7-2012

Tail Assisted Dynamic Self Righting: Full Derivations

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Recommended Citation

Aaron M. Johnson, Tomas Libby, Evan Chang-Siu, Masayoshi Tomizuka, R J. Full, and Daniel E. Koditschek, "Tail Assisted Dynamic Self Righting: Full Derivations", . July 2012.

A. M. Johnson, T. Libby, E. Chang-Siu, M. Tomizuka, R. J. Full and D. E. Koditschek, Tail Assisted Dynamic Self Righting: Full Derivations, tech. rep., University of Pennsylvania (2012).

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Abstract
This technical report is a companion document to the CLAWAR 2012 paper of the same name, for which we explicitly write out a full derivation of the kinematics and dynamics. Please refer to that document for motivation, experimentation, and discussion.

Keywords
Tails, Angular Momentum, Motor Selection, Legged Robots

Disciplines
Robotics

Comments
A. M. Johnson, T. Libby, E. Chang-Siu, M. Tomizuka, R. J. Full and D. E. Koditschek, Tail Assisted Dynamic Self Righting: Full Derivations, tech. rep., University of Pennsylvania (2012).

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Tail Assisted Dynamic Self Righting: Full Derivations

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1. Introduction

This technical report is a companion document to the CLAWAR 2012 paper of the same name,[1] for which we explicitly write out a full derivation of the kinematics and dynamics. Please refer to that document for motivation, experimentation, and discussion. Equations numbered as (A.1) are new, while equations numbered as (1) match that document. Reference and footnote citations will not necessarily be equivalent.

2. Kinematics

Consider a simple planar model of a tailed robot in an aerial maneuver, representing the tail by a point mass held by a massless rigid rod, and the rest of the vehicle represented by a single rigid body (Figure 1). The tail is a point mass of m_t attached via a length l_t to a pivot l_b away from the body center of mass, and that the body has mass m_b and inertia I_b about its center of mass. The body and tail centers of mass are separated by a length l with angle θ_a, which is dependent on the internal angle $\theta_r = \theta_b - \theta_t$ (note that Figure 1 has a negative θ_r).

2.1. *Frames of Reference and Nonholonomic Constraints*

For ease of analysis we now define a system reference frame whose origin is placed at the centroid of the combined body-tail mechanism, denoted $\mathbf{r}_{com}$,

$$\mathbf{r}_{com} = \frac{m_b \mathbf{r}_b + m_t \mathbf{r}_t}{m_b + m_t} \equiv 0. \tag{A.1}$$

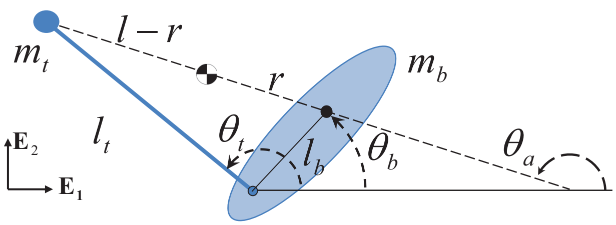

Fig. 1. Planar two body model.

where $\mathbf{r}_b$ and $\mathbf{r}_t$ are the position vectors of the body and tail, respectively. The orientation of this frame, expressed by the orthonormal vectors, $\{\mathbf{e}_r, \mathbf{e}_s\}$ is determined so that the body mass center lies along the vector $\mathbf{e}_r$ with magnitude $r \in \mathbb{R}^+$, as seen in Figure 1,

$$-m_t(l-r) + m_b r = 0; \qquad r = \frac{m_t}{m_b + m_t} l. \tag{A.2}$$

From the definition of the center of mass and the basis vectors $\mathbf{e}_r = \cos\theta_a \mathbf{E}_1 + \sin\theta_a \mathbf{E}_2$, and $\mathbf{e}_s = -\sin\theta_a \mathbf{E}_1 + \cos\theta_a \mathbf{E}_2$ it can be seen that

$$\mathbf{r}_b = r\mathbf{e}_r; \qquad \mathbf{r}_t = -(l-r)\mathbf{e}_r. \tag{A.3}$$

