HETEROTIC CHEN-RUAN COHOMOLOGY

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ABSTRACT

HETEROTIC CHEN-RUAN COHOMOLOGY

Ryan Manion

Tony Pantev

We extend the construction of the Chen-Ruan cohomology in the setting of heterotic string theory. We show that it properly reduces to the Chen-Ruan cohomology in the case where the gauge bundle E is chosen to be the tangent bundle TX and examine its basic properties, followed by demonstrating nontrivial examples and computations. The second portion of this work examines the extension of the anomaly cancellation condition for gerbes through an extended example. Namely, we use Fourier-Mukai transforms and the methods of [13] to set up a construction of bundles over a gerbe which should be non-anomalous.

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Chapter 1

Introduction

String theory proposes that rather than considering the usual four space-time dimensions to model the universe, we instead consider spaces of the form $\mathbb{R}^4 \times X$ where X is some compact manifold which must satisfy some stringent geometric conditions. The physical states comprising the "spectrum" of the theory manifest themselves mathematically as cohomology classes of certain sheaves on this compact piece of space-time X. The sheaves come from the local sections of vector bundles over X, whose construction must again satisfy various geometric conditions arising from physical requirements. Mathematically such constraints correspond to statements about characteristic classes. Both matter and force particles are encoded in the various vibrational modes of the string propagating in this background. One direction in which to extend the possibilities of such a theory would be to replace the space X by a stack $\mathfrak{X}$. Stacks naturally arise in quantum gauge theories and are mathematical constructs which formalize the notion of an internal symmetry at each point of space-time. Much work has been done to understand what physically relevant theories arise from string propagating on stacks. The stacks of particular interest are gerbes due to the fact that their categories of sheaves are slightly larger than that of the space which they lie over. Considering the case when $\mathfrak{X}$ is a gerbe has the potential to give us entirely new quantum physics.

The geometric objects of interest in our physical theory are a space (or orbifold, or stack) denoted X (or $\mathfrak{X}$ when an orbifold or stack) along with a vector bundle ("gauge bundle") E over it. The particles manifest themselves as cohomology classes of the sheaf of sections of bundles associated to this bundle E. In the case of a non-stacky space X, we need to impose the following conditions coming from physics:

$$\left.\begin{aligned} \omega_X &\cong \det(E^*) \\ \mathrm{ch}_2(TX) &= \mathrm{ch}_2(E) \end{aligned}\right\} \text{(Anomaly cancellation)}$$

$$\left.\begin{array}{c} E \text{ is stable} \\ E \text{ has a structure group } E8 \times E8 \\ X \text{ is a Calabi-Yau manifold} \end{array}\right\} \text{(Supersymmetry)}$$

We will primarily focus on the first two of these conditions for this work. There exist characterstic classes for vector bundles over general smooth differentiable stacks [6], so the anomaly cancellation condition has an obvious candidate for a stacky interpretation. In the case of a global quotient stack $\mathfrak{X} = [X/G]$ these are precisely the equivariant characteristic classes. For stacks and gerbes the notion of characteristic class can be refined in various ways. Of particular interest are the “Chern classes with values in representations”, or “Chern reps” $\mathrm{ch}_i^{\mathrm{rep}}$, coming from the work of Segal and Toën [21] [22]. The $\mathrm{ch}_i^{\mathrm{rep}}$’s are precisely the characteristic classes needed to formulate the various index and Riemann-Roch theorems for stacks. Requiring equality of $\mathrm{ch}_i^{\mathrm{rep}}$’s in the anomaly cancellation is a stronger statement - it would imply the equivalence of the stacky ch_2’s.

This thesis is organized as follows. In chapter 2 we review and set notation for the language of stacks and gerbes, as well as the Riemann-Roch theorem for stacks. In chapter 3 we analyze a construction of the Heterotic Chen-Ruan cohomology (suggested by Sharpe). We prove that the proposal is independent of the presentation of the stack, its invariance under Serre duality, and discuss certain obstructions to its existence and other properties. We follow this with examples which demonstrate both computational methods and evidence of existence of examples, which are nontrivial to construct. In chapter 4 we review and set notation for the language of Fourier-Mukai transforms and elliptic fibrations in preparation for chapter 5. In chapter 5, following the work of [13], we use these transforms in a specific computation which assists us in finding what the anomaly cancellation condition should be for a gerbe through a concrete example. This is done by using a gerbey version of the spectral construction of [15]. The spectral construction is a common method to construct bundles of a desired geometric type on elliptically fibered spaces $f : X \to Y$. This is done by applying a relative (meaning fiberwise) Fourier-Mukai transform, which exchanges vector bundles (locally free sheaves) with sheaves (more generally complexes of sheaves) supported on a proper closed subspace which forms a branched cover of the base Y. More precisely, the transform is an equivalence of triangulated categories from $D^b(X)$ to itself. This was initially done with the additional assumption that the fibration f had a section, which allowed one to construct a sheaf over all of X inducing the transform. Dropping the assumption of this section existing necessarily changes the target space for the Fourier-Mukai transform from $D^b(X)$ to $D^b(\mathfrak{X})$ for a gerbe $\mathfrak{X}$. In chapter 5 we describe a space X which has two distinct elliptic fibrations, one with a section and one without. This allows us to perform two Fourier-Mukai dualities. Beginning with a bundle satisfying anomaly cancellation on X, we transform via the fibration with a section to obtain spectral data on X, which we transform via the fibration without a section to obtain a bundle on a gerbe $\mathfrak{X}$. This bundle should serve as a gerbey prototype for a non-anomalous bundle - or **omalous** bundle as defined in [12] - and thus it is of interest to determine its characteristic classes at the level of Chern reps.

Chapter 2

Stacks and Gerbes

This chapter we review and set notation for the language of stacks. An excellent introductory reference is [23] a more comprehensive introduction is [18], and an important historical initial application of stacks is [10].

2.1 Stacks, definitions

2.1.1 Stacks categorically

The definition of a stack is a purely categorical notion, which we briefly recall. Given a site $\mathfrak{C}$ and a fibered category $\mathfrak{X}$ over it with functor $\pi : \mathfrak{X} \to \mathfrak{C}$, one calls $\mathfrak{X}$ a stack over $\mathfrak{C}$ if it satisfies certain descent conditions on the categories $\mathfrak{X}(A)$ for each $A \in \mathrm{Ob}(\mathfrak{C})$. Here $\mathfrak{X}(A)$ denotes the "fiber" category (which will always be assumed to be a groupoid):

$$\mathrm{Ob}(\mathfrak{X}(A)) := \{X \in \mathfrak{X} | \pi(X) = A\}$$

$$\mathrm{Hom}_{\mathfrak{X}(A)}(X_1, X_2) := \{f : X_1 \to X_2 | \pi(f) = \mathrm{id}_A\}$$

Further, for each $A \in \mathrm{Ob}(\mathfrak{C})$ one has the fibered category $(\mathfrak{C}/A)$ defined as:

$$\mathrm{Ob}(\mathfrak{C}/A) := \{f : B \to A\}$$

$$\mathrm{Hom}_{(\mathfrak{C}/A)}(f_1 : B_1 \to A, f_2 : B_2 \to A) := \{g : B_1 \to B_2 | f_1 = f_2 \circ g\}$$

This assignment is clearly functorial, and it gives us a fully faithful embedding of $\mathfrak{C}$ into the (2-)category $(St/\mathfrak{C})$ of stacks over $\mathfrak{C}$ (we will assuming the topology to be subcanonical). This is a consequence of:

Lemma 2.1.1. [Yoneda's lemma] *The functor* $\Phi : \mathfrak{C} \to (St/\mathfrak{C})$ *given by* $\Phi(A) = (\mathfrak{C}/A)$ *gives us the following equivalence of categories for any stack* $\mathfrak{X}$*:*

$$Hom_{(St/\mathfrak{C})}((\mathfrak{C}/A), \mathfrak{X}) \cong \mathfrak{X}(A)$$

In other words, we can study the stack in its entirety by analyzing the maps from spaces A (identified with $(\mathfrak{C}/A)$, an abuse of notation which we make from this point on) mapping into $\mathfrak{X}$. We also frequently make use of the word "space" to describe objects of our base category $\mathfrak{C}$.

The category of stacks forms a 2-category in which 2-fiber products exist. One can then make sense of a morphism from a space $f : A \to \mathfrak{X}$ having the usual geometric properties (flatness, properness, etc). This is done by imposing the said geometric condition on the base changed morphism of f along any map from a space $g : B \to \mathfrak{X}$. That the pullback is a space is an extra condition on f: One calls f a **representable** morphism if the fiber product $A \times_{\mathfrak{X}} B$ is a space for any such g. More generally, one calls a morphism of stacks $f : \mathfrak{Y} \to \mathfrak{X}$ representable if $\mathfrak{Y} \times_{\mathfrak{X}} B$ is a space for any mapping from a space $g : B \to \mathfrak{X}$.

For the purposes of this paper, our base category will be (An/$\mathbb{C}$) the category of complex analytic spaces over $\mathbb{C}$ equipped with the topology induced by defining covers to be surjective families of étale morphisms.

2.1.2 Stacks geometrically

In order to do geometry, one introduces algebraic (and Deligne-Mumford) stacks.

Definition 2.1.2. *A stack $\mathfrak{X}$ is* **algebraic** *[resp.* **Deligne-Mumford***] if there exists a space X with a surjective, smooth [resp. étale] morphism $f : X \to \mathfrak{X}$ such that for any space Y, any morphism $g : Y \to \mathfrak{X}$ is representable. The space X (paired with f) is called an* **atlas** *for $\mathfrak{X}$.*

One useful way to study algebraic stacks is via groupoid objects in our base category $\mathfrak{C}$. Given a stack $\mathfrak{X}$ with atlas $f : X \to \mathfrak{X}$, one obtains a space $X \times_{\mathfrak{X}} X$ along with the following morphisms:

1. Two projections $s, t : X \times_{\mathfrak{X}} X \to X$ called the "source" and "target" maps.

$$s(x, y) = y \qquad t(x, y) = x$$

2. The interchange of factors $i : X \times_{\mathfrak{X}} X \to X \times_{\mathfrak{X}} X$, called the "inversion" map.

$$i(x, y) = (y, x)$$

3. The diagonal morphism $e : X \to X \times_{\mathfrak{X}} X$ called the "identity" map.

$$e(x) = (x, x)$$

4. The projection onto the first and third factors $m : X \times_{\mathfrak{X}} X \times_{\mathfrak{X}} X \to X \times_{\mathfrak{X}} X$ called the "multiplication" morphism.

$$m(x, y, z) = (x, z)$$

These morphisms give the structure of a groupoid with objects the points of X and morphisms the points of $X \times_{\mathfrak{X}} X$. In other words, they satisfy the expected compatabilities. Moreover, for any space A one can form the groupoid with objects $X(A) = \mathrm{Hom}(A, X)$ and morphisms $(X \times_{\mathfrak{X}} X)(A) = \mathrm{Hom}(A, X \times_{\mathfrak{X}} X)$. One can construct a fibered category $\lfloor X \times_{\mathfrak{X}} X \rightrightarrows X \rfloor$ such that over each A in $\mathfrak{C}$, we have this groupoid. This fibered category need not satisfy the descent conditions, but there is a canonical stack it maps to called the **stackification** of the fibered category, denoted $[X \times_{\mathfrak{X}} X \rightrightarrows X]$. This stack is isomorphic to our original stack $\mathfrak{X}$, thus we can study $\mathfrak{X}$ by studying the groupoid.

Let us make this correspondence more precise. One calls the tuple of data $(X, X \times_{\mathfrak{X}} X, s, t, i, e, m)$ a groupoid object in $\mathfrak{C}$. A morphism between two groupoid objects $(U_1, R_1, s_1, t_1, i_1, e_1, m_1)$ and $(U_2, R_2, s_2, t_2, i_2, e_2, m_2)$ consists of a pair of morphisms $f : U_1 \to U_2$ and $g : R_1 \to R_2$ which induce a functor between the corresponding groupoids. One calls a morphism coming from (f, g) a **Morita equivalence** if f is smooth and surjective, and the following commutative diagram is cartesian in $\mathfrak{C}$:

$$\begin{tikzcd} R_1 \arrow[r, "(s_1,t_1)"] \arrow[d, "g"] & U_1 \times U_1 \arrow[d, "f"] \\ R_2 \arrow[r, "(s_2,t_2)"] & U_2 \times U_2 \end{tikzcd}$$

These two conditions correspond to the functor at the level of groupoids being essentially surjective and fully-faithful, respectively. One calls two groupoids Morita equivalent if there exists a chain of Morita equivalences connecting them. We now recall the following correspondence:

$$\{\text{Isomorphism classes of stacks}\} \cong \{\text{Morita equivalence classes of groupoids}\}$$

This correspondence takes a stack $\mathfrak{X}$ with atlas $X \to \mathfrak{X}$ to the groupoid $(X, X \times_{\mathfrak{X}} X, s, t, i, e, m)$ above. The reverse direction takes a groupoid (U, R, s, t, i, e, m) to the stackification $[R \rightrightarrows U]$ of the fibered category described above. For this reason, we will focus purely on groupoid objects in the category of complex analytic spaces for the remainder of this work. Groupoids are frequently denoted simply by $R \rightrightarrows U$, with the existence of the other structure morphisms to be implied. In what follows we focus on formulating all of our theorems and constructions in the language of groupoids for concreteness and for preparation for doing computations at the groupoid level.

2.1.3 Quotient stacks

Given a group G acting on a space X, one can construct the quotient stack from the so-called transformation groupoid:

$$R \rightrightarrows U := (X \times G) \rightrightarrows X$$

$s : X \times G \to X$	$t : X \times G \to X$	$m : (X \times G) \times_{s,X,t} (X \times G) \to X \times G$
$s(x,g) = x$	$t(x,g) = g \cdot x$	$m((x,g),(y,h)) = (y, gh)$

$e : X \to X \times G$	$i : X \times G \to X \times G$
$e(x) = (x, 1)$	$i(x,g) = (g \cdot x, g^{-1})$

We denote this stack by $[X/G]$. This stack has the property that the atlas morphism $X \to [X/G]$ gives X the structure of a G-torsor over $[X/G]$. In other words, this quotient always behaves as nicely as possible.

