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# Abstraction and Control for Groups of Fully-Actuated Planar Robots

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## Abstract

This paper shows how a large number of robots can be coordinated by designing control laws on a small dimensional manifold, independent on the number and ordering of the robots. The small dimensional description of the team has a product structure of a Lie group, which captures the dependence of the ensemble on world frame, and a shape manifold, which is an intrinsic description of the team. We design decoupled controls for group and shape. The individual control laws which are mapped to the desired collective behavior can be realized by feedback depending only on the current state of the robot and the state on the small dimensional manifold, so that the robots have to broadcast their states and only have to listen to some coordinating agent with small bandwidth.

## Comments

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## Abstraction and Control for Groups of Fully-Actuated Planar Robots

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**Abstract**—This paper shows how a large number of robots can be coordinated by designing control laws on a small dimensional manifold, independent on the number and ordering of the robots. The small dimensional description of the team has a product structure of a Lie group, which captures the dependence of the ensemble on world frame, and a shape manifold, which is an intrinsic description of the team. We design decoupled controls for group and shape. The individual control laws which are mapped to the desired collective behavior can be realized by feedback depending only on the current state of the robot and the state on the small dimensional manifold, so that the robots have to broadcast their states and only have to listen to some coordinating agent with small bandwidth.

### I. INTRODUCTION

We approach the problem of controlling a large number of robots required to accomplish a mission as a group. For example, consider the task of moving hundreds of robots from arbitrary initial positions through a tunnel while staying grouped so that the distance between each pair does not exceed a certain value. The simplest solution, generating motion plans or control laws for each robot, is obviously not feasible from a computational viewpoint. It is desired to have a certain level of abstraction: the motion generation/control problem should be solved in a lower dimensional space which captures the behavior of the group and the nature of the task.

The robots can be required to form a *virtual structure*. In this case, the problem is reduced to a left invariant control system on  $SE(l)$  ( $l = 1, 2$ ), and the individual trajectories are  $SE(l)$  - orbits [1]. The literature on stabilization and control of virtual structures is rather extensive. Most of the recent works model formations using *formation graphs*, which are graphs whose nodes capture the individual agent kinematics or dynamics, and whose edges represent inter-agent constraints that must be satisfied [6]. Characterizations of rigid formations can be found in [4], [1]. The controllers guaranteeing local asymptotic stability of a given rigid formation are derived using Lyapunov energy-type functions [6]. Examples of such functions include positive definite convex *formation functions* [3] and biologically inspired *artificial potential functions* [5]. The global minima of such functions exhibit  $SE(l)$ ,  $l = 1, 2, 3$  symmetry and also expansion/contraction

symmetries, which can be used to decouple the mission control problem into a formation keeping subproblem and a maneuver subproblem [5].

The virtual structure approach is not appropriate for many applications, including obstacle avoidance, tunnel passing, etc. Also, the rigid formulation is based on identified robots, which makes the obtained control laws and motion plans invalid in the case of individual failures. Moreover, the rigidity constraint induces an inherent coupling between the control systems on the symmetry group and the shape space. For example, in [5], the authors have to limit the speed of convergence on the symmetry group so that, while moving as a group, the individual agents do not leave the local regions of attractions guaranteeing convergence to the desired shape.

We propose an abstraction based on the definition of a map  $\phi$  from the configuration space  $Q$  of the robots to a lower dimensional *abstract manifold*  $A$ . We focus on planar kinematic fully actuated robots and require the abstract manifold to have a product structure  $A = G \times S$ , where  $G$  is a *Lie group* which captures the dependence of the problem on the chosen world coordinate frame and  $S$  is a *shape manifold*, which is an intrinsic description of the team. We also impose that the map  $\phi$  is so that each abstract variable can be controlled independently, so that the user can easily design controllers to only change the shape for example, and keep the group variable fixed. In this paper,  $G$  is  $SE(2)$  and  $S$  gives a description of the distribution of the robots along the axes of a virtual frame whose pose on the world frame evolves on  $G$ . The task to be accomplished by the team suggests a natural feedback control system on the abstract manifold. We show that the individual control laws which are mapped to the desired abstract behavior can be realized by feedback depending only on the current state of the robot and the state on the abstract manifold, so that the robots have to broadcast their states and only have to listen to some coordinating agent with small bandwidth.

