_*$ and $\phi_{\rm kink}$ for $R > R_*$). From this we see that the inverse length scale $1/R$ plays the role of an order parameter of a phase transition, much like temperature T in finite temperature field theory. For a very small radius R (analogous to high T) the system is stable only around its trivial solution, with all symmetries restored. As the radius increases (analogous to T decreasing) the system can undergo a phase transition, which could be of either first or second order. The analogy with temperature, however, is not meant to be taken literally. For whereas the temperature in any 4D effective cosmology must be monotonically decreasing for most of its history, the orbifold radius R could in principle increase, decrease, or oscillate on very long time scales, depending on the dynamics of the stabilization mechanism (which we have so far ignored).

III. KINKS ON A WARPED BACKGROUND

We now extend previous investigations to the case of a scalar field in a warped extra dimension, while neglecting any backreaction on the warping from the scalar field itself. In this case one includes the effects of the curved metric along the extra dimension on the scalar field solutions

while still ignoring the dynamics of the gravitational sector. We therefore consider the action

$$S = \int d^5x \sqrt{-g} \left[\frac{1}{2} g^{MN} (\partial_M \phi) \partial_N \phi - V(\phi) \right], \quad (6)$$

where the form of the metric is now taken to be

$$ds^2 = e^{-2\sigma(y)} \gamma_{\mu\nu}(x) dx^\mu dx^\nu - dy^2, \quad (7)$$

and where $\sigma(y)$ is the warp factor and $\gamma_{\mu\nu}$ the 4D metric on slices of constant y. The purpose of considering scalar field configurations on a fixed background is to explore whether our previous results continue to hold in the presence of a warped background in a regime where we still have semi-analytical control over the solutions. We postpone a discussion of the full dynamical problem, including the backreaction on the metric due to the presence of the scalar field, until the next section.

A. Kink scalar in an AdS$_5$ background

In the original Randall-Sundrum (RS) model [9], the metric takes the form (7) with $\sigma(y) = k|y|$ and $\gamma_{\mu\nu} = \eta_{\mu\nu}$, where k has dimensions of mass and is related to the 5D cosmological constant of AdS$_5$. In this background any static nontrivial field configurations $\bar\phi(y)$ are solutions of

$$\bar\phi'' - 4k\bar\phi' - \frac{\partial V}{\partial \phi}\bigg|_{\bar\phi} = 0. \quad (8)$$

Scalar perturbations around this kink background, $\varphi(x, y) = \phi(x, y) - \bar\phi(y)$, can be decomposed as

$$\varphi(x, y) = \sum_n \varphi_x^{(n)}(x) \varphi_y^{(n)}(y) \quad (9)$$

such that the normal modes $\varphi_x^{(n)}(x)$ and $\varphi_y^{(n)}(y)$ are solutions of

$$^{(4)}\square \varphi_x + m_n^2 \varphi_x = 0 \quad (10)$$

$$\varphi_y'' - 4k\varphi_y' - (\mu^2(y) - m_n^2 e^{2ky})\varphi_y = 0, \quad (11)$$

where $\mu^2 \equiv \frac{\partial^2 V}{\partial \phi^2}\big|_{\bar\phi}$ and $^{(4)}\square \equiv \eta^{\mu\nu} \partial_\mu \partial_\nu$. Taking the derivative of the kink equation (8) gives

$$\varphi_M'' - 4k\varphi_M' - \mu^2(y)\varphi_M = 0, \quad (12)$$

where we have defined $\varphi_M \equiv \bar\phi'$. Thus φ_M is a massless solution $(m_n^2 = 0)$ of the perturbation equation (11), although it satisfies mixed boundary conditions rather than the Dirichlet boundary conditions imposed on φ_y.

At this point we are already able to state a new result of this work, which is an extension of the previous result related to the impossibility of having stable kink solutions with nodes inside the interval. Suppose that $\bar\phi(y)$ happens to have δ nodes inside the interval. We have just shown that $\bar\phi' \equiv \varphi_M$ will solve the equation for a massless excitation,

but with mixed boundary conditions. Since $\bar\phi$ has δ nodes, $\bar\phi' = \varphi_M$ must have $\delta + 1$ nodes inside the interval. The following inequalities relating the eigenvalues λ_n^D for the Dirichlet case and the eigenvalues λ_n^M for a general mixed boundary condition case [49] hold from Sturm-Liouville theory

$$\lambda_n^D \leq \lambda_{n+2}^M \leq \lambda_{n+2}^D. \quad (13)$$

Since we have $\lambda_{\delta+1}^M = 0$ (i.e., the eigenvalue of the solution with $\delta + 1$ nodes), we can immediately deduce that the mass-squared of the lowest excitation of the Dirichlet problem must be negative since $\lambda_{\delta-1}^D \leq \lambda_{\delta+1}^M = 0$ with $\delta \geq 1$.

Proposition 3.—In a warped background on a slice of AdS$_5$, any static solution to Eq. (8), with $\delta > 0$ nodes inside the interval, is always unstable.

However, for nodeless static solutions (when $\delta = 0$) the results for the flat case obtained in [45,46] cannot be extended here. Lacking a general stability condition, we will instead propose a weaker sufficient stability condition for these and other more generic solutions in the next subsection.

B. Kink scalar on a general background

In a general warped background with metric ansatz (7) the equation for a static scalar background configuration is

$$\bar\phi'' - 4\sigma'\bar\phi' - \frac{\partial V}{\partial \phi}\bigg|_{\bar\phi} = 0. \quad (14)$$

In this situation, although we have been unable to extend the stability theorems found earlier, we are still able to find a general *sufficient* condition for perturbative stability of the background configurations.

Small perturbations around the background $\bar\phi(y)$ may be defined as in (9). The spectrum of these perturbations consists of solutions to the eigenvalue problem

$$\varphi_y(y_1) = \varphi_y(y_2) = 0 \quad (15)$$

$$\varphi_y'' - 4\sigma'\varphi_y' - [\mu^2(y) - m_n^2 e^{2\sigma(y)}]\varphi_y = 0. \quad (16)$$

A useful form of this equation is obtained by performing a change of variables $e^{\sigma(y)}dy = dz$ and defining $\sigma(z) = -\frac{2}{3}\ln(J(z))$ and $W(z) = \mu^2(z)e^{-2\sigma(y(z))}$ to yield

$$\frac{(J\varphi)''}{J\varphi} - \frac{J''}{J} - (W(z) - m_n^2) = 0. \quad (17)$$

To proceed, we make use of the following integral inequality [50]. For any function $f(z)$, such that $f(a) = f(b) = 0$, and with n nodes within the interval $[a, b]$, there exists $\rho \in \mathcal{R}$ such that

$$\int_a^b e^{-\rho f''(z)/f(z)} dz \geq (n + 1)e\sqrt{\rho \pi}. \quad (18)$$

Applied to (17), this implies

$$e^{\rho m_n^2} \int_a^b e^{-\rho[(J''/J)+W(z)]} dz \geq (n+1)e\sqrt{\rho\pi}, \qquad (19)$$

the logarithm of which yields

$$m_n^2 \geq \frac{1}{\rho} \ln[(n+1)e\sqrt{\rho\pi}] - \frac{1}{\rho} \ln\left(\int_a^b e^{-\rho[(J''/J)+W(z)]} dz\right), \qquad (20)$$

which is a lower bound for the eigenvalues in terms of the background quantities $\sigma(z)$ and $\mu(z)$ (which are contained in J and W). In the case of the lowest eigenvalue we have

$$m_0^2 \geq \frac{1}{\rho} \ln(e\sqrt{\rho\pi}) - \frac{1}{\rho} \ln\left(\int_a^b e^{-\rho[(J''/J)+W(z)]} dz\right) \quad (21)$$

and so a sufficient condition for perturbative stability $(m_0^2 \geq 0)$ is

$$\int_a^b e^{-\rho[(J''/J)+W(z)]} dz \leq e\sqrt{\rho\pi}. \qquad (22)$$

We may formulate this explicitly in terms of the warp factor $\sigma(z)$ so that finally, a static solution $\bar{\phi}(z)$ of (14), obeying Dirichlet boundary conditions, is stable if

$$\int_a^b e^{-\rho[-(3/2)\sigma''+(9/4)\sigma'^2+\mu^2(z)e^{-2\sigma}]} dz \leq e\sqrt{\rho\pi}, \qquad (23)$$

where $\mu^2(z) \equiv \frac{\partial^2 V}{\partial \phi^2}|_{\bar{\phi}(z)}$ and where the actual value of ρ is that which extremizes the right-hand side. This is a sufficient condition, but not a necessary one. In order to demonstrate how effective this *weaker* stability condition can be, we now turn to a simple example in which the condition can actually be evaluated.

