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Consistent Abstractions of Affine Control Systems

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Abstract
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Keywords
Abstraction, affine control systems, hierarchies, local accessibility

Comments
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Consistent Abstractions of Affine Control Systems

George J. Pappas, *Member, IEEE*, and Slobodan Simić

***Abstract*—In this paper, we consider the problem of constructing abstractions of affine control systems that preserve reachability properties, and, in particular, local accessibility. In this framework, showing local accessibility of the higher level, *abstracted* model is *equivalent* to showing local accessibility of the, more detailed, lower level model. Given an affine control system and a smooth surjective map, we present a canonical construction for extracting an affine control system describing the trajectories of the abstracted variables. We then obtain conditions on the abstraction maps that render the original and abstracted system equivalent from a local accessibility point of view. Such consistent hierarchies of accessibility preserving abstractions of nonlinear control systems are then considered for various classes of affine control systems including linear, bilinear, drift free, and strict feedback systems.**

***Index Terms*—Abstraction, affine control systems, hierarchies, local accessibility.**

I. Introduction

A NATURAL approach for reducing the complexity of large scale systems places a hierarchical structure on the system architecture. For example, in the common two-layer planning and control hierarchies, the planning level uses a coarser system model than the lower control level. One of the main challenges in hierarchical systems is the extraction of a hierarchy of models at various levels of abstraction while preserving properties of interest.

Abstraction is also important in the analysis of complex systems. In the area of formal verification of concurrent systems, problems of exponential complexity are frequently encountered, and hierarchical system abstractions are used for complexity reduction [9], [16], [17]. For example, in order to verify that a given large scale system satisfies certain properties, one tries to extract a simpler but qualitatively equivalent abstracted system. Checking the desired property on the abstracted system should be *equivalent* or *sufficient* to checking the property on the original system. Depending on the property, special quotient systems which preserve the property of interest are constructed.

As a result, the notion of *abstraction* refers to grouping the system states into equivalence classes. A *hierarchy* can be thought of as a finite sequence of abstractions. *Consistent* abstractions are property preserving abstractions. Depending on the cardinality of the resulting quotient space we may have *discrete* or *continuous* abstractions. With this notion of abstraction, the abstracted system is defined as the quotient system dynamics. In this spirit, abstractions of purely *discrete*-event systems have been formally considered in the computer science community [9], [16] based on the fundamental work of [17]. Similar work for discrete event systems has been also considered in the control community [7], [29], [30]. A related research area considers equivalent discrete abstractions of continuous or hybrid systems [2], [8], [14] as well as sufficient discrete abstractions of hybrid systems [4], [10], [23].

In previous work, we have focused on extracting *continuous abstractions* from continuous systems. In particular, in [21], a hierarchical framework for continuous control systems was conceptualized and formally proposed. In [20], easily checkable characterizations were obtained for constructing controllability preserving abstractions of linear control systems. This immediately resulted in a hierarchical controllability algorithm from which we recovered the best known controllability algorithm from numerical linear algebra [11], [15]. In the same spirit, in [19] we characterized stabilizability preserving abstractions of linear systems. The resulting hierarchical stabilizability algorithm recovers the stabilizability algorithm in [24].

In this paper, we extend our hierarchical approach to a significant class of nonlinear control systems that consists of affine control systems on smooth manifolds.[1] In particular, we address the following problem.

Problem 1.1: Given an *affine* control system

$$\dot{x} = F(x) + G(x)u \quad x \in \mathbb{R}^n \quad u \in \mathbb{R}^m \tag{1}$$

and a smooth, surjective map $y = \Phi(x)$, where $\Phi: \mathbb{R}^n \to \mathbb{R}^p$, $p \leq n$, construct a control system

$$\dot{y} = f(y) + g(y)v \quad y \in \mathbb{R}^p \quad v \in \mathbb{R}^k \tag{2}$$

which can produce as trajectories all functions of the form $y(t) = \Phi(x(t))$, where $x(t)$ is a trajectory of (1). Furthermore, characterize smooth maps Φ for which (1) is locally accessible (controllable) if and only if (2) is locally accessible (controllable).

The surjective map Φ partitions the state space into equivalence classes. System (2) will be referred to as the *abstraction* of the more detailed model (1). It should be noted that the notion of abstraction in this paper is quite different from previous notions of state aggregation [5], [13], [26], and the more established notion of approximate model reduction [3], [28]. In model reduction, the input and output of the system are fixed, while the state

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[1] A preliminary version of this work for analytic, drift-free systems appeared in [22].

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dimension is reduced. The abstraction problem that we formulate does not require the input of the two systems to be the same. This is typical in planning and control hierarchies where, for example, the input at the kinematic level may be a velocity input, whereas the input at the dynamic level may be a torque input.

In [20], we extended the geometric notion of Φ-related vector fields to control systems, which allowed us to push forward control systems through maps and obtain well defined control systems describing the abstracted dynamics. The fact that the abstraction map Φ sends trajectories of (1) to trajectories of (2) enabled us to propagate reachable sets from system (1) to system (2). Furthermore, in [20], we were able to provide constructive formulas for constructing linear abstractions of linear control systems.

In this paper, we provide a constructive method for extracting abstractions for affine control systems on smooth manifolds. Our method is the natural nonlinear generalization of the linear method provided in [20]. Furthermore, the method is natural in the sense that it constructs the *smallest* Φ-related or abstracted control system. In addition, our method is structure preserving in the sense that the affine structure of our control systems is preserved throughout the abstraction process. Therefore, by repeating our construction, we can obtain a hierarchy, that is a finite sequence, of affine abstractions.

We then consider the problem of constructing abstractions while preserving the property of local accessibility [18]. We determine conditions on the map Φ under which local accessibility of the abstracted system (2) is equivalent to local accessibility of (1). Such conditions greatly reduce the complexity of determining local accessibility properties of nonlinear control systems, since rather than checking controllability of a large scale nonlinear system, we can construct a hierarchy of consistent abstractions and then check the local accessibility of systems which are much smaller in size. A property preserving hierarchy will then propagate the desired property along the sequence of abstractions from the simplest abstracted model to the original complex system.

The structure of this paper is as follows. In Section II, we review the results in [20] in the setting of linear systems. In Section III, we review some differential geometric concepts that are used in the paper, whereas in Section IV, we review some results from [20] that are used in this paper. In Section V, we provide methods for constructing abstractions of affine control systems. In Section VI, we characterize abstractions that preserve the property of local accessibility. This leads to hierarchical accessibility criteria which are considered for various classes of affine systems in Section VII. Finally, Section VIII discusses interesting directions for further research.

II. Linear Abstractions

The main goal of this paper is to obtain nonlinear analogues of the results in [20]. We start our review of the results in [20] with a formal definition of linear abstractions.

Definition 2.1 [Linear Abstractions ([20])]: Consider the linear control systems

$$\begin{aligned}(\Sigma_1)\quad \dot{x} =& Ax + Bu \quad x \in \mathbb{R}^n \quad u \in \mathbb{R}^m\\ (\Sigma_2)\quad \dot{y} =& Fy + Gv \quad y \in \mathbb{R}^p \quad v \in \mathbb{R}^k\end{aligned}$$

and a surjective map $y = Cx$. Then control system Σ_2 is called a C-abstraction or abstraction of system Σ_1 if system Σ_2 can produce as trajectories all functions of the form $y(t) = Cx(t)$, where $x(t)$ is a trajectory of system Σ_1.

The above definition of abstraction relates the trajectories of the two systems. Note that system Σ_2 must capture all (output) trajectories of system Σ_1, but may also generate more trajectories. At the level of vector fields, we have the following notion.

Definition 2.2 (C-Related Linear Systems): Consider the linear time-invariant control systems

$$\begin{aligned}(\Sigma_1)\quad \dot{x} =& Ax + Bu \quad x \in \mathbb{R}^n \quad u \in \mathbb{R}^m\\ (\Sigma_2)\quad \dot{y} =& Fy + Gv \quad y \in \mathbb{R}^p \quad v \in \mathbb{R}^k\end{aligned}$$

and the linear, surjective map $y = Cx$. Then, Σ_2 is C-related to Σ_1 if for all $x \in \mathbb{R}^n$, $u \in \mathbb{R}^k$, there exists $v \in \mathbb{R}^l$ such that

$$C(Ax + Bu) = FCx + Gv.$$

The notion of C-related control systems simply states that system Σ_2 must be able to generate (using its control input $v \in \mathbb{R}^l$), the image under C of any tangent vector that system Σ_1 may generate at any point $x \in \mathbb{R}^n$, and given any control input $u \in \mathbb{R}^k$. The connection between C-abstractions and C-related systems is given by the following theorem.

Theorem 2.3 (C-Abstractions and C-Related Systems [20]) : Consider the linear time-invariant control systems

$$\begin{aligned}(\Sigma_1)\quad \dot{x} =& Ax + Bu \quad x \in \mathbb{R}^n \quad u \in \mathbb{R}^m\\ (\Sigma_2)\quad \dot{y} =& Fy + Gv \quad y \in \mathbb{R}^p \quad v \in \mathbb{R}^k\end{aligned}$$

and the linear, surjective map $y = Cx$. Then, Σ_2 is a C-abstraction of Σ_1 if and only if Σ_2 is C-related to Σ_1.