The total angular momentum, H_0, about the system's center of mass (obtained by adding the tail's point-mass component to the body's point-mass component along with the body's angular momentum about its own origin) is,

$$H_0\mathbf{E}_3 = I_b\dot{\theta}_b\mathbf{E}_3 + \mathbf{r}_b \times (m_b\dot{\mathbf{r}}_b) + \mathbf{r}_t \times (m_t\dot{\mathbf{r}}_t), \tag{A.4}$$

where $\mathbf{E}_3 = \mathbf{E}_1 \times \mathbf{E}_2$ exits the page. The tail and body velocities expressed in the moving system frame now read

$$\dot{\mathbf{r}}_b = r\dot{\theta}_a\mathbf{e}_s; \qquad \dot{\mathbf{r}}_t = -(l-r)\dot{\theta}_a\mathbf{e}_s \tag{A.5}$$

Inserting (A.3) and (A.5) into (A.4) and using (A.2),

$$H_0 = m_b r^2\dot{\theta}_a + m_t(l-r)^2\dot{\theta}_a + I_b\dot{\theta}_b \tag{A.6}$$

$$= \left(m_b r^2 + m_t r^2 - 2m_t l r + m_t l^2\right)\dot{\theta}_a + I_b\dot{\theta}_b \tag{A.7}$$

$$= \left(\frac{(m_b+m_t)m_t^2 l^2}{(m_b+m_t)^2} - \frac{2m_t^2 l^2}{m_b+m_t} + \frac{(m_b+m_t)m_t l^2}{m_b+m_t}\right)\dot{\theta}_a + I_b\dot{\theta}_b \tag{A.8}$$

$$= \frac{m_b m_t}{m_t+m_b} l^2\dot{\theta}_a + I_b\dot{\theta}_b, \tag{A.9}$$

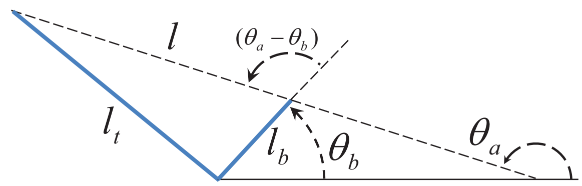

Fig. 2. Geometry of internal angles. The angle opposite l is $-\theta_r$, and the angle opposite l_t is $\pi - (\theta_a - \theta_b)$.

yields a scalar equation for the total angular momentum about the COM. Due to the absence of external moments, the total angular momentum about the COM is constant, and (A.9) imposes a non-holonomic constraint on the dynamics. Note that the version of this equation given in,[1]

$$H_0 = I_a\dot{\theta}_a + I_b\dot{\theta}_b; \qquad I_a = \frac{m_b m_t}{m_b + m_t} l_t^2, \tag{1}$$

is actually in reference to the linear version of (A.9), where $l = l_t$, as described in the next paragraph, rather than the general case where l can vary with configuration[a].

2.2. *Generalized Coordinates*

We find it convenient to adopt θ_b, θ_t as our generalized coordinates, and will now rewrite the noholonomic constraint in those terms. First consider the simplifying assumption that the tail pivots at the body center of mass ($l_b = 0$), then $\dot{\theta}_a = \dot{\theta}_t$, $l = l_t = const.$, and the constraint is linear in segment angular rates. We choose a time scale γ (units of $1/s$, see further discussion in the next section) such that $t^* = \gamma t$, where the * indicates dimensionless values. Substituting the dimensionless derivatives $\dot{\theta}_i = \gamma\dot{\theta}_i^*$, simplifying and dividing by I_b yields a dimensionless version of Eqn. (1),

$$\bar{H}_0 = \varepsilon\,\dot{\theta}_t^* + \dot{\theta}_b^*; \qquad \varepsilon = \frac{I_a}{I_b}; \tag{2}$$

where we define ε to be tail *effectiveness* (generalizing the previous definition[2]) and $\bar{H}_0 = H_0/(\gamma I_b)$ is a dimensionless momentum.