Consider a flat metric, where $\sigma(y) \equiv 0$, and a trivial background scalar configuration, i.e., $\bar{\phi}(y) \equiv 0$, but where the 5D scalar potential is allowed to have a tachyonic mass. In this case Eq. (16) becomes

$$\varphi_y'' - (\mu^2 - m_n^2)\varphi_y = 0. \qquad (24)$$

If φ_y has Dirichlet boundary conditions, the solutions to this problem are

$$\varphi_y = \sin(\sqrt{m_n^2 - \mu^2}y), \qquad (25)$$

where $m_n^2 - \mu^2 = n^2\pi^2/L^2$ and $L = b - a$ is the size of the extra dimension. The mass of the lightest mode is $m_0^2 = \mu^2 + \pi^2/L^2$ and so the condition for stability is $m_0 \leq 0$, which means that the bulk scalar mass μ^2 can be negative, but not arbitrarily so:

$$\mu^2 \geq -\pi^2/L^2. \qquad (26)$$

Therefore in this case (where $\sigma' = \sigma'' = 0$), our *sufficient* condition (23) becomes

$$e^{-\rho\mu^2} \int_a^b dz \leq e\sqrt{\rho\pi}, \qquad (27)$$

which leads to

$$\mu^2 \geq \frac{1}{\rho} \ln\left(\frac{L}{e\sqrt{\rho\pi}}\right). \qquad (28)$$

The value of ρ that extremizes this bound is $\rho = \pi L^2/e$, and so our weaker bound is

$$\mu^2 \geq -\frac{e\pi}{2}\frac{1}{L^2}. \qquad (29)$$

This result is a factor of $2\pi/e$ weaker than the exact bound (26). Nevertheless this result is nontrivial as it clearly demonstrates that it is possible to have negative bulk masses and retain a stable system.[1]

IV. KINKS IN GRAVITATING WARPED EXTRA DIMENSIONS

So far we have examined static scalar field configurations in a fixed background. We have found that some of the results that were shown to hold in a flat extra-dimensional background continue to hold in a fixed warped background, and we have found useful generalizations of other results. We now want to include the dynamics of the gravitational sector and explore how these results can be extended when the gravitational backreaction is included. Therefore we now seek nontrivial static field configurations in which the warp factor has its own dynamics determined by the 5D Einstein equations.

As soon as we include a dynamical gravitational sector, we are required to worry about stabilization of the extra dimension. In the above discussion we assumed that the extra dimension was stabilized and that the dynamics of the stabilization mechanism were frozen out. Here we want to include the backreaction of any matter fields on the 5D metric, and so we must include the dynamics of stabilization. A natural question to ask is whether the kink fields of interest could provide a stabilization mechanism. Unfortunately, in [47] it was shown that when one considers static solutions for both the warp factor and a single scalar field, the lightest scalar perturbative mode (the radion) will be tachyonic whenever the derivative of the scalar profile vanishes inside the interval. In other words, the system is unstable whenever the scalar field profile passes through an extremum in the bulk. This means that if we insist on obtaining a nontrivial configuration for a single scalar field with Dirichlet boundary conditions, we are guaranteed to obtain a tachyonic radion and the extra dimension will be unstable. To address this issue we will add extra scalar fields whose purpose will be to stabilize the radion as in [48].

The resulting system becomes considerably more difficult to analyze than the case with only one bulk scalar field,

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[1]The stability conditions of the trivial vacuum in the presence of negative bulk mass terms in an extra-dimensional scalar field theory have been analyzed and generalized to general warped backgrounds in [51].

particularly with regard to questions about stability. On the other hand, the case with three or more scalar fields is formally no more difficult to analyze than the case with only two scalar fields. Hence we will keep our treatment general to include an arbitrary number of scalar fields χ_a $(a, b = 1, \ldots, \mathcal{N})$, although when we consider particular examples below, we will specialize to the case with only two scalar fields (a kink field and a nonkink field). For simplicity we will assume throughout that the scalar fields are only coupled gravitationally.

We therefore consider the 5D action for gravity and $\mathcal{N}$ free scalar fields

$$
S = -\frac{M_*^3}{2} \int d^5x \sqrt{-g}[\mathcal{R} - 2\Lambda]
$$
$$
+ \int d^5x \sqrt{-g} \left[\sum_{a=1}^{\mathcal{N}} \frac{1}{2} g^{MN} (\partial_M \chi_a)(\partial_N \chi_a) - W(\chi_a) \right.
$$
$$
\left. - \sum_{i=1,2} \lambda_i(\chi_a)\delta(y - y_i) \right],
\tag{30}
$$

where $M_* \equiv (8\pi G)^{-1/3}$, G is the 5D Newton constant, $\mathcal{R}$ is the 5D Ricci scalar, and Λ is the 5D cosmological constant. The full scalar potential in the bulk is $W(\chi_a)$, and the brane potentials are $\lambda_i(\chi_a)$. As before, we take the 5D line element of the form

$$
ds^2 = e^{-2\sigma(y)}\gamma_{\mu\nu}(x)dx^\mu dx^\nu - dy^2,
\tag{31}
$$

where $\gamma_{\mu\nu}$ is the induced metric on the 4D hypersurfaces of constant y, which foliate the extra dimension. The 5D Einstein and field equations are

$$
\sigma'' - \sigma'^2 + \frac{\Lambda}{6} = \frac{1}{2M_*^3} \left(\sum_a^{\mathcal{N}} \frac{1}{2} \chi_a'^2 + \frac{1}{3} W(\chi_a) \right.
$$
$$
\left. + \frac{2}{3} \sum_{i=1,2} \lambda_i(\chi_a)\delta(y - y_i) \right)
\tag{32}
$$

$$
\sigma'^2 - \frac{\Lambda}{6} + \frac{{}^{(4)}\mathcal{R}}{12} e^{2\sigma} = \frac{1}{6M_*^3} \left(\sum_a^{\mathcal{N}} \frac{1}{2} \chi_a'^2 - W(\chi_a) \right)
\tag{33}
$$

$$
\chi_a'' - 4\sigma'\chi_a' - \frac{\partial W}{\partial \chi_a} - \sum_{i=1,2} \frac{\partial \lambda_i}{\partial \chi_a} \delta(y - y_i) = 0,
\tag{34}
$$

where ${}^{(4)}\mathcal{R}$ is the 4D Ricci scalar associated with the induced 4D metric $\gamma_{\mu\nu}$, which we have left arbitrary. The boundary conditions for the system are determined by Israel junction conditions at each brane. These are obtained by integrating the equations of motion over an infinitesimally small interval across each brane, giving

$$
[\sigma']_{y_i} \equiv \lim_{\epsilon \to 0} [\sigma'(y_i + \epsilon) - \sigma'(y_i - \epsilon)] = \frac{1}{3M_*^3} \lambda_i(\chi_a)|_{y_i}
$$
$$
\tag{35}
$$

$$
[\chi_a']_{y_i} \equiv \lim_{\epsilon \to 0} [\chi_a'(y_i + \epsilon) - \chi_a'(y_i - \epsilon)] = \frac{\partial \lambda_i}{\partial \chi_a}\bigg|_{y_i}.
\tag{36}
$$

These yield $\mathcal{N}$ conditions on each brane, which is exactly the number of data that need to be specified in order for Eqs. (32) and (34) to form a well-posed problem.