Given C-abstractions and C-related systems, it is clearly advantageous to work with C-related systems since they potentially offer algebraic methods for constructing abstractions. In particular, the following proposition gives us a canonical construction in order to generate C-related linear abstractions.

Theorem 2.4 [Canonical Construction ([20])] : Consider the linear system

$$(\Sigma_1)\quad \dot{x} = Ax + Bu$$

and a surjective map $y = Cx$. Let

$$(\Sigma_2)\quad \dot{y} = Fy + Gv$$

be the system where

$$\begin{aligned}F =& CAC^+\\ G =& [CB \quad CAv_1 \quad \ldots \quad CAv_r]\end{aligned}$$

where C^+ is the Moore–Penrose pseudoinverse of C, and $v_1, \ldots, v_r$ span Ker(C). Then Σ_2 is C-related to Σ_1.

Note that by Proposition 2.5, given *any* linear control system, and *any* full-row rank matrix C, there always exists another linear control system which is C-related to it. In addition to propagating trajectories from the original to the abstracted system, we are also interested in propagation of other properties such as controllability. From linear systems theory we know

that the reachable space from the origin for system (Σ_1) is given by $\mathcal{R}(A,B) = \mathrm{Im}[B \quad AB \quad \ldots \quad A^{n-1}B]$. In particular, system Σ_1 is controllable if and only if $\mathcal{R}(A,B) = \mathbb{R}^n$. As an immediate corollary of Theorem 2.3 we obtain that $C\mathcal{R}(A,B) \subseteq \mathcal{R}(F,G)$, and, in particular, if Σ_1 is controllable then Σ_2 is controllable.

In order to propagate controllability from the abstracted linear system Σ_2 to the original system Σ_1, conditions must be placed on the abstracting map $y = Cx$, resulting in *consistent* abstractions [20]. With respect to controllability, the following theorem characterizes consistent linear abstractions.

Theorem 2.5 [Controllability Preserving Abstractions ([20])]: Consider the linear system

$$(\Sigma_1) \quad \dot{x} = Ax + Bu$$

and surjective map $y = Cx$. Let

$$(\Sigma_2) \quad \dot{y} = Fy + Gv$$

be the C-related system where

$$\begin{aligned} F =& CAC^+ \\ G =& [CB \quad CAv_1 \quad \ldots \quad CAv_r] \end{aligned}$$

where C^+ is the Moore–Penrose pseudoinverse of C and $v_1, \ldots, v_r$ span Ker(C). Furthermore, if

$$\mathrm{Ker}(C) \subseteq \mathcal{R}(A,B)$$

then Σ_1 is controllable if and only if Σ_2 is controllable.

The condition Ker$(C) \subseteq \mathcal{R}(A,B)$ suggests that in order to abstract away some dynamics (captured by Ker(C)) while preserving controllability, one would have to ensure the ignored dynamics are controllable. From the assumptions of Theorem 2.5, it is easy to see that a controllability preserving linear abstraction always exists if $B \neq 0$, since we can always choose matrix C satisfying Ker$(C) = \mathrm{Im}[B]$. Therefore the controllability preserving condition serves as a guideline for choosing our abstracting matrix C.

The goal of this paper is to develop similar results for nonlinear, affine control systems of the form $\dot{x} = f(x) + g(x)u$. In particular, we are interested in generalizing the canonical construction of Theorem 2.4 for affine control systems. Furthermore, given that most results for nonlinear systems are local in nature, rather than propagating global controllability, we focus on the property of local accessibility, and obtain the nonlinear analogue of Theorem 2.5. In order to achieve this, we must rely on the differential geometric methods for accessibility of nonlinear systems.

III. Geometric Preliminaries

We begin by recalling some definitions from differential geometry ([1], [18]). Let M be a differentiable manifold, and denote by T_pM the tangent space of M at $p \in M$. Let $TM = \bigcup_{p \in M} T_pM$ be the tangent bundle of M, and let π be the canonical projection map $\pi: TM \to M$. Recall, for instance, that $T_p\mathbb{R}^n = \mathbb{R}^n$, and that $T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$. Throughout the paper, the reader can keep $\mathbb{R}^n$ as a model manifold without loss of any of the main ideas. Given a smooth map $\Phi: M \to N$ between smooth manifolds M and N, the tangent map $T_p\Phi: T_pM \to T_{\Phi(p)}N$ pushes forward tangent vectors from T_pM to $T_{\Phi(p)}N$. The union of all tangent maps $T_p\Phi$ is denoted by $T\Phi$. Recall that if both M and N are euclidean spaces, then $T\Phi$ is just the total derivative of Φ. In this paper, we will be concerned with maps $\Phi: M \to N$ which are surjective submersions. In such cases, we will think of N as an embedded submanifold of M. As a model example to keep in mind, take $M = \mathbb{R}^n$, $N = \mathbb{R}^k = \mathbb{R}^k \times \{\mathbf{0}\} \subset \mathbb{R}^n$, where $k < n$, and Φ is the projection to the first k coordinates.

A vector field on a manifold M is a smooth map $X: M \to TM$ which assigns to each point $p \in M$ a tangent vector $X(p) \in T_pM$. An integral curve of a vector field X is a smooth curve $c: I \subseteq \mathbb{R} \to M$ that satisfies $T_tc = X(c(t))$ for all $t \in I$. Given two vector fields X and Y on M, by $[X,Y]$ we denote their usual Lie bracket.

A distribution Δ on M assigns to each $p \in M$ a subspace of T_pM. A distribution generated by vector fields $X_1, \ldots, X_k$ is given by $\Delta = \mathrm{span}\{X_1, X_2, \ldots, X_k\}$. The dimension of Δ at $p \in M$, denoted by $\dim(\Delta(p))$, is then $\dim(\mathrm{span}\{X_1(p), X_2(p), \ldots, X_k(p)\})$. Regular distributions require the dimension of the distribution to be independent of $p \in M$. A vector field X belongs to a distribution Δ if $X(p) \in \Delta(p)$ at each $p \in M$.

Given two smooth distributions Δ_1 and Δ_2, we define the distribution $[\Delta_1, \Delta_2]$ by declaring $[\Delta_1, \Delta_2](p)$ to be the subspace of T_pM generated by vectors of the form $[X_1, X_2](p)$ where X_1, X_2 are smooth vector fields belonging in Δ_1 and Δ_2 respectively. Given a distribution Δ, Lie(Δ) is the Lie algebra generated by Δ. It is obtained by taking the span of iterated Lie brackets of vector fields in Δ.

Given a vector field X on manifold M and a smooth map $\Phi: M \to N$, not necessarily a diffeomorphism, the push forward of X by $T\Phi$ is generally not a well-defined vector field on N. This leads to the concept of Φ-related vector fields.

Definition 3.1 (Φ-Related Vector Fields [1], [18]): Let X and Y be vector fields on manifolds M and N, respectively, and $\Phi: M \to N$ be a smooth map. Then, Y is Φ-related to X if for every $p \in M$

$$T\Phi(X(p)) = Y(\Phi(p)). \tag{3}$$

If Φ is a smooth surjection from M to N, then given a vector field X on a manifold M, the push forward of X by $T\Phi$ is a well defined vector field on N only if $T_{p_1}\Phi(X(p_1)) = T_{p_2}\Phi(X(p_2))$ whenever $\Phi(p_1) = \Phi(p_2)$ for any two points $p_1, p_2 \in M$. The following well-known theorem gives us a condition on the integral curves of two Φ-related vector fields.

Theorem 3.2 (Φ-Related Vector Fields [1], [18]): Let X and Y be vector fields on M and N respectively and let $\Phi: M \to N$ be a smooth map. Then, vector fields X and Y are Φ-related if and only if for every integral curve c of X, $\Phi(c)$ is an integral curve of Y.

Even though Φ-relatedness of vector fields is a rather restrictive condition, this is not the case for control systems. In order to have global definitions of control systems ([6], [18]), we shall need the concept of fiber bundles.

Definition 3.3 (Fiber Bundles): A fiber bundle is a quintuple $(B, M, \pi, U, \{O_i\}_{i\in I})$ where B, M, U are smooth manifolds called the total space, the base space and the standard fiber, respectively. The map $\pi: B \to M$ is a surjective submersion and $\{O_i\}_{i\in I}$ is an open cover of M such that for every $i \in I$ there exists a diffeomorphism $\Psi_i: \pi^{-1}(O_i) \to O_i \times U$ satisfying $\pi_o \circ \Psi_i = \pi$ where π_o is the projection from $O_i \times U$ to O_i (local triviality condition). The submanifold $\pi^{-1}(p)$ is called the fiber at $p \in M$.

If all the fibers are vector spaces of constant dimension, then the fiber bundle is called a vector bundle. If all the fibers are affine spaces then the fiber bundle is called an affine bundle.