2.2 Gerbes, inertia stacks, and descent

2.2.1 Sheaves on stacks

Given a groupoid $R \rightrightarrows U$ representing a stack $\mathfrak{X}$, one possible way to describe sheaves on the stack is via descent. A sheaf on $\mathfrak{X}$ will be the data of a sheaf $\mathscr{E} \in \boldsymbol{Sh}(U)$ on the atlas and an isomorphism $\varphi : s^*\mathscr{E} \to t^*\mathscr{E}$ satisfying the cocycle condition $p_1^*\varphi \circ p_2^*\varphi = m^*\varphi$ on $R \times_{s,U,t} R$, where p_i is the projection onto the i-th factor. (Note that for quotient stacks $[X/G]$, this translates to the fact that a sheaf on $[X/G]$ is equivalently defined as a G-equivariant sheaf on X.) We will frequently make use of this equivalence of categories:

$$\boldsymbol{Sh}([X/G]) \cong \boldsymbol{Sh}_G(X) = (G - \text{equivariant sheaves})$$

Of particular interest is the case when X is a point, which this reduces to:

$$\boldsymbol{Sh}([\mathrm{pt}/G]) \cong \boldsymbol{Sh}_G(\mathrm{pt}) = (G - \text{representations})$$

To describe the pullback of a sheaf along any morphism $f : A \to \mathfrak{X}$ from a space A, consider the diagram:

$$\begin{tikzcd}
(A \times_{\mathfrak{X}} U) \times_A (A \times_{\mathfrak{X}} U) \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & U \times_{\mathfrak{X}} U \arrow[d, shift left] \arrow[d, shift right] \\
A \times_{\mathfrak{X}} U \arrow[r] \arrow[d] & U \arrow[d] \\
A \arrow[r] & \mathfrak{X}
\end{tikzcd}$$

Pulling back the sheaf $\mathscr{E}$ to $A \times_{\mathfrak{X}} U$ and pulling back φ, we obtain by descent a sheaf on A when $\mathfrak{X}$ is an algebraic stack.

A morphism between sheaves $(\mathscr{F}, \varphi)$ and $(\mathscr{E}, \psi)$ consists of a map:

$$\alpha : \mathscr{F} \to \mathscr{E}$$

Such that the following diagram commutes:

$$\begin{tikzcd}
s^*\mathscr{F} \arrow[r, "\varphi"] \arrow[d, "s^*\alpha"] & t^*\mathscr{F} \arrow[d, "t^*\alpha"] \\
s^*\mathscr{E} \arrow[r, "\psi"] & t^*\mathscr{E}
\end{tikzcd}$$

2.2.2 Gerbes

Here we introduce the stacks of particular interest, namely gerbes. These arise in taking quotients by non-effective group actions. First, a general definition:

Definition 2.2.1. *We say that a stack $\mathfrak{Y}$ is a* **gerbe** *over a stack $\mathfrak{X}$ if there is morphism of stacks $f : \mathfrak{Y} \to \mathfrak{X}$ satisfying:*

1. *The map f is an epimorphism.*
2. *The diagonal map $\Delta : \mathfrak{Y} \to \mathfrak{Y} \times_{\mathfrak{X}} \mathfrak{Y}$ is an epimorphism.*

Definition 2.2.2. *A morphism of stacks $f : \mathfrak{Y} \to \mathfrak{X}$ over a site $\mathfrak{C}$ is an* **epimorphism** *if for any object $x \in \mathfrak{X}(U)$, there exists a cover $\{U_i\}_{i \in I}$ of U in $\mathfrak{C}$, and objects $y_i \in \mathfrak{Y}(U_i)$ such that $f(y_i) \cong x|_{U_i}$ for all i.*

Typically one is interested in gerbes over spaces, namely a stack $\mathfrak{X}$ with a map to a space M over which it is a gerbe. This will be the type of gerbes we study, because as we will see, their geometry is not too different from that of the space M.

If $\mathfrak{X} \to M$ is a gerbe, then the two conditions of definition 2.2.1 imply that the objects of $\mathfrak{X}$ are all locally isomorphic, and the categories $\mathfrak{X}(U)$ are locally nonempty. In particular, the automorphism groups of any two objects in $\mathfrak{X}(U)$ for some U are isomorphic locally, and encoded by a sheaf of groups on the base M. For our purposes we only consider the case where this is a constant sheaf of groups. In terms of the representing groupoids, one could think of a gerbe over a space as simply as adding in additional automorphisms to each object without adding in new objects - see section 2.3.

2.2.3 Inertia stacks

Our next definition is that of the inertia stack. It is necessary for the construction of the Grothendieck-Riemann-Roch theorem for stacks. It naturally arises in the index theory of orbifolds and its cohomology (with coefficients in $\mathbb{C}$) is recovers precisely the Chen-Ruan Cohomology [9] and its additive structure. Intuitively, it is the space of "loops" in your stack, with each object of the inertia stack corresponding to an automorphism of an object of $\mathfrak{X}$. In this section we explicitly construct the groupoid of the inertia stack for any stack, and demonstrate a canonical automorphism which acts on any sheaf over the inertia stack, which will be vital in what follows.

Definition 2.2.3. *Given a stack $\mathfrak{X}$, the* **inertia stack** $I_{\mathfrak{X}}$ *is defined to be the pull-back under the diagonal morphisms:*

$$\begin{tikzcd} I_{\mathfrak{X}} \arrow[r] \arrow[d] & \mathfrak{X} \arrow[d, "\Delta"'] \\ \mathfrak{X} \arrow[r, "\Delta"] & \mathfrak{X} \times \mathfrak{X} \end{tikzcd}$$

One can show that $I_{\mathfrak{X}} \to \mathfrak{C}$ is a fibered category which is equivalent (as a fibered category over $\mathfrak{C}$) to a category whose objects and morphisms can be described as follows:

$$\begin{aligned} \mathrm{Ob}(I_{\mathfrak{X}}) &= \big\{(x,\sigma) | x \in \ \mathrm{Ob}(\mathfrak{X}), \sigma \in \mathrm{Aut}(x)\big\} \\ \mathrm{Hom}_{I_{\mathfrak{X}}}\big((x,\sigma),(y,\tau)\big) &= \big\{\alpha \in \mathrm{Hom}_{\mathfrak{X}}(x,y) | \alpha \circ \sigma = \tau \circ \alpha\big\} \end{aligned} \tag{2.2.1}$$

Given a sheaf $\mathscr{F}$ on the stack $\mathfrak{X}$, the pullback of the sheaf to $I_{\mathfrak{X}}$ will have a natural automorphism acting on it. This is because giving an object of $I_{\mathfrak{X}}(U)$ is the same as giving an object of $\mathfrak{X}(U)$ along with an automorphism of this object by above, which is the same as giving a morphism $f : U \to \mathfrak{X}$ along with an automorphism of f. This will induce an autmorphism of the pullback sheaf $f^*\mathscr{F}$ to U for any such f, as desired. We make this more precise below.

We can explicitly write out a groupoid presentation for the inertia stack of a stack $\mathfrak{X}$ with atlas $f : X \to \mathfrak{X}$. First construct the standard groupoid from this atlas: $(X \times_{\mathfrak{X}} X \rightrightarrows X)$. There is a general recipe for constructing the groupoid representing the fiber product of two groupoids over another (see section 9 of [20]). Tracing through the details, one finds that the atlas F, or "space of automorphisms," is:

$$F := X \times_{\Delta, X\times X, (s,t)} (X \times_{\mathfrak{X}} X)$$

Note that F is a closed subscheme of $X \times_{\mathfrak{X}} X$, so we may apply the structure morphisms of the groupoid to elements of it. One can then define a conjugation action of the morphisms on these automorphisms. This gives us an action map:

$$a : (X \times_{\mathfrak{X}} X) \times_{s,X,p_X} F \to F$$

$$a(r, f) := rfr^{-1} \text{ (More precisely, } m(r, m(f, i(r))))$$

Here p_X denotes the projection from F onto the X in its first coordinate. This action takes us from an automorphism of $s(r)$ to one of $t(r)$ as can been seen pictorially via the simple diagram:

$$\begin{tikzcd} \bullet_{s(r)} \arrow[loop above, "f"] \arrow[r, bend left, "r"] & \bullet_{t(r)} \arrow[l, bend left, "r^{-1}"] \end{tikzcd}$$

One can check can now construct the groupoid representing $I_{\mathfrak{X}}$ as follows, where we denote our source and target maps via a projection onto F denoted π_F and the above action morphism a rather than s and t:

$$I_{\mathfrak{X}} \cong [(X \times_{\mathfrak{X}} X) \times_{s,X,p_X} F \rightrightarrows F]$$

$\pi_F : (X \times_{\mathfrak{X}} X) \times_{s,X,p_X} F \to F$	$a : (X \times_{\mathfrak{X}} X) \times_{s,X,p_X} F \to F$
$\pi_F(r, f) = f$	$a(r, f) = rfr^{-1}$

$m : [(X \times_{\mathfrak{X}} X) \times_{s,X,p_X} F] \times_{\pi_F,F,a} [(X \times_{\mathfrak{X}} X) \times_{s,X,p_X} F] \to [(X \times_{\mathfrak{X}} X) \times_{s,X,p_X} F]$
$m((r_1, f_1), (r_2, f_2)) = (r_1 r_2, f_2)$

$e : F \to (X \times_{\mathfrak{X}} X) \times_{s,X,p_X} F$	$i : (X \times_{\mathfrak{X}} X) \times_{s,X,p_X} F \to (X \times_{\mathfrak{X}} X) \times_{s,X,p_X} F$
$e(f) = (1, f)$	$i(r, f) = (r^{-1}, rfr^{-1})$

We have omitted the m and i morphisms defining composition and inversion on the above groupoid presentation for $I_{\mathfrak{X}}$ for notational ease. We can express the morphism $\pi : I_{\mathfrak{X}} \to \mathfrak{X}$ at the level of groupoids as follows:

$$\begin{tikzcd} F \arrow[rr, "p_X"] & & X \\ (X \times_{\mathfrak{X}} X) \times_{s,X,p_X} F \arrow[u, shift left, "\pi_F"'] \arrow[u, shift right, "a"'] \arrow[rr, "p_{X \times_{\mathfrak{X}} X}"] & & X \times_{\mathfrak{X}} X \arrow[u, shift right, "s"] \arrow[u, shift left, "t"'] \end{tikzcd} \tag{2.2.2}$$

A sheaf on $\mathfrak{X}$ will be given by a pair $(\mathscr{F}, \varphi)$ of $\mathscr{F} \in \boldsymbol{Sh}(X)$ and an isomorphism $\varphi : s^*\mathscr{F} \to t^*\mathscr{F}$ satisfying the cocycle condition. Because the pair $(p_X, p_{X \times_{\mathfrak{X}} X})$ is a morphism of groupoids, we have then that the pair $(p_X^*\mathscr{F}, p_{X \times_{\mathfrak{X}} X}^*\varphi)$ defines a sheaf on $I_{\mathfrak{X}}$. Moreover, we have a non-trivial (meaning not equal to e) morphism from the atlas of $I_{\mathfrak{X}}$ to the relations, defined as:

$$\gamma : F \to (X \times_{\mathfrak{X}} X) \times_{s,X,p_X} F \tag{2.2.3}$$

$$\gamma(f) := (f, f)$$

Note that $\pi_F \circ \gamma = a \circ \gamma = \mathrm{id}_F$, so that one may consider γ as mapping an object of $I_{\mathfrak{X}}$ to an automorphism of that object. This will be the automorphism acting on each sheaf over $I_{\mathfrak{X}}$. This is the content of the following theorem:

Theorem 2.2.4. *Let $\mathfrak{X}$ be a stack with atlas $f : X \to \mathfrak{X}$ and corresponding groupoid $X \times_{\mathfrak{X}} X \rightrightarrows X$. Then for any sheaf $(\mathscr{F}, \varphi) \in \boldsymbol{Sh}(X \times_{\mathfrak{X}} X \rightrightarrows X) \cong \boldsymbol{Sh}(\mathfrak{X})$, there is a canonical nontrivial automorphism of the pullback sheaf $(p_X^* \mathscr{F}, p_{X \times_{\mathfrak{X}} X}^* \varphi) \in \boldsymbol{Sh}((X \times_{\mathfrak{X}} X) \times_{s,X,p_X} F \rightrightarrows F) \cong \boldsymbol{Sh}(I_{\mathfrak{X}})$ via the projection $\pi : I_{\mathfrak{X}} \to \mathfrak{X}$. It is induced at the groupoid level on the atlas of $I_{\mathfrak{X}}$ via the morphism γ defined in 2.2.3:*

$$\gamma^* p_{X \times_{\mathfrak{X}} X}^* \varphi : \underbrace{\gamma^* p_{X \times_{\mathfrak{X}} X}^* s^* \mathscr{F}}_{\cong p_X^* \mathscr{F}} \to \underbrace{\gamma^* p_{X \times_{\mathfrak{X}} X}^* t^* \mathscr{F}}_{\cong p_X^* \mathscr{F}}$$

This morphism satisfies the required commutativity condition to induce a morphism of sheaves on the stack $I_{\mathfrak{X}}$.