Note that the above boundary value problem consists of a system of coupled nonlinear differential equations. Finding solutions analytically for such a setup is highly unlikely, although it is still possible to proceed in the opposite direction, i.e., given a particular analytical solution one can obtain the setup from which it originates. To do so, one relies on the powerful method of the superpotential [52–54], which can be useful even for two or more scalar fields (see, for example, [42] in the context of soft-wall models). However, even if one solution is constructed in this way, there is no guarantee that this is the only solution with the same action. We will now describe how to look for all possible solutions of a given action using a combination of numerical and graphical techniques.

A. Multiple solutions

Whenever there is more than one static solution to the above boundary value problem *with the same action*, we say that multiple solutions exist. In general, the bulk scalar fields can have Dirichlet boundary conditions, Neumann boundary conditions, or more general mixed boundary conditions. Here we focus on the case where we have one kink field ϕ (obeying Dirichlet boundary conditions), with the remaining $\mathcal{N} - 1$ fields χ_a having Neumann or mixed boundary conditions. When the profiles of these extra fields are monotonic, they will tend to stabilize the extra dimension, whereas if their profiles have vanishing derivatives inside the interval, they will tend to destabilize the extra dimension [47]. Despite this subtlety, we will generically refer to the nonkink fields as "stabilization" fields.

To find solutions we proceed as follows: We specify the Lagrangian in the bulk and on one of the branes, and we numerically solve an initial value problem to determine the profiles of the fields along the extra dimension. Dirichlet boundary conditions are imposed on the kink field ϕ at the initial brane by demanding that it vanish there. For this to hold, we assume the kink field has a sufficiently heavy brane mass so that it decouples from the stabilization fields on the branes. As a result, the kink field disappears from the junction conditions (35) and (36), which then yield only $\mathcal{N}$ conditions on the initial brane. This leaves $\mathcal{N} + 1$ initial conditions that need to be specified, which we take to be the boundary values for the derivatives ϕ', $\chi_1' \cdots \chi_{\mathcal{N}-1}'$, and σ'. After solving the initial value problem for a given choice of initial conditions, we impose Dirichlet boundary conditions on ϕ at the final boundary by locating the second brane at a point where the profile of ϕ vanishes. In general, the profile will vanish at several

points along the extra dimension, and one may study kinks with the desired number of nodes by choosing the location of the second brane accordingly. Here, as in the flat case, we are primarily interested in nodeless kink solutions, and we therefore place the second brane at the first zero of the profile function.

We now have a solution to a *boundary* value problem whose boundary conditions on the second brane are not yet known. We parametrize the brane potential on the second brane $\lambda_2(\chi_a)$ in terms of P parameters α_b (for example, the brane tension Σ_2, the brane mass term m_a^2 of each scalar, the quartic coupling of each scalar, etc.)

$$\lambda_2(\chi_a) = f(\Sigma_2, m_1^2, m_2^2, \ldots, m_{\mathcal{N}}^2, \ldots). \qquad (37)$$

Then the junction conditions (35) and (36) at the second brane ($i = 2$) give $\mathcal{N}$ linear equations for the P unknowns α_b. By evaluating the fields on the second brane, and using the parametrization in (37), we then invert the $\mathcal{N}$ junction conditions to determine the α_b. If this is possible, then the solution to our initial value problem is also a solution to a corresponding boundary value problem. From this we see that we must have $P \geq \mathcal{N}$ in order to guarantee that the field configuration we obtained is the solution to a corresponding boundary value problem. If $P = \mathcal{N}$, the α_b are uniquely determined, and there is a unique Lagrangian for which the above field configuration is a solution. On the other hand if $P > \mathcal{N}$, some of the α_b are arbitrary and so there is a family of solutions for these final-boundary conditions. In that case there is a family of Lagrangians which yield the obtained field configuration, and one can proceed by focusing on one member of this family. If $P < \mathcal{N}$, the linear system of parameters α_b may be overdetermined, in which case the obtained field configuration is not a solution to any corresponding boundary value problem.

We can find additional solutions by changing the initial-boundary conditions and repeating the above process. Note that by freely varying the field derivatives (ϕ', $\chi_1' \ldots \chi_{\mathcal{N}-1}'$) at the initial brane and determining the remaining quantities from the junction conditions, it is possible to leave the initial-brane potential unchanged. This is necessary in order that the action remains unchanged (it is not sufficient because part of the action is determined by the final-brane potential). A solution and the resulting final-boundary conditions (the α_b) are then found as before. Since each set of initial shooting values yields a set of α_b, each α_b is a function of the $\mathcal{N}$ initial-boundary derivatives. Each α_b therefore defines an $\mathcal{N}$-dimensional surface whose level-surfaces can be projected onto the $\phi'(y_1)$-$\chi_a'(y_1)$ parameter space (which is an $\mathcal{N}$-dimensional space). In the above construction there are P such quantities, and so P level-surfaces intersect at every point in this parameter space, representing one solution for this action. The question of whether multiple solutions exist for the same action is equivalent to the

question of whether the same P surfaces simultaneously intersect at more than one point in the parameter space.

We will now show how this works in two simple examples. In both cases, we will consider a kink field ϕ in addition to just one stabilization field χ, with no interaction terms among them in the scalar potential. In both examples there will be regions of parameter space in which two distinct static configurations are possible for the same action.

B. Example 1: Quartic potential

In both of the following examples we consider a Lagrangian for two scalar fields

$$\mathcal{L}_{\text{matter}} = \frac{1}{2} g^{MN} (\partial_M \phi) \partial_N \phi - V(\phi) - \sum_{i=1,2} \beta_i(\phi) \delta(y - y_i)$$

$$+ \frac{1}{2} g^{MN} (\partial_M \chi) \partial_N \chi - U(\chi) - \sum_{i=1,2} \lambda_i(\chi) \delta(y - y_i),$$

$$(38)$$

where ϕ is the kink field and χ is the stabilization field with potentials

$$U(\chi) = \frac{1}{2} m_\chi^2 \chi^2 \qquad (39)$$

$$\lambda_i(\chi) = M_*^{-1} \left(\frac{1}{2} \mu_i^2 \chi^2 + \Sigma_i \right). \qquad (40)$$

The fact that the second brane potential for χ is parametrized in terms of two parameters, μ_2^2 and Σ_2, will allow us to find unique solutions to the boundary conditions on the second brane. The junction conditions (35) and (36) become

$$\sigma'(y_i) = (-1)^{i-1} \frac{1}{6M_*^4} \left(\frac{1}{2} \mu_i^2 \chi^2(y_i) + \Sigma_i \right) \qquad (41)$$

$$\chi'(y_i) = (-1)^{i-1} \frac{1}{2} \mu_i^2 \chi(y_i). \qquad (42)$$

On the second brane ($i = 2$) these can be inverted to give

$$\mu_2^2 = -2 \frac{\chi'(y_2)}{\chi(y_2)} \qquad (43)$$

$$\Sigma_2 = -6M_*^4 \sigma(y_2) + \chi'(y_2)\chi(y_2) \qquad (44)$$

so that once we determine the fields on the second brane, we can extract the boundary conditions (and therefore the Lagrangian) to which those fields are a solution.