The tangent bundle of a smooth manifold is an example of a fiber (vector) bundle. Some others are as follows.

Example 3.4 (Trivial Fiber Bundle): If $B = M \times U$ and $\pi: B \to M$ is the projection to the second coordinate, $(x, u) \mapsto x$, then the five-tuple $(B, M, \pi, U, \{M\})$ is called the *trivial fiber bundle* over M with fiber U. For example, the 2-torus is a trivial fiber bundle over the circle S^1 with fiber S^1. Locally, every fiber bundle looks like the trivial one [1].

Example 3.5 (Distributions): Every distribution Δ can be regarded as a vector bundle by taking B to be the union of all $\Delta(p)$ and defining the projection by $\pi(v) = p$ whenever $v \in \Delta(p)$. The fiber is $\mathbb{R}^k$, where $k = \dim \Delta$. The local triviality condition means that Δ is locally spanned by k linearly independent vector fields.

If $\mathcal{A}$ is an affine bundle on M, then locally there exist a vector field X_0 and a distribution Δ such that $\mathcal{A} = X_0 + \Delta$. If Δ is generated by vector fields $X_1, \ldots, X_k$ then $\mathcal{A} = X_0 + \text{span}\{X_1, X_2, \ldots, X_k\}$. Formally, B is the union of all affine spaces $X_0(p) + \Delta(p)$, $\pi(v) = p$ for all $v \in \mathcal{A}(p)$, the fiber U is an arbitrary but fixed affine k-dimensional subspace of $\mathbb{R}^n$ where $n = \dim M$.

Example 3.6: Consider the following (affine) control system on $M = \mathbb{R}^2$:

$$\begin{aligned}\dot{x} &= 1 - uy\\ \dot{y} &= ux.\end{aligned}$$

Then at each point (x_0, y_0), the set of all possible tangent directions is a straight line in $\mathbb{R}^2$ (considered as the tangent space to M at (x_0, y_0)) given by the equation $x_0(x-1) + y_0 y = 0$. Note that this line does not pass through the origin which is why it forms an affine subspace. Here, $X_0 = (1, 0)^T$ and $X_1(x, y) = -(y, x)^T$.

We will denote the Lie algebra generated by $\mathcal{A}$ by Lie($\mathcal{A}$). It is obtained by taking the span of all iterated Lie brackets of vector fields in $\mathcal{A}$. For simplicity, we will abuse the notation and use Lie($\mathcal{A}$) also to denote the distribution given by $\{X(p): X \in \text{Lie}(\mathcal{A})\}$.

IV. Control System Abstractions

Definition 3.1 and Theorem 3.2 capture the essence of Problem 1.1, but for vector fields. The restrictive nature of Theorem 3.2 is due to the deterministic nature of vector fields. The nondeterministic nature of control systems, however, allows us to remove such restrictions. In [20], Definition 3.1 and Theorem 3.2 were extended to control systems. We now briefly review some of the results of those papers. We first begin with a global definition of control systems.

Definition 4.1 (Control Systems [6], [18]) : A control system $S = (U, F)$ consists of a fiber bundle $\pi: U \to M$ and a smooth map $F: U \to TM$ which is fiber preserving, that is $\pi' \circ F = \pi$ where $\pi': TM \to M$ is the tangent bundle projection. Given a control system $S = (U, F)$, the control bundle $\mathcal{B}$ of S is naturally defined pointwise by $\mathcal{B}(p) = F(\pi^{-1}(p))$ for all $p \in M$. A control system is called affine if the control bundle $\mathcal{B}$ is an affine bundle.

The base manifold M of the control bundle is the state space and the fibers $\pi^{-1}(p)$ can be thought of as the state dependent control spaces. Given the state p and the input, the map F selects a tangent vector from $\mathcal{B}(p) \subseteq T_pM$. The notion of trajectories of control systems in this context is now given.

Definition 4.2 (Trajectories of Control Systems): A smooth curve $c: I \to M$ is called a trajectory of the control system $S = (U, F)$ if there exists a curve $c^U: I \to U$ satisfying

$$\begin{aligned}\pi(c^U) &= c\\ T_t c &= F(c^U).\end{aligned}$$

In local coordinates, Definition 4.2 simply says that a trajectory of a control system is a curve $x(t)$ for which there exists a function $u(t)$ satisfying, $\dot{x}(t) = F(x(t), u(t))$. Note that even though Definition 4.2 assumes c to be smooth, the bundle curve c^U is not necessarily smooth. The definition, therefore, allows nonsmooth control inputs as long as the projection $\pi(c^U) = c$ is smooth.

We now consider abstractions of control systems. An *abstraction* is a map $\Phi: M \to N$ which we will assume to be a surjective, smooth submersion.[2] We can now define Φ-related control systems in a manner similar to Definition 3.1 for vector fields.

Definition 4.3 (Φ-Related Control Systems) : Let $S_M = (U_M, F_M)$ with $\pi_M: U_M \to M$ and $S_N = (U_N, F_N)$ with $\pi_N: U_N \to N$ be two control systems. Let $\Phi: M \to N$ be a smooth map. Let $\mathcal{B}_M$ and $\mathcal{B}_N$ be the control bundles associated with control systems S_M and S_N respectively. Then S_N is Φ-related to S_M if for every $p \in M$

$$T_p\Phi(\mathcal{B}_M(p)) \subseteq \mathcal{B}_N(\Phi(p)). \tag{4}$$

Control system S_N will be referred to as an *abstraction* of control system S_M ([20]). Note that many control systems S_N may be Φ-related to S_M as the set of tangent vectors on N that must be captured, can be generated using many control parameterizations.

It is straightforward to show that Φ-relatedness of control systems indeed generalizes Definition 3.1 [20]. Furthermore, if $\mathcal{B}_{N_1}$ and $\mathcal{B}_{N_2}$ satisfy condition (4), then $\mathcal{B}_{N_1} \cap \mathcal{B}_{N_2}$ also satisfies condition (4). This suggests that there exists a *minimal* system S_M, up to control parameterization, that is Φ-related to S_M. The

[2]Note that any map Φ gives rise to an equivalence relation by defining states x and y equivalent if $\Phi(x) = \Phi(y)$. In order for the resulting quotient space to have a manifold structure, the equivalence relation must be regular [1].

minimal system naturally corresponds to the case where condition (4) becomes an equality, or equivalently when the following diagram commutes:

$$\begin{array}{ccc} M & \stackrel{\Phi}{\to} & N \\ \mathcal{B}_M \downarrow & & \downarrow \mathcal{B}_N \\ \mathcal{F}(M) & \stackrel{T\Phi}{\to} & \mathcal{F}(N), \end{array} \tag{5}$$

where $\mathcal{F}(M)$ is the space of fiber subbundles of TM.

In contrast to the restrictive conditions of Theorem 3.2, the following straightforward proposition, shows that every control or dynamical system is Φ-related to some control system for any map Φ.

Proposition 4.4 ([20]): Given any control system $S_M = (U_M, F_M)$ and any smooth map $\Phi: M \to N$, then there always exists a control system $S_N = (U_N, F_N)$ which is Φ-related to S_M.

The following theorem generalizes Theorem 3.2 to control systems.

Theorem 4.5 (Φ-Related Control Systems [20]) : Let $S_N = (U_N, F_N)$ and $S_M = (U_M, F_M)$ be two control systems and $\Phi: M \to N$ be a smooth map. Then S_N is Φ-related to S_M if and only if for every trajectory c_M of S_M, $\Phi(c_M)$ is a trajectory of S_N.

Because of Theorem 4.5, throughout this paper, we can equivalently say that S_N is an abstraction of S_M or that S_N is Φ-related to S_M. If Σ_{S_M} and Σ_{S_N} denote all trajectories of control systems S_M and S_N, respectively, then Theorem 4.5 simply states that S_N is Φ-related to S_M if and only if $\Phi(\Sigma_{S_M}) \subseteq \Sigma_{S_N}$. The abstracted system therefore overapproximates the abstracted trajectories of the original system which may result in trajectories that the abstracted system S_N may generate but are infeasible in the original model S_M.

Even though Definition 4.3 and Theorem 4.5 for control systems remove the tight restrictions of Definition 3.1 and Theorem 3.2 for dynamical systems, the challenge now becomes providing methods for constructing abstractions of control systems. This is the objective of Section V.

V. Abstraction Construction

The results we reviewed in Section IV were true for general control bundles, including affine bundles. In this section, we present a canonical way of constructing abstractions for affine control systems. Therefore, from this point on, we assume that all objects are smooth and all control bundles are affine.