### II. DEFINITIONS AND PROBLEM FORMULATION

Consider  $N$  kinematically controlled robots with states  $q_i$  belonging to manifold  $Q_i$  and control spaces  $U_i$ . For planar fully actuated agents, the states are position vectors

$q_i \in Q_i = \mathbf{R}^2$ ,  $i = 1, \dots, N$  with respect to some world frame  $\{W\}$ , and the controls  $u_i \in U_i = \mathbf{R}^2$ :

$$\dot{q}_i = u_i \quad (1)$$

Collecting all the robot states together, we get a  $2N$ -dimensional control system

$$\dot{q} = u \quad (2)$$

where  $q \in Q = \prod_{i=1}^N Q_i = \mathbf{R}^{2N}$ ,  $u \in U = \prod_{i=1}^N U_i = \mathbf{R}^{2N}$  and the canonical projections:

$$\pi_i(q) = q_i, d\pi_i(u) = u_i \quad (3)$$

The motion (behavior) of the ensemble of robots is determined if the corresponding velocities are specified:

*Definition 1 (Behavior):* Any vector field  $X_Q \in TQ$  is called a behavior.

Given a large number of robots evolving on the configuration space  $Q$ , we want to solve motion generation / control problems on a smaller dimensional space, which captures the essential features of the group, according to the class of tasks to be accomplished. We want the dimension of the control problem to be independent of the number of agents and also independent on possible ordering of the robots. These requirements will provide control laws which are robust to individual failures and also good scaling properties. We also need to make sure that, after solving the task on the small dimensional space, we can go back and generate control laws for the individual agents. All these ideas lead to the following definitions:

*Definition 2 (Abstraction):* Any submersion

$$\phi : Q \rightarrow A, \phi(q) = a \quad (4)$$

is called an abstraction if it is invariant to permutations of the robots and the dimension  $n$  of  $A$  is not dependent on the number of robots  $N$ .  $A$  and  $a$  are called abstract manifold and abstract state, respectively.

We require that  $A$  have a product structure

$$A = G \times S, a = (g, s), \phi = (\phi_g, \phi_s) \quad (5)$$

where  $G$  is a Lie group. An arbitrary  $g \in G$  is called *group*, or *pose* and an  $s \in S$  is called *shape*. The main idea is to have a control suited description of the team of robots  $a$  in terms of the pose  $g$  of a virtual structure, which captures the dependence of the team on the world frame  $\{W\}$ , plus a shape  $s$ , which is decoupled from  $g$ , and therefore, an intrinsic property of the formation. In other words, if  $\bar{g}$  is an arbitrary element of  $G$ , we require the map  $\phi$  to satisfy

$$\phi(q) = (g, s) \Rightarrow \phi(\bar{g}q) = (\bar{g}g, s) \quad (6)$$

where  $\bar{g}q$  represents the action of the group element  $\bar{g}$  on the configuration  $q \in Q$  and  $\bar{g}g$  represents the left translation of  $g$  by  $\bar{g}$  using the composition rule on the group  $G$ . Since we only approach planar robots in this

paper,  $G$  is  $SE(2)$ .  $\bar{g}q$  represents a rigid displacement of all the robots by  $\bar{g}$ . (6) is a left invariance - type property of the map  $\phi$ , which gives invariance of our to be designed control laws to the pose of the world frame  $\{W\}$ . Indeed, if the world frame  $\{W\}$  is displaced by  $\bar{g}$ , the shape  $s$  is not affected while the pose  $g$  is left translated by  $\bar{g}$ .

Instead of designing high dimensional behaviors  $X_Q$ , we want to be able to describe collective behaviors in terms of time - parameterized curves on the small dimensional abstract manifold  $A$ .

*Definition 3 (Abstract behavior):* Any vector field  $X_A \in TA$  is called an abstract behavior.