The only things left to specify are the bulk potential for the kink field and the initial-boundary conditions. In this first example, we take the kink potential to be

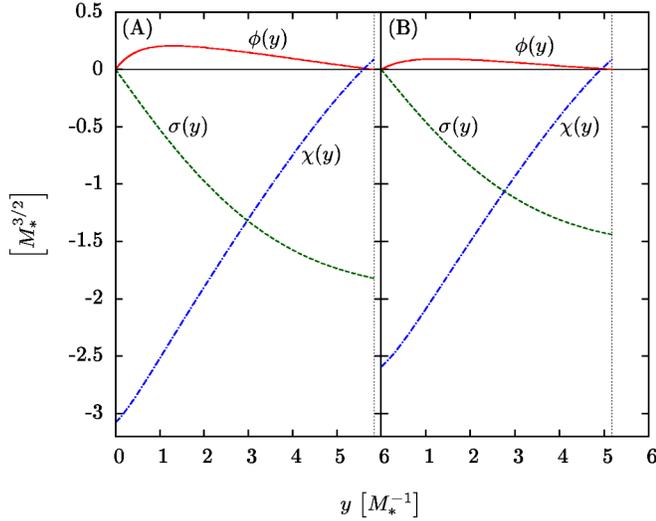$$V(\phi) = -\frac{1}{2} m_\phi^2 \phi^2 + \frac{1}{4} \lambda \phi^4. \qquad (45)$$

FIG. 4 (color online). Profiles of the scalar backgrounds $\phi(y)$ and $\chi(y)$ as well as the warp factor $\sigma(y)$, showing the two possible solutions (panels 4(a) and 4(b)) to the same boundary value problem defined by the physical parameters $m_\chi^2 = -0.5M_*^2$, $\mu_1^2 = -0.25M_*^2$, $\mu_2^2 = -8M_*^2$, $\Sigma_1 = -2M_*^4$, $\Sigma_2 = 0.52M_*^4$, $m_\phi^2 = 0.5M_*^2$, $\lambda = 2M_*^{-1}$, and $\Lambda = 0$.

FIG. 5 (color online). Level-curves of μ_2^2 and Σ_2 in the ϕ_1'-χ_1' parameter space, for example, 1 with $m_\chi^2 = -0.5M_*^2$, $\mu_1^2 = -0.25M_*^2$, $\Sigma_1 = -2M_*^4$, $m_\phi^2 = 0.5M_*^2$, $\lambda = 2M_*^{-1}$, and $\Lambda = 0$. Circled are two points in the ϕ_1'-χ_1' parameter space with the same values of μ_2^2 and Σ_2, corresponding to two solutions with the same Lagrangian (plotted in Fig. 4).

Taking the initial brane to be located at $y = 0$, we find solutions to the initial-boundary value problem at this brane with Dirichlet boundary conditions imposed on the field ϕ. Examples of nodeless solutions to the initial value problem are shown in Fig. 4. To ensure that ϕ obeys Dirichlet boundary conditions on the second brane, we locate the second brane at the first point (other than $y = 0$) where the profile of ϕ vanishes (the vertical dashed lines in Fig. 4).[2] Once the position of the second brane is identified, the final-boundary conditions are determined from (43) and (44). By varying the initial shooting conditions, $\phi_1' \equiv \phi'(y_1)$ and $\chi_1' \equiv \chi'(y_1)$, and repeating this process of finding solutions, identifying the location of the second brane, and determining the final-boundary conditions, we generate level-curves of μ_2^2 and Σ_2. These are plotted in Fig. 5. Notice that most μ_2^2 contours cross each Σ_2 just once, signifying that there is a single solution for the corresponding action with the kink potential of Eq. (45). However, some contours cross each other more than once (see, for example, the circles in Fig. 5.) Furthermore there is only a finite region in the ϕ_1'-χ_1' parameter space where solutions exist. If either $|\phi_1'|$ or $|\chi_1'|$ are increased sufficiently, the solution to the initial value problem blows up. In that case the *boundary* value problem has no solution, since a second boundary where $\phi = 0$ does not exist. It is therefore possible to scan the

entire allowed ϕ_1'-χ_1' space and examine whether multiple solutions with the same action exist.

C. Example 2: Higher-order potential

In this second example we take a slightly more complicated kink potential

$$V(\phi) = -\frac{1}{2}m_\phi^2\phi^2 - \frac{1}{4}\lambda\phi^4 + \frac{1}{6}\xi\phi^6. \qquad (46)$$

The other potentials and boundary conditions are the same as in the previous example, the only difference being the dynamical evolution of the system due to the new potential $V(\phi)$. We choose this potential because, contrary to the potential in our first example, in the limit of weak gravity and flat spacetime, it leads to multiple solutions to the same boundary value problem [45,46] due to the nonlinear nature of the equations.

In our more general setting, including gravity and a stabilization field, we find numerically that there exists more than one solution for the same Lagrangian in a large portion of the parameter space. In Fig. 6 we show two such solutions.

Note that these solutions would be extremely difficult to discover by randomly guessing initial-boundary conditions. To be more methodical we follow the same procedure as before to find level-curves of the final-boundary conditions, shown in Fig. 7. Again, solutions to a particular action will be given by the intersection of the appropriate contours for the brane mass-squared μ_2^2 and brane tension Σ_2. As can be seen, there are regions in which some contours intersect at more than one point, showing that