In the general case, when the tail pivot is some distance from the center of mass of the body ($l_b \neq 0$), l varies with configuration, and $\dot{\theta}_a$ depends

[a] I_a is conventionally called the *reduced mass* moment of inertia in the linear case. To consider a non-point mass tail, another term $I_t\dot{\theta}_t$ must be added to H_0.

on $\dot{\theta}_r$, $\dot{\theta}_b$ and $\dot{l}$. Applying the law of cosines about $-\theta_r$ (see Figure 2),

$$l^2 = l_t^2 + l_b^2 - 2l_t l_b \cos\theta_r, \tag{A.10}$$

and differentiating with respect to time,

$$2l\dot{l} = 2l_t l_b \sin\theta_r \dot{\theta}_r, \tag{A.11}$$

yields an expression for $\dot{l}$. Repeating this for the angle opposite l_t (which is $\pi - \theta_a + \theta_b$) yields:

$$l_t^2 = l^2 + l_b^2 + 2ll_b \cos(\theta_a - \theta_b) \tag{A.12}$$

$$0 = 2l\dot{l} + 2l_b\dot{l}\cos(\theta_a - \theta_b) - 2ll_b \sin(\theta_a - \theta_b)(\dot{\theta}_a - \dot{\theta}_b) \tag{A.13}$$

where $\cos(\theta_a - \theta_b)$ and $\sin(\theta_a - \theta_b)$ are readily found via the law of sines,

$$\frac{\sin(\theta_a - \theta_b)}{l_t} = \frac{\sin(-\theta_r)}{l} \tag{A.14}$$

$$\sin(\theta_a - \theta_b) = -\frac{l_t}{l}\sin\theta_r, \tag{A.15}$$

and by equating l_b with sum of the projections of l and l_t,

$$l_b = l_t \cos\theta_r - l\cos(\theta_a - \theta_b) \tag{A.16}$$

$$\cos(\theta_a - \theta_b) = \frac{l_t}{l}\cos\theta_r - \frac{l_b}{l}. \tag{A.17}$$

Thus starting with (A.13) and applying (A.11),

$$0 = 2l\dot{l}\left(1 + \frac{l_b}{l}\cos(\theta_a - \theta_b)\right) - 2ll_b \sin(\theta_a - \theta_b)(\dot{\theta}_a - \dot{\theta}_b) \tag{A.18}$$

$$0 = 2l_t l_b \sin\theta_r \dot{\theta}_r \left(1 + \frac{l_b}{l}\cos(\theta_a - \theta_b)\right) - 2ll_b \sin(\theta_a - \theta_b)(\dot{\theta}_a - \dot{\theta}_b) \tag{A.19}$$

and then (A.17),

$$0 = 2l_t l_b \sin\theta_r \dot{\theta}_r \left(1 + \frac{l_b}{l}\left(\frac{l_t}{l}\cos\theta_r - \frac{l_b}{l}\right)\right) - 2ll_b \sin(\theta_a - \theta_b)(\dot{\theta}_a - \dot{\theta}_b) \tag{A.20}$$

$$0 = \frac{2l_t l_b \sin\theta_r \dot{\theta}_r}{l^2}\left(l^2 + l_b l_t \cos\theta_r - l_b^2\right) - 2ll_b \sin(\theta_a - \theta_b)(\dot{\theta}_a - \dot{\theta}_b) \tag{A.21}$$

and (A.10),

$$0 = \frac{2l_t l_b \sin\theta_r \dot{\theta}_r}{l^2}\left(l_t^2 - l_b l_t \cos\theta_r\right) - 2ll_b \sin(\theta_a - \theta_b)(\dot{\theta}_a - \dot{\theta}_b) \tag{A.22}$$

and (A.15),

$$0 = \frac{2l_t l_b \sin\theta_r \dot{\theta}_r}{l^2}\left(l_t^2 - l_b l_t \cos\theta_r\right) + 2l_b l_t \sin\theta_r(\dot{\theta}_a - \dot{\theta}_b) \tag{A.23}$$

$$0 = 2l_t l_b \sin\theta_r \left(\frac{\dot{\theta}_r}{l^2}\left(l_t^2 - l_b l_t \cos\theta_r\right) + (\dot{\theta}_a - \dot{\theta}_b)\right) \tag{A.24}$$

$$0 = \frac{\dot{\theta}_r}{l^2}\left(l_t^2 - l_b l_t \cos\theta_r\right) + (\dot{\theta}_a - \dot{\theta}_b) \tag{A.25}$$

where the last simplification holds when $2l_t l_b \sin\theta_r \neq 0$. This can be rearranged to get an expression for $\dot{\theta}_a$,

$$\dot{\theta}_a = -\frac{\dot{\theta}_r}{l^2}\left(l_t^2 - l_b l_t \cos\theta_r\right) + \dot{\theta}_b \tag{A.26}$$

$$\dot{\theta}_a = -\frac{l_t^2}{l^2}\left(1 - \frac{l_b}{l_t}\cos\theta_r\right)\dot{\theta}_r + \dot{\theta}_b. \tag{A.27}$$