Let  $d\phi$  denote the differential (tangent) of the map  $\phi$ . Note that the submersion condition in Definition 2 guarantees the surjectivity of the differential  $d\phi$  at any  $q \in Q$ , which will guarantee the existence of vector fields  $X_Q$  pushed forward to any abstract behavior  $X_A$ .

The abstraction  $\phi$  gives a decomposition of the space of behaviors on  $Q$  into behaviors which can be "seen" in the abstract manifold  $A$  and behaviors which cannot be seen in  $A$ .

*Definition 4 (Detectable behaviors):* A behavior  $X_Q \in TQ$  which is mapped to a non-zero abstract behavior  $X_A \in TA$  is called a detectable behavior. A behavior which is not detectable is called non-detectable.

In this paper, we will not allow individual motions which cannot be captured in  $A$ . However, non-detectable behaviors can be useful to accommodate other specifications. For example, an abstract behavior  $X_A$  could specify the time - evolution of the pose (group part  $g$ ) and semi-axes (shape part  $s$ ) of an ellipsoid with the guarantee that all the robots are inside it. The behavior  $X_Q$  could be the sum of the detectable behavior which produces the desired  $X_A$  plus a non-detectable part (not affecting the abstract behavior) which could accomplish the specification that the area inside the ellipsoid is uniformly occupied by the robots. We are now able to formulate the main problem:

*Problem 5 (Abstract control):* Determine physically meaningful formation abstractions  $\phi$ , abstract behaviors  $X_A$ , and corresponding individual robot control laws  $u_i$ , satisfying the following requirements:

- (i) The abstract state  $a$  is at rest if and only if all the robots  $q_i$  are at rest.
- (ii) The energy spent by the individual robots to produce a desired abstract behavior  $X_A$  is kept to a minimum.
- (iii) The abstract manifold  $A$  has a product structure (5) and  $\phi$  satisfies the left invariance property (6).
- (iv) The control systems on the group  $G$  and shape  $S$  are decoupled.
- (v) If the state  $a$  of the abstract manifold is bounded, then the state of each robot  $q_i$  is bounded.
- (vi) The amount of inter - robot communication in the overall control architecture is limited.

Requirement (i) from Problem 5 guarantees that each individual motion on  $Q_i$  can be “seen” in the small dimensional manifold  $A$  and, therefore, can be “penalized” by control. This is equivalent to the detectability of the corresponding behavior  $X_Q$ . If requirement (iv) is satisfied, then one can design control laws for the interest variables on  $a$  separately, *e.g.*, change the pose of the formation  $g$  while preserving the shape  $s$ . The other requirements in Problem 5 are self-explanatory.

### III. APPROACH

In this section we characterize the solution to Problem 5. First, note that the map  $\phi$  gives a foliation of the configuration space  $Q$ . We assume that the abstract manifold has the desired product structure  $A = G \times S$ . Let  $\Omega_g$  be the codistribution spanned by the differential one forms obtained by differentiating each component of  $\phi_g$ . Similarly,  $\Omega_s$  is the codistribution determined by  $\phi_s$ . Let  $\Delta_g$  and  $\Delta_s$  denote the corresponding annihilating distributions, *i.e.*,  $\Omega_g(\Delta_g) = 0$ ,  $\Omega_s(\Delta_s) = 0$ . Let  $\Delta_g^\perp$  and  $\Delta_s^\perp$  be the orthogonal distributions (in some metric on  $Q$ ) to  $\Delta_g$  and  $\Delta_s$ , respectively. If  $X_q$  denotes the non-zero value of the vector field  $X_Q$  at the point  $q \in Q$ , then

$$X_q \in \Delta_g^\perp \quad (7)$$

guarantees that, on the abstract manifold, at  $a = \phi(q) = (g, s)$ ,  $g$  changes in time. Similarly,

$$X_q \in \Delta_s^\perp \quad (8)$$

corresponds to a change in the shape variable  $s$ . The set of detectable behaviors at  $q \in Q$  is given by  $\Delta_g^\perp \oplus \Delta_s^\perp$ . Requirement (i) from Problem 5 can therefore be written as:

$$X_q \in \Delta_g^\perp \oplus \Delta_s^\perp \quad (9)$$

In other words, system (2) is forbidden to move on a leaf  $\phi = \text{const.}$  (motion which could not be “observed” on the abstract manifold  $A$ ) if and only if (9) is satisfied.