Proof. Given a sheaf $(\mathscr{E}, \psi)$ over the groupoid defining $I_{\mathfrak{X}}$, consider the pullback $\gamma^* \psi$:

$$\begin{tikzcd}
\gamma^* \pi_F^* \mathscr{E} \arrow[r, "\gamma^* \psi"] \arrow[d, equal] & \gamma^* a^* \mathscr{E} \arrow[d, equal] \\
\mathscr{E} & \mathscr{E}
\end{tikzcd}$$

In other words, $\gamma^* \psi \in \mathrm{Aut}(\mathscr{E})$. In order to show it defines an automorphism of the sheaf on $I_{\mathfrak{X}}$, one must further have that the following diagram commutes:

$$\begin{tikzcd}
\pi_F^* \mathscr{E} \arrow[r, "\psi"] \arrow[d, "\pi_F^* \gamma^* \psi"'] & a^* \mathscr{E} \arrow[d, "a^* \gamma^* \mathscr{E}"] \\
\pi_F^* \mathscr{E} \arrow[r, "\psi"] & a^* \mathscr{E}
\end{tikzcd}$$

The commutativity of the above diagram follows from the cocycle condition which ψ satisfies. Looking at the above diagram stalkwise, one finds morphisms Υ_1 and Υ_2 which one can use to pull back the cocycle condition of ψ to prove this. These maps are defined as follows:

$$\Upsilon_i : [(X \times_{\mathfrak{X}} X) \times_{s,X,p_X} F] \to [(X \times_{\mathfrak{X}} X) \times_{s,X,p_X} F] \times_{\pi_F,F,a} [(X \times_{\mathfrak{X}} X) \times_{s,X,p_X} F]$$

$$\Upsilon_1(r, f) := ((r, f), (f, f))$$

$$\Upsilon_2(r, f) := ((rfr^{-1}, rfr^{-1}), (r, f))$$

Further we define a morphism β as follows:

$$\beta : [(X \times_{\mathfrak{X}} X) \times_{s,X,p_X} F] \to [(X \times_{\mathfrak{X}} X) \times_{s,X,p_X} F]$$

$$\beta(r, f) := (rf, f)$$

Then pulling back the cocycle condition $p_1^*\psi \circ p_2^*\psi = m^*\psi$ via both Υ_i, and noting the following isomorphisms:

$$\Upsilon_1^* p_1^* \psi \cong \psi \qquad \Upsilon_1^* p_2^* \psi \cong \pi_F^* \gamma^* \psi \qquad \Upsilon_1^* m^* \psi \cong \beta^* \psi$$

$$\Upsilon_2^* p_1^* \psi \cong a^* \gamma^* \psi \qquad \Upsilon_2^* p_2^* \psi \cong \psi \qquad \Upsilon_2^* m^* \psi \cong \beta^* \psi$$

We have that the following two diagrams commute:

$$\begin{tikzcd}
\pi_F^*\mathscr{E} \arrow[d, "\pi_F^*\gamma^*\psi"'] \arrow[dr, "\beta^*\psi"] & \\
\pi_F^*\mathscr{E} \arrow[r, "\psi"] & a^*\mathscr{E}
\end{tikzcd}
\qquad
\begin{tikzcd}
\pi_F^*\mathscr{E} \arrow[r, "\psi"] \arrow[dr, "\beta^*\psi"'] & a^*\mathscr{E} \arrow[d, "a^*\gamma^*\mathscr{E}"] \\
& a^*\mathscr{E}
\end{tikzcd}$$

Combining these two commutative triangles yields the desired commutative diagram. $\square$

One might wonder if the above automorphism depends on the choice of presentation for $\mathfrak{X}$. If one has a Morita equivalence between groupoids representing $\mathfrak{X}$, then one induces a Morita equivalence between the induced groupoids representing $I_{\mathfrak{X}}$, and under this presentation the two constructions of the morphism γ will correspond exactly and induce identical automorphisms on any sheaf. Alternatively, one case phrase the existence of this automorphism purely categorically by using the categories described in (2.2.1). From this perspective, clearly any object $(x, \sigma) \in \mathrm{Ob}(I_{\mathfrak{X}})$ has an automorphism induced via σ. Again, our goal is concreteness and computability (for later chapters), so we make each construction as explicit as possible.

2.2.4 Inertia stacks of quotient stacks

Of particular interest are stacks of the form $\mathfrak{X} = [X/G] = [X \times G \rightrightarrows X]$. Using the previous section, one finds that the induced groupoid for $I_{\mathfrak{X}}$ is also a quotient groupoid in this case:

$$I_{[X/G]} = [F/G] = [F \times G \rightrightarrows F]$$

Where:

$$F = \{(x, g) | g \cdot x = x\} \subseteq X \times G$$

And the group G acts on F via:

$$h \cdot (x, g) := (h \cdot x, hgh^{-1})$$

If we further assume that the group G is discrete, or more weakly that the set of $g \in G$ with $X^g \neq \emptyset$ is discrete, then the space F decomposes:

$$F = \bigsqcup_{g \in G} X^g \times \{g\}$$

Since the G action is via conjugation on the second component, the quotient stack breaks up into a disjoint union of stacks indexed by the conjugacy classes of G which we will denote $Conj(G)$. Let $[\![g]\!]$ denote the conjugacy class of g. Then one can show that:

$$I_{[X/G]} \cong \bigsqcup_{[\![g]\!] \in Conj(G)} [X^g/C(g)] = \bigsqcup_{[\![g]\!] \in Cong(G)} [X^g \times C(g) \rightrightarrows X^g] \tag{2.2.4}$$

Here $C(g)$ denotes the centralizer of g. Note that we have chosen a representative of each conjugacy class for this presentation, but the isomorphism class of $I_{[X/G]}$ is independent of this choice. Simply noting that when G is abelian then $C(g) = G$ for all $g \in G$ gives us the following:

Corollary 2.2.5. *Let G be an abelian group acting on a space X such that the subgroup $H \subseteq G$ defined by $H = \{g \in G \,|\, g \cdot x = x \text{ for some } x \in X\}$ is discrete. Then the inertia stack of $[X/G]$ is:*

$$I_{[X/G]} = \bigsqcup_{g \in G} [X^g/G]$$

Given a sheaf $(\mathscr{F}, \varphi)$ on $[X/G]$, in other words a G-equivariant sheaf, the pull-back to the inertia stack $I_{[X/G]}$ corresponds simply to the induced $C(g)$-equivariant sheaves $\mathscr{F}|_{X^g}$ for each g. One finds that with this groupoid presentation of $I_{[X/G]}$, the corresponding morphism $\gamma : X^g \to X^g \times C(g)$ of Lemma 2.2.4 is simply $\gamma(x) = (x, g)$. In other words, the induced automorphism of the $C(g)$-equivariant sheaf $\mathscr{F}|_{X^g}$ is the one coming from the g-action of the equivariant structure. The element g fixes the entire space X^g but may act nontrivially on the sheaf $\mathscr{F}|_{X^g}$.

2.3 $\mathscr{O}^*$ gerbes over spaces

Given an abelian group A and a class $\alpha \in H^2(X, A)$, one can construct the corresponding A-banded gerbe over X as follows. Choose an Leray open covering $\mathfrak{U} = \sqcup U_i$ of X so that we have $H^k(U_I, A) = 0$ for all multi-indices I and $k \geq 1$. Then we have $\alpha = [\alpha_{ijk}] \in \check{H}^2_{\mathfrak{U}}(X, A) \cong H^2(X, A)$ for some representing cocycle α_{ijk} relative to the cover $\mathfrak{U}$. Then one can construct the following groupoid:

$$\coprod_{i,j\in I} U_{ij} \times A \rightrightarrows \coprod_{i\in I} U_i$$

$\mathfrak{s} : \coprod_{i,j\in I} U_{ij} \times A \to \coprod_{i\in I} U_i$ $\mathfrak{s}(x_{ij}, \lambda) = x_j$	$\mathfrak{t} : \coprod_{i,j\in I} U_{ij} \times A \to \coprod_{i\in I} U_i$ $\mathfrak{t}(x_{ij}, \lambda) = x_i$

$\mathfrak{m} : (\coprod_{i,j\in I} U_{ij} \times A) \times_{\mathfrak{s}, \sqcup_{i\in I} U_i, \mathfrak{t}} (\coprod_{i,j\in I} U_{ij} \times A) \to \coprod_{i,j\in I} U_{ij} \times A$ $\mathfrak{m}((x_{ij}, \lambda), (x_{jk}, \mu)) = (x_{ik}, \alpha_{ijk}(x_{ijk})\lambda\mu)$

$\mathfrak{e} : \coprod_{i\in I} U_i \to \coprod_{i,j\in I} U_{ij} \times A$ $\mathfrak{e}(x_i) = (x_{ii}, 1)$	$\mathfrak{i} : \coprod_{i,j\in I} U_{ij} \times A \to \coprod_{i,j\in I} U_{ij} \times A$ $\mathfrak{i}(x_{ij}, \lambda) = (x_{ji}, \lambda^{-1})$

This groupoid has a natural morphism to the groupoid $\bigsqcup_{ij} U_{ij} \rightrightarrows \bigsqcup_i U_i$ representing X:

$$\begin{tikzcd} \bigsqcup_i U_i \arrow[rr, "\mathrm{id}"] & & \bigsqcup_i U_i \\ \bigsqcup_{ij} U_{ij} \times A \arrow[u, shift right, "s"'] \arrow[u, shift left, "t"] \arrow[rr, "\pi_U"] & & \bigsqcup_{ij} U_{ij} \arrow[u, shift right, "s"'] \arrow[u, shift left, "t"] \end{tikzcd} \tag{2.3.1}$$

This gives the space the structure of an A-banded gerbe over X - notice that π_U is an A-torsor. Given a sheaf $(\mathscr{E}, \varphi)$ on the groupoid $\bigsqcup_{i,j\in I} U_{ij} \times A \rightrightarrows \bigsqcup_{i\in I} U_i$, one can attempt to push the isomorphism down along the A-torsor π_U by descent, which must necessarily involve fixing equivariant structures. More precisely, we have that:

$$\varphi : \pi_U^* s^* \mathscr{E} \to \pi_U^* t^* \mathscr{E}$$

We can choose a character $\chi_1 \in \hat{A}$ in order to fix an equivariant structure on $\pi_U^* s^* \mathscr{E}$, then also choose $\chi_2 \in \hat{A}$ in order to determine an equivariant structure on $\pi_U^* t^* \mathscr{E}$. The morphism φ will thus only descend if it is equivariant with respect to the character $\chi_1^{-1}\chi_2$. We must split the bundle $\mathscr{E}$ up into summands for which φ is equivariant with respect to a given character, and then descend each of these subbundles separately. The twisted multiplication map will descend as an obstruction to the cocycle condition of our descent data on the base groupoid. What we will obtain are **twisted sheaves** in the sense of [8] with twisting class $\chi(\alpha_{ijk}) \in H^2_{\mathfrak{U}}(X, A)$. In other words, sheaves $\mathscr{E}_i \in \boldsymbol{Sh}(U_i)$ and isomorphisms $\psi_{ij} : \mathscr{E}_j|_{U_{ij}} \to \mathscr{E}_i|_{U_{ij}}$ satsifying:

$$\psi_{ij} \circ \psi_{jk} \circ \psi_{ki} = \chi(\alpha_{ijk})\mathrm{id}$$

One can construct slightly more general presentations for gerbes of a given class $\alpha \in H^2(X, A)$. Suppose we choose a cover in which $\alpha|_{U_i} = 0$, so that we can find cocycles $\beta_i \in C^1_{\mathfrak{U}}(U_i, A)$ such that $\partial\beta_i = \alpha|_{U_i}$. Then we have $\partial\left(\beta_i|_{U_{ij}} - \beta_j|_{U_{ij}}\right) = 0$, thus representing a class in $H^1_{\mathfrak{U}}(U_{ij}, A)$, or in other words, an A-torsor over each U_{ij}. The above case (2.3.1) occurs when these A-torsors are trivial, and in such a case one could also arrive at the twisted sheaves by simply pulling back φ via a section of this A-torsor. But, in the more general situation where this torsor has no section, the descent methods above work as well. One would still call such sheaves α-twisted sheaves.

The sheaves on such a A-gerbe over a space X are thus naturally graded by the characters $\hat{A}$. Of particular interest to us will be gerbes with structure group $\mathbb{C}^*$ which we call $\mathscr{O}^*$-gerbes. The sheaves on them (more generally their derived categories) thus have a natural $\hat{\mathbb{C}}^*$=$\mathbb{Z}$-grading on them, which we call the **weight**. The sheaves of weight k on an $\mathscr{O}^*_X$-gerbe $\mathfrak{X} \to X$ have a nice interpretation [13] which goes as follows. Choose an open cover $\mathfrak{U} = \{U_i\}_{i\in I}$ of X such that $\alpha|_{U_i} = 0$ for all i, and let T define the $\mathbb{C}^*$ torsor over U_{ij} as discussed above. Then we have the following map of groupoids corresponding to $\mathfrak{X} \to X$:

$$\begin{tikzcd}
\bigsqcup_i U_i \arrow[r, "\mathrm{id}"] & \bigsqcup_i U_i \\
T \arrow[u, shift left, "\mathfrak{t}"'] \arrow[u, shift right, "\mathfrak{s}"] \arrow[r, "\pi"] & \bigsqcup_{ij} U_{ij} \arrow[u, shift left, "t"'] \arrow[u, shift right, "s"]
\end{tikzcd}$$

Then let L denote the line bundle obtained by taking the associated vector bundle to T under the tautological representation:

$$L := (T \times \mathbb{C}^*)\Big/(t\cdot\lambda, z) \sim (t, \lambda\cdot z) \text{ for all } \lambda \in \mathbb{C}^*$$

Then a sheaf of weight k on $\mathfrak{X}$ is the same as a pair $(\mathscr{E}, \varphi)$ where $\mathscr{E} \in \boldsymbol{Sh}(\sqcup U_i)$

along with an isomorphism:

$$\varphi : s^*\mathscr{E} \to t^*\mathscr{E} \otimes \mathscr{L}^{\otimes k}$$

Satisfying an analogous cocycle condition. Such sheaves form a category, where a morphism between two sheaves $(\mathscr{E}, \varphi)$ to $(\mathscr{F}, \psi)$ of weight k consists of a map $f : \mathscr{E} \to \mathscr{F}$ such that:

$$\begin{tikzcd}
s^*\mathscr{E} \arrow[r, "s^*f"] \arrow[d, "\varphi"] & s^*\mathscr{F} \arrow[d, "\psi"] \\
t^*\mathscr{E} \otimes \mathscr{L}^{\otimes k} \arrow[r, "t^*f\otimes \mathrm{id}"] & t^*\mathscr{F} \otimes \mathscr{L}^{\otimes k}
\end{tikzcd}$$

2.4 GRR for stacks

Töen [22] generalized the Grothendieck-Riemann-Roch theorem to the case of stacks, which will be relevant to our future calculations. Here we review it and set notation. We feel compelled to also mention the work of Edidin [14], who proved the Hirzebruch-Riemann-Roch theorem using localization theorems in equivariant Chow rings in the spirit of Atiyah-Bott [4].