This expression enables expansion of (A.9) into a nonlinear version of (1),

$$H_0 = \frac{m_b m_t}{m_t + m_b} l^2 \left(-\frac{l_t^2}{l^2}\left(1 - \frac{l_b}{l_t}\cos\theta_r\right)\dot{\theta}_r + \dot{\theta}_b\right) + I_b\dot{\theta}_b \tag{A.28}$$

$$H_0 = -I_a\left(1 - \frac{l_b}{l_t}\cos\theta_r\right)\dot{\theta}_r + \left(\frac{m_b m_t}{m_t + m_b} l^2 + I_b\right)\dot{\theta}_b \tag{A.29}$$

when combined with (A.10),

$$H_0 = -I_a\left(1 - \frac{l_b}{l_t}\cos\theta_r\right)\dot{\theta}_r + \left(I_a(1 + \frac{l_b^2}{l_t^2} - 2\frac{l_b}{l_t}\cos\theta_r) + I_b\right)\dot{\theta}_b, \tag{A.30}$$

and with the definition of $\theta_r = \theta_b - \theta_t$, yields the angular momentum in link coordinates,

$$H_0 = I_a\left(1 - \frac{l_b}{l_t}\cos\theta_r\right)\dot{\theta}_t + \left(I_a(\frac{l_b^2}{l_t^2} - \frac{l_b}{l_t}\cos\theta_r) + I_b\right)\dot{\theta}_b. \tag{A.31}$$

Dividing by I_b, and simplifying leads to the nonlinear version of (2),

$$\bar{H}_0 = \varepsilon(1 - \lambda\cos\theta_r)\dot{\theta}_t^* + (\varepsilon(\lambda^2 - \lambda\cos\theta_r) + 1)\dot{\theta}_b^*; \qquad \lambda = l_b/l_t. \tag{3}$$

This equation governs both stabilization and zero angular momentum maneuvering. In the latter case, $\bar{H}_0 = 0$, and by maintaining our definition of effectiveness in righting as the ratio of segment speeds, the configuration-dependent *non-linear effectiveness* is,

$$\varepsilon_n = -\frac{\dot{\theta}_b^*}{\dot{\theta}_t^*} = \frac{\varepsilon(1 - \lambda\cos(\theta_r))}{\varepsilon(\lambda^2 - \lambda\cos(\theta_r)) + 1}. \tag{4}$$

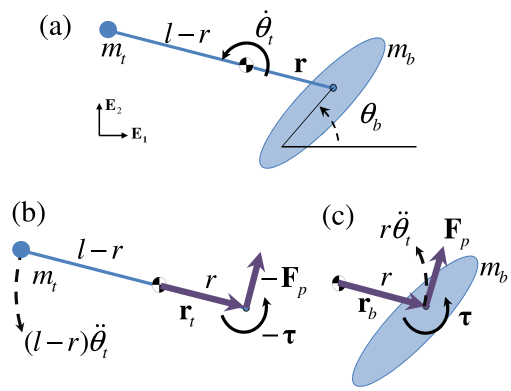

Fig. 3. Two link system with tail joint at COM of body.

3. Dynamics and Power Scaling for a Free Fall Task

While all isometrically scaled robots will maneuver with similar kinematics given enough time, a real terrestrial robot is constrained by the duration of its aerial phase (fall, leap, or other dynamic behavior) and this imposes a new set of requirements on a new set of parameters that specify the tail actuation power train.

3.1. *Linear Dynamics for a Simplified Design*

Consider a maneuverability task specified by the requirement to reorient the body through a fixed angle $\theta_b = \theta_0$, in a desired time $t = t_0$. Here we develop the linear (assuming $\lambda = 0$) dynamics associated with a bang-bang style of control (i.e., accelerating and then decelerating the tail with maximal available torque) imparted by a conventionally power-limited actuator (i.e., whose maximal available torque must decrease linearly with speed[3]).

The system dynamics are calculated by taking the time derivative of the angular momentum of each link about the COM, $\dot{\mathbf{H}}_i = \tau \mathbf{E}_3 + \mathbf{r}_i \times \mathbf{F}_p$ for $i \in \{b, t\}$. The first term on the right is the motor torque and the second term represents the torque due to the pin joint force, $\mathbf{F}_p$, applied at a position, $\mathbf{r}_i$, from the COM of the system (see Figure 3).