The decoupling between the control of the group  $G$  and the shape  $S$  of  $A$  (requirement (iv) of Problem 5) is achieved if the distributions  $\Delta_g^\perp$  and  $\Delta_s^\perp$  are orthogonal. In this case, control vectors satisfying (7) will produce a change in the pose of the group  $g$  while the shape  $s$  is maintained constant. On the other hand, controls satisfying (8) will change the shape of a stationary formation. Complete decoupling of the control variables in  $A$  is guaranteed if, in addition, orthogonal control directions are chosen as basis for  $\Delta_g^\perp$  and  $\Delta_s^\perp$ .

For (v), note that Problem 5 can actually be seen as an input - output linearization problem for the control system (2) with output  $a = \phi(q)$ . The vector field  $X_A$  guarantees some desired behavior of the output  $a$ , which will, of course, guarantee its boundness. Now the hardest problem, as usual in input - output linearization, is calculating and

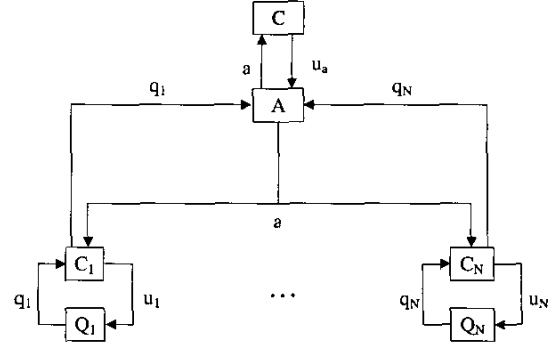


Fig. 1. Overall control architecture

stabilizing the internal dynamics. To avoid this, we define the output map so that bounds on output would easily imply bounds on the state, so it will not be necessary to explicitly calculate the internal dynamics.

From now on, we will assume that  $Q$  is equipped with an Euclidean metric. On the energy spent by the individual robots to realize a given formation behavior  $X_A$ , (requirement (ii) in Problem 5) first note that since  $\phi$  is a submersion,  $\phi_1, \dots, \phi_n$  are functionally independent, or, equivalently,  $d\phi = (d\phi_1, \dots, d\phi_n)$  is full row rank ( $d\phi_i$  should be interpreted as rows giving the coordinates of the corresponding differential one forms, and  $d\phi$  is the row span of  $d\phi_i$ 's). Then, in the assumed Euclidean metric, the minimum norm vector  $X_Q^*$  which is pushed forward to an arbitrary  $X_A$ , *i.e.*,  $d\phi X_Q^* = X_A$  is given by

$$X_Q^* = d\phi^T (d\phi d\phi^T)^{-1} X_A \quad (10)$$

On the other hand, if  $Q$  is equipped with an Euclidean metric, then  $\Delta_g^\perp$  and  $\Delta_s^\perp$  are spanned by vectors with coordinates given by the coordinates of the differential forms associated to  $d\phi_g$  and  $d\phi_s$ . Therefore  $X_Q^*$  satisfies (9). Moreover, if  $d\phi_i$ 's are orthogonal and  $X_A, X_Q^*$  are written in coordinates, we have

$$\dot{q}^* = \sum_{i=1}^n \frac{d\phi_i^T}{d\phi_i d\phi_i^T} \dot{a}_i \quad (11)$$

from which the decoupling of the control variables on  $F$  is obvious.

To limit the amount of inter - robot communication in the overall control scheme, we propose an architecture where the control law of a robot only depends on its own state and the low dimensional state of the team from the group manifold:

$$u_i = u_i(q_i, a) \quad (12)$$

Pictorially, the desired control architecture combining abstraction and partial state feedback features is given in Figure 1.

#### IV. ABSTRACTION

In this section we define a physically significant abstraction (4) with a product structure (5). The proof of left invariance is omitted and can be found in [2]. We first show that the abstraction satisfies requirements (iv) and (vi) from Problem 5. Then we construct individual control laws in accordance with requirements (i) and (ii). Satisfaction of requirement (v) is proved in Section V.