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[2] For certain initial conditions, the profile of ϕ will blow up before it vanishes for a second time. When this happens the initial conditions used do not lead to a solution of our boundary value problem.
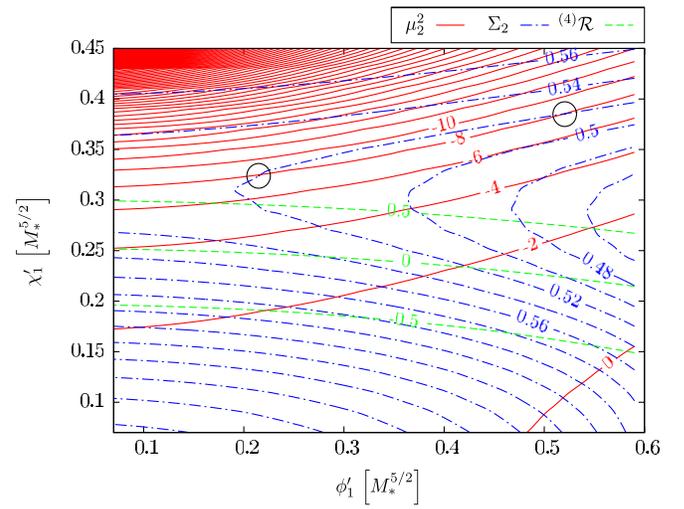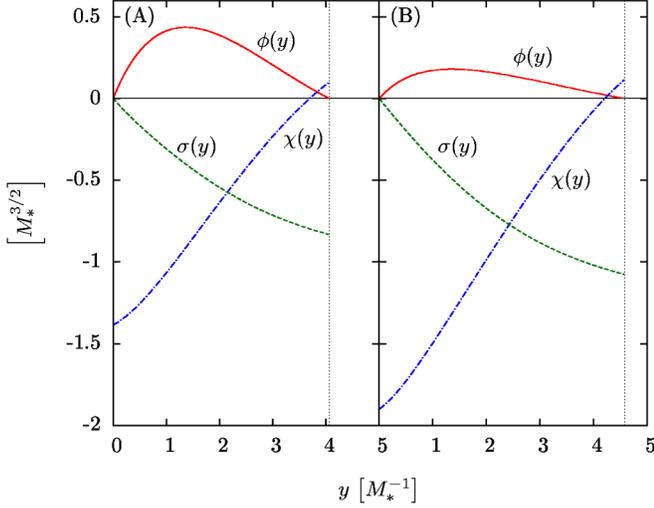
FIG. 6 (color online). Profiles of the scalar backgrounds $\phi(y)$ and $\chi(y)$ as well as the warp factor $\sigma(y)$, showing the two possible solutions (panels 6(a) and 6(b)) to the same boundary value problem defined by the physical parameters $m_\chi^2 = -0.5M_*^2$, $\mu_1^2 = -0.25M_*^2$, $\mu_2^2 = -5M_*^2$, $\Sigma_1 = -2M_*^4$, $\Sigma_2 = 0.56M_*^4$, $m_\phi^2 = 0.5M_*^2$, $\lambda = 2M_*^{-1}$, $\xi = 6M_*^{-4}$, and $\Lambda = 0$.
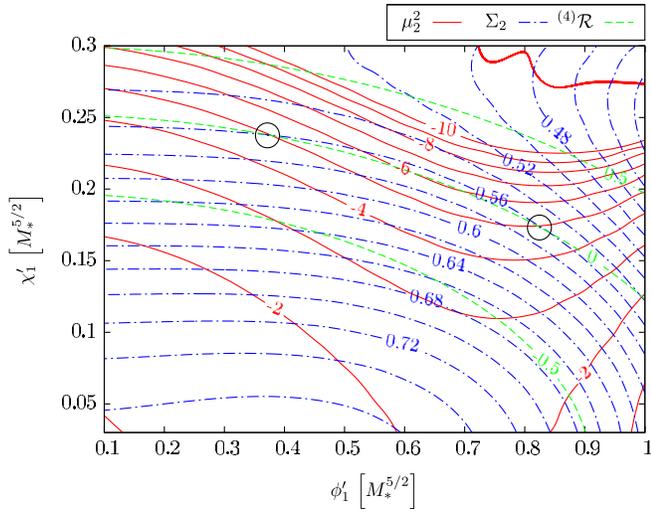

FIG. 7 (color online). Level-curves of μ_2^2 and Σ_2 in the ϕ'-χ' parameter space, for example, 2 with $m_\chi^2 = -0.5M_*^2$, $\mu_1^2 = -0.25M_*^2$, $\Sigma_1 = -2M_*^4$, $m_\phi^2 = 0.5M_*^2$, $\lambda = 2M_*^{-1}$, $\xi = 6M_*^{-4}$ and $\Lambda = 0$. Circled are two points in the ϕ_1'-χ_1' parameter space with the same values of μ_2^2 and Σ_2, corresponding to two solutions with the same Lagrangian. These solutions are plotted in Fig. 6.

multiple solutions for the same action are possible, as expected. In particular, we again circle two such points, corresponding to the solutions plotted in Figs. 6. As an interesting remark, note that both of these particular solutions happen to lie near the region of parameter space where the 4D cosmological constant vanishes.

V. STABILITY OF SOLUTIONS

Having shown how different nontrivial static field configurations exist in warped extra dimensions, the next question to ask is whether these solutions are stable. As we reviewed in Sec. II, in the case of flat extra dimensions there exist [46] techniques for determining the stability of such solutions. Indeed, for certain potentials, in that case perturbative stability can be determined analytically. Unfortunately, in the case of warped extra dimensions, the question of stability is complicated by the presence of multiple scalar fields and their coupled dynamics. Here we begin to study the perturbative stability of these kinked configurations. We derive the linearized equations and reformulate the problem in terms of a matrix Sturm-Liouville problem. However, the full analysis requires matrix Sturm-Liouville methods which we omit and leave for future work. Although incomplete, our approach here will be completely general for the stability of any coupled, multiscalar field system, in the presence of a warped extra dimension. In particular, although we have been interested in kinklike field configurations with Dirichlet boundary conditions, in this section we leave the boundary conditions in a general form.

We begin by expanding the metric to first order. Instead of the coordinates in (7), in this section it will be more convenient to choose coordinates so that the metric takes the form

$$ds^2 = a^2(y)(\gamma_{\mu\nu}(x)dx^\mu dx^\nu - dy^2). \tag{47}$$

Working in the generalized longitudinal gauge (see the appendix for details), we introduce scalar perturbations Φ and Ψ and write the perturbed metric as

$$ds^2 = a^2(y)[(1 + 2\Phi(x, y))\gamma_{\mu\nu}(x)dx^\mu dx^\nu \\ - (1 + 2\Psi(x, y))dy^2]. \tag{48}$$

Next we expand the $\mathcal{N}$ scalar fields to first-order in small perturbations $\xi_a(x, y)$

$$\chi_a(x, y) = \bar{\chi}_a(y) + \xi_a(x, y), \tag{49}$$

and compute the linearized Einstein equations, yielding $\mathcal{N} + 1$ dynamical equations for $\mathcal{N} + 1$ scalar fields ($\mathcal{N}$ fundamental scalars and one graviscalar, or radion). Since only $\mathcal{N}$ of these equations are independent, the Einstein constraint equations are used to eliminate one of the scalar fields in terms of the others (see the ppendix for details). The resulting $\mathcal{N}$ independent equations are

$$
{}^{(4)}\Box\Psi - \Psi'' - \left(9\mathcal{H} - 2a^2\frac{1}{\bar{\chi}_{\mathcal{N}}'}\frac{\partial W}{\partial\chi_{\mathcal{N}}}\Big|_{\bar{\chi}}\right)\Psi'
$$
$$
- \left(12\mathcal{H}^2 + 4\mathcal{H}' - \frac{1}{2}{}^{(4)}\mathcal{R} - 4a^2\mathcal{H}\frac{1}{\bar{\chi}_{\mathcal{N}}'}\frac{\partial W}{\partial\chi_{\mathcal{N}}}\Big|_{\bar{\chi}}\right)\Psi
$$
$$
= -\frac{4a^2}{3M_*^3}\sum_{a=1}^{\mathcal{N}-1}\left(\frac{\partial W}{\partial\chi_a}\Big|_{\bar{\chi}} - \frac{\bar{\chi}_a'}{\bar{\chi}_{\mathcal{N}}'}\frac{\partial W}{\partial\chi_{\mathcal{N}}}\Big|_{\bar{\chi}}\right)\xi_a \tag{50}
$$

$$^{(4)}\Box\xi_a - \xi_a'' - 3\mathcal{H}\xi_a' - a^2\frac{\partial^2 W}{\partial\chi_a^2}\bigg|_{\bar{\chi}}\xi_a$$

$$= -3\bar{\chi}_a'\Psi' - 2a^2\frac{\partial W}{\partial\chi_a}\bigg|_{\bar{\chi}}\Psi, \qquad (51)$$

where $\mathcal{H} \equiv \frac{a'}{a}$, $^{(4)}\Box \equiv \gamma^{\mu\nu}\partial_\mu\partial_\nu$, and in Eq. (51), as throughout, we have assumed that there are no direct couplings between the 5D scalar fields in the scalar potential (in the appendix we derive the general form of these equations when couplings between the fields are included). These dynamical equations can be written more compactly as

$$\Box\Psi + \mathcal{D}_1^y\Psi = \mathcal{D}_2^y\xi \qquad (52)$$

$$\Box\xi + \mathcal{D}_3^y\xi = \mathcal{D}_4^y\Psi, \qquad (53)$$

where ξ has suppressed discrete indices which run over the $\mathcal{N} - 1$ fundamental scalar fields, Ψ is the graviscalar, the $\mathcal{D}_i^y$ are y-dependent differential operators (i.e., linear differential operators having y-dependent coefficients and acting only on functions of y) also with suppressed discrete indices, and $\Box \equiv \gamma^{\mu\nu}\partial_\mu\partial_\nu$ is the 4D wave operator.

