Is Foster's Reactance Theorem Satisfied in Double-Negative and Single-Negative Media?[1]

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Abstract

In this Letter, we address the issue of Foster's reactance theorem for the material media in which one or both of the material parameters ε and μ may possess negative real parts. We demonstrate that this theorem is indeed satisfied for a one-port termination filled with a lossless metamaterial with negative real permittivity and permeability, known as a "double-negative (DNG)" medium, or with a lossless metamaterial with either negative real permittivity or negative real permeability, which can be named as a "single-negative (SNG)" medium. However, in the case of DNG media when the reactive input impedance of such a termination is compared with that of its counterpart filled with a conventional lossless "double-positive (DPS)" material, it is found that the two reactances have opposite signs. Similar conclusions can be made for a termination filled with a lossless epsilon-negative (ENG) material when it is compared with that of its mu-negative (MNG) counterpart. Therefore, if a one-port termination filled with a lossless DPS or ENG material possesses inductive input reactance, when the same termination is filled instead with a lossless DNG or MNG material, its input reactance may be capacitive.

Introduction

Composite materials with negative real permittivity *and* permeability in a certain band of frequency, known under various names such as "left-handed (LH)" media (see e.g., [1]-[5]), "double-negative (DNG)" media [6], and "backward-wave (BW)" media [8], have recently attained considerable attention and interest (see e.g., [1]-[13]). The idea dates back to 1967 when Veselago theoretically studied time-harmonic monochromatic plane wave propagation in a material whose permittivity and permeability were assumed to be simultaneously negative [1]. Shelby, Smith, and Schultz recently constructed such a composite medium for the microwave regime, and experimentally showed the presence of anomalous refraction in this medium [5].

Recently as a potential application of these DNG metamaterials we introduced theoretically an idea for compact one-dimensional (1-D) cavity resonators, in which a pair of DNG and DPS slabs was inserted [11]. An interesting issue was raised by Munk about the applicability of Foster's reactance theorem for the DNG metamaterials [14]. As is well known, if one has a one-port reactive lossless termination in a microwave network, the input impedance Z_{in} of such a one-port termination is purely imaginary, i.e., $Z_{in} = jX_{in}$, (and of course $Y_{in} = jB_{in}$), and thus it is purely reactive. (Here the time dependence of $\exp(j\omega t)$ is assumed.) Foster's reactance theorem states that in general for such a one-port reactive termination [15] , we have

$$\frac{\partial X_{in}}{\partial \omega} > 0 \qquad \text{and} \qquad \frac{\partial B_{in}}{\partial \omega} > 0 \qquad\qquad (1)$$

As mentioned in [15], this implies that the poles and zeros of a reactance (or a susceptance) function must alternate along the frequency axis. The important issue is to find out whether the DNG, epsilon-negative (ENG), or mu-negative (MNG) metamaterials also satisfy Foster's Reactance Theorem. Here, by the term "ENG" we mean materials in which the permittivity has the negative real part, while the real part of permeability is positive, at a given frequency band, and by the term "MNG" we mean materials in which the real part of permeability is negative but the permittivity has a positive real part at a certain range of frequency. In a recent symposium, we presented a portion of our findings in this matter for DNG metamaterials [13]. Here in this Letter, we describe some details of our results for the DNG, as well as ENG and MNG cases.

The Case of DNG, ENG, and MNG Materials

Consider an arbitrarily-shaped cavity with a perfectly conducting wall filled with an isotropic lossless DNG, ENG, or MNG metamaterial, and assume an open port at the wall of this cavity. The open port is connected to a cylindrical waveguide having a cross section similar to the shape of the open port. This waveguide has also perfectly conducting walls and is filled with the same metamaterial as in the cavity. (See Fig. 1) This would form a one-port lossless termination, in which one of these lossless metamaterials is inserted. If one follows the mathematical steps of Foster's reactance theorem described in [15], one obtains
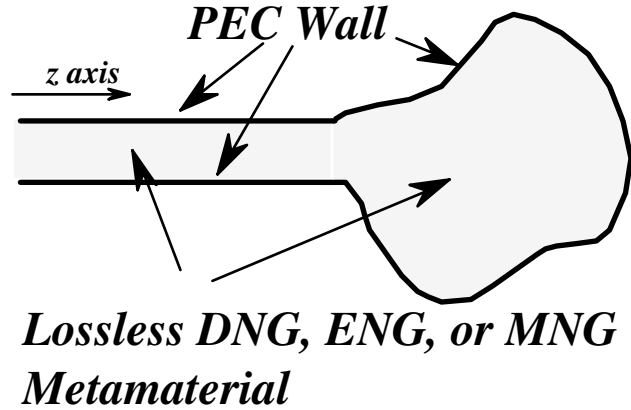