For an arbitrary configuration  $q \in Q$ , the group part  $g$  of the abstract state  $a$  is defined by  $g = (R, \mu) \in G = SE(2)$ . Let

$$\mu = \frac{1}{N} \sum_{i=1}^N q_i \in \mathbb{R}^2 \quad (13)$$

Define

$$r_i = [x_i, y_i]^T = R^T (q_i - \mu), \quad i = 1, \dots, N \quad (14)$$

The rotation part  $R \in SO(2)$  is defined by the following equation

$$\sum_{i=1}^N x_i y_i = 0 \quad (15)$$

In this paper we restrict our attention to a 2 - dimensional shape  $s = [s_1, s_2]$  defined by

$$s_1 = \frac{1}{N-1} \sum_{i=1}^N x_i^2, \quad s_2 = \frac{1}{N-1} \sum_{i=1}^N y_i^2 \quad (16)$$

Since  $SO(2)$  is 1-dimensional, the dimension of the abstract manifold  $A$  is  $n = 5$ , independent of the number of robots  $N$ . Also it is obvious that our definitions (13), (15), (16) of group and shape are invariant to permutations of robots, as required by Definition 2. The submersion condition will be studied later in this section.

Before we show that the abstraction  $\phi$  defined above solves Problem 5, we study its physical significance.

##### A. Significance

There are two slightly different interpretations of the abstraction defined by (13), (14), (15), and (16). In this paper, we only discuss one of them. The interested reader is referred to [2] for more information. Let

$$\Sigma = \frac{1}{N-1} \sum_{i=1}^N (q_i - \mu)(q_i - \mu)^T \quad (17)$$

$\mu$  and  $\Sigma$  given by (13) and (17) can be interpreted as sample mean and covariance of a normally distributed random variable with realizations  $q_i$ .  $R$  in (15) is the rotation that diagonalizes the covariance and  $s_1, s_2$  are the eigenvalues of the covariance matrix. This means that, for a large number of normally distributed robots,  $\mu, R, s_1$  and  $s_2$  give the pose and semi-axes of a concentration ellipsoid. Specifically, the contours of constant probability

$p$  for normally distributed points in plane with mean  $\mu$  and covariance  $\Sigma$  are described by

$$(x - \mu)^T \Sigma^{-1} (x - \mu) = c, \quad c = -2 \ln(1 - p) \quad (18)$$

The ellipse in (18), called *equiprobability* or *concentration ellipse*, has the property that  $p$  percent of the points are inside it, and can be therefore used as a spanning region for our robots, under the assumption that they are normally distributed. Therefore we have:  *$p$  percent of a large number  $N$  of normally distributed robots described by a 5 - dimensional abstract variable  $a = (g, s) = (R, \mu, s_1, s_2)$  is enclosed in an ellipse centered at  $\mu$ , rotated by  $R \in SO(2)$  in the world frame  $\{W\}$  and with semi-axes  $\sqrt{cs_1}$  and  $\sqrt{cs_2}$ , where  $c$  is given by (18).*

Even though the normal distribution assumption might seem very restrictive, we show in [2] that it is enough that the robots be normally distributed in the initial configuration. Our controls laws will preserve the normal distribution.

##### B. Group and shape control

In this section, under the assumption that the configuration space  $Q$  is equipped with an Euclidean metric, we construct detectable behaviors and decoupled control systems for group and shape, in accordance with requirements (i) and (iv) from Problem 5. The calculations are rather involved and are omitted. The interested reader is referred to [2]. Let

$$E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad (19)$$

and

$$\begin{aligned} H_1 &= I_2 + R^2 E_2, & H_2 &= I_2 - R^2 E_2, \\ H_3 &= R^2 E_1 \end{aligned} \quad (20)$$

where  $I_2$  is the  $2 \times 2$  identity matrix. Assume  $R \in SO(2)$  is parameterized by  $\theta \in (-\pi/2, \pi/2)$ . Then, in coordinates,  $a = (\mu, \theta, s_1, s_2)$ . The control distributions corresponding to group  $\Delta_g^\perp$  and shape  $\Delta_s^\perp$  as defined in Section III, are given by