The boundary conditions are determined by integrating the equations of motion across each brane. Integrating Eqs. (50) and (51), these are found to be

$$[\Psi']_{y_i} - 2a^2\frac{1}{\bar{\chi}_\mathcal{N}'}\frac{\partial W}{\partial\chi_\mathcal{N}}\bigg|_{\bar{\chi}}\Psi'|_{y_i} - 4a^2\mathcal{H}\frac{1}{\bar{\chi}_\mathcal{N}'}\frac{\partial W}{\partial\chi_\mathcal{N}}\bigg|_{\bar{\chi}}\Psi|_{y_i}$$

$$= \frac{4a^2}{3M_*^3}\sum_{a=1}^{\mathcal{N}-1}\left(\frac{\partial W}{\partial\chi_a}\bigg|_{\bar{\chi}} - \frac{\bar{\chi}_a'}{\bar{\chi}_\mathcal{N}'}\frac{\partial W}{\partial\chi_\mathcal{N}}\bigg|_{\bar{\chi}}\right)\xi_a|_{y_i} \qquad (54)$$

$$[\xi_a']_{y_i} + a^2\frac{\partial^2 W}{\partial\chi_a^2}\bigg|_{\bar{\chi}}\xi_a|_{y_i} = 2a^2\frac{\partial W}{\partial\chi_a}\bigg|_{\bar{\chi}}\Psi|_{y_i}. \qquad (55)$$

These boundary conditions can be put in the form

$$\Psi'(x, y_i) = A_1(y_i)\Psi(x, y_i) + A_2(y_i)\xi(x, y_i) \qquad (56)$$

$$\xi'(x, y_i) = B_1(y_i)\Psi(x, y_i) + B_2(y_i)\xi(x, y_i), \qquad (57)$$

where $A_{1,2}$ and $B_{1,2}$ are functions of y_i, defined via (55), and we have used (56) in (54) to obtain (57).

The plan now is to perform a separation of variables in order to obtain a Sturm-Liouville eigenvalue problem, and then to analyze this eigenvalue problem to determine stability of the system. Because the 5D equations of motion of the scalar perturbations are coupled, the correct separation of variables ansatz is a coupled one

$$\Psi^{(n)}(x, y) = \Psi_y^{(n)}(y)u^{(n)}(x) \qquad (58)$$

$$\xi^{(n)}(x, y) = \xi_y^{(n)}(y)u^{(n)}(x), \qquad (59)$$

where $u^{(n)}(x)$ is the nth 4D Kaluza-Klein physical mode and $\Psi_y^{(n)}(y)$ and $\xi_y^{(n)}(y)$ are the wave functions. Plugging

this ansatz into Eqs. (52) and (53) leads to the following coupled equations:

$$\Psi_y(y)\Box u(x) + u(x)\mathcal{D}_1^y\Psi_y(y) = u(x)\mathcal{D}_2^y\xi_y(y) \qquad (60)$$

$$\xi_y(y)\Box u(x) + u(x)\mathcal{D}_3^y\xi_y(y) = u(x)\mathcal{D}_4^y\Psi_y(y). \qquad (61)$$

The separation of variables thus yields a 4D wave equation for $u(x)$

$$\Box u(x) + m_u^2 u(x) = 0 \qquad (62)$$

and a system of two coupled differential equations

$$\mathcal{D}_1^y\Psi_y(y) - m_u^2\Psi_y(y) = \mathcal{D}_2^y\xi_y(y) \qquad (63)$$

$$\mathcal{D}_3^y\xi_y(y) - m_u^2\xi_y(y) = \mathcal{D}_4^y\Psi_y(y) \qquad (64)$$

with boundary conditions for the profiles

$$\Psi_y'(y_i) = A_1(y_i)\Psi_y(y_i) + A_2(y_i)\xi_y(y_i) \qquad (65)$$

$$\xi_y'(y_i) = B_1(y_i)\Psi_y(y_i) + B_2(y_i)\xi_y(y_i). \qquad (66)$$

The system of Eqs. (63) and (64) constitute an eigenvalue problem. The stability of the static background around which we have added scalar perturbations therefore depends on the existence, or absence, of a negative eigenvalue m_u^2 associated with a solution to Eqs. (63) and (64).

This situation is somewhat unusual, since generally the Kaluza-Klein eigenvalue problem arising from dimensional reduction consists of a single second-order differential equation, which can be put in standard Sturm-Liouville form. Analyzing that Sturm-Liouville eigenvalue problem is straightforward, since, in particular, it is known that the eigenvalues are bounded from below, and that the eigenfunction corresponding to the smallest eigenvalue has no zeros within the interval. Therefore, the question of stability in practical terms becomes the search for a solution to the Kaluza-Klein equation such that it contains no nodes. Its associated eigenvalue will be the lightest possible eigenvalue and, if positive, the system will have no classical instabilities.

In the present case, however, the Kaluza-Klein problem is a system of coupled differential equations. Consequently, matrix Sturm-Liouville techniques are required. In order to analyze stability further, one must extend the theory of oscillations and the concept of nodes of solutions to a higher dimensional problem. Such an analysis, although rather involved, is underway, and will be presented in a future work.

VI. DISCUSSION AND OUTLOOK

Braneworld theories generally lead to scalar degrees of freedom that propagate in the extra-dimensional bulk. Understanding the vacuum structure of these models in the presence of bulk scalar fields is therefore a prerequisite to fully appreciating their phenomenological possibilities.

Furthermore, bulk scalars may provide a useful way to localize fermions and build braneworld models purely with field theory (e.g., fat branes and soft walls).

In this work, we have studied the vacuum structure of braneworld models with one warped extra dimension and multiple bulk scalar fields. In particular we have focused on static configurations along the extra space coordinate where one of the fields—with Dirichlet boundary conditions—acquires a nontrivial kinklike profile. To find these solutions one needs to solve both the Einstein and the scalar field equations. In the limit of a flat 5D metric and weak gravity such solutions are known to exist, and the problem of finding all possible static configurations as well as determining their perturbative stability has been addressed and solved [45,46]. Here we have built upon this previous work to determine how warping along the extra dimension affects the existence and stability of these kink-like solutions.

When considering a fixed warped background, it was sufficient to look for nontrivial solutions for a single scalar field. In this case, neglecting any backreaction of the scalar field on the gravitational dynamics, we found that such kinklike solutions do indeed exist. As in the case of a flat extra dimension, we were able to prove that any kinklike solution with nodes in the bulk is unstable. Thus we have focused on nodeless kink solutions and the trivial solution. However, in contrast to the flat case, in the presence of warping we were only able to find a sufficient condition for determining the stability of these solutions. We were therefore unable to analytically determine stability for nontrivial solutions in a warped background, even when that background is fixed (e.g., in the Randall-Sundrum model with no backreaction). Instead we were forced to determine stability numerically.