The pin force is the sole force acting on both links, therefore the value of $\mathbf{F}_p$ is equal to the product of each links' mass and acceleration,

$$\|\mathbf{F}_p\|_2 = m_b r |\ddot{\theta}_t| = m_t (l_t - r) |\ddot{\theta}_t|, \qquad \text{(A.32)}$$

where $\|\cdot\|_2$ is the 2-norm. Thus the body dynamics are

$$\dot{\mathbf{H}}_b = \tau\mathbf{E}_3 + \mathbf{r}_b \times \mathbf{F}_p \tag{A.33}$$

$$I_b\ddot{\theta}_b + m_b r^2\ddot{\theta}_t = \tau + m_b r^2\ddot{\theta}_t \tag{A.34}$$

$$I_b\ddot{\theta}_b = \tau. \tag{A.35}$$

Since the motor torque and pin force act with opposite direction for the tail link their signs change in the tail dynamics,

$$\dot{\mathbf{H}}_t = -\tau\mathbf{E}_3 - \mathbf{r}_t \times \mathbf{F}_p \tag{A.36}$$

$$m_t(l_t - r)^2\ddot{\theta}_t = -\tau - m_t r(l_t - r)\ddot{\theta}_t \tag{A.37}$$

$$I_a\ddot{\theta}_t = -\tau. \tag{A.38}$$

3.2. *Link Response with a Conventional Actuator*

Consider a motor-like actuator with a torque function linearly dependent on speed, applying torque to the tail joint as excited by its fixed (maximal) terminal voltage. The electromagnetic motor operates along its speed-torque curve,

$$\tau = \tau_m\left(1 - \frac{\dot{\theta}_b - \dot{\theta}_t}{\omega_m}\right) \tag{A.39}$$

where τ_m is the stall torque and ω_m is the no-load speed of the motor after the gear box.

Recall from Section 2 that the velocities of the body angle, inner angle, and tail angle are kinematically related by $\dot{\theta}_r = \dot{\theta}_b - \dot{\theta}_t = (1 + 1/\varepsilon)\dot{\theta}_b$, and therefore the full dynamics of any one angle can be determined by the dynamics of any of them by

$$\ddot{\theta}_b = \frac{\ddot{\theta}_r}{1+\frac{1}{\varepsilon}} \qquad \ddot{\theta}_t = -\frac{\ddot{\theta}_r}{1+\varepsilon}. \tag{A.40}$$

By substituting (A.35) into (A.39),

$$I_b\ddot{\theta}_b = \tau_m\left(1 - \frac{\dot{\theta}_r}{\omega_m}\right). \tag{A.41}$$

Finally, using the left hand equality of (A.40), $\ddot{\theta}_r$ is,

$$\ddot{\theta}_r = \frac{\tau_m(1+1/\varepsilon)}{I_b\omega_m}\left(\omega_m - \dot{\theta}_r\right). \tag{A.42}$$

To fully non-dimensionalize the dynamics we again choose a time scale γ such that $t^* = \gamma t$, where the * indicates dimensionless values. Since we seek to specify the entire power train, we find it convenient to decouple the roles of the actuator and the transmission in achieving this task by parametrization with respect to peak mechanical power, $P = \tau_m \omega_m/4$, (whose product form cancels the appearance of the gear ratio) and no-load speed, ω_m (whose linear dependence upon the gear ration makes it a useful surrogate for the transmission). Substituting the dimensionless derivatives $\dot{\theta}_i = \gamma \dot{\theta}_i^*$ and $\ddot{\theta}_i = \gamma^2 \ddot{\theta}_i^*$ yields,

$$\gamma^2 \ddot{\theta}_r^* = \frac{4P(1+1/\varepsilon)}{I_b \omega_m^2} \left(\omega_m - \gamma \dot{\theta}_r^* \right). \tag{A.43}$$