Given a sheaf $\mathscr{E}$ on a stack $\mathfrak{X}$, let $\pi : I_{\mathfrak{X}} \to \mathfrak{X}$ be the projection from its inertia stack. Let T denote the "translation morphism":

$$T : K_0(I_{\mathfrak{X}}) \to K_0(I_{\mathfrak{X}})$$

Given a class $[V] \in K_0(I_{\mathfrak{X}})$, restricting it to a component $\alpha \subset I_{\mathfrak{X}}$ there is a natural finite cyclic group of automorphisms of $[V]|_\alpha$ coming from Theorem 2.2.4. Let $i \in I$

index the eigenspaces and eigenvalues, then define T on this component by:

$$T(\sum_i [V|_\alpha]_i) := \sum_i [V|_\alpha]_i \otimes \lambda_i$$

Then define $\mathrm{ch}^{\mathrm{rep}}$ to be the following composition:

$$K_0(\mathfrak{X}) \xrightarrow{\pi^*} K_0(I_{\mathfrak{X}}) \xrightarrow{T} K_0(I_{\mathfrak{X}}) \otimes \mathbb{C} \xrightarrow{\mathrm{ch}\otimes 1} H^\bullet(I_{\mathfrak{X}}, \mathbb{C})$$

This given morphism to the cohomology is not functorial over proper pushforward, but can be corrected to make it so. To this end, for each component $\alpha \subset I_{\mathfrak{X}}$, consider the inclusion $\alpha \to \mathfrak{X}$ so that we can define the conormal bundle $N^*_{\alpha/\mathfrak{X}} \in K_0(\alpha)$. One defines the lambda operation for any stack $\mathfrak{Y}$ as:

$$\lambda_{-1} : K_0(\mathfrak{Y}) \to K_0(\mathfrak{Y})$$

$$\lambda_{-1}([V]) = 1 - [V] + [\wedge^2 V] - \ldots + (-1)^r [\wedge^r V]$$

Here V is a vector bundle of rank r. Then we can define the following correction term for each component of the inertia stack:

$$\mathrm{ch}(T \circ \lambda_{-1}(N^*_{\alpha/\mathfrak{X}}))^{-1} \mathrm{Td}(T\alpha) \in H^\bullet(\alpha, \mathbb{C}) \subset H^\bullet(I_{\mathfrak{X}}, \mathbb{C})$$

Denote by $\mathrm{TD}(\mathfrak{X}) \in H^\bullet(I_{\mathfrak{X}}, \mathbb{C})$ the class which restricts to the above class on each component of the inertia stack. Then we have the following:

Definition 2.4.1. *Let $\mathfrak{X}$ be a Deligne-Mumford stack and $[V] \in K_0(\mathfrak{X})$. We define the Riemann-Roch morphism:*

$$\mathfrak{R} : K_0(\mathfrak{X}) \to H^\bullet(I_{\mathfrak{X}}, \mathbb{C})$$

$$\mathfrak{R}([V]) := ch^{rep}([V])\, TD(\mathfrak{X})$$

This morphism satisfies the desired functoriality properties for arbitrary (not necessarily representable!) morphisms of stacks (see [22]), and will thus enable one to compute cohomological Fourier-Mukai transforms for Deligne-Mumford stacks.

Chapter 3

Additive Analogues of Chen-Ruan Cohomology

This chapter will describe a novel orbifold sheaf cohomology theory that generalizes Chen-Ruan orbifold cohomology [2] [9]. The orbifold sheaf cohomology described herein is motivated by physics, specifically by massless spectra in orbifolds of heterotic strings, for which reason we call it **Heterotic Chen-Ruan (HCR) cohomology**. Just as Chen-Ruan cohomology is the cohomology theory pertinent to the A model topological field theory on orbifolds, HCR is the cohomology theory pertinent to a heterotic analogue of the A model, known as the A/2 model [3]. This chapter will focus on understanding the additive part of the cohomology ring. Multiplicative structures on smooth manifolds are defined by what is known as quantum sheaf cohomology [12]. Multiplicative structures in the HCR cohomology on orbifolds are left to future work.

3.1 HCR Cohomology Definition

The initial input data for the physics inspired construction is that of a finite abelian group G which acts on a space X along with a G-equivariant vector bundle E over X. On this data we impose the following **anomaly cancellation conditions**:

$$\mathrm{ch}_2^G(E) = \mathrm{ch}_2^G(TX) \qquad \det(E^*) \cong K_X \tag{3.1.1}$$

The notation signifies we are using G-equivariant characteristic classes, and the isomorphism in 3.1.1 is a G-equivariant isomorphism. Note that as a consequence of the second condition, we have that $\mathrm{c}_1^G(E) = \mathrm{c}_1^G(TX)$, which would be the natural equivariant analog of the usual anomaly cancellation conditions. This stronger condition we impose is used to construct what is called the A/2 model. Given the above data, for each $g \in G$ such that the fixed locus $X^g = \{x \in X \mid g \cdot x = x\}$ is nonempty, we can restrict both the bundle E and the tangent bundle TX to X^g. We have that if $x \in X^g$ then for any $h \in G$ we have since G is abelian:

$$g \cdot (h \cdot x) = h \cdot (g \cdot x) = h \cdot x$$

In other words, the action of G preserves the fixed locus X^g. This action could be nontrivial on the fibers of the bundles E and TX restricted to X^g, however. So, if we let $t_g := |g| < \infty$, then over each $x \in X^g$, the fibers E_x and TX_x are $\mathbb{Z}/t_g\mathbb{Z} \cong \langle g \rangle$

representations, and thus split into eigenspaces:

$$E|_{X^g} = E_0^g \oplus E_1^g \oplus \cdots \oplus E_{t_g-2}^g \oplus E_{t_g-1}^g$$

$$TX|_{X^g} = T_0^g \oplus T_1^g \oplus \cdots \oplus T_{t_g-2}^g \oplus T_{t_g-1}^g$$

Where g acts via $\zeta_{t_g}^i$ on E_i^g and T_i^g, where $\zeta_{t_g} = e^{\frac{2\pi i}{t_g}}$. One might need to do this over each connected component of X^g, but for notational simplicity we will assume that each fixed locus is connected so that the above splitting occurs over all of X^g. We also define $d_i^g := \text{rank}(E_i^g)$. Further, consider the following subsets of $\mathbb{Q}$

$$M_j(g) := \exp^{-1}(\zeta_{t_g}^{-j}) = \mathbb{Z} - \frac{j}{t_g} \text{ (for } j = 0, 1, \dots, t_g - 1)$$

$$N_j(g) := \exp^{-1}(\zeta_{t_g}^{j}) = \mathbb{Z} + \frac{j}{t_g} \text{ (for } j = 0, 1, \dots, t_g - 1)$$

Here $\exp(\theta) := e^{2\pi i\theta}$. These correspond to "energy levels" of certain particles. Now define the set of maps $\Phi(g)$ which count the allowable multiplicities of particles with given energy levels:

$$\Phi(g) \subset \text{Hom}((M_0(g))_{<0}, \{0, 1, \dots, d_0^g\}) \times \ldots \times \text{Hom}((M_{t_g-1}(g))_{<0}, \{0, 1, \dots, d_{t_g-1}^g\})$$

$$\times \text{Hom}((N_0(g))_{<0}, \{0, 1, \dots, d_0^g\}) \times \ldots \times \text{Hom}((N_{t_g-1}(g))_{<0}, \{0, 1, \dots, d_{t_g-1}^g\})$$

An element $(p_*, q_*) = (p_0, \dots, p_{t_g-1}, q_1, \dots, q_{t_g-1}) \in \Phi(g)$ if and only if it satisfies the following condition:

$$\sum_{j=0}^{t_g-1} \left[\sum_{m \in (M_j(g))_{<0}} m p_j(m) + \sum_{n \in (N_j(g))_{<0}} n q_j(n) \right] = E_{L,g} \tag{3.1.2}$$

One should consider the functions p_* and q_* as mapping energy levels to their multiplicities, which are bounded by the ranks of the eigenbundles E_i^g. Then the left-hand side of 3.1.2 can be understood as a "total energy" value. The constant $E_{L,g}$, called the "vacuum energy," is coming from the following.

Definition 3.1.1. *Let $\eta_i^T \in [0,1)$ denote $exp^{-1}(\lambda_i)$ where the λ_i are the eigenvalues of g acting on $TX|_{X^g}$. Similarly we define $\eta_i^E \in [0,1)$ via the g-action on $E|_{X^g}$. Then we define:*

$$\Lambda_T^g := \sum_{i=1}^{rk(TX)} \left(\frac{1}{2}\eta_i^T(1-\eta_i^T) \right) \qquad \Lambda_E^g := \sum_{i=1}^{rk(E)} \left(\frac{1}{2}\eta_i^E(1-\eta_i^E) \right)$$

*The **vacuum energy** is then defined to be the rational number:*

$$E_{L,g} := \Lambda_T^g - \Lambda_E^g$$

For physical reasons, we only consider cases in which $E_{L,g} \leq 0$.

The subscript "L" of $E_{L,g}$ means "left-moving" in the context of physics, to distinguish it from other energies associated to the theory.

Remark 3.1.2. Note that the sum in the left-hand side of (3.1.2) lies in $\frac{\mathbb{Z}}{t_g}$, while the vacuum energy $E_{L,g}$ is in $\frac{\mathbb{Z}}{t_g^2}$. Thus, it is entirely possible for the set $\Phi(g)$ to be empty.

We are now able to define the HCR cohomology.

Definition 3.1.3. *Let X be a G-space for an abelian group G with a G-equivariant vector bundle E over it satisfying the anomaly cancellation conditions 3.1.1. Denote* *by $\mathscr{E}$ the corresponding sheaf of sections of E. Now we define for each $g \in G$ the sheaf cohomology group $H^\bullet_g :=$*

$$H^\bullet\left(X^g, \bigoplus_{(p_*,q_*)\in\Phi(g)} \bigotimes_{j=0}^{t_g-1}\left[\bigotimes_{m\in(M_j(g))_{<0}} \wedge^{p_j(m)}(\mathscr{E}_j^g)^* \bigotimes_{n\in(N_l(g))_{<0}} \wedge^{q_j(n)}(\mathscr{E}_j^g)\right]\otimes V(\mathscr{E})^g\right)^G$$

$$V(\mathscr{E})^g := \wedge^\bullet(\mathscr{E}_0^g)^* \otimes \sqrt{K_{X^g}\otimes det(\mathscr{E}_0^g)} \tag{3.1.3}$$

In the case that the vacuum energy of a component is zero, we define:

$$H^\bullet_g := H^\bullet\left(X^g, V(\mathscr{E})^g\right)^G$$

Here K_{X^g} is the canonical bundle of X^g, and there is an implicit sum over all values of $\bullet$ in all occurences. The definition of $V(\mathscr{E})^g$ involves the choice of a G-equivariant square root bundle (which may be obstructed). We then define the ***Heterotic Chen-Ruan cohomology*** *(or* ***HCR cohomology****) to be:*

$$H^\bullet_{HCR} := \bigoplus_{g\in G} H^\bullet_g$$

Remark 3.1.4. We work with the assumption that the bundle whose sheaf cohomology we are computing is G-equivariant, so there is a natural G-action on the cohomology group. As mentioned above, we require that the bundle $\sqrt{K_{X^g}\otimes \det(\mathscr{E}_0^g)}$ exists and also has a G-equivariant structure, in other words we need an equivariant square root. Defining the HCR cohomology thus involves a choice of square root for each $g \in G$. Since $K_{X^g}\otimes \det(\mathscr{E}_0^g) = K_{X^{g^{-1}}}\otimes \det(\mathscr{E}_0^{g^{-1}})$, we may choose the same square root and equivariant structure for each pair $\{g, g^{-1}\}$. This assumption will be implicit in what follows, and allows one to assume then that:

$$V(\mathscr{E})^g = V(\mathscr{E})^{g^{-1}} \tag{3.1.4}$$

This obstruction to the equivariant square root bundle is measured by a sheaf cohomology class on the stack $[X/G]$, which we will see soon.

3.2 Basic Properties

In this section we examine the properties of the HCR cohomology defined in the previous section.

3.2.1 HCR Cohomology on Stacks

The definition of the HCR cohomology in the previous section is naturally formulated in terms of sheaf cohomology on a quotient stack $[X/G]$ for a finite abelian group. Recall that for a global quotient stack $\mathfrak{X} = [X/G]$ by a finite group with $\mathscr{E} \in \boldsymbol{Sh}(\mathfrak{X})$, we can compute the cohomologies $H^i(\mathfrak{X}, \mathscr{E})$ by considering $\mathscr{E} \in \boldsymbol{Sh}_G(X)$ and using the isomorphism:

$$H^i([X/G], \mathscr{E}) \cong H^i(X, \mathscr{E})^G \tag{3.2.1}$$

This can be proven by using a Leray spectral sequence to the X-fibration $\pi : [X/G] \to [\mathrm{pt}/G]$. Then we have that:

$$\Gamma_{\mathfrak{X}} = \Gamma_{[\mathrm{pt}/G]} \circ \pi_* \Rightarrow \mathbb{R}\Gamma_{\mathfrak{X}} = \mathbb{R}\Gamma_{[\mathrm{pt}/G]} \circ \mathbb{R}\pi_*$$

The spectral sequence associated to the composition of derived functors is then:

$$E_2^{p,q} = \mathbb{R}^p\Gamma_{[\mathrm{pt}/G]}\left(\mathbb{R}^q\pi_*\mathscr{E}\right) \to H^{p+q}(\mathfrak{X}, \mathscr{E})$$

A sheaf on $[\mathrm{pt}/G]$ is equivalent to a G-representation, in which case $\Gamma_{[\mathrm{pt}/G]}$ is the functor of taking invariants. Thus the derived functors compute the group cohomology of this G-module:

$$E_2^{p,q} = H^p(G, H^q(X, \mathscr{E})) = \begin{cases} H^q(X, \mathscr{E})^G & \text{if } p = 0 \\ 0 & \text{if } p \neq 0 \end{cases}$$

This follows because all higher group cohomology of a finite group is torsion. More generally, for any reductive group G we have the same collapsing of the E_2 page.