Including the dynamics of the gravitational sector forces the inclusion of additional scalar fields whose purpose is to stabilize the size of the extra dimension. In that case we were again able to find nontrivial kinklike configurations, except now for a coupled multiple-field system. We have described a general graphical technique to find all possible static configurations of the background equations with one kink-scalar field and an arbitrary number of additional stabilization fields. The technique amounts to generating solution surfaces by varying the shooting parameters needed to solve the coupled system of equations. This technique also allows us to look for multiple solutions with the same action. We have demonstrated how to implement this technique in two simple examples, where we considered one kink field and one stabilization field in the presence of gravity. As in the flat case, when the potential for the kink field is a higher-order polynomial (leading to higher-order nonlinearity in the field equations), we found that multiple solutions may exist for the same action. Interestingly, however, we also found multiple solutions for the same action when the kink potential was a fourth-

order polynomial, which differs from the result obtained in a flat background.

We have addressed the issue of stability only partially. We have derived the full 5D perturbative equations, including gravitational perturbations, for multiple scalar fields in the presence of a warped extra dimension, and with general boundary conditions. The system of equations constitute a matrix eigenvalue problem, which must be analyzed using an extension of the usual theorems coming from oscillation theory or Sturm-Liouville eigenvalue problems. Such techniques exist in the mathematical literature but due to the complexity of the task, we have left the numerical analysis of the general case for a later work.

APPENDIX: SCALAR PERTURBATIONS IN THE GENERALIZED LONGITUDINAL GAUGE

In this appendix we derive the linearized 5D Einstein and field equations for scalar perturbations in the bulk. We linearize around a background metric of the form

$$ds^2 = a^2(y)(\gamma_{\mu\nu}(x)dx^\mu dx^\nu - dy^2). \tag{A1}$$

The background Einstein and field equations in these coordinates are

$$\mathcal{H}' - \frac{\Lambda}{6}a^2 = -\frac{\kappa^3}{2}\left(\sum_a \frac{1}{2}\chi_a'^2 + \frac{1}{3}a^2 W(\chi_a) \right.$$
$$\left. + \frac{2}{3}a^2\sum_i \lambda_i(\chi_a)\delta(y - y_i)\right) \tag{A2}$$

$$\mathcal{H}^2 - \frac{\Lambda}{6}a^2 + \frac{{}^{(4)}\mathcal{R}}{12} = \frac{\kappa^3}{6}\left(\sum_a \frac{1}{2}\chi_a'^2 - a^2 W(\chi_a)\right) \tag{A3}$$

$$\chi_a'' + 3\mathcal{H}\chi_a' - a^2\frac{\partial W}{\partial\chi_a} - a^2\sum_i\frac{\partial\lambda_i}{\partial\chi_a}\delta(y - y_i) = 0, \tag{A4}$$

where $\mathcal{H} \equiv \frac{a'}{a}$, ${}^{(4)}\mathcal{R}$ is the 4D Ricci scalar with respect to the background 4D metric $\gamma_{\mu\nu}$. To first order in scalar perturbations, the 5D metric can be written

$$ds^2 = a^2(y)[\{(1 + 2\Phi(x, y))\gamma_{\mu\nu}(x) + 2E(x, y)_{|\mu\nu}\}dx^\mu dx^\nu$$
$$+ 2B(x, y)_{|\mu}dx^\mu dy - (1 + 2\Psi(x, y))dy^2], \tag{A5}$$

where | indicates a covariant derivative with respect to the

4D slices of the bulk. Choosing to work in the generalized longitudinal gauge, we set $B = E = 0$, and the linearized metric simplifies to

$$ds^2 = a^2(y)[(1 + 2\Phi(x, y))\gamma_{\mu\nu}(x)dx^\mu dx^\nu$$
$$- (1 + 2\Psi(x, y))dy^2]. \qquad (A6)$$

We also expand the scalar fields to first order

$$\chi_a(x, y) = \bar{\chi}_a(y) + \xi_a(x, y), \qquad (A7)$$

where the fields $\bar{\chi}_a$ obey the background equations of motion (A2)–(A4) above and $\xi_a(x, y)$ are small perturbations. The linearized Einstein and field equations are

$$2\Phi + \Psi = 0 \qquad (A8)$$

$$\Phi' - \mathcal{H}\Psi = -\frac{1}{3M_*^3}\sum_a \bar{\chi}'_a \xi_a \qquad (A9)$$

$$^{(4)}\Box(2\Phi + \Psi) - 4\Phi'' + 8\mathcal{H}'\Psi + 8\mathcal{H}^2\Psi$$
$$+ 4\mathcal{H}(\Psi' - 3\Phi') + \frac{2}{3}{}^{(4)}\mathcal{R}\Phi$$
$$= \frac{4}{3M_*^3}\sum_a\left(\bar{\chi}'_a\xi'_a - \bar{\chi}'^2_a\Psi + a^2\frac{\partial W}{\partial\chi_a}\Big|_{\bar{\chi}}\xi_a\right) \qquad (A10)$$

$$^{(4)}\Box\Phi - 4\mathcal{H}\Phi' + 4\mathcal{H}^2\Psi + \frac{1}{3}{}^{(4)}\mathcal{R}\Psi$$
$$= -\frac{1}{3M_*^3}\sum_a\left(\bar{\chi}'_a\xi'_a - \bar{\chi}'^2_a\Psi - a^2\frac{\partial W}{\partial\chi_a}\Big|_{\bar{\chi}}\xi_a\right) \qquad (A11)$$

$$^{(4)}\Box\xi_a - \xi''_a - 3\mathcal{H}\xi'_a + a^2\sum_b\frac{\partial^2 W}{\partial\chi_a\partial\chi_b}\Big|_{\bar{\chi}}\xi_b$$
$$= -2\bar{\chi}''_a\Psi - \bar{\chi}'_a(\Psi' - 4\Phi' + 6\mathcal{H}\Psi), \qquad (A12)$$

where $^{(4)}\Box \equiv \gamma^{\mu\nu}\partial_\mu\partial_\nu$ is the 4D wave operator. Applying the constraint equation (A8) to Eqs. (A9)–(A12), and making use of the background equations (A2)–(A4), yields

$$\Psi' + 2\mathcal{H}\Psi = \frac{2}{3M_*^3}\sum_a \bar{\chi}'_a\xi_a \qquad (A13)$$

$$\Psi'' + 5\mathcal{H}\Psi' + \left(4\mathcal{H}' + 4\mathcal{H}^2 - \frac{1}{6}{}^{(4)}\mathcal{R} + \frac{2}{3M_*^3}\sum_a\bar{\phi}'^2_a\right)\Psi$$
$$= \frac{2}{3M_*^3}\sum_a\left(\bar{\chi}'_a\xi'_a + a^2\frac{\partial W}{\partial\chi_a}\Big|_{\bar{\chi}}\xi_a\right) \qquad (A14)$$

$$^{(4)}\Box\Psi - 4\mathcal{H}\Psi' - \left(8\mathcal{H}^2 - \frac{1}{3}{}^{(4)}\mathcal{R} - \frac{2}{3M_*^3}\sum_a\bar{\chi}'^2_a\right)\Psi$$
$$= \frac{2}{3M_*^3}\sum_a\left(\bar{\chi}'_a\xi'_a - a^2\frac{\partial W}{\partial\chi_a}\Big|_{\bar{\chi}}\xi_a\right) \qquad (A15)$$

$$^{(4)}\Box\xi_a - \xi''_a - 3\mathcal{H}\xi'_a + \sum_b\left(a^2\frac{\partial^2 W}{\partial\chi_a\partial\chi_b}\Big|_{\bar{\chi}} - \frac{2}{M_*^3}\bar{\chi}'_a\bar{\chi}'_b\right)\xi_b$$
$$= -2\left(3\mathcal{H}\bar{\chi}'_a - a^2\frac{\partial W}{\partial\chi_a}\Big|_{\bar{\chi}}\right)\Psi. \qquad (A16)$$