Choosing $\gamma := \frac{4P(1+1/\varepsilon)}{I_b \omega_m^2}$, and grouping the leading terms as $\alpha := \frac{\omega_m^3 I_b}{4P(1+1/\varepsilon)} = \omega_m/\gamma$, simplifies the dynamics to a single, dimensionless equation governing a reorientation with a motor under fixed (presumably maximum) voltage with $\bar{H}_0 = 0$,

$$\ddot{\theta}_r^* = \alpha - \dot{\theta}_r^* \tag{A.44}$$

i.e. to the inner joint the system looks like a simple inertial load and so the dynamics are a dimensionless version of (A.41). Assuming the initial conditions of $\dot{\theta}_r^*(0) = 0$ and $\theta_r(0) = 0$ admit the solution to this differential equation,

$$\ddot{\theta}_r^*(t^*) = \alpha \exp(-t^*) \tag{A.45}$$

$$\dot{\theta}_r^*(t^*) = \alpha(1 - \exp(-t^*)) \tag{A.46}$$

$$\theta_r(t^*) = \alpha(-1 + t^* + \exp(-t^*)). \tag{A.47}$$

In general we will most likely care more about rotations of θ_b or θ_t, but they are easy to derive (assuming initial conditions of $\theta_b = \theta_t = 0$ and $\dot{\theta}_b = \dot{\theta}_t = 0$)

$$\theta_b = \frac{\theta_r}{1 + \frac{1}{\varepsilon}} \qquad \theta_t = -\frac{\theta_r}{1+\varepsilon} \tag{A.48}$$

and similarly for all derivatives.

Therefore the closed form trajectory of the body angle is,

$$\theta_b = f_b(t^*) := \frac{\omega_m}{\gamma(1 + \frac{1}{\varepsilon})}(-1 + t^* + \exp(-t^*)) \tag{5}$$

Thus for a fixed system and time, $t_0^* = \gamma t_0$, we see that the robot body has rotated $\theta_b = f_b(\gamma t_0)$. Conversely if we desire a body rotation of θ_0, we must solve the implicit function $t = f_b^{-1}(\theta_0)/\gamma$ to find the time required. We can also turn this problem around and ask for a given task specification, a θ_0 body rotation in t_0, what are the constraints on the system parameters?

3.3. *Optimal Gearing for the Simplified Design*

To solve these problems, it will be convenient to introduce another grouping of system parameters, $\beta = \frac{4P(1+1/\varepsilon)}{I_b}$ (so that $\gamma = \beta/\omega_m^2$ and $\alpha = \omega_m^3/\beta$), decoupling the motor and system specification (now represented by β) from the transmission (now represented by ω_m).

We would like to eliminate ω_m from consideration, i.e. choose the "best" ω_m. Depending on the task, there are two optimal values for ω_m to consider — in the first case (i) there is a fixed $t := t_0$, and we would like to find the minimal β (i.e. minimal P, maximal ε, or maximal I_b), while the second case (ii) the sytem parameter $\beta := \beta_0$ is fixed (i.e. P, ε, and I_b are all fixed), and we would like to find the ω_m that minimizes the completion time t. In this section we will consider (i), while in the next section we will consider (ii).

Define a function $c(\omega_m, t, \beta)$ by expanding and regrouping (5),

$$0 = -\beta\theta_0 + \frac{\omega_m^3}{1+\frac{1}{\varepsilon}}\left(-1 + \frac{t\beta}{\omega_m^2} + \exp\left(-\frac{t\beta}{\omega_m^2}\right)\right) := c(\omega_m, t, \beta) \tag{A.49}$$

Since we wish to eliminate ω_m as a design parameter, we will consider certain properties of one of these implicit functions, either $t = s_t(\omega_m)$ or $\beta = s_\beta(\omega_m)$, that enforces one of the constraints,

$$(i) \qquad c(\omega_m, t_0, s_\beta(\omega_m)) \equiv 0, \tag{A.50}$$

$$(ii) \qquad c(\omega_m, s_t(\omega_m), \beta_0) \equiv 0, \tag{A.51}$$

with the aim of optimizing either (i) the the motor's mechanical power output via the system parameter, β, or (ii) the time duration of the task. Namely we will choose ω_m so as to minimize either (i) s_β or (ii) s_t by finding the critical points of the two different functions as follows.