$$\Delta_g^\perp = \text{span}\{X_q^\mu, X_q^\theta\}, \quad \Delta_s^\perp = \text{span}\{X_q^{s_1}, X_q^{s_2}\} \quad (21)$$

where

$$X_q^\mu = \begin{bmatrix} I_2 \\ \vdots \\ I_2 \end{bmatrix}, \quad X_q^\theta = \begin{bmatrix} H_3(q_1 - \mu) \\ \vdots \\ H_3(q_N - \mu) \end{bmatrix} \quad (22)$$

and

$$X_q^{s_1} = \begin{bmatrix} H_1(q_1 - \mu) \\ \vdots \\ H_1(q_N - \mu) \end{bmatrix}, \quad X_q^{s_2} = \begin{bmatrix} H_2(q_1 - \mu) \\ \vdots \\ H_2(q_N - \mu) \end{bmatrix} \quad (23)$$

Therefore, in accordance with (9), requirement (i) of Problem 5 is satisfied if we restrict the behaviors to the

detectable set  $\Delta_g^\perp \oplus \Delta_s^\perp$  given by (21), (22), and (23). It can be shown that the control distributions  $\Delta_g^\perp$  and  $\Delta_s^\perp$  are orthogonal, so decoupled control systems can be designed for group and shape, in accordance with requirement (iv) of Problem 5. The minimum norm vector  $\dot{q}$  on  $TQ$  which is pushed forward to a vector field  $\dot{a}$  in  $TA$  (as described in (11)) is given by

$$\dot{q} = X_q^\mu \dot{\mu} + \frac{s_1 - s_2}{s_1 + s_2} \dot{\theta} X_q^\theta + \frac{s_1}{4s_1} X_q^{s_1} + \frac{s_2}{4s_2} X_q^{s_2} \quad (24)$$

Note that the controls  $\dot{\mu}$ ,  $\dot{\theta}$ ,  $s_1$ ,  $s_2$  act on orthogonal directions so one can explicitly control each of the formation variables without affecting the others.

We define the individual controls as projections  $d\pi_i$  of the minimum norm vector (24):

$$u_i = \dot{q}_i = \dot{\mu} + \frac{s_1 - s_2}{s_1 + s_2} H_3(q_i - \mu) \dot{\theta} + \frac{1}{4s_1} H_1(q_i - \mu) s_1 + \frac{1}{4s_2} H_2(q_i - \mu) s_2 \quad (25)$$

*Remark 6:* The overall control architecture implementing (25) fits the structure in Figure 1. Each robot  $i$  needs to implement controller  $C_i$ , which is only dependent on its own state  $q_i$  and the small dimensional abstract state  $a$ . Also, each robot has to send its own state to the abstract control system, which calculates and then broadcasts the updated abstract state. Therefore, robot  $i$  only has to broadcast its 2-dimensional state  $q_i$  and listen to a the 5-dimensional abstract state  $a$ , independent on the number of robots  $N$ .

*Remark 7:* It can be shown that the submersion condition in Definition 2 is equivalent to  $s_1 \neq 0$  and  $s_2 \neq 0$ , which is also equivalent to the well definition of the control laws (25). The abstract behavior on  $A$  should be designed so that  $s_1 > 0$  and  $s_2 > 0$ , for all  $t$ .  $s_1 = 0$  and  $s_2 = 0$  physically correspond to degenerate situations when all the robots become collinear.

*Remark 8:* Control law (25) corresponds to an affine transformation. Therefore, properties like collinearity, ratios of distances on lines, and parallelism are preserved and control law (25) can be used for formations in which these are desired. Even more interesting, it is known that affine transformations preserve the normal distribution. This means that if the robots are initially normally distributed, by applying the control laws (25), they remain normally distributed. The 5 - dimensional abstract state, interpreted as sample mean  $\mu$  and sample covariance  $\Sigma$ , gives us control over the pose, aspect ratio and size of the concentration ellipsoid as defined in Section IV-A.