Because of equation (3.2.1), we may rewrite the equation 3.1.3 defining the HCR cohomology in terms of sheaf cohomology on a stack. We denote by Ψ_g the sheaf occuring in the $H_g^\bullet$ summand of the HCR cohomology 3.1.3, and Ψ the sheaf on the inertia stack whose restriction to each component $[X^g/G]$ corresponds to Ψ_g:

$$\begin{aligned} H^\bullet_{HCR} &= \bigoplus_{g\in G} H_g^\bullet \\ &= \bigoplus_{g\in G} H^\bullet(X^g, \Psi_g)^G \\ &= \bigoplus_{g\in G} H^\bullet([X^g/G], \Psi_g) \\ &= H^\bullet(\bigsqcup_{g\in G}[X^g/G], \Psi) \\ &= H^\bullet(I_{[X/G]}, \Psi) \end{aligned}$$

This hints at a way to generalize the definition of the HCR cohomology for more general stacks. One must just formulate the bundle Ψ without reference to some globally acting group G. This is done by using sheaf automorphism constructed in theorem 2.2.4. Given any stack $\mathfrak{X}$ such that for all $x \in \mathrm{Ob}(\mathfrak{X})$, we have $|\mathrm{Aut}(x)| < \infty$. Then pulling back a sheaf $\mathscr{E}$ on $\mathfrak{X}$ to $I_{\mathfrak{X}}$, we can use that the aforementioned automorphism is finite order to deduce that our sheaf decomposes into eigensheaves, and thus come to a definition of the sheaf Ψ above:

Theorem 3.2.1. *Let $\mathscr{E}$ be a sheaf on a stack $\mathfrak{X}$ such that the automorphism group of any object in $\mathfrak{X}$ is finite, and that $\mathscr{E}$ satisfies the anomaly cancellation conditions:*

$$ch_2(\mathscr{E}) = ch_2(T\mathfrak{X}) \qquad\qquad det(\mathscr{E}^*) \cong K_{\mathfrak{X}}$$

Let $\pi : I_{\mathfrak{X}} \to \mathfrak{X}$ denote the usual projection from the inertia stack to $\mathfrak{X}$. Suppose $I_{\mathfrak{X}} = \sqcup\alpha$ is a decomposition of the inertia stack into connected components. Then there exists a definition of the HCR cohomology for the pair $\mathscr{E}$ and $\mathfrak{X}$ which specializes to the previous definition in the case of a global quotient stack by a finite abelian group, which will be of the form:

$$H^\bullet_{HCR} = \bigoplus_{\alpha \subseteq I_{\mathfrak{X}}} H^\bullet_\alpha = \bigoplus_{\alpha \subseteq I_{\mathfrak{X}}} H^\bullet(\alpha, \Psi_\alpha)$$

Proof. The main idea is the following. We show that the sheaf automorphism $\mu \in \mathrm{Aut}(\pi^*\mathscr{E})$ constructed in Theorem 2.2.4, when restricted to a component α, is of finite order t_α. Then one can decompose $\pi^*\mathscr{E}|_\alpha$ according to eigensheaves and construct a sheaf Ψ_α analogous to 3.1.3.

In more detail, suppose we use the notation and groupoid presentations from (2.2.2), so that we have a pair $(\mathscr{E}, \varphi)$ defining a sheaf on the groupoid presentation for $\mathfrak{X}$. Then:

$$\mu = \gamma^* p^*_{X \times_{\mathfrak{X}} X} \varphi : p_X^* \mathscr{E} \to p_X^* \mathscr{E}$$

We claim that the fact that the stabilizers are finite implies that this automorphism μ has finite order, and thus splits the sheaf into eigenbundles under its action. For any $n \in \mathbb{N}$, consider the following equality of sheaf morphisms:

$$p_1^* \varphi \circ p_2^* \varphi \circ \ldots \circ p_n^* \varphi = m_n^* \varphi \tag{3.2.2}$$

These are maps between sheaves on the n-fold fiber product:

$$(X \times_{\mathfrak{X}} X) \times_{s,X,t} (X \times_{\mathfrak{X}} X) \times_{s,X,t} \ldots \times_{s,X,t} (X \times_{\mathfrak{X}} X)$$

We define p_i to be the projection onto the i-th component and m_n to be the multiplication of all of the elements together. Define the following map:

$$\Delta_n : F \to (X \times_{\mathfrak{X}} X) \times_{s,X,t} (X \times_{\mathfrak{X}} X) \times_{s,X,t} \ldots \times_{s,X,t} (X \times_{\mathfrak{X}} X)$$

$$\Delta_n(f) = (f, f, \ldots, f)$$

Applying Δ_n^* to (3.2.2) and using that $\Delta_n^* p_i^* = \xi^*$ where $\xi : F \hookrightarrow X \times_{\mathfrak{X}} X$, and $\Delta_n^* m_n^* = \theta_n^*$ where $\theta_n(f) = f^n$, we obtain:

$$(\xi^* \varphi)^n = \theta_n^* \varphi \tag{3.2.3}$$

Notice that since $p_{X \times_{\mathfrak{X}} X} \circ \gamma = \xi$ that in fact $\xi^* \varphi = \mu$, thus by (3.2.3) we have that $\mu^n = \theta_n^* \varphi$. Further, for every $f \in F$ there is some n such that $f^n = 1$, or more precisely $f = (e \circ p_X)(f)$. Define the following subset of F:

$$F_n := \theta_n^{-1}(e(X))$$

As $p_X : F \to X$ is a bundle of finite groups (the stabilizer subgroup G_x is over each x) with section e, the subset $e(X)$ is both open and closed in F, and thus so also is each F_n. If $Z \subseteq F$ is a connected component, then $Z \subseteq F_{n(Z)}$ for some $n(Z)$, and correspondingly $\theta_{n(Z)}|_Z = e \circ p_X$. Thus $\theta_{n(Z)}^* \varphi|_Z = p_X^* e^* \varphi|_Z = \mathrm{id}$ since $e^*\varphi = \mathrm{id}$. We have thus shown that $(\mu|_Z)^{n(Z)} = \mathrm{id}$. It follows that on the stack $I_{\mathfrak{X}}$, over each connected component $\alpha \subseteq I_{\mathfrak{X}}$ there is some smallest $t_\alpha \in \mathbb{N}_{>0}$ such that $(\mu|_\alpha)^{t_\alpha} = \mathrm{id}$. Thus $\pi^*\mathscr{E}|_\alpha$ has a natural base-preserving $\mathbb{Z}/t_\alpha\mathbb{Z}$-action on it, under which we can decompose it (as well as the tangent bundle) into eigenbundles exactly as before:

$$E|_\alpha = E_0^\alpha \oplus E_1^\alpha \oplus \cdots \oplus E_{t_\alpha-2}^\alpha \oplus E_{t_\alpha-1}^\alpha$$

$$T\mathfrak{X}|_\alpha = T_0^\alpha \oplus T_1^\alpha \oplus \cdots \oplus T_{t_\alpha-2}^\alpha \oplus T_{t_\alpha-1}^\alpha$$

Here $\mu|_\alpha$ acts via $\zeta_{t_\alpha}^j$ on E_j^α and T_j^α where $\zeta_{t_\alpha} = e^{\frac{2\pi i}{t_\alpha}}$. From here one can compute the vacuum energy $E_{L,\alpha}$ for each component using the same prescription. One then constructs the bundles of interest on each component α with the same recipe as before. Again, take special note that the construction involves the choice of a square root $\sqrt{K_\alpha \otimes \det(\mathscr{E}_0^\alpha)}$ for each α, whose obstruction to existence is an obstruction to the HCR cohomology existing, and whos various choices will result in different HCR cohomologies. $\square$

Thus we can make sense of the HCR cohomology for any appropriate sheaf $\mathscr{E}$ on any stack with finite stabilizers.

Remark 3.2.2. One must again be wary of the obstructions to the HCR cohomology existing, namely in the existence of the square root bundles $\sqrt{K_\alpha \otimes \det(\mathscr{E}_0^\alpha)}$ on each component α. These obstructions can be succinctly stated in terms of the Kummer sequence on each component of the stack:

$$1 \longrightarrow \mu_2 \longrightarrow \mathscr{O}_\alpha^* \xrightarrow{(\_)^2} \mathscr{O}_\alpha^* \longrightarrow 1$$

Here the obstruction to the existence of the square root of $[\mathscr{L}] \in H^1(\alpha, \mathscr{O}_\alpha^*)$ lies in $H^2(\alpha, \mu_2)$ and the square roots themselves are only unique up to a class in $H^1(\alpha, \mu_2)$.

3.2.2 Reduction to Chen-Ruan

As a special case of the HCR cohomology, consider when $E = T\mathfrak{X}$. Then the anomaly cancellation conditions hold for trivial reasons. In this case, we have the following:

Theorem 3.2.3. *Suppose that $E = T\mathfrak{X}$, where $\mathfrak{X}$ is a compact, Kahler orbifold. Then we recover the additive Chen-Ruan cohomology through constructing the HCR cohomology:*

$$H^\bullet_{HCR} \cong H^\bullet_{CR}$$

Proof. Because the bundles are identical, over each component α of $I_{\mathfrak{X}}$ we have the η_i^T's and the η_i^E's are logarithms of eigenvalues of the same bundles, and thus

$E_{L,\alpha} = 0$. It follows that $(p_*, q_*) \in \Phi(\alpha)$ if and only if all of the functions $\{p_i, q_i\}_{i=0}^{t_\alpha - 1}$ are identically zero. Moreover, the trivial eigenspace T_0^α of $T\mathfrak{X}|_\alpha$ is simply the tangent space $T\alpha$ of the component α viewed as a closed substack of $\mathfrak{X}$. So we have:

$$\begin{aligned}
H^\bullet_{HCR} &= \bigoplus_{\alpha \subseteq I_{\mathfrak{X}}} H^\bullet\left(\alpha, \wedge^\bullet (T_0^\alpha)^* \otimes \sqrt{K_\alpha \otimes \det T_0^\alpha}\right) \\
&= \bigoplus_{\alpha \subseteq I_{\mathfrak{X}}} H^\bullet\left(\alpha, \wedge^\bullet (T\alpha)^* \otimes \sqrt{K_\alpha \otimes \det(T\alpha)}\right) \\
&= \bigoplus_{\alpha \subseteq I_{\mathfrak{X}}} H^\bullet\left(\alpha, \wedge^\bullet \Omega_\alpha\right) \\
&= \bigoplus_{\alpha \subseteq I_{\mathfrak{X}}} \bigoplus_{k=0}^{\dim(\alpha)} \bigoplus_{p+q=k} H^{p,q}(\alpha) \\
&= \bigoplus_{\alpha \subseteq I_{\mathfrak{X}}} \bigoplus_{k=0}^{\dim(\alpha)} H^k(\alpha, \mathbb{C}) \qquad \text{(For compact Kahler orbifolds)} \\
&= \bigoplus_{k=0}^{\dim(I_{\mathfrak{X}})} H^k(I_{\mathfrak{X}}, \mathbb{C})
\end{aligned}$$

Thus we recover the Chen-Ruan cohomology groups with their additive structure, as desired. Note that we chose a square root of the bundle $K_\alpha \otimes \det(T\alpha) \cong \mathscr{O}_\alpha$ to simply be $\mathscr{O}_\alpha$ itself, while we could have substituted different 2-torsion bundles on α and obtained a different spectrum. $\square$

3.2.3 Invariance Under Serre Duality

Note that the inertia stack $I_{\mathfrak{X}}$ has a naturally defined involution taking a component α to α^{-1}. More precisely, recall that objects in the category $I_{\mathfrak{X}}$ are of the form (x, σ) for $x \in \mathrm{Ob}(\mathfrak{X})$ and $\sigma \in \mathrm{Aut}(x)$. Then the involution ι is defined simply by $\iota(x,\sigma) = (x,\sigma^{-1})$. At the level of groupoids using our atlas from (2.2.2), one can see the involution as a pair of morphisms (β_0, β_1) forming an automorphism of groupoids:

$$\begin{tikzcd} F \arrow[r, "\beta_0"] & F \\ (X \times_{\mathfrak{X}} X) \times_{s,X,p_X} F \arrow[u, shift left, "\pi_F"] \arrow[u, shift right, "a"'] \arrow[r, "\beta_1"] & (X \times_{\mathfrak{X}} X) \times_{s,X,p_X} F \arrow[u, shift left, "\pi_F"] \arrow[u, shift right, "a"'] \end{tikzcd} \tag{3.2.4}$$

$$\beta_0(f) = f^{-1} \qquad \beta_1(r,f) = (r, f^{-1})$$

So we denote these paired components (which are equal if and only if $|\alpha| = 2$) by α and α^{-1}. Then the statement we will prove in this section is as follows:

Claim. Serre duality induces isomorphisms:

$$H^{\bullet}_{\alpha} \xrightarrow{\cong} \left(H^{\bullet}_{\alpha^{-1}}\right)^{*}$$

Thus the entire spectrum $H^{\bullet}_{HCR} \oplus {H^{\bullet}_{HCR}}^{*}$ is invariant under the isomorphisms induced above.