Let $S_M = (U_M, F_M)$ be a control system on a manifold M. Denote the affine control bundle of S_M by $\mathcal{A}_M$. This is an affine subbundle of TM, so there exists a vector field X_M on M and a distribution Δ_M on M such that

$$\mathcal{A}_M = X_M + \Delta_M.$$

We say that Δ_M is the distribution *associated with* $\mathcal{A}_M$. Let $\Phi: M \to N$ be a surjective submersion, where N is an embedded submanifold of M. Denote by K the vector subbundle of TM defined as

$$K = \mathrm{Ker}(T\Phi) = \{v \in TM: T\Phi(v) = 0\}. \tag{6}$$

Since Φ is a submersion, the distribution K has constant dimension $k = m - n$ everywhere, where $m = \dim M$ and $n = \dim N$. Furthermore, K is an integrable distribution. Denote the foliation that is tangent to K by $\mathcal{K}$.

Our goal is to construct the smallest control system S_N on N which is Φ-related to S_M. We will accomplish this by constructing the smallest K-invariant affine subbundle $\tilde{\mathcal{A}}_M$ of TM containing $\mathcal{A}_M$ whose associated distribution contains K, and taking S_N to be any control system whose control bundle equals $T\Phi(\tilde{\mathcal{A}}_M)$.

A fiber bundle $\mathcal{B}$ over M is called *X-invariant*, for some smooth vector field X with local flow ϕ_t, if $T_p\phi_t(\mathcal{B}(p)) \subset \mathcal{B}(\phi_t(p))$, for all $p \in M$ and $t \in \mathbb{R}$ for which both sides are defined. For a distribution K, we say that $\mathcal{B}$ is *K-invariant*, if it is X-invariant for every vector field X in K.

Proposition 5.1: Let $\mathcal{A}$ be an affine subbundle of TM, where $\mathcal{A}(p) = X_0(p) + \Delta(p)$, for some vector field X_0 on M and distribution Δ on M. Let X be a vector field on M. Then $\mathcal{A}$ is X-invariant if and only if

$$[X, \mathcal{A}] \subset \Delta.$$

Proof: ($\Rightarrow$): Assume $\mathcal{A}$ is X-invariant. Denote the local flow of X by ϕ_t and let Y be any vector field in Δ. Then, for every $p \in M$ and t

$$T\phi_{-t}\{X_0(\phi_t(p)) + Y(\phi_t(p))\} \in \mathcal{A}(p).$$

Subtracting $X_0(p)+Y(p)$ from the left hand side, dividing by t, and letting $t \to 0$, we obtain $[X, X_0+Y](p) \in \Delta(p)$. Therefore, $[X, \mathcal{A}] \subset \Delta$.

($\Leftarrow$): Since $[X, \Delta] \subset [X, \mathcal{A}] \subset \Delta$, by a standard result in differential geometry [18], it follows that the distribution Δ is X-invariant. Similarly, we obtain that the distribution $\mathbb{R}X_0 \oplus \Delta$ is X-invariant. Therefore, for every $p \in M$ and $t \in \mathbb{R}$ for which $\phi_t(p)$ is defined

$$T\phi_{-t}(X_0(\phi_t(p)) = \alpha(p,t)X_0(p) \bmod \Delta$$

for some real-valued function α. That is, $T\phi_{-t}(X_0(\phi_t(p)) - \alpha(p,t)X_0(p) \in \Delta(p)$. Since $(d/dt)|_0 T\phi_{-t}(X_0(\phi_t(p)) = [X, X_0](p) \in \Delta(p)$, it is easy to see that $\dot{\alpha}(p,0) = 0$, for all p. However, α is a 1-cocyle over the flow of X, i.e., $\alpha(p, t+s) = \alpha(p,t)\alpha(\phi_t(p), s)$, so $\dot{\alpha}(p,t) = \alpha(p,t)\dot{\alpha}(\phi_t(p), 0) = 0$. Since $\alpha(p,0) = 1$, it follows that $\alpha(p,t)$ is identically equal to one. This implies that $T\phi_t(X_0) \in \mathcal{A}$, as desired. □

Definition 5.2 (Canonical Construction in M) : Given $\mathcal{A}_M$ and K as above, let $\tilde{\Delta}_M$ be the smallest K-invariant distribution containing Δ_M, K, and $[K, X_M]$ (see Fig. 1). Therefore, $\tilde{\Delta}_M$ is generated by

$$L_M \cup [K, L_M] \cup [K, [K, L_M]] \cup \cdots \tag{7}$$

where $L_M = \Delta_M \cup K \cup [K, X_M]$. Define the $\tilde{\mathcal{A}}_M$ as

$$\tilde{\mathcal{A}}_M = X_M + \tilde{\Delta}_M. \tag{8}$$

The affine bundle $\tilde{\mathcal{A}}_M$ is called the *canonical bundle associated with* $\mathcal{A}_M$ and K.

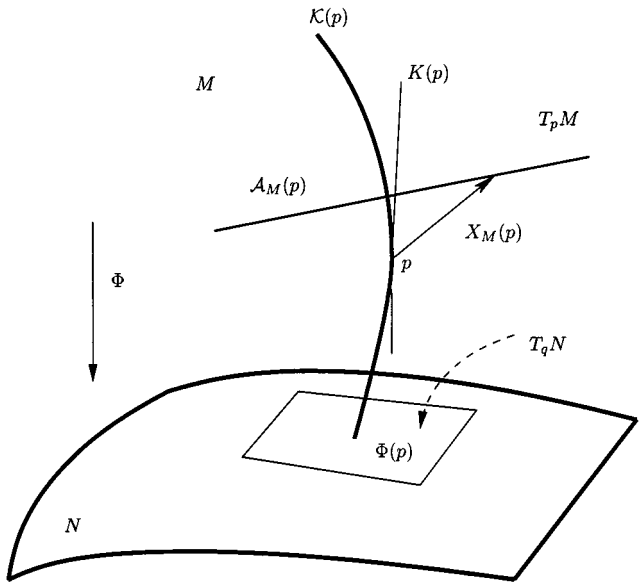

Fig. 1. Construction of $\tilde{\mathcal{A}}_M$.

The following proposition establishes the invariance properties needed for our construction.

Proposition 5.3 (K-Invariance and Φ-Relatedness):

a) The affine bundle $\tilde{\mathcal{A}}_M$ of Definition 5.2 contains $\mathcal{A}_M$, it is K-invariant, and its associated distribution $\tilde{\Delta}_M$ contains K. Moreover, it is the smallest affine bundle with these properties.

b) If $\Phi(p_0) = \Phi(p_1)$ for some $p_0, p_1 \in M$, then

$$(T_{p_0}\Phi)(\tilde{\mathcal{A}}_M(p_0)) = (T_{p_1}\Phi)(\tilde{\mathcal{A}}_M(p_1)).$$

Proof:

a) Clearly $\mathcal{A}_M \subset \tilde{\mathcal{A}}_M$. K-invariance follows from the construction of $\tilde{\Delta}_M$, the inclusion

$$[K, \tilde{\mathcal{A}}_M] = [K, X_M] + [K, \tilde{\Delta}_M] \subset \tilde{\Delta}_M$$

and Proposition 5.1.

To show that $\tilde{\mathcal{A}}_M$ is the smallest affine bundle with these properties, let $\mathcal{A}$ be another K-invariant affine bundle containing $\mathcal{A}_M$ whose associated distribution contains K. Then $\mathcal{A} = X_M + \Delta$, for some distribution Δ containing Δ_M. By K-invariance, $[K, \Delta] \subset [K, \mathcal{A}] \subset \Delta$, so Δ is K-invariant. Similarly, $[K, X_M] \subset \Delta$. Since $\tilde{\Delta}_M$ is by construction the smallest K-invariant distribution containing Δ_M, K, and $[K, X_M]$, it follows that $\tilde{\Delta}_M \subset \Delta$, hence $\tilde{\mathcal{A}}_M \subset \mathcal{A}$.

b) By the Frobenius Theorem, locally each leaf of $\mathcal{K}$ is a plane $x_{k+1} = \text{constant}, \ldots, x_m = \text{constant}$, N is the plane $\mathbb{R}^n \times \mathbf{0}$ in $\mathbb{R}^m$, and Φ is the projection $(x_1, \ldots, x_m) \mapsto (x_1, \ldots, x_n, 0, \ldots, 0)$. Assume p_0 and p_1 both lie in one such foliation chart [1] of $\mathcal{K}$. Since $\tilde{\mathcal{A}}_M$ is K-invariant and (in the same chart) $T\Phi = \text{diag}(I_{n\times n}, \mathbf{0})$, where $I_{n\times n}$ is the $n \times n$ identity matrix, it is easy to see that (b) holds. If p_1 is not in the same foliation chart as p_0, we can apply a similar argument to a finite sequence of foliation charts covering a path (in the leaf $\mathcal{K}(p_0)$) connecting p_0 and p_1. □

The above proposition ensures well posedness of the following definition which summarizes our canonical construction for extracting affine abstractions from affine control systems.