## V. ABSTRACT BEHAVIOR

Assume the goal is to move the robots from arbitrary initial positions  $q_i(0)$  to final rest positions of desired mean  $\mu^d$ , orientation  $\theta^d$ , and shape  $s_1^d, s_2^d$ .

An obvious choice of the control vector field  $\dot{a} = [\dot{\mu}, \dot{\theta}, s_1, s_2]$  on the abstract manifold  $A$  is

$$\begin{aligned} \dot{\mu} &= K_\mu(\mu^d - \mu), & \dot{\theta} &= k_\theta(\theta^d - \theta) \\ s_1 &= k_{s_1}(s_1^d - s_1), & s_2 &= k_{s_2}(s_2^d - s_2) \end{aligned} \quad (26)$$

where  $K_\mu \in \mathbb{R}^{2 \times 2}$  is a positive definite matrix and  $k_\theta, k_{s_{1,2}} > 0$ .

More generally, the task might require the robots to follow a desired trajectory  $a^d(t) = [\mu^d(t), \theta^d(t), s_1^d(t), s_2^d(t)]$  on  $A$ . A control vector field on  $A$  can be of the form:

$$\begin{aligned} \dot{\mu} &= K_\mu(\mu^d(t) - \mu(t)) + \dot{\mu}^d(t) \\ \dot{\theta} &= k_\theta(\theta^d(t) - \theta(t)) + \dot{\theta}^d(t) \\ s_1 &= k_{s_1}(s_1^d(t) - s_1(t)) + \dot{s}_1^d(t) \\ s_2 &= k_{s_2}(s_2^d(t) - s_2(t)) + \dot{s}_2^d(t) \end{aligned} \quad (27)$$

Note that (26) (or (27)) only guarantees the desired behavior on the abstract manifold  $A$ . If the imposed trajectory  $a^d(t)$  is bounded at all times, it is easy to see that  $a(t)$  is bounded. For the problem to be well defined, we still need to make sure that the internal states are bounded (requirement (v) of Problem 5). We have:

*Proposition 9:* If  $a$  is bounded, then so are  $q_i$ .

*Proof:* The proof is based on the triangle inequality for norms and can be found in [2]. ■

In the stabilization to a point case, the boundness and globally asymptotic convergence to the desired values of the abstract variables are guaranteed by (26). Proposition 9 proves the boundness of the internal dynamics. We still need to study the equilibria and regions of convergence for each robot. We have the following Proposition:

*Proposition 10:* For any  $\mu^d, \theta^d, s_1^d, s_2^d$ , the closed loop system (25), (26) globally asymptotically converges to the equilibrium manifold  $\mu = \mu^d, \theta = \theta^d, s_1 = s_1^d, s_2 = s_2^d$ .

*Proof:* The proof, given in [2], is based on the Global Invariant Set Theorem (LaSalle) and the triangle inequality for norms. ■

## VI. SIMULATION RESULTS: TUNNEL PASSING

Consider the task of driving “almost all” of  $N = 100$  robots through a tunnel of given geometry, and spread out at its end. Assuming that the robots are normally distributed in the initial configuration, then they remain normally distributed by applying the control laws (25), according to Remark 8. If 99 percent is an acceptable quantization of “almost all”, according to Section IV-A, the problem can be reduced to a 5-dimensional control problem for a concentration ellipsoid of probability  $p = 0.99$ . We divide the tunnel passing task into three subtasks: (1) gather the robots in front of the tunnel, (2) drive the robots through the tunnel, and (3) spread out at its end.

For subtask (1), we use the globally stabilizing controllers (25), (26). We chose  $\mu^d = [3 \ 23]$ ,  $\theta^d = 0$ ,  $s_1^d = 10.8574$ ,  $s_2^d = 0.3518$ . The shape corresponds to semi-axes of  $\sqrt{cs_1^d} = 10$  and  $\sqrt{cs_2^d} = 1.8$  along  $x$  and  $y$ , respectively.