This preservation under Serre duality is of critical significance to the physical theory. We will demonstrate the above for any vector bundle E over $\mathfrak{X}$. First we re-interpret the set $\Phi(\alpha)$ combinatorially, namely by considering elements as corresponding to integer partitions. Recall that $(p_*, q_*) \in \Phi(\alpha)$ if and only if it satisfies equation (3.1.2). Multiplying both sides of (3.1.2) by $-t_\alpha$ and changing variables to $\mathsf{m} = -t_\alpha m$ and $\mathsf{n} = -t_\alpha n$:

$$\sum_{j=0}^{t_\alpha - 1} \left[\sum_{\mathsf{m} \in (t_\alpha \mathbb{Z} + j)_{>0}} \mathsf{m} p_j(-\frac{\mathsf{m}}{t_\alpha}) + \sum_{\mathsf{n} \in (t_\alpha \mathbb{Z} - j)_{>0}} \mathsf{n} q_j(-\frac{\mathsf{n}}{t_\alpha}) \right] = -t_\alpha E_{L,\alpha} =: F_\alpha \geq 0$$

The above can be considered as an integer partition of the non-negative integer F_α by integers indexed by m and n, with multiplicities being defined by the values of the functions p_j and q_j. In particular, the multiplicities of each integer occuring in each partition is constrained by values determined from the $\{d_j^\alpha\}_j$.

To be more precise, let $a \in \mathbb{Z}_{>0}$. Note that the above sum can be written as:

$$\sum_{j=0}^{t_\alpha - 1}\left[\sum_{\substack{\mathsf{m}\equiv j\\ \mathsf{m}>0}} \mathsf{m}p_j(-\frac{\mathsf{m}}{t_\alpha}) + \sum_{\substack{\mathsf{n}\equiv -j\\ \mathsf{n}>0}} \mathsf{n}q_j(-\frac{\mathsf{n}}{t_\alpha})\right] = F_\alpha$$

Where $\equiv$ denotes congruence modulo t_α. The following is true for each such a:

- If $a \equiv 0$ then the multiplicity of a occuring in a partition $(p_*, q_*) \in \Phi(\alpha)$ is $p_0(-\frac{a}{t_\alpha}) + q_0(-\frac{a}{t_\alpha}) \leq 2d_0^\alpha$.
- If $a \equiv j \equiv -(t_\alpha - j)$ for $j = 1, 2, \ldots, t_\alpha - 1$, then the multiplicity of a occuring in a partition $(p_*, q_*) \in \Phi(\alpha)$ is $p_j(-\frac{a}{t_\alpha}) + q_{t_\alpha - j}(-\frac{a}{t_\alpha}) \leq d_j^\alpha + d_{t_\alpha - j}^\alpha$.

In particular, we can identify elements of $\Phi(\alpha)$ with partitions of F_α which are split into two partitions P and Q. We make this into a definition for convenience.

Definition 3.2.4. *Let $N \in \mathbb{N}$. Then a* ***segregated partition*** *of N is a partition $\sum_{i=1}^{k} n_k = N$ along with the data of two subsets $P, Q \subseteq \{1, 2, \ldots, k\}$ such that $P \bigcap Q = \emptyset$ and $P \bigcup Q = \{1, 2, \ldots, k\}$ indexing two disjoint subsets of the partition.*

We now define a function:

$$\mathbb{X}_\alpha : \Phi(\alpha) \to \{\text{Segregated partitions of } F_\alpha\}$$

By defining $\mathbb{X}_\alpha(p_*, q_*)$ to be equal to:

$$P = \left\{ \begin{array}{c} \text{1 with multiplicity} \\ p_1\left(-\frac{1}{t_\alpha}\right) \end{array}, \begin{array}{c} \text{2 with multiplicity} \\ p_2\left(-\frac{2}{t_\alpha}\right) \end{array}, \ldots, \begin{array}{c} \mathsf{m} \text{ with multiplicity} \\ p_j\left(-\frac{\mathsf{m}}{t_\alpha}\right) \\ \text{if } \mathsf{m} \equiv j \\ j \in \{0, 1, \ldots, t_\alpha - 1\} \end{array} \right\}$$

$$Q = \left\{ \begin{array}{c} \text{1 with multiplicity} \\ q_{t_\alpha - 1}\left(-\frac{1}{t_\alpha}\right) \end{array}, \begin{array}{c} \text{2 with multiplicity} \\ q_{t_\alpha - 2}\left(-\frac{2}{t_\alpha}\right) \end{array}, \ldots, \begin{array}{c} \mathsf{n} \text{ with multiplicity} \\ q_{t_\alpha - j}\left(-\frac{\mathsf{n}}{t_\alpha}\right) \\ \text{if } \mathsf{n} \equiv j \\ j \in \{0, 1, \ldots, t_\alpha - 1\} \end{array} \right\}$$

Then the combinatorial description of $\Phi(\alpha)$ above can be stated as:

Lemma 3.2.5. *The function $\mathbb{X}_\alpha$ gives a bijection between $\Phi(\alpha)$ and the set of all segregated partitions of F_α such that the following hold:*

$\mathsf{m} \equiv 0 \implies$ *(multiplicity of m in P) $\leq d_0^\alpha$ and (multiplicity of m in Q) $\leq d_0^\alpha$*

$\mathsf{m} \equiv 1 \implies$ *(multiplicity of m in P) $\leq d_1^\alpha$ and (multiplicity of m in Q) $\leq d_{t_\alpha - 1}^\alpha$*

$\mathsf{m} \equiv 2 \implies$ *(multiplicity of m in P) $\leq d_2^\alpha$ and (multiplicity of m in Q) $\leq d_{t_\alpha - 2}^\alpha$*

$\vdots$

$\mathsf{m} \equiv j \implies$ *(multiplicity of* m *in* P*)* $\leq d_j^\alpha$ *and (multiplicity of* m *in* Q*)* $\leq d_{t_\alpha - j}^\alpha$

for $j = 1, \ldots, t_\alpha - 1$

Proof. This is simply a restatement of the condition (3.1.2). □

This combinatorial description of $H_\alpha^\bullet$ will make it easier to compare it with the HCR cohomology coming from α^{-1}. Under the identification $\iota : \alpha \cong \alpha^{-1}$, we can identify $E|_\alpha$ with $E|_{\alpha^{-1}} \cong \iota^*(E|_\alpha)$. We have the following decomposition of $E|_{\alpha^{-1}}$ relative to the α^{-1} action:

$$\begin{array}{cccccccccc} E|_{\alpha^{-1}} = & E_0^{\alpha^{-1}} & \oplus & E_1^{\alpha^{-1}} & \oplus & E_2^{\alpha^{-1}} & \oplus \ldots \oplus & E_{t_\alpha - 1}^{\alpha^{-1}} \\ & \| & & \| & & \| & & \| \\ & E_0^\alpha & & E_{t_\alpha - 1}^\alpha & & E_{t_\alpha - 2}^\alpha & & E_1^\alpha \end{array} \tag{3.2.5}$$

If we define the ranks as before, the above equality implies that:

$$d_0^{\alpha^{-1}} = d_0^\alpha \text{ and } d_i^{\alpha^{-1}} = d_{t_\alpha - i}^\alpha \text{ for } i \in \{1, \ldots, t_\alpha - 1\}$$

Using these equalities above we have $\Phi(\alpha^{-1})$ being defined as the subset (note that $t_{\alpha^{-1}} = t_\alpha = |\alpha|$ which implies that $M_j(\alpha^{-1}) = M_j(\alpha)$, $N_j(\alpha^{-1}) = N_j(\alpha)$):

$$\begin{aligned}\Phi(\alpha^{-1}) \subset & \mathrm{Hom}((M_0(\alpha))_{<0}, \{0,1,\dots,d_0^\alpha\})\times \\ & \mathrm{Hom}((M_1(\alpha))_{<0}, \{0,1,\dots,d_{t_\alpha-1}^\alpha\})\times \\ & \vdots \\ & \mathrm{Hom}((M_{t-1}(\alpha))_{<0}, \{0,1,\dots,d_1^\alpha\})\times \\ & \mathrm{Hom}((N_0(\alpha))_{<0}, \{0,1,\dots,d_0^\alpha\})\times \\ & \mathrm{Hom}((N_1(\alpha))_{<0}, \{0,1,\dots,d_{t_\alpha-1}^\alpha\})\times \\ & \vdots \\ & \mathrm{Hom}((N_{t-1}(\alpha))_{<0}, \{0,1,\dots,d_1^\alpha\})\end{aligned}$$

Where $(\mathfrak{p}_*, \mathfrak{q}_*) = (\mathfrak{p}_0, \dots, \mathfrak{p}_{t-1}, \mathfrak{q}_1, \dots, \mathfrak{q}_{t-1}) \in \Phi(\alpha^{-1})$ if and only if it satisfies equation (3.1.2):

$$\sum_{j=0}^{t_\alpha-1}\left[\sum_{m\in(M_j(\alpha))_{<0}} m\mathfrak{p}_j(m) + \sum_{n\in(N_j(\alpha))_{<0}} n\mathfrak{q}_j(n)\right] = E_{L,\alpha^{-1}} = E_{L,\alpha}$$

Once again multiplying the above by $-t_\alpha$ and using the change of variables $\mathsf{m} := -t_\alpha m$, $\mathsf{n} := -t_\alpha n$ we obtain the condition that:

$$\sum_{j=0}^{t_\alpha-1}\left[\sum_{\substack{\mathsf{m}\equiv j\\ \mathsf{m}>0}} \mathsf{m}\mathfrak{p}_j(-\frac{\mathsf{m}}{t_\alpha}) + \sum_{\substack{\mathsf{n}\equiv -j\\ \mathsf{n}>0}} \mathsf{n}\mathfrak{q}_j(-\frac{\mathsf{n}}{t_\alpha})\right] = F_\alpha$$

We can thus again consider $(\mathfrak{p}_*, \mathfrak{q}_*)$ as a segregated partition of F_α with almost mirror conditions as to those on element of $\Phi(\alpha)$. We make this precise below, by again first defining a function:

$$\mathbb{Y}_\alpha : \Phi(\alpha^{-1}) \to \{\text{Segregated partitions of } F_\alpha\}$$

By defining $\mathbb{Y}_\alpha(\mathfrak{p}_*, \mathfrak{q}_*)$ to be equal to:

$$P = \left\{ \begin{array}{c} \text{1 with multiplicity} \\ \mathfrak{p}_{t_\alpha - 1}\left(-\frac{1}{t_\alpha}\right) \end{array}, \begin{array}{c} \text{2 with multiplicity} \\ \mathfrak{p}_{t_\alpha - 2}\left(-\frac{2}{t_\alpha}\right) \end{array}, \ldots, \begin{array}{c} \mathsf{n} \text{ with multiplicity} \\ \mathfrak{p}_{t_\alpha - j}\left(-\frac{\mathsf{n}}{t_\alpha}\right) \\ \text{if } \mathsf{m} \equiv j \\ j \in \{0, 1, \ldots, t_\alpha - 1\} \end{array} \right\}$$

$$Q = \left\{ \begin{array}{c} \text{1 with multiplicity} \\ \mathfrak{q}_1\left(-\frac{1}{t_\alpha}\right) \end{array}, \begin{array}{c} \text{2 with multiplicity} \\ \mathfrak{q}_2\left(-\frac{2}{t_\alpha}\right) \end{array}, \ldots, \begin{array}{c} \mathsf{m} \text{ with multiplicity} \\ \mathfrak{q}_j\left(-\frac{\mathsf{m}}{t_\alpha}\right) \\ \text{if } \mathsf{m} \equiv j \\ j \in \{0, 1, \ldots, t_\alpha - 1\} \end{array} \right\}$$

Then we accordingly have:

Lemma 3.2.6. *The function $\mathbb{Y}_\alpha$ gives a bijection between $\Phi(\alpha^{-1})$ and the set of all segregated partitions of F_α such that the conditions of Lemma 3.2.5 hold.*

Proof. This is simply a restatement of the condition (3.1.2). □

Now we have the desired correspondence:

Corollary 3.2.7. *The composition:*

$$\mathbb{X}_\alpha^{-1} \circ \mathbb{Y}_\alpha : \Phi(\alpha^{-1}) \to \Phi(\alpha)$$

Defines a bijection of sets, and is given explicitly by:

$$(\mathfrak{p}_*, \mathfrak{q}_*) \mapsto (p_*, q_*)$$

$$p_0\left(-\frac{\mathsf{m}}{t_\alpha}\right) = \mathfrak{q}_0\left(-\frac{\mathsf{m}}{t_\alpha}\right) \text{ and } q_0\left(-\frac{\mathsf{m}}{t_\alpha}\right) = \mathfrak{p}_0\left(-\frac{\mathsf{m}}{t_\alpha}\right)$$

And for $j = 1, \ldots, t_\alpha - 1$:

$$p_j\left(-\frac{\mathsf{m}}{t_\alpha}\right) = \mathfrak{q}_{t_\alpha - j}\left(-\frac{\mathsf{m}}{t_\alpha}\right) \text{ and } q_j\left(-\frac{\mathsf{m}}{t_\alpha}\right) = \mathfrak{p}_{t_\alpha - j}\left(-\frac{\mathsf{m}}{t_\alpha}\right)$$

With this we can now prove Serre duality for arbitrary orbifolds/DM stacks with dualizing complex $\omega_\alpha[\dim(\alpha)]$ for each component α of the inertia stack. In this case Serre duality on each component $\alpha \subseteq I_{\mathfrak{X}}$ is of the form $H^i(\alpha, \mathscr{E}) \cong \mathrm{Ext}^{n-i}(\mathscr{E}, \omega_\alpha)^*$ as in the classical case, see Theorem 2.22 in [19].