Definition 5.4 (Canonical Construction on N) : Let $S_M = (U_M, F_M)$ be a control system on a manifold M with affine control bundle

$$\mathcal{A}_M = X_M + \Delta_M.$$

Let $\Phi: M \to N$ be a surjective submersion, where N is an embedded submanifold of M. Denote by K the vector subbundle of TM defined by (6). Define the affine distribution $\tilde{\mathcal{A}}_M$ by

$$\tilde{\mathcal{A}}_M = X_M + \tilde{\Delta}_M$$

where $\tilde{\Delta}_M$ is generated by (7). The affine bundle $\mathcal{A}_N$ on N defined by

$$\mathcal{A}_N(q) = T_p\Phi(\tilde{\mathcal{A}}_M(p))$$

for any $p \in \Phi^{-1}(q)$, is said to be canonically Φ-related to $\mathcal{A}_M$. Any control system $S_N = (U_N, F_N)$ with control bundle $\mathcal{A}_N$ is said to be canonically Φ-related to S_M.

Theorem 5.5 (Canonically Φ-Related Systems) : The bundle $\mathcal{A}_N$ of Definition 5.4 is the smallest bundle on N which is Φ-related to $\mathcal{A}_M$.

Proof: That $\mathcal{A}_N$ is Φ-related to $\mathcal{A}_M$ follows from its construction and Proposition 5.3. To show that it is the smallest, assume $\mathcal{B}$ is another bundle on N Φ-related to $\mathcal{A}_M$. Let $\tilde{\mathcal{B}} = (T\Phi)^{-1}(\mathcal{B})$. Then $\tilde{\mathcal{B}}$ clearly contains $\mathcal{A}_M$ and is K-invariant. Therefore, by Proposition 5.3, $\tilde{\mathcal{A}}_M \subset \tilde{\mathcal{B}}$. It is then immediate that $\mathcal{A}_N = T\Phi(\tilde{\mathcal{A}}_M) \subset T\Phi(\tilde{\mathcal{B}}) = \mathcal{B}$, which proves the minimality of $\mathcal{A}_N$. □

Definitions 5.2 and 5.4, and Theorem 5.5 provide us with a constructive method to construct Φ-related systems. Furthermore, the construction is natural since it generates the smallest such system. We shall apply the canonical construction to various classes of affine systems in Section VII. In Section VI, we consider the relationship between Φ-related control systems regarding accessibility and reachability properties.

VI. Accessibility Equivalence

In addition to constructing abstractions of nonlinear systems, we are also interested in preserving properties of interest between the original and abstracted model. In [20], we focused on controllability of linear control systems. In this paper, we focus on local accessibility for affine control systems.

We first recall some standard definitions for reachable sets. Consider a control system $S_M = (U_M, F_M)$, let $V \subset M$ be a neighborhood of $p \in M$, and consider time $T > 0$. The reachable set from p at time T, denoted $\mathcal{R}^V_{S_M}(p, T)$, is the set of points that can be reached from p with trajectories of S_M that remain within V for all $0 < t \leq T$. In our definition of control systems, the reachable set is formally expressed as follows.

Definition 6.1 (Reachable Sets [18]) : Let $S_M = (U_M, F_M)$ be a control system on a manifold M. Given a neighborhood V

of $p \in M$, define the reachable set in time $T > 0$ as

$$\begin{aligned}\mathcal{R}^V_{S_M}(p,T) = \{&p' \in M \mid \exists\, c^U\colon [0,T] \to U_M \text{ with} \\ &c = \pi(c^U) \quad c(0) = p \quad c(T) = p' \text{ and} \\ &\forall\, 0 < t \le T, \quad c(t) \in V\}. \end{aligned} \tag{9}$$

The reachable set from p up to time T is defined as

$$\mathcal{R}^V_T(p, S_M) = \bigcup_{0<\tau\le T} \mathcal{R}^V_{S_M}(p,\tau). \tag{10}$$

Using the above definition of reachable sets, we can now define various notions of local accessibility.

Definition 6.2 (Local Accessibility [18]): A control system $S_M = (U_M, F_M)$ on a manifold M is said to be the following.

a) Locally accessible at $p \in M$ if for every neighborhood V of p and every $T > 0$, $\mathcal{R}^V_T(p, S_M)$ contains a nonempty, open set of M.
b) Locally accessible if it is locally accessible at every $p \in M$.
c) Symmetrically locally accessible at $p \in M$ if it is locally accessible at $p \in M$, and $\mathcal{R}^V_T(p, S_M)$ contains an open neighborhood of p.
d) Symmetrically locally accessible if it is symmetrically locally accessible at every $p \in M$.
e) Controllable if for every $p \in M$, $\mathcal{R}^M_\infty(p, S_M) = M$.

The following theorem allows us to check accessibility properties of control systems by simply checking the rank of certain distributions.

Theorem 6.3 (Rank Conditions [18]) : Consider a control system $S_M = (U_M, F_M)$ on an n-dimensional manifold M, and let $\mathcal{B}_M$ be the associated control bundle. Let $\mathcal{C}_M = \text{Lie}(\mathcal{B}_M)$ be the accessibility Lie algebra generated by $\mathcal{B}_M$. Then

a) if $\dim(\mathcal{C}_M(p)) = n$, then S_M is locally accessible at $p \in M$;
b) if $\dim(\mathcal{C}_M(p)) = n$ for all $p \in M$, then S_M is locally accessible;
c) if $\dim(\mathcal{C}_M(p)) = n$ and $\mathcal{B}_M(p)$ is symmetric at p, that is if $X_p \in \mathcal{B}_M(p)$ then $-X_p \in \mathcal{B}_M(p)$, then S_M is symmetrically locally accessible at $p \in M$;
d) if $\dim(\mathcal{C}_M(p)) = n$ and $\mathcal{B}_M(p)$ is symmetric for all $p \in M$, then S_M is symmetrically locally accessible;
e) if $\dim(\mathcal{C}_M(p)) = n$, $\mathcal{B}_M(p)$ is symmetric for all $p \in M$, and M is a connected manifold, then S_M is controllable.

We now focus on our problem of interest, namely the propagation of accessibility properties from the original to the abstracted system, and vice versa. One way is immediately given to us by Theorem 4.5 which propagates trajectories from the original to the abstracted system.

Theorem 6.4 (Accessibility Propagation): Let a control system $S_N = (B_N, F_N)$ be Φ-related to a control system $S_M = (B_M, F_M)$ with respect to some surjective submersion $\Phi\colon M \to N$. Then, for all $p \in M$

$$\Phi\left(\mathcal{R}^V_{S_M}(p,T)\right) \subseteq \mathcal{R}^{\Phi(V)}_{S_N}(\Phi(p),T) \tag{11}$$

$$\Phi\left(\mathcal{R}^V_T(p,S_M)\right) \subseteq \mathcal{R}^{\Phi(V)}_T(\Phi(p),S_N). \tag{12}$$

Therefore

a) if S_M is locally accessible at $p \in M$, then S_N is locally accessible at $\Phi(p) \in N$;
b) if S_M is locally accessible, then S_N is locally accessible;
c) if S_M is symmetric locally accessible at $p \in M$, then S_N is symmetric locally accessible at $\Phi(p) \in N$;
d) if S_M is symmetric locally accessible, then S_N is symmetric locally accessible;
e) if S_M is controllable, then S_N is controllable.

Proof: Consider any $p \in M$ and let $p' \in \mathcal{R}^V_{S_M}(p,T)$. By assumption there exists trajectory $c^U_M\colon [0,T] \to U_M$ of S_M with $c_M = \pi_M\left(c^U_M\right)$, $c_M(0) = p$, $c_M(T) = p'$ and for all $0 < t \le T$ we have $c_M(t) \in V$. Since S_N is Φ-related to S_M, by Theorem 4.5 there exists trajectory $c^U_N\colon [0,T] \to U_N$ of S_N with $c_N = \pi_N\left(c^U_N\right)$ and $c_N(t) = \Phi(c_M(t))$. Therefore, $c_N(0) = \Phi(p)$, $c_N(T) = \Phi(p')$, and $c_N(t) = \Phi(c_M(t)) \in \Phi(V)$ for all $0 < t \le T$. Thus, $\Phi(p') \in \mathcal{R}^{\Phi(V)}_{S_N}(\Phi(p),T))$ which proves (11). Having established (11), then (12) as well as a), b), c), d), and e) follow immediately using straightforward topological arguments. □

Note that Theorem 6.4 is true for any map Φ as long as it is a smooth surjective submersion. Furthermore, Theorem 6.4 holds for any two Φ-related systems, not only for the canonical construction of Definition 5.4. A different but equivalent proof of Theorem 6.4 would propagate the accessibility Lie algebra of S_M through the epimorphism $T\Phi$.

Whereas Theorem 6.4 propagates accessibility from the original to the abstracted system, from a hierarchical perspective, the reverse question is the complexity reducing direction. In other words, checking accessibility of the abstracted system should be equivalent to checking accessibility of the original, more complicated, system. We shall call such property preserving abstractions *consistent abstractions*.

This question will be answered for the canonical construction of Definition 5.4. We begin with the following proposition.