First, for case (i), t_0 is fixed and we are trying to find the value of ω_m that minimizes s_β,

$$0 = D_{\omega_m} c + D_\beta c D_{\omega_m} s_\beta \tag{A.52}$$

$$0 = \frac{\partial c}{\partial \omega_m} + \frac{\partial c}{\partial s_\beta}\frac{d s_\beta}{d\omega_m} \tag{A.53}$$

$$\frac{d s_\beta}{d\omega_m} = \frac{\frac{\partial c}{\partial \omega_m}}{\frac{\partial c}{\partial s_\beta}} := 0 \tag{A.54}$$

$$0 = \frac{\partial c}{\partial \omega_m} \tag{A.55}$$

where the simplification in A.55 holds for $\frac{\partial c}{\partial s_\beta} \neq 0$ which is true when $-\theta_0 + \omega_m t_0(1 - \exp\left(-\frac{t_0 s_\beta}{\omega_m^2}\right))/(1 + \frac{1}{\varepsilon}) \neq 0$, i.e. $\theta_0 \neq 0$. Expanding the partial derivative,

$$0 = \frac{\partial c}{\partial \omega_m} = \frac{1}{1 + \frac{1}{\varepsilon}} \left(-3\omega_m^2 + t_0 s_\beta + \left(3\omega_m^2 + 2\omega_m^3 t_0 s_\beta \omega_m^{-3}\right) \exp\left(-\frac{t_0 s_\beta}{\omega_m^2}\right)\right) \tag{A.56}$$

$$0 = (-3 + \frac{t_0 s_\beta}{\omega_m^2}) + \left(3 + 2\frac{t_0 s_\beta}{\omega_m^2}\right) \exp\left(-\frac{t_0 s_\beta}{\omega_m^2}\right) \tag{A.57}$$

$$0 = (-3 + k) + (3 + 2k)\exp(-k); \qquad k := \frac{t_0 s_\beta}{\omega_m^2}, \tag{A.58}$$

where there is only a single positive value $k := k_0 \approx 2.15$ for which (A.58) holds (numerically computed). To get the optimal s_β and ω_m using (A.49) and (A.58),

$$k_0 = 2.15 = \frac{t_0 s_\beta}{\omega_m^2}, \qquad \omega_m = \sqrt{\frac{t_0 s_\beta}{k_0}} \tag{A.59}$$

$$s_\beta \theta_0 = \left(\frac{t_0 s_\beta}{k_0}\right)^{3/2} \left(\frac{-1 + k_0 + \exp(-k_0)}{1 + \frac{1}{\varepsilon}}\right) \tag{A.60}$$

$$\theta_0 = \frac{t_0^{3/2} s_\beta^{1/2} k_1}{1 + \frac{1}{\varepsilon}} \tag{A.61}$$

$$s_\beta = \frac{\theta_0^2 \left(1 + \frac{1}{\varepsilon}\right)^2}{t_0^3 k_1^2} \tag{A.62}$$

$$\omega_m = \frac{\theta_0 \left(1 + \frac{1}{\varepsilon}\right)}{t_0 k_0^{1/2} k_1}, \tag{A.63}$$

where $k_1 = [k_0^{-3/2}(-1 + k_0 + \exp(-k_0))] \approx 0.402$. Note that (A.62) is the minimal β needed to complete a θ_0 body rotation in t_0 time, and that the optimal motor speed given in (A.63) is proportional to the angular change desired over the time desired.

Expanding s_β in (A.61) yields,

$$\theta_0 = \frac{2t_0^{3/2} P^{1/2} k_1}{(1 + 1/\varepsilon)^{1/2} I_b^{1/2}}, \tag{6}$$

and thus the minimum power required is,

$$P = \frac{\theta_0^2}{4t_0^3 k_1^2}(1 + 1/\varepsilon) I_b, \tag{7}$$

but of wider interest may be power density, $P_d = P/m$.

This relationship reveals an important constraint on dynamic tail reorientation: the effect of robot size. Consider a robot isometrically scaled by a length scale L. Then mass m we will scale by the cube of length L and $I_b \propto L^5$. If the robot were required to reorient through the same angle in the same time regardless of size, then by substitution into Eq. (7) we would require power density $P_d \propto L^2$. However, a larger robot will fall slower relative to its length. Considering a free falling distance $h \propto L$ implies that the time available $t \propto L^{1/2}$. Therefore, from Eq. (7) the power density,

$$P_d \propto \frac{1}{L^3} \frac{1}{L^{3/2}} L^5 = L^{1/2}, \tag{8}$$

scales as the square root of length. This indicates that inertial reorientation gets more expensive at large size scales; larger robots may suffer reduced performance, or must dedicate a growing portion of total body mass to tail actuation. However, the robots in this paper span a characteristic length range of almost four fold without dramatic differences in ability; in this case, variance in motor power density may trump scaling.