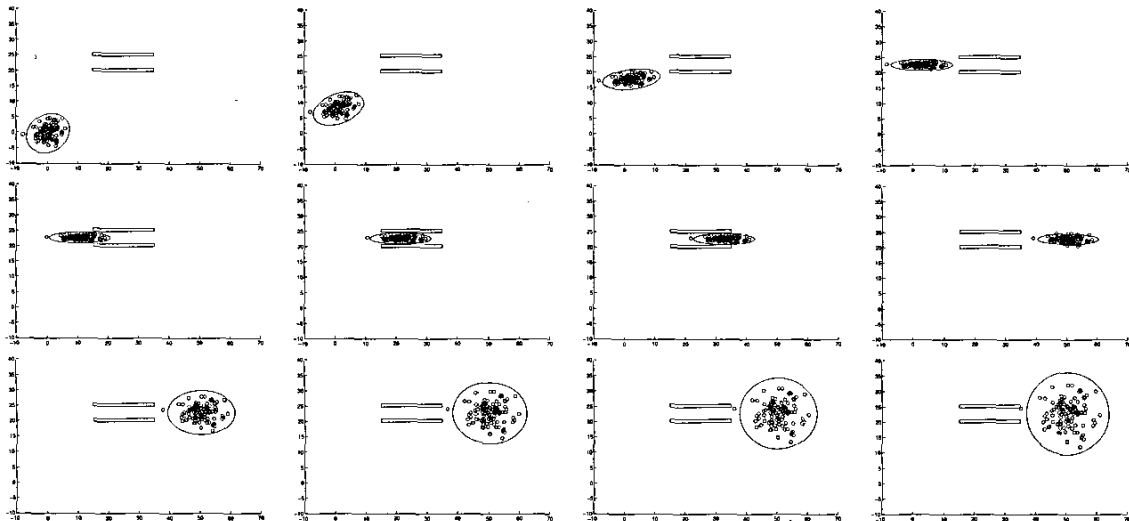


Fig. 2. 99 of 100 normally distributed robots are driven through a tunnel by designing 5 - dimensional controls for the corresponding equiprobability ellipse. Pose and shape can be controlled separately.

The abstract controller parameters were  $K_\mu = 2I_2$ ,  $k_\theta = 2$ ,  $k_{s_1} = k_{s_2} = 2$ . Note that in this first subtask both shape and pose are controlled. The produced motion is shown in the first row of Figure (2).

No shape and orientation control is necessary to accomplish subtask (2). We use trajectory following controllers of type (27) on  $A$  to move the ellipse through the tunnel. If we want to uniformly move the ellipse at  $[50 \ 23]$  in 1 second while keeping shape and orientation constant, we only have to control  $\mu_x$ , therefore  $\dot{\mu}_y = \dot{\theta} = \dot{s}_1 = \dot{s}_2 = 0$ . We use  $\mu_x^d(t) = (1-t)3 + t50$  (therefore  $\dot{\mu}_x^d(t) = 47$ ). The second row of Figure (2) shows four instants of the generated trajectories. As expected, shape and orientation is preserved, therefore illustrating the control decoupling proved in Section IV-B.

For the third subtask, we illustrate control of shape decoupled from pose, which is maintained constant. We again use the globally stabilizing controllers (25), (26) with  $\dot{\mu} = 0$ ,  $\dot{\theta} = 0$ ,  $s_1^d = s_2^d = 20$ ,  $k_{s_1} = k_{s_2} = 2$ . The obtained expansion is shown in the last row of Figure (2).

## VII. CONCLUSION AND FUTURE WORK

We propose a control method for a large number of robots based on an abstraction of the team to a small dimensional manifold with a product structure of a Lie group and a shape space. The task to be accomplished by the team suggests a natural feedback control system on the manifold. We focus on planar fully actuated robots and show that the group and shape variables can be controlled separately. The individual control laws which are mapped to the desired behavior of the team can be realized by

feedback depending only on the robots' current state and the small dimensional state on the abstract manifold. Future work will be directed towards incorporating more shape variables, include under-actuation constraints in the abstraction, extending the results to 3-D environments, and implementing the obtained control architectures in our blimp - car experimental platform.

## VIII. REFERENCES

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