Fig. 1. Geometry of the problem. A one-port termination filled with an isotropic lossless double-negative (DNG) ($\varepsilon < 0$, $\mu < 0$), epsilon-negative (ENG) ($\varepsilon < 0$, $\mu > 0$), or mu-negative (MNG) ($\varepsilon > 0$, $\mu < 0$) metamaterial.

$$\int_{\text{terminal plane}} \left(E \times \frac{\partial H^*}{\partial \omega} + \frac{\partial E^*}{\partial \omega} \times H \right) \cdot n \, dS = -j \int_V \left(H \cdot H^* \frac{\partial(\omega\mu)}{\partial \omega} + E \cdot E^* \frac{\partial(\omega\varepsilon)}{\partial \omega} \right) dV \qquad (2)$$

(as Eq. (4.24a) in Ref. [15]) where the "terminal plane" is any transverse plane in the waveguide at some $z = z_o$ along the z axis (this is the "port" to this one-port termination.), V is the volume inside the cavity including the part of the waveguide up to the terminal plane, n is a unit inward normal to the boundary of the volume V, and the superscript (*) indicates complex conjugation.

The mathematical steps leading to this equation do not depend on the sign of ε and/or μ. If the lossless material filling this structure is a DNG medium, we assume that the cross sectional size of the waveguide is chosen such that only the dominant propagating mode can propagate in the waveguide section. However, if the material is assumed to be an ENG or an MNG medium, then there will be evanescent modes in this waveguide. In either case, if the terminal plane is far enough away from the opening at the cavity wall, the dominant mode of the waveguide, either the propagating or the evanescent mode, is assumed to be the only mode present at the terminal plane, and this mode forms the standing wave in this waveguide either in the form of trigonometric sinusoidal function (when a DNG medium fills the region) or hyperbolic sinusoidal function (when an ENG or an MNG medium fills the region). At the terminal plane, we define the transverse input impedance Z_{in} as

$$Z_{in} \boldsymbol{H}_{trans} = \hat{z} \times \boldsymbol{E}_{trans} \tag{3}$$

with $\boldsymbol{E}_{trans}$ and $\boldsymbol{H}_{trans}$ being the transverse components of the electric and magnetic fields at the terminal plane, and $\hat{z}$ being the unit vector along the waveguide axis, i.e., z axis. Since this one-port termination is assumed to be lossless, the transverse input impedance is purely imaginary, i.e., $Z_{in} = jX_{in}$. When we substitute Eq. (3) into the left hand side of Eq. (2), after some mathematical manipulations, we can express the left hand side of Eq. (2) as

$$\int_{\text{terminal plane}} \left(\boldsymbol{E} \times \frac{\partial \boldsymbol{H}^*}{\partial \omega} + \frac{\partial \boldsymbol{E}^*}{\partial \omega} \times \boldsymbol{H} \right) \cdot \boldsymbol{n} \, dS = -j \frac{\partial X_{in}}{\partial \omega} \int_{\text{terminal plane}} \left| \boldsymbol{H}_{trans} \right|^2 dS . \tag{4}$$

Considering Eq. (2) and (4), we find, similar to the proof of Foster's reactance theorem given in [13], that

$$\text{sign of } \frac{\partial X_{in}}{\partial \omega} = \text{ sign of } \left(W_m + W_e \right) \tag{5}$$

where W_m and W_e are the time-averaged total energy stored in the lossless termination volume V, and are given as

$$W_m = \tfrac{1}{4} \int_V \boldsymbol{H} \cdot \boldsymbol{H}^* \frac{\partial \left(\omega \mu \right)}{\partial \omega} dV , \quad \text{and} \quad W_e = \tfrac{1}{4} \int_V \boldsymbol{E} \cdot \boldsymbol{E}^* \frac{\partial \left(\omega \varepsilon \right)}{\partial \omega} dV \tag{6}$$