Proposition 6.5: Consider an affine control system $S_M = (U_M, F_M)$ and its associated affine control bundle $\mathcal{A}_M$ on a manifold M. Let $\Phi\colon M \to N$ be a surjective submersion where N is an embedded submanifold of M. Use Definition 5.4 to construct control system $\tilde{S}_M$ on M with control bundle $\tilde{\mathcal{A}}_M$, and $S_N = (U_N, F_N)$ on N that is canonically Φ-related to S_M. Furthermore, assume that

$$\text{Ker}(T\Phi) \subseteq \text{Lie}(\mathcal{A}_M).$$

Then, the following hold.

a) $\text{Lie}(\tilde{\mathcal{A}}_M) = \text{Lie}(\mathcal{A}_M)$.
b) For every $p \in M$, open set $V \subset M$, and $T > 0$

$$\mathcal{R}^V_T(p, S_M) = \mathcal{R}^V_T(p, \tilde{S}_M).$$

c) For every $q \in N$ we have

$$\mathcal{A}_N(q) = \tilde{\mathcal{A}}_M(q) \cap T_q N.$$

d) For every $q \in N$, open set $W \subset N$, and $T > 0$

$$\mathcal{R}^W_T(q, S_N) = \mathcal{R}^{\Phi^{-1}(W)}_T(q, \tilde{S}_M) \cap N.$$

e) For every $q \in N$, open set $W \subset N$, and $T > 0$

$$\mathcal{R}_T^W(q, S_N) = \mathcal{R}_T^{\Phi^{-1}(W)}(q, S_M) \cap N.$$

Proof:

a) Since $K \subseteq \text{Lie}(\mathcal{A}_M)$, we have $\tilde{\mathcal{A}}_M \subseteq \text{Lie}(\mathcal{A}_M)$, which implies that $\text{Lie}(\tilde{\mathcal{A}}_M) \subseteq \text{Lie}(\mathcal{A}_M)$. The opposite inclusion follows from $\mathcal{A}_M \subseteq \tilde{\mathcal{A}}_M$.

b) Follows from a).

c) Recall that $K \subset \tilde{\Delta}_M$ and $T_q\Phi$ is the projection $T_qM \to T_qN$ in the direction of K. Then, c) follows without difficulty from these facts.

d) Let $q \in N$ be arbitrary and suppose $q' \in \mathcal{R}_T^W(q, S_N)$. Then there exists an S_N-trajectory c from q to q' with $c(t) \in W$ for $0 < t \leq T$. By c), c is also an $\tilde{S}_M$-trajectory and it clearly lies in $W \subset N$. Thus, $q' \in \mathcal{R}_T^W(q, \tilde{S}_M) \subset \mathcal{R}_T^{\Phi^{-1}(W)}(q, \tilde{S}_M)$ which proves one direction.

Now suppose that $q' \in \mathcal{R}_T^{\Phi^{-1}(W)}(q, \tilde{S}_M) \cap N$. Then there exists an $\tilde{S}_M$-trajectory c (not necessarily in N) from q to q' with $c(t) \in \Phi^{-1}(W)$ for all $0 < t \leq T$. But then $\Phi(c)$ is an S_N-trajectory where $q, q' \in N$, $\Phi(c)$ connects p and q, and $\Phi(c)(t) \in W$ for all $0 < t \leq T$. Therefore, $q' \in \mathcal{R}_T^W(q, S_N)$ which completes the proof.

e) Follows from b) and d). □

The following theorem is an immediate consequence of the preceding result.

Theorem 6.6 (Accessibility Equivalence): Consider an affine control system $S_M = (U_M, F_M)$ and its associated affine control bundle $\mathcal{A}_M$ on a manifold M, and let $\Phi: M \to N$ be a surjective submersion. Use Definition 5.4 to construct a control system $S_N = (U_N, F_N)$ on N that is canonically Φ-related to S_M. Furthermore, assume that

$$\text{Ker}(T\Phi) \subseteq \text{Lie}(\mathcal{A}_M). \tag{13}$$

Then

a) S_N is locally accessible at $q \in N$ if and only if S_M is locally accessible at every $p \in \Phi^{-1}(q)$;

b) S_N is locally accessible if and only if S_M is;

c) S_N is symmetric locally accessible at $q \in N$ if and only if S_M is symmetric locally accessible at every $p \in \Phi^{-1}(q)$;

d) S_N is symmetrically locally accessible if and only if S_M is;

e) S_N is controllable if and only if S_M is.

Therefore, if S_N is Φ-related to S_M using the canonical construction described in Definition 5.4, and condition (13) is satisfied, then S_N is a consistent abstraction of S_M.

Condition (13) can be used in guiding the selection of the abstraction mapping $\Phi(x)$. Note that (13) can always be satisfied as long as inputs exist. For example, for the affine control system

$$\dot{x} = F(x) + G(x)u$$

we can always choose a map $\Phi(x)$ whose derivative satisfies the condition $\text{Ker}(T\Phi) = G(x)$, as long as $G(x)$ does not vanish. In this case, we are only ignoring directions that are *directly* controlled, therefore controllable, and condition (13) is automatically satisfied. The fact that the presence of control makes consistent abstraction possible clearly demonstrates the complexity reducing properties of control systems.

VII. Corollaries

In this section, we illustrate the construction of Definition 5.4 and apply Theorem 6.6 for various classes of affine control systems. We begin by recovering the results for linear systems that were obtained in [20].

A. Linear Systems

Consider the linear system

$$\dot{x} = Ax + \sum_{i=1}^{m} b_i u_i \in Ax + \text{span}\{b_1, \ldots, b_m\} \tag{14}$$

where $x \in \mathbb{R}^n$, and $b_1, \ldots, b_m \in \mathbb{R}^n$ are constant input vector fields. Suppose our abstraction maps are surjective linear maps $y = \Phi(x) = Cx$. Then C has full-row rank, the tangent map $T\Phi$ is simply C, and $K = \text{Ker}(T\Phi) = \text{Ker}(C)$. Consider

$$\begin{aligned} K =& \text{span}\{v_1, \ldots, v_k\}, \\ \mathcal{A}_M(x) =& Ax + \text{span}\{b_1, \ldots, b_m\}. \end{aligned}$$

The construction of Definition 5.4 results in

$$\begin{aligned} L_M =& \text{span}\{b_1, \ldots, b_m\} \\ & \cup \text{ span}\{Av_1, \ldots, Av_k\} \\ & \cup \text{ span}\{v_1, \ldots, v_k\} \\ \tilde{\mathcal{A}}_M(x) =& Ax + \text{span}\{b_1, \ldots, b_m\} \\ & + \text{span}\{v_1, \ldots, v_k\} \\ & + \text{span}\{Av_1, \ldots, Av_k\}. \end{aligned}$$

Higher order Lie brackets in (7) are clearly zero. The affine distribution $\mathcal{A}_N(y)$ at $y = Cx$ is

$$\begin{aligned} \mathcal{A}_N(y) = CAx + \text{span}\{Cb_1, \ldots, Cb_m\} \\ + \text{span}\{CAv_1, \ldots, CAv_k\}. \end{aligned}$$

for *any* $x \in C^{-1}(y)$. Since C has full row rank, we can choose $x = C^+y$ where $C^+ = C^T(CC^T)^{-1}$ is the Moore–Penrose pseudoinverse of C. Therefore, the canonically C-related system for any linear surjective map $y = Cx$ is

$$\begin{aligned} \mathcal{A}_N(y) = CAC^+y + \text{span}\{Cb_1, \ldots, Cb_m\} \\ + \text{span}\{CAv_1, \ldots, CAv_k\} \end{aligned}$$

or more compactly

$$\dot{y} = Fy + Gv \tag{15}$$

where

$$\begin{aligned} F =& CAC^+ \\ G =& [\, Cb_1 \quad \ldots \quad Cb_m \quad CAv_1 \quad \ldots \quad CAv_k \,]. \end{aligned}$$

In order to propagate accessibility properties, the linear abstraction map must satisfy the consistency condition (13) which in the linear context becomes

$$\begin{aligned}\mathrm{Ker}(C) \subseteq &\mathrm{Lie}(Ax, b_1, \ldots, b_m)\\ &= \mathrm{span}[\,B \quad AB \quad \ldots \quad A^{n-1}B\,] + \mathrm{span}\{Ax\}.\end{aligned} \tag{16}$$

Condition (16) can always be satisfied as long as $B \neq 0$ since we can always choose $\mathrm{Ker}(C) = \mathrm{span}[B]$. In order words, we can always obtain accessibility preserving abstractions as long as there are control inputs. Under these conditions, Theorem 6.6 directly implies that local accessibility of (15) is *equivalent* to local accessibility of (14). In fact, from Theorem 2.5, condition (16) propagates not only local accessibility, but also global controllability [20].