Theorem 3.2.8. ***[Serre duality]*** *Serre duality induces an isomorphism:*

$$H^\bullet_\alpha \to {H^\bullet_{\alpha^{-1}}}^*$$

Proof. First note that $V(\mathscr{E})^\alpha = \wedge^\bullet(E_0^\alpha)^* \otimes \sqrt{K_\alpha \otimes \det(E_0^\alpha)}$ has the property that:

$$[V(\mathscr{E})^\alpha]^* \otimes K_\alpha \cong V(\mathscr{E})^\alpha$$

This will be used below when applying Serre duality. The following long calculation can be summarized in the following steps:

Line 1 $\to$ Line 2: Serre duality.

Line 2 $\to$ Line 3: Apply Corollary 3.2.7 to rewrite.

Line 3 → Line 4: Change variables $k := t_\alpha - j$ and use the identification of eigenbundles coming from equation (3.2.5).

$$
\begin{aligned}
H^\bullet_\alpha &= H^\bullet\left(\alpha, \bigoplus_{(p_*,q_*)\in\Phi(\alpha)} \bigotimes_{j=0}^{t_\alpha-1}\left[\bigotimes_{\substack{\mathsf{m}\equiv j\\ \mathsf{m}>0}} \wedge^{p_j(-\frac{\mathsf{m}}{t_\alpha})}(\mathscr{E}_j^\alpha)^* \bigotimes_{\substack{\mathsf{n}\equiv -j\\ \mathsf{n}>0}} \wedge^{q_j(-\frac{\mathsf{n}}{t_\alpha})}(\mathscr{E}_j^\alpha)\right] \otimes V(\mathscr{E})^\alpha\right)\\
&\cong H^\bullet\left(\alpha, \bigoplus_{(p_*,q_*)\in\Phi(\alpha)} \bigotimes_{j=0}^{t_\alpha-1}\left[\bigotimes_{\substack{\mathsf{m}\equiv j\\ \mathsf{m}>0}} \wedge^{p_j(-\frac{\mathsf{m}}{t_\alpha})}(\mathscr{E}_j^\alpha) \bigotimes_{\substack{\mathsf{n}\equiv -j\\ \mathsf{n}>0}} \wedge^{q_j(-\frac{\mathsf{n}}{t_\alpha})}(\mathscr{E}_j^\alpha)^*\right] \otimes [V(\mathscr{E})^\alpha]^* \otimes K_\alpha\right)^*\\
&= H^\bullet\Bigg(\alpha, \bigoplus_{(\mathfrak{p}_*,\mathfrak{q}_*)\in\Phi(\alpha^{-1})} \left[\bigotimes_{\substack{\mathsf{m}\equiv 0\\ \mathsf{m}>0}} \wedge^{\mathfrak{q}_0(-\frac{\mathsf{m}}{t_\alpha})}(\mathscr{E}_0^\alpha) \bigotimes_{\substack{\mathsf{n}\equiv 0\\ \mathsf{n}>0}} \wedge^{\mathfrak{p}_0(-\frac{\mathsf{n}}{t_\alpha})}(\mathscr{E}_0^\alpha)^*\right] \otimes\\
&\qquad \bigotimes_{j=1}^{t_\alpha-1}\left[\bigotimes_{\substack{\mathsf{m}\equiv j\\ \mathsf{m}>0}} \wedge^{\mathfrak{q}_{t_\alpha-j}(-\frac{\mathsf{m}}{t_\alpha})}(\mathscr{E}_j^\alpha) \bigotimes_{\substack{\mathsf{n}\equiv -j\\ \mathsf{n}>0}} \wedge^{\mathfrak{p}_{t_\alpha-j}(-\frac{\mathsf{n}}{t_\alpha})}(\mathscr{E}_j^\alpha)^*\right] \otimes V(\mathscr{E})^\alpha\Bigg)^*\\
&\cong H^\bullet\Bigg(\alpha, \bigoplus_{(\mathfrak{p}_*,\mathfrak{q}_*)\in\Phi(\alpha^{-1})} \left[\bigotimes_{\substack{\mathsf{m}\equiv 0\\ \mathsf{m}>0}} \wedge^{\mathfrak{q}_0(-\frac{\mathsf{m}}{t_\alpha})}(\mathscr{E}_0^{\alpha^{-1}}) \bigotimes_{\substack{\mathsf{n}\equiv 0\\ \mathsf{n}>0}} \wedge^{\mathfrak{p}_0(-\frac{\mathsf{n}}{t_\alpha})}(\mathscr{E}_0^{\alpha^{-1}})^*\right] \otimes\\
&\qquad \bigotimes_{k=1}^{t_\alpha-1}\left[\bigotimes_{\substack{\mathsf{m}\equiv -k\\ \mathsf{m}>0}} \wedge^{\mathfrak{q}_k(-\frac{\mathsf{m}}{t_\alpha})}(\mathscr{E}_k^{\alpha^{-1}}) \bigotimes_{\substack{\mathsf{n}\equiv k\\ \mathsf{n}>0}} \wedge^{\mathfrak{p}_k(-\frac{\mathsf{n}}{t_\alpha})}(\mathscr{E}_k^{\alpha^{-1}})^*\right] \otimes V(\mathscr{E})^{\alpha^{-1}}\Bigg)^*\\
&= (H^\bullet_{\alpha^{-1}})^*
\end{aligned}
$$

□

3.3 HCR Cohomology With 0-Dimensional Inertia

Note that the inertia stack $I_{\mathfrak{X}}$ always has a connected component which is isomorphic to the original stack $\mathfrak{X}$. The simplest nontrivial case for a computation of HCR cohomology would be the case where $\mathfrak{X}$ is a Deligne-Mumford stack such that the other components of the inertia stack $I_{\mathfrak{X}}$ are 0-dimensional.

Theorem 3.3.1. *Let $\mathfrak{X}$ be a Deligne-Mumford stack. Then if its decomposition into connected components $I_{\mathfrak{X}} = \mathfrak{X} \sqcup (\sqcup_i \mathfrak{Y}_i)$ is such that $dim(\mathfrak{Y}_i) = 0$ for all i, then we have:*

$$I_{\mathfrak{X}} = \mathfrak{X} \sqcup \left(\bigsqcup_{i \in I} [pt/G_i] \right) \tag{3.3.1}$$

Where G_i are all finite groups.

Proof. Since all Deligne-Mumford stacks are locally of the form $[X/G]$ for some finite group G (see Lemma 2.2.3 of [1]), if we let M denote the coarse moduli space of $\mathfrak{X}$, then there exists an étale covering $\sqcup_a M_a$ of M such that we have the following diagram, both squares being cartesian:

$$\begin{tikzcd}
\sqcup_a I_{[U_a/G_a]} \arrow[r] \arrow[d] & I_{\mathfrak{X}} \arrow[d] \\
\sqcup_a [U_a/G_a] \arrow[r] \arrow[d] & \mathfrak{X} \arrow[d] \\
\sqcup_a M_a \arrow[r] & M
\end{tikzcd}$$

Here $|G_a| < \infty$ for all a. By possibly refining the cover $\sqcup M_a$, we may assume that it is an open cover of M. Moreover we have that by equation (2.2.4) that:

$$
\begin{aligned}
I_{[U_a/G_a]} &= \bigsqcup_{[\![g]\!] \in \mathrm{Conj}(G_a)} [U_a^g/C(g)] \\
&= [U_a/G_a] \sqcup \left(\bigsqcup_{\substack{[\![g]\!] \in \mathrm{Conj}(G_a) \\ [\![g]\!] \neq [\![1]\!]}} [U_a^g/C(g)] \right)
\end{aligned}
$$

Note that we implicitly have chosen representatives for each conjugacy class. Without loss we may assume that if g is chosen to represent $[\![g]\!]$, then g^{-1} is chosen to represent $[\![g^{-1}]\!]$, if they are distinct classes. Since $\dim([U_a^g/C(g)]) = 2\dim(U_a^g) - [\dim(U_a^g) + \dim(C(g))] = 0$ and $C(g)$ is finite, we have that $\dim(U_a^g) = 0$. Thus, it breaks into a disjoint union of $C(g)$-orbits which we will index by $J_{(a,[\![g]\!])}$. For each orbit $j \in J_{(a,[\![g]\!])}$, fix a representative x_j. Then we have:

$$
\begin{aligned}
I_{[U_a/G_a]} &= [U_a/G_a] \sqcup \left(\bigsqcup_{\substack{[\![g]\!] \in \mathrm{Conj}(G_a) \\ [\![g]\!] \neq [\![1]\!]}} \bigsqcup_{j \in J_{(a,[\![g]\!])}} [\mathrm{Orbit}(x_j)/C(g)] \right) \\
&\cong [U_a/G_a] \sqcup \left(\bigsqcup_{\substack{[\![g]\!] \in \mathrm{Conj}(G_a) \\ [\![g]\!] \neq [\![1]\!]}} \bigsqcup_{j \in J_{(a,[\![g]\!])}} [\mathrm{pt}/C(g)_{x_j}] \right)
\end{aligned}
$$

Where here $C(g)_{x_j}$ denotes the stabilizer subgroup of x_j. Note that since they are components of fibers of open substacks, the $[\mathrm{pt}/C(g)_{x_j}]$ are open substacks of $I_{\mathfrak{X}}$. They are also closed substacks and thus represent connected components of $I_{\mathfrak{X}}$. $\square$

We now discuss the computation of the HCR cohomology in the case of 0-dimensional inertia, when we will have equation (3.3.1). Begin with a vector bundle E over such $\mathfrak{X}$ satisfying anomaly cancellation. We will have one summand of the HCR cohomology for each of the above components of the inertia stack. Over the "identity component" isomorphic to $\mathfrak{X}$, the vacuum energy is zero for this component so we simply obtain:

$$H^\bullet\left(\mathfrak{X}, \wedge^\bullet \mathscr{E}_0^* \otimes \sqrt{K_{\mathfrak{X}} \otimes \det(\mathscr{E}_0)}\right) = H^\bullet\left(\mathfrak{X}, \wedge^\bullet \mathscr{E}^* \otimes \sqrt{\mathscr{O}_{\mathfrak{X}}}\right)$$

This follows by the fact that $\mathscr{E}_0 = \mathscr{E}$ as we are splitting E into eigenbundles of the action of the identity element, and anomaly cancellation trivializes the bundle under the square root. We choose $\mathscr{O}_{\mathfrak{X}}$ to be this square root in what follows.

Restricting E to the substack $[\mathrm{pt}/G_i] \subset \mathfrak{X}$ we have $E|_{[\mathrm{pt}/G_i]} \in \boldsymbol{Sh}([\mathrm{pt}/G_i])$ is a G_i-representation. Moreover, each component $[\mathrm{pt}/G_i]$ of $I_{\mathfrak{X}}$ comes equipped with a natural $g \in G_i$ coming from the proof of Theorem 3.3.1 (identifying $G_i \cong C(g)_{x_j}$) or equivalently the map γ defined in equation (2.2.3). We will use the notation $[\mathrm{pt}/G_i]_g$ to denote the corresponding component of $I_{\mathfrak{X}}$. Note that the natural involution $\iota : I_{\mathfrak{X}} \to I_{\mathfrak{X}}$ defined in the diagram (3.2.4) will map between the components $[\mathrm{pt}/G_i]_g$ and $[\mathrm{pt}/G_i]_{g^{-1}}$ (or acts as the identity when $g = g^{-1}$). If we let $t_g := |g|$, then the E and $T\mathfrak{X}$ split into eigenbundles of the g-action:

$$E|_{[\mathrm{pt}/G_i]_g} = E_0^{i,g} \oplus E_1^{i,g} \oplus \ldots \oplus E_{t_g-1}^{i,g}$$

$$T\mathfrak{X}|_{[\mathrm{pt}/G_i]_g} = T_0^{i,g} \oplus T_1^{i,g} \oplus \ldots \oplus T_{t_g-1}^{i,g}$$

Further we define:

$$d_0^{i,g} := \mathrm{rank}(E_0^{i,g})\,,\ d_1^{i,g} := \mathrm{rank}(E_1^{i,g})\,,\ \ldots\,,\ d_{t_g-1}^{i,g} := \mathrm{rank}(E_{t_g-1}^{i,g})$$

Now the HCR cohomology is of the form

$$\begin{aligned}
H^{\bullet}_{HCR} &= H^{\bullet}(\mathfrak{X}, \wedge^{\bullet}\mathscr{E}^{*}) \oplus \left[\bigoplus_{i\in I}\bigoplus_{g\in G_i} H^{\bullet}([\mathrm{pt}/G_i]_g, \Psi_{i,g})\right] \\
&= H^{\bullet}(\mathfrak{X}, \wedge^{\bullet}\mathscr{E}^{*}) \oplus \left[\bigoplus_{i\in I}\bigoplus_{g\in G_i} H^{\bullet}(\mathrm{pt}, \Psi_{i,g})^{G_i}\right] \\
&= H^{\bullet}(\mathfrak{X}, \wedge^{\bullet}\mathscr{E}^{*}) \oplus \left[\bigoplus_{i\in I}\bigoplus_{g\in G_i} H^{0}(\mathrm{pt}, \Psi_{i,g})^{G_i}\right] \\
&= H^{\bullet}(\mathfrak{X}, \wedge^{\bullet}\mathscr{E}^{*}) \oplus \left[\bigoplus_{i\in I}\bigoplus_{g\in G_i} \Psi_{i,g}^{G_i}\right]
\end{aligned}$$