We obtain a 5D wavelike equation for Ψ by subtracting (A14) from (A15) to give

$$^{(4)}\Box\Psi - \Psi'' - 9\mathcal{H}\Psi' - \left(4\mathcal{H}' + 12\mathcal{H}^2 - \frac{1}{2}{}^{(4)}\mathcal{R}\right)\Psi$$
$$= -\frac{4a^2}{3M_*^3}\sum_a\frac{\partial W}{\partial\chi_a}\Big|_{\bar{\chi}}\xi_a. \qquad (A17)$$

Equations (A16) and (A17) comprise $\mathcal{N} + 1$ dynamical equations for the $\mathcal{N} + 1$ perturbation variables Ψ and ξ_a. However, since these variables are connected through the constraint (A13), only $\mathcal{N}$ of them are independent. Therefore we may use (A13) to eliminate one of the variables in terms of the others. Choosing to eliminate the $\mathcal{N}$th scalar field, $\xi_\mathcal{N}$, in terms of $\xi_{a<\mathcal{N}}$ and Ψ, Eq. (A13) gives

$$\xi_\mathcal{N} = -\frac{1}{\bar{\chi}'_\mathcal{N}}\left(\sum_{b=1}^{\mathcal{N}-1}\bar{\chi}'_b\xi_b - \frac{3M_*^3}{2}(\Psi' + 2\mathcal{H}\Psi)\right). \qquad (A18)$$

(Note that we cannot eliminate Ψ in terms of the scalar fields ξ_a, since this requires an integration over unknown functions. This is understandable since doing so would amount to reducing the problem to one in flat spacetime, which ought to be impossible.) Substituting this into (A16) and (A17) and rearranging gives

$$^{(4)}\Box\Psi - \Psi'' - \left(9\mathcal{H} - 2a^2\frac{1}{\bar{\chi}'_\mathcal{N}}\frac{\partial W}{\partial\chi_\mathcal{N}}\Big|_{\bar{\chi}}\right)\Psi'$$
$$- \left(12\mathcal{H}^2 + 4\mathcal{H}' - \frac{1}{2}{}^{(4)}\mathcal{R} - 4a^2\mathcal{H}\frac{1}{\bar{\chi}'_\mathcal{N}}\frac{\partial W}{\partial\chi_\mathcal{N}}\Big|_{\bar{\chi}}\right)\Psi$$
$$= -\frac{4a^2}{3M_*^3}\sum_{a=1}^{\mathcal{N}-1}\left(\frac{\partial W}{\partial\chi_a}\Big|_{\bar{\chi}} - \frac{\bar{\chi}'_a}{\bar{\chi}'_\mathcal{N}}\frac{\partial W}{\partial\chi_\mathcal{N}}\Big|_{\bar{\chi}}\right)\xi_a \qquad (A19)$$

$$^{(4)}\Box\xi_a - \xi''_a - 3\mathcal{H}\xi'_a$$
$$+ a^2\sum_{b=1}^{\mathcal{N}-1}\left(\frac{\partial^2 W}{\partial\chi_b\partial\chi_a}\Big|_{\bar{\chi}} - \frac{\bar{\chi}'_b}{\bar{\chi}'_\mathcal{N}}\frac{\partial^2 W}{\partial\chi_\mathcal{N}\partial\chi_a}\Big|_{\bar{\chi}}\right)\xi_b$$
$$= 3\left(\bar{\chi}'_a - \frac{M_*^3 a^2}{2}\frac{1}{\bar{\chi}'_\mathcal{N}}\frac{\partial^2 W}{\partial\chi_\mathcal{N}\partial\chi_a}\Big|_{\bar{\chi}}\right)\Psi'$$
$$+ 2a^2\left(\frac{\partial W}{\partial\chi_a}\Big|_{\bar{\chi}} - \frac{3M_*^3\mathcal{H}}{2}\frac{1}{\bar{\chi}'_\mathcal{N}}\frac{\partial^2 W}{\partial\chi_\mathcal{N}\partial\chi_a}\Big|_{\bar{\chi}}\right)\Psi. \qquad (A20)$$

We may write this more compactly as

$$^{(4)}\Box\Psi + \mathcal{D}_1^y\Psi = \sum_{a=1}^{\mathcal{N}-1}(\mathcal{D}_2^y)_a\xi_a \qquad (A21)$$

$$^{(4)}\Box\xi_a + \sum_{b=1}^{\mathcal{N}-1}(\mathcal{D}_3^y)_{ab}\xi_b = (\mathcal{D}_4^y)_a\Psi, \qquad (A22)$$

where

$$
\begin{aligned}
\mathcal{D}_1^y &\equiv -\partial_y^2 - \left(9\mathcal{H} - 2a^2\frac{1}{\bar{\chi}_{\mathcal{N}}'}\frac{\partial W}{\partial\chi_{\mathcal{N}}}\Big|_{\bar{\chi}}\right)\partial_y \\
&\quad - \left(12\mathcal{H}^2 + 4\mathcal{H}' - \frac{1}{2}{}^{(4)}\mathcal{R}\right. \\
&\quad \left. - 4a^2\mathcal{H}\frac{1}{\bar{\chi}_{\mathcal{N}}'}\frac{\partial W}{\partial\chi_{\mathcal{N}}}\Big|_{\bar{\chi}}\right) \\
(\mathcal{D}_2^y)_a &\equiv -\frac{4a^2}{3M_*^3}\left(\frac{\partial W}{\partial\chi_a}\Big|_{\bar{\chi}} - \frac{\bar{\chi}_a'}{\bar{\chi}_{\mathcal{N}}'}\frac{\partial W}{\partial\chi_{\mathcal{N}}}\Big|_{\bar{\chi}}\right) \\
(\mathcal{D}_3^y)_{ab} &\equiv -\delta_{ab}(\partial_y^2 + 3\mathcal{H}\partial_y) + \mathcal{M}_{ab} \\
\mathcal{M}_{ab} &\equiv a^2\left(\frac{\partial^2 W}{\partial\chi_b\partial\chi_a}\Big|_{\bar{\chi}} - \frac{\bar{\chi}_b'}{\bar{\chi}_{\mathcal{N}}'}\frac{\partial^2 W}{\partial\chi_{\mathcal{N}}\partial\chi_a}\Big|_{\bar{\chi}}\right) \\
(\mathcal{D}_4^y)_a &\equiv 3\left(\bar{\chi}_a' - \frac{M_*^3 a^2}{2}\frac{1}{\bar{\chi}_{\mathcal{N}}'}\frac{\partial^2 W}{\partial\chi_{\mathcal{N}}\partial\chi_a}\Big|_{\bar{\chi}}\right)\partial_y \\
&\quad + 2a^2\left(\frac{\partial W}{\partial\chi_a}\Big|_{\bar{\chi}} - \frac{3M_*^3\mathcal{H}}{2}\frac{1}{\bar{\chi}_{\mathcal{N}}'}\frac{\partial^2 W}{\partial\chi_{\mathcal{N}}\partial\chi_a}\Big|_{\bar{\chi}}\right).
\end{aligned}
\qquad (A23)
$$

If we suppress the discrete indices in Eqs. (A21) and (A22) they take an even simpler form

$$^{(4)}\Box\Psi + \mathcal{D}_1^y\Psi = \mathcal{D}_2^y\xi \qquad (A24)$$

$$^{(4)}\Box\xi + \mathcal{D}_3^y\xi = \mathcal{D}_4^y\Psi. \qquad (A25)$$

This is the generic form that the scalar perturbation equations of motion take for $\mathcal{N}-1$ coupled scalar fields and a graviscalar.

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