B. Bilinear Systems

Consider the bilinear system

$$\dot{x} = Ax + \sum_{i=1}^{m} (B_i \cdot x) u_i \tag{17}$$

where $x \in \mathbb{R}^n$, and $A, B_1, \ldots, B_m \in \mathbb{R}^{n\times n}$. Note that the reachable set from the origin is only the origin. Suppose our aggregation map is again linear $y = \Phi(x) = Cx$ and surjective. Then

$$\begin{aligned} K &= \mathrm{span}\{v_1, \ldots, v_k\},\\ \mathcal{A}_M &= Ax + \mathrm{span}\{B_1 x, \ldots, B_m x\}.\end{aligned}$$

The canonical construction results in

$$\begin{aligned} L_M(x) &= \mathrm{span}\{B_1 x, \ldots, B_m x\}\\ &\quad + \mathrm{span}\{v_1, \ldots, v_k\}\\ &\quad + \mathrm{span}\{Av_1, \ldots, Av_k\},\\ \tilde{\mathcal{A}}_M(x) &= Ax + \mathrm{span}\{B_1 x, \ldots, B_m x\}\\ &\quad + \mathrm{span}\{v_1, \ldots, v_k\}\\ &\quad + \mathrm{span}\{Av_1, \ldots, Av_k\}\\ &\quad + \mathrm{span}\{B_1 v_1, \ldots, B_1 v_k\}\\ &\quad \vdots\\ &\quad + \mathrm{span}\{B_m v_1, \ldots, B_m v_k\}.\end{aligned}$$

Second-order Lie brackets between L_M and K are zero. Since $v_1, \ldots, v_k \in \mathrm{Ker}(C)$, and choosing $x = C^+ y$ results in an affine bundle $\mathcal{A}_N$ defined by

$$\begin{aligned} \mathcal{A}_N(y) &= CAC^+ y + \mathrm{span}\left\{CB_1 C^+ y, \ldots, CB_m C^+ y\right\}\\ &\quad + \mathrm{span}\{CAv_1, \ldots, CAv_k\}\\ &\quad + \mathrm{span}\{CB_1 v_1, \ldots, CB_1 v_k\}\\ &\quad \vdots\\ &\quad + \mathrm{span}\{CB_m v_1, \ldots, CB_m v_k\}.\end{aligned}$$

Therefore. the canonically C-related system is

$$\begin{aligned}\dot{y} = CAC^+ y &+ \sum_{i=1}^{m} (CB_i C^+ y) v_i\\ &+ \sum_{i=1}^{k} (CAv_i) w_k + \sum_{i=1}^{m}\sum_{j=1}^{k} (CB_i v_j) z_{ij}.\end{aligned}$$

In order to propagate accessibility properties, the linear abstraction map must satisfy the consistency condition (13)

$$\mathrm{Ker}(C) \subseteq \mathrm{Lie}(Ax, B_1 x, \ldots, B_m x). \tag{18}$$

The Lie algebra $\mathrm{Lie}(Ax, B_1 x, \ldots, B_m x)$ of bilinear systems is spanned by $[A, B_i]x$, $[B_i, B_j]x$ and higher order matrix brackets. Unfortunately, at $x = 0$ we have $\mathrm{Lie}(Ax, B_1 x, \ldots, B_m x) = 0$, and therefore, a consistent abstraction is obtained only on $\mathbb{R}^n \setminus \{0\}$. This is not necessarily the case, however, if one considers bilinear systems of the form

$$\dot{x} = Ax + \sum_{i=1}^{m} (B_i \cdot x) u_i + \sum_{i=1}^{k} d_i w_i$$

in which case one can consistently abstract some dynamics on $\mathbb{R}^n$ by choosing $\mathrm{Ker}(C) \subseteq \mathrm{span}\{d_1, \ldots, d_k\}$.

C. Drift Free Systems

As a special case of affine control systems, consider the so-called drift free systems

$$\dot{x} = \sum_{i=1}^{m} g_i(x) u_i \in \mathrm{span}\{g_1(x), \ldots, g_m(x)\} \tag{19}$$

where $g_1(x), \ldots, g_m(x)$ are smooth vector fields on $\mathbb{R}^n$. In this case, the canonical construction of Definition 5.4 is simplified as the drift term $X_M = 0$. Therefore, rather than dealing with affine bundles, we now work with standard distributions. This results in the following construction.

Definition 7.1 (Canonical Construction on N): Let $S_M = (U_M, F_M)$ be a drift-free control system on a manifold M with distribution Δ_M. Let $\Phi: M \to N$ be a surjective submersion, where N is an embedded submanifold of M. Denote by K the vector subbundle of TM defined as

$$K = \mathrm{Ker}(T\Phi) = \{v \in TM : T\Phi(v) = 0\}.$$

Define the distribution $\tilde{\Delta}_M$ which is generated by

$$\Delta_M \cup K \cup [K, \Delta_M] \cup [K, [K, \Delta_M]] \cup \cdots. \tag{20}$$

The distribution Δ_N on N defined by

$$\Delta_N(q) = T_p \Phi(\tilde{\Delta}_M(p))$$

for any $p \in \Phi^{-1}(q)$, is canonically Φ-related to Δ_M. Any control system $S_N = (U_N, F_N)$ with distribution Δ_N is said to be canonically Φ-related to S_M.

The canonical construction of Definition 5.4 ensures that the abstraction of an affine control system is affine. Similarly, the

canonical construction of Definition 7.1 ensures that the abstraction of drift free control systems is also drift free. As an example, consider the unicycle model

$$\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{\theta}\end{bmatrix} = \begin{bmatrix}\cos(\theta)\\ \sin(\theta)\\ 0\end{bmatrix} v + \begin{bmatrix}0\\ 0\\ 1\end{bmatrix}\omega \tag{21}$$

and consider the abstracting map which simply ignores θ, that is $\Phi(x,y,\theta) = (x,y)$. The construction of Definition 7.1 results in

$$\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \begin{bmatrix}\cos(\theta)\\ \sin(\theta)\end{bmatrix} v_1 + \begin{bmatrix}\sin(\theta)\\ -\cos(\theta)\end{bmatrix} v_2$$

for *any* choice of θ. Choosing $\theta = 0$ results in

$$\begin{aligned}\dot{x} =& v_1\\ \dot{y} =& v_2.\end{aligned} \tag{22}$$

Note that the canonical construction preserves the drift free structure of the system. Furthermore, since

$$\mathrm{Ker}(T\Phi) = \mathrm{span}\left\{\begin{bmatrix}0\\0\\1\end{bmatrix}\right\} \subseteq \mathrm{span}\left\{\begin{bmatrix}\cos(\theta)\\ \sin(\theta)\\ 0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$$

system (22) is a consistent abstraction of the unicycle model (21). Therefore the unicycle model (21) is locally accessible if and only if system (22) is locally accessible, which is trivially true. The above abstraction of the nonholomic unicycle by a two dimensional integrator is exactly in the spirit of [25], where topological properties for collision avoidance of the models are also considered in detail.

D. Strict Feedback Systems

Consider the class of strict feedback systems used in backstepping designs [12], which have the following block triangular structure

$$\begin{aligned}\dot{x}_1 =& f_1(x_1) + g_1(x_1)x_2\\ \dot{x}_2 =& f_2(x_1,x_2) + g_2(x_1,x_2)x_3\\ &\vdots\\ \dot{x}_n =& f_n(x_1,x_2,\ldots,x_n) + g_n(x_1,x_2,\ldots,x_n)u\end{aligned} \tag{23}$$

where $x_i \in \mathbb{R}^{n_i}$ and all maps f_i, g_i are smooth. For notational simplicity, we present the canonical construction for $n = 2$, that is

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}f_1(x_1) + g_1(x_1)x_2\\ f_2(x_1,x_2)\end{bmatrix} + \begin{bmatrix}0\\ g_2(x_1,x_2)\end{bmatrix} u \tag{24}$$

and therefore, the affine bundle is

$$\begin{aligned}\mathcal{A}_M(x_1,x_2) =& \begin{bmatrix}f_1(x_1) + g_1(x_1)x_2\\ f_2(x_1,x_2)\end{bmatrix}\\ &+ \mathrm{span}\left\{\begin{bmatrix}0\\ g_2(x_1,x_2)\end{bmatrix}\right\}.\end{aligned} \tag{25}$$

Suppose our abstraction map is the a simple projection $y = \Phi(x_1,x_2) = x_1$. Then

$$\begin{aligned}y =& [I \quad 0]\begin{bmatrix}x_1\\ x_2\end{bmatrix} = x_1\\ K =& \mathrm{Ker}(T\Phi) = \mathrm{span}\left\{\begin{bmatrix}0\\ I\end{bmatrix}\right\}.\end{aligned} \tag{26}$$

The canonical construction results in $L_M = \Delta_M \cup K \cup [K, X_M]$, where

$$\begin{aligned}\Delta_M =& \mathrm{span}\left\{\begin{bmatrix}0\\ g_2(x_1,x_2)\end{bmatrix}\right\}\\ [K, X_M] =& \left[\begin{bmatrix}0\\ I\end{bmatrix}, \begin{bmatrix}f_1 + g_1\cdot x_2\\ f_2\end{bmatrix}\right] = \begin{bmatrix}g_1\\ \frac{\partial f_2}{\partial x_2}\end{bmatrix}\end{aligned}$$

and therefore, $[K, L_M]$ consists of

$$\begin{aligned}[K,\Delta_M] =& \mathrm{span}\left\{\begin{bmatrix}0\\ \frac{\partial g_2}{\partial x_2}\end{bmatrix}\right\}\\ [K,[K,X_M]] =& \mathrm{span}\left\{\begin{bmatrix}0\\ \frac{\partial^2 f_2}{\partial x_2^2}\end{bmatrix}\right\}\\ [K,K] =& 0.\end{aligned}$$