It is important to note that DNG, ENG, or MNG materials are inherently quite dispersive. So although ε and/or μ can be negative at a certain given band of frequency in such media, these parameters do vary with frequency quite noticeably. In fact, for the time-averaged stored electric and magnetic energy density to be positive, $\frac{\partial(\omega\mu)}{\partial\omega}$ and $\frac{\partial(\omega\varepsilon)}{\partial\omega}$ should be positive. Therefore, we find that

$$\frac{\partial X_{in}}{\partial\omega} > 0 \tag{7}$$

which implies that Foster's reactance theorem is indeed satisfied for the lossless DNG, ENG, or MNG materials, similar to the case of conventional DPS materials with real positive permittivity and permeability. However, the following important question may be asked: Although $\frac{\partial X_{in}}{\partial\omega} > 0$ for these lossless materials, what is the sign of $\frac{\partial X_{in}}{\partial z}$ for a one-port termination filled with either of DNG, ENG, or MNG material? This is an important distinction one has to consider between a DPS and a DNG medium, or between an ENG and a MNG material. To answer this question, without loss of generality we assume that the dominant mode of the waveguide is a TE mode. (A similar proof can be given for the case of a TM mode.) For this dominant mode, the transverse component of the magnetic field at the terminal plane can be written as

$$\boldsymbol{H}_{trans} = \frac{1}{-j\omega\mu}\hat{\boldsymbol{z}} \times \frac{\partial\boldsymbol{E}_{trans}}{\partial z} \tag{8}$$

If we substitute the above relation into Eq. (3) and note that $Z_{in} = jX_{in}$, we get

$$X_{in}\,\hat{\boldsymbol{z}} \times \frac{\partial\boldsymbol{E}_{trans}}{\partial z} = -\omega\mu\,\hat{\boldsymbol{z}} \times \boldsymbol{E}_{trans} \tag{9}$$

For the DNG case, we have a standing wave for the dominant propagating mode in the waveguide at the terminal plane, i.e., at the "port", and thus the transverse electric field components can be written in the general form as

$$\boldsymbol{E}_{trans} = \boldsymbol{f}(x, y)\sin(\beta z + \phi) \tag{10}$$

where $f(x,y)$ is a transverse vector with two components in the x-y plane, as a function of x and y coordinates, β is the longitudinal wavenumber for the dominant TE mode, and ϕ is an arbitrary phase. For the case of ENG or MNG materials, we have evanescent waves in both directions inside the waveguide, and as a result the transverse electric field components can be expressed as

$$E_{trans} = g(x,y)\sinh(\alpha z + \gamma) \tag{11}$$

where $g(x,y)$, similar to $f(x,y)$, is also a transverse vector in the x-y plane, α is the evanescent decay constant for the dominant TE evanescent mode, and γ is an arbitrary constant. Substituting Eq. (10) and Eq. (11) into Eq. (9), we get, respectively

$$X_{in}\,\beta\hat{z}\times f(x,y)\cos(\beta z+\phi) = -\omega\mu\hat{z}\times f(x,y)\sin(\beta z+\phi) \qquad \text{for the DPS or DNG case} \tag{12}$$

$$X_{in}\,\alpha\hat{z}\times g(x,y)\cosh(\alpha z+\gamma) = -\omega\mu\hat{z}\times g(x,y)\sinh(\alpha z+\gamma) \qquad \text{for the ENG or MNG case} \tag{13}$$

From Eq. (12), we notice that when we compare the case of a conventional DPS medium with that of a DNG medium, the sign of X_{in} would flip (since the sign of μ changes), regardless of what sign we choose for β. So if at the terminal plane of the one-port termination filled with a conventional lossless DPS medium with $\mu_1 > 0$ and $\varepsilon_1 > 0$ we have an inductive reactance, when we exchange the filling medium with a lossless DNG material with $\mu_2 = -\mu_1$ and $\varepsilon_2 = -\varepsilon_1$, the input reactance at the same terminal plane will be capacitive. This implies that whatever the sign of $\frac{\partial X_{in}}{\partial z}$ is for the former, $\frac{\partial X_{in}}{\partial z}$ has an opposite sign for the latter. Therefore, although Foster's reactance theorem is satisfied for lossless DNG, ENG, and MNG metamaterials, as is for the conventional materials, the rate of change of reactance X_{in} with respect to the location of the terminal plane, i.e., $\frac{\partial X_{in}}{\partial z}$, differs for the case of DNG metamaterials. A similar argument can be said about exchanging an ENG with an MNG material, as a medium filling such a one-port termination. As can be seen from Eq. (13), when the one-port termination is filled with a lossless ENG medium with $\mu_3 > 0$ and $\varepsilon_3 < 0$, the sign of X_{in} would be opposite to that of X_{in} of the case where the termination is filled with a lossless MNG with $\mu_4 = -\mu_3 < 0$ and $\varepsilon_4 = -\varepsilon_3 > 0$. Therefore, it is important to emphasize that although Foster's reactance theorem is satisfied for

the lossless DNG, ENG, and MNG media, one should remember that $\frac{\partial X_{in}}{\partial z} \neq \frac{\partial X_{in}}{\partial \omega}$, which is consistent with the fact that these materials are dispersive media.

Acknowledgements

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