4. XRL Tail Design

See[1] for a full discussion of the tail design, here we will simply derive (9). As in case (ii) from the previous section, here the power is given for each motor and we now seek to determine the minimal completion time as a function of peak power (parametrized by morphology) rather than the inverse function as above. The time requirement t_0 from the previous section can be thought of a constraint, while here for those motors that meet that constraint we want to consider the fastest completion time as a metric.

Recall that, for case (ii), β_0 is fixed and we are trying to find the value of ω_m that minimizes s_t,

$$0 = D_{\omega_m} c + D_t c D_{\omega_m} s_t \tag{A.64}$$

$$0 = \frac{\partial c}{\partial \omega_m} + \frac{\partial c}{\partial s_t} \frac{d s_t}{d \omega_m} \tag{A.65}$$

$$\frac{d s_t}{d \omega_m} = \frac{\frac{\partial c}{\partial \omega_m}}{\frac{\partial c}{\partial s_t}} := 0 \tag{A.66}$$

$$0 = \frac{\partial c}{\partial \omega_m}, \tag{A.67}$$

where the simplification in A.66 holds for $\frac{\partial c}{\partial s_t} \neq 0$ which is true when $\omega_m \beta_0 \left(1 - \exp\left(-\frac{s_t \beta_0}{\omega_m^2}\right)\right) / \left(1 + \frac{1}{\varepsilon}\right) \neq 0$, i.e. $\beta_0 \neq 0$, $\omega_m \neq 0$, and $s_t \neq 0$.

Therefore, expanding the solution to $0 = \frac{\partial c}{\partial \omega_m}$ as in (A.55), and as in (A.58),

$$0 = (-3 + k) + (3 + 2k)\exp(-k); \qquad k := \frac{s_t \beta_0}{\omega_m^2}. \tag{A.68}$$

The optimal s_t and ω_m using (A.49) and (A.68),

$$k_0 = 2.15 = \frac{s_t \beta_0}{\omega_m^2}, \qquad \omega_m = \sqrt{\frac{s_t \beta_0}{k_0}} \tag{A.69}$$

$$\beta_0 \theta_0 = \left(\frac{s_t \beta_0}{k_0}\right)^{3/2} \left(\frac{-1 + k_0 + \exp(-k_0)}{1 + \frac{1}{\varepsilon}}\right) \tag{A.70}$$

$$\theta_0^2 = \frac{s_t^3 \beta_0 k_1^2}{1 + \frac{1}{\varepsilon}} \tag{A.71}$$

$$s_t = \left(\frac{\theta_0^2 \left(1 + \frac{1}{\varepsilon}\right)^2}{\beta_0 k_1^2}\right)^{1/3} \tag{A.72}$$

$$\omega_m = \frac{\left(\frac{\theta_0^2 \left(1 + \frac{1}{\varepsilon}\right)^2}{\beta_0 k_1^2}\right)^{1/6} \beta_0^{1/2}}{k_0^{1/2}} = \left(\frac{\theta_0 \beta_0 \left(1 + \frac{1}{\varepsilon}\right)}{k_1 k_0^{3/2}}\right)^{1/3}. \tag{A.73}$$

Note that (A.72) is the fastest a robot with system parameters β_0 can complete a θ_0 body rotation, and that the optimal motor speed given in (A.73) scales with the cube root of the system parameters and desired angular change, which makes α directly proportional to the system parameters and desired angular offset.

Therefore when calculating the performance of a given motor and system, the optimal no load speed (after gear ratio) and resulting completion time functions are,

$$\omega_m = \left(\frac{\theta_0 \beta_0 \left(1 + \frac{1}{\varepsilon}\right)}{k_1 k_0^{3/2}}\right)^{1/3}; \qquad t = \left(\frac{\theta_0^2}{4 P k_1^2}(1 + 1/\varepsilon) I_b\right)^{1/3}. \tag{9}$$

Note that the optimal ω_m given in (9) from[1] was slightly incorrect and that it did not define β_0.

Acknowledgments

This was supported primarily by the ARL/GDRS RCTA and the NSF CiBER-IGERT under Award DGE-0903711. The authors would like to thank Praveer Nidamaluri for building the X-RHex tail, as well as David Hallac, Justin Starr, Avik De, Ryan Knopf, Mike Choi, Joseph Coto, and Adam Farabaugh for help with the robot and experiments.

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