Clearly, $[K, L_M] \subseteq K$. Higher order Lie brackets, even though nonzero, also belong to $K = \mathrm{Ker}(T\Phi)$. Therefore, the construction results in

$$\begin{aligned}\tilde{\mathcal{A}}_M =& \begin{bmatrix}f_1(x_1) + g_1(x_1)x_2\\ f_2(x_1,x_2)\end{bmatrix} + \mathrm{span}\left\{\begin{bmatrix}0\\ g_2(x_1,x_2)\end{bmatrix}\right\}\\ &+ \mathrm{span}\left\{\begin{bmatrix}g_1\\ \frac{\partial f_2}{\partial x_2}\end{bmatrix}\right\} + \mathrm{span}\left\{\begin{bmatrix}0\\ I\end{bmatrix}\right\}\\ &+ \mathrm{span}\left\{\begin{bmatrix}0\\ \frac{\partial g_2}{\partial x_2}\end{bmatrix}\right\} + \mathrm{span}\left\{\begin{bmatrix}0\\ \frac{\partial^2 f_2}{\partial x_2^2}\end{bmatrix}\right\}.\end{aligned}$$

Pushing forward $\tilde{\mathcal{A}}_M(x_1,x_2)$ through $T\Phi$ results in

$$\mathcal{A}_N(x_1) = f_1(x_1) + g_1(x_1)x_2 + \mathrm{span}\{g_1(x_1)\}$$

for *any* $(x_1,x_2) \in \Phi^{-1}(x_1)$. Choosing $(x_1,0) \in \Phi^{-1}(x_1)$ results in the following abstracted system:

$$\begin{aligned}\dot{x}_1 =& f_1(x_1) + \mathrm{span}\{g_1(x_1)\}\\ =& f_1(x_1) + g_1(x_1)x_2\end{aligned} \tag{27}$$

where x_2 is now thought of as a virtual input. The above calculation also shows that for strict feedback systems, if x_2 is to be abstracted, then one can simply eliminate the differential equation associated with x_2. Therefore, the triangular nature of strict feedback systems make the computations for the canonical construction very simple.

In order to propagate accessibility, the consistency condition (13) must be satisfied. This means that

$$\begin{aligned}&\mathrm{Ker}(T\Phi) \subseteq \mathrm{Lie}(\mathcal{A}_M)\\ \Rightarrow\ &\mathrm{span}\left\{\begin{bmatrix}0\\ I\end{bmatrix}\right\} \subseteq \mathrm{Lie}\left(\begin{bmatrix}f_1 + g_1x_2\\ f_2\end{bmatrix}, \begin{bmatrix}0\\ g_2\end{bmatrix}\right).\end{aligned} \tag{28}$$

From (28), it is clear that if $g_2(x_1, x_2) \neq 0$ for all x_1, x_2, then the consistency condition is trivially satisfied and the the local accessibility of (24) is equivalent to the local accessibility of (27). If $g_2(x_1, x_2) = 0$ for some x_1, x_2, then the consistency condition may be satisfied by $f_2(x_1, x_2)$ or by using higher order Lie brackets. For example, the first-order Lie bracket contains

$$\text{span}\left\{\begin{bmatrix} g_1 g_2 \\ \frac{\partial f_2}{\partial x_2} g_2 - \frac{\partial g_2}{\partial x_2} f_2 - \frac{\partial g_2}{\partial x_1} f_1 - \frac{\partial g_2}{\partial x_1} g_1 x_2 \end{bmatrix}\right\}.$$

Therefore, the consistency condition is automatically satisfied if

$$\text{span}\left\{g_2, f_2, \frac{\partial f_2}{\partial x_2} g_2 - \frac{\partial g_2}{\partial x_2} f_2 - \frac{\partial g_2}{\partial x_1} f_1 - \frac{\partial g_2}{\partial x_1} g_1 x_2\right\} = \mathbb{R}^{n_2}$$

for all $x_1, x_2 \in \mathbb{R}^{n_1+n_2}$. If this is not satisfied, then higher order Lie brackets may be used.

Some classes of strict feedback systems deserve special attention.

E. Nonlinear Systems With Appended Linear Dynamics

Consider the following class of systems:

$$\begin{aligned} \dot{x}_1 &= f(x_1) + g(x_1) x_2 \\ \dot{x}_2 &= A x_2 + B u \end{aligned} \tag{29}$$

where $x_1 \in \mathbb{R}^{n_1}$, $x_2 \in \mathbb{R}^{n_2}$, $u \in \mathbb{R}^{n_u}$, f, g are smooth maps, and A, B are matrices of appropriate size. Such systems frequently arise in mechanical systems with nonlinear kinematics but linear actuator dynamics. In studying the local accessibility of such systems, rather than computing the full-blown accessibility Lie algebra, one would like to decompose the analysis in order to reduce the complexity.

System (29) can be thought of as a strict feedback system with considerably more structure since $f_2(x_1, x_2) = Ax_2$ and $g_2(x_1, x_2) = B$. Consider again the simple projection map $\Phi(x_1, x_2) = x_1$ which ignores the linear dynamics. The canonical construction of Theorem 5.5 proceeds in the same way as for strict feedback systems and results in the Φ-related system

$$\dot{x}_1 = f(x_1) + g(x_1) x_2 \tag{30}$$

where $x_2 \in \mathbb{R}^{n_2}$ is now an input.

Local accessibility of (30) is equivalent to the local accessibility of (29) if the consistency condition (13) is satisfied. The special structure of system (29), and some algebra reveals the following consistency condition:

$$\begin{aligned} \text{Ker}(T\Phi) &= \text{span}\left\{\begin{bmatrix} 0 \\ I \end{bmatrix}\right\} \\ &\subseteq \text{span}\left\{\begin{bmatrix} \text{irrelevant terms} \\ B \quad AB \quad \ldots \quad A^{n_2} B \end{bmatrix}\right\}. \end{aligned}$$

In other words, if the pair (A, B) is controllable, then we can simply ignore the linear part of the system, and local accessibility of (30) is equivalent to the local accessibility of (29). Therefore, the accessibility properties of system (29) truly decompose to the controllability property of the linear subsystem, and the accessibility property of the nonlinear subsystem.

F. Linear Systems With Appended Nonlinearities

Conversely, consider the following class of systems:

$$\begin{aligned} \dot{x}_1 &= A x_1 + B x_2 \\ \dot{x}_2 &= f_2(x_2) + g_2(x_2) u \end{aligned} \tag{31}$$

where $x_1 \in \mathbb{R}^{n_1}$, $x_2 \in \mathbb{R}^{n_2}$, $u \in \mathbb{R}^{n_u}$, f, g are smooth maps, and A, B are matrices of appropriate dimension. In this case, the abstracting map $\Phi(x_1, x_2) = x_1$ ignores the nonlinear part of the system. System (31) can be thought of as system in strict feedback form with special structure. Therefore, the canonical construction results in the abstracted model

$$\dot{x}_1 = A x_1 + B x_2. \tag{32}$$

Again the structure of (29) and some algebra lead to the following form for the consistency condition:

$$\begin{aligned} \text{Ker}(T\Phi) &= \text{span}\left\{\begin{bmatrix} 0 \\ I \end{bmatrix}\right\} \\ &\subseteq \text{span}\left\{\begin{bmatrix} \text{irrelevant terms} \\ \text{Lie}(f_2, g_2) \end{bmatrix}\right\}. \end{aligned}$$

Therefore, if the nonlinear subsystem is locally accessible, that is $\text{Lie}(f_2, g_2) = \mathbb{R}^{n_2}$, then the local accessibility of the nonlinear system (31) is equivalent to the controllability of the linear system (32).

VIII. Conclusion

In this paper, consistent abstractions of affine control systems were considered. In particular, we provided constructive methods for abstracting affine control systems with respect to smooth surjective maps. Our construction is structure preserving in the sense that affine control systems are abstracted by affine control systems. Furthermore, we characterized abstraction maps that result in preserving the property of local accessibility from the abstracted model to the original model. Our framework was then applied to various classes of nonlinear control systems including linear, bilinear, drift free, and strict feedback systems.

We believe that there is a clear research agenda which focuses on classes of systems as well as properties of interest and characterizes the abstracting maps that preserve the properties of interest for the particular class under consideration. For example, obtaining consistent abstractions for nonlinear systems with respect to stabilizability would be helpful in better understanding backsteppable systems. For hierarchical controller design, refining the controller design from the abstracted level to the more complicated model is a challenge. For linear systems, this was recently achieved in [19] from which we can extract as a special case the the hierarchical stabilization algorithm of [24]. Other properties of interest include trajectory optimality, preserving Hamiltonian structure [27], and the propagation of state and input constraints.

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