In other words, other than the term from the identity component it is simply the sum of the trivial subrepresentation of the representations $\Psi_{i,g}$. To determine this representation, recall that there was an obstruction/choice involving the square root bundle $\sqrt{K_{[\mathrm{pt}/G_i]} \otimes \det(E_0^g)}$. Here, $K_{[\mathrm{pt}/G_i]}$ is trivial (as a G_i-representation), but the determinant bundle $\det(E_0^g)$ could be given by a nontrivial character $\eta_{i,g} : G_i \to \mathbb{C}^*$ such that $\eta_{i,g}(g) = 1$. Suppose a choice $\chi_{i,g}$ exists such that $\chi_{i,g}^2 = \eta_{i,g}$. Keeping in mind the assumption from equation (3.1.4) of remark 3.1.4, we assume also that $\chi_{i,g^{-1}} = \chi_{i,g}$, where $\chi_{i,g^{-1}}$ is the square root of the isomorphic representation lying over the component $[\mathrm{pt}/G_i]_{g^{-1}}$. Then considering $\Psi_{i,g}$ as a G_i-representation, we have:

$$\begin{aligned}
\Psi_{i,g}^{G_i} &= \left[\Upsilon_{i,g} \otimes \wedge^{\bullet}(E_0^{i,g})^{*} \otimes \sqrt{K_{[\mathrm{pt}/G_i]} \otimes \det(E_0^{i,g})}\right]^{G_i} \\
&\cong \left[\Upsilon_{i,g} \otimes \wedge^{\bullet}(E_0^{i,g})^{*} \otimes \chi_{i,g}\right]^{G_i} \\
&\cong \left[\Upsilon_{i,g} \otimes \wedge^{\bullet}(E_0^{i,g})^{*}\right]^{\bar{\chi}_{i,g}}
\end{aligned}$$

Where the superscript $\bar{\chi}_{i,g}$ denotes the operation of projecting to the subrepresentation consisting of all copies of the G_i-representation $\bar{\chi}_{i,g}$ inside $\Upsilon_{i,g} \otimes \wedge^\bullet (E_0^{i,g})^*$. Here $\Upsilon_{i,g}$ denotes the bundle obtained via the formula (3.1.3), a construction which we will review in this context. For each pair (i, g) we can compute the vacuum energy (3.1.1) $E_{L,i,g}$ by using the logarithms of the eigenvalues of the g-action on both E and $T\mathfrak{X}$ restricted to $[\mathrm{pt}/G_i]_g$. If $E_{L,i,g} \in \mathbb{Z}[\frac{1}{t_g}]_{\leq 0}$ then we wish to find all the segregated partitions of $F_{i,g} := -t_g E_{L,i,g} \in \mathbb{Z}_{\geq 0}$ such that they satisfy the conditions of lemma 3.2.5, in other words:

Definition 3.3.2. *Let $\mathbb{S}_{i,g}$ denote the set of segregated partitions of $F_{i,g}$ into two subsets P and Q such that:*

$\mathsf{m} \equiv 0 \implies$ *(multiplicity of* m *in P)* $\leq d_0^{i,g}$ *and (multiplicity of* m *in Q)* $\leq d_0^{i,g}$

$\mathsf{m} \equiv 1 \implies$ *(multiplicity of* m *in P)* $\leq d_1^{i,g}$ *and (multiplicity of* m *in Q)* $\leq d_{t_g-1}^{i,g}$

$\mathsf{m} \equiv 2 \implies$ *(multiplicity of* m *in P)* $\leq d_2^{i,g}$ *and (multiplicity of* m *in Q)* $\leq d_{t_g-2}^{i,g}$

$\vdots$

$\mathsf{m} \equiv j \implies$ *(multiplicity of* m *in P)* $\leq d_j^{i,g}$ *and (multiplicity of* m *in Q)* $\leq d_{t_g-j}^{i,g}$

for $j = 0, \ldots, t_g - 1$, and all congruences are modulo t_g.

We will describe $\Upsilon_{i,g}$ in terms of $\mathbb{S}_{i,g}$. Denote elements of $\mathbb{S}_{i,g}$ by:

$$(\underbrace{1, 2, 5, 6, 6}_{\text{Elements of P}} \mid \underbrace{2, 2, 4}_{\text{Elements of Q}})_{i,g} \in \mathbb{S}_{i,g}$$

Following the equation 3.1.3, we arrive at the following summands of $\Upsilon_{i,g}(n_\bullet | m_\bullet)$ for each element $(n_\bullet | m_\bullet)_{i,g} \in \mathbb{S}_{i,g}$. Given a partition:

$$(n_1, n_2, \ldots, n_a \,|\, m_1, m_2, \ldots, m_b)_{i,g} \in \mathbb{S}_{i,g}$$

Begin with the trivial 1-dimensional G_i-representation. For each integer $n \in P$ occuring with multiplicity k, we tensor our previous representation with $\wedge^k(E_n^{i,g})^*$ where we reduce n modulo t_g to lie in $\{0, 1, \ldots, t_g - 1\}$. For each integer $m \in Q$ occuring with multiplicity k, we tensor our previous representation with $\wedge^k(E_{-m}^{i,g})$ where we reduce $-m$ modulo t_g to lie in $\{0, 1, \ldots, t_g - 1\}$. After running through the entire partition we arrive at our desired summand $\Upsilon_{i,g}(n_\bullet|m_\bullet)$:

$$\Upsilon_{i,g} = \bigoplus_{(n_\bullet|m_\bullet)_{i,g} \in \mathbb{S}_{i,g}} \Upsilon_{i,g}(n_\bullet|m_\bullet) \tag{3.3.2}$$

Example 3.3.3. Suppose that $t_g = 4$, $F_{i,g} = 32$, and the ranks of the $d_\bullet^{i,g}$ are such that we have the following allowable segregated partition of 32:

$$(1, 2, 2, 6 \,|\, 1, 1, 1, 4, 7, 7)_{i,g} \in \mathbb{S}_{i,g}$$

Then the corresponding summand of $\Upsilon_{i,g}$ would be:

$$\Upsilon_{i,g}(1, 2, 2, 6 \,|\, 1, 1, 1, 4, 7, 7) = \underbrace{(E_1^{i,g})^*}_{1 \in P} \otimes \underbrace{\wedge^2(E_2^{i,g})^*}_{2,2 \in P} \otimes \underbrace{(E_2^{i,g})^*}_{6 \in P} \otimes \underbrace{\wedge^3 E_3^{i,g}}_{1,1,1 \in Q} \otimes \underbrace{E_0^{i,g}}_{4 \in Q} \otimes \underbrace{\wedge^2 E_1^{i,g}}_{7,7 \in Q}$$

Remark 3.3.4. Note that Serre duality can be observed at the level of partitions in this notation by noting that:

$$(n_1, n_2, \ldots, n_a \,|\, m_1, m_2, \ldots, m_b)_{i,g} \in \mathbb{S}_{i,g}$$

If and only if

$$(m_1, m_2, \ldots, m_b \,|\, n_1, n_2, \ldots, n_a)_{i,g^{-1}} \in \mathbb{S}_{i,g^{-1}}$$

One may check that the corresponding summands of $\Upsilon_{i,g}$ and $\Upsilon_{i,g^{-1}}$ defined above are indeed paired under Serre duality as in theorem 3.2.8.

We have thus shown:

Algorithm 3.3.5. Let E be a bundle over $\mathfrak{X}$ with 0-dimensional inertia as in equation (3.3.1). The one compute the HCR cohomology in the following steps:

(I) Choose a square root $\chi_{i,g} = \sqrt{\det(E_0^{i,g})}$ for all pairs $\{g, g^{-1}\} \in G_i$. If no such exists, the HCR cohomology is not defined.

(II) For each (i, g), compute the vacuum energy $E_{L,i,g}$. If it is in $\mathbb{Z}[\frac{1}{t_g}]_{\leq 0}$ then define $F_{i,g} := -t_g E_{L,g,i} \in \mathbb{Z}_{\geq 0}$.

(III) For each (i, g) such that step (II) yielded an $F_{i,g}$, construct the set of segregated partitions $\mathbb{S}_{i,g}$ of $F_{i,g}$. Use the correspondence in (3.3.2) to construct $\Upsilon_{i,g}$.

(IV) Finally arrive at:

$$H^\bullet_{HCR} = H^\bullet(\mathfrak{X}, \wedge^\bullet \mathscr{E}^*) \oplus \left[\bigoplus_{i \in I} \bigoplus_{g \in G_i} \left[\Upsilon_{i,g} \otimes \wedge^\bullet (E_0^{i,g})^* \right]^{\bar{\chi}_{i,g}} \right] \tag{3.3.3}$$

Note that in the case where $\chi_{i,g} = 1$ is the trivial character for all (i, g), we are simply taking the G_i-invariants and the above formula reduces to:

$$H^\bullet_{HCR} = H^\bullet(\mathfrak{X}, \wedge^\bullet \mathscr{E}^*) \oplus \left[\bigoplus_{i \in I} \bigoplus_{g \in G_i} \left[\Upsilon_{i,g} \otimes \wedge^\bullet (E_0^{i,g})^* \right]^{G_i} \right] \tag{3.3.4}$$

Note also that our freedom of choosing each $\chi_{i,g}$ allows us to multiply our choices by any 2-torsion character $\xi_{i,g} \in \hat{G}_i$, in which case the HCR cohomology becomes:

$$H^\bullet_{HCR} = H^\bullet(\mathfrak{X}, \wedge^\bullet \mathscr{E}^*) \oplus \left[\bigoplus_{i \in I} \bigoplus_{g \in G_i} \left[\Upsilon_{i,g} \otimes \wedge^\bullet (E_0^{i,g})^* \right]^{\bar{\chi}_{i,g} \bar{\xi}_{i,g}} \right]$$

Given the above algorithm, we can show the following lemma. Recall that a priori, for each component $[\mathrm{pt}/G_i]_g$ of the inertia stack $I_{\mathfrak{X}}$ we have a corresponding automorphism of any sheaf on it of order t_g and a vacuum energy $E_{L,i,g}$ which we require to be in $\mathbb{Z}[\frac{1}{t_g}]$. In the case of 0-dimensional inertia, we have the following:

Lemma 3.3.6. *Suppose that E is a bundle satisfying anomaly cancellation over a stack $\mathfrak{X}$ with 0-dimensional inertia. Then if $[pt/G_i]_g$ is a component of the inertia stack such that $E|_{[pt/G_i]}$ yields a nonzero contribution to the HCR cohomology, then the corresponding vacuum energy $E_{L,i,g}$ lies in $\mathbb{Z}[\frac{1}{2}]$. If in addition we have that $|g|$ is odd, then $E_{L,i,g} \in \mathbb{Z}$.*

Proof. By (3.3.3) we have the following corresponding contribution to $H^{\bullet}_{HCR}$:

$$\left[\Upsilon_{i,g} \otimes \wedge^{\bullet}(E_0^{i,g})^* \otimes \chi_{i,g}\right]^{G_i} \neq \{0\}$$

This thus implies in particular that there is a nontrivial subspace on which g acts trivially. Let $t_g := |g|$ and choose $\zeta = e^{\frac{2\pi i}{t_g}}$. We examine with what weight (meaning power of ζ) g acts on the above representation. By construction, g acts with weight $t_g E_{L,i,g}$ on $\Upsilon_{i,g}$. Also, g acts with weight 0 on $\wedge^{\bullet}(E_0^{i,g})^*$, and depending on the given square root we have $\chi_{i,g}(g) = \pm 1$, or in other words g acts with weight 0 or $\frac{t_g}{2}$ (in which case t_g must be even). By assumption, we have a nontrivial subspace of weight 0, meaning that either:

$$t_g E_{L,i,g} \equiv 0 \,(\mathrm{mod}\ t_g) \ \text{ or } \ t_g E_{L,i,g} + \frac{t_g}{2} \equiv 0 \,(\mathrm{mod}\ t_g)$$

Which would imply then that:

$$E_{L,i,g} \in \mathbb{Z} \ \text{ or } \ E_{L,i,g} \in \mathbb{Z}\left[\frac{1}{2}\right]$$

The second case was only possible in the event that t_g were even, thus if it were odd then the vacuum energy must be integral. $\square$

3.4 Examples

This section serves as both proof that nontrivial examples of the HCR cohomology construction exist as well as a demonstration for how one would compute the HCR cohomology using Algorithm 3.3.5.

Example 3.4.1. Let X denote an abelian variety with $\dim_{\mathbb{C}}(X) = 4$. Let $G = \mathbb{Z}/2\mathbb{Z} = \langle g \rangle$ act on X via inversion, thus giving us $2^8 = 256$ fixed points. Let $\mathscr{O}_X$ denote the structure sheaf with its canonical G-equivariant structure, and $\mathscr{O}_X^\chi$ the same sheaf with the equivariant structure twisted by the nontrivial character $\chi : G \to \mathbb{C}^*$. Then let $E = (\mathscr{O}_X^\chi)^{\oplus 28}$. We compute the HCR cohomology. First we verify that anomaly cancellation holds:

$$\mathrm{ch}_2^G(X) = \mathrm{ch}_2^G((\mathscr{O}_X^\chi)^{\oplus 4}) = 4\mathrm{ch}_2^G(\mathscr{O}_X^\chi) = 2\mathrm{c}_1^G(\mathscr{O}_X^\chi)^2 = \mathrm{c}_1^G(\mathscr{O}_X^\chi)\mathrm{c}_1^G(\mathscr{O}_X) = 0 \quad (3.4.1)$$

This follows because $\mathscr{O}_X^\chi \otimes \mathscr{O}_X^\chi \cong \mathscr{O}_X$ as G-equivariant bundles, then taking first Chern classes to get:

$$2\mathrm{c}_1^G(\mathscr{O}_X^\chi) = \mathrm{c}_1^G(\mathscr{O}_X) = 0$$

Where the we have used that $\mathscr{O}_X$ is trivial equivariantly. Further, we have:

$$K_X = \det({\mathscr{O}_X^\chi}^{\oplus 4}) \cong \mathscr{O}_X \quad (3.4.2)$$

Now we compute for E:

$$\mathrm{ch}_2^G(E) = 28\mathrm{ch}_2^G(\mathscr{O}_X^\chi) = 14\mathrm{c}_1^G(\mathscr{O}_X^\chi)^2 = 7\mathrm{c}_1^G(\mathscr{O}_X^\chi)\mathrm{c}_1^G(\mathscr{O}_X) = 0 \text{ matching equation (3.4.1)}$$

Further:

$$\det(E) = \det((\mathscr{O}_X^\chi)^{\oplus 28}) \cong \mathscr{O}_X$$

$$\text{So then: } \det(E^*) \cong \mathscr{O}_X \text{ matching equation (3.4.2)}$$

Thus anomaly cancellation holds.

(I) Let $\{x_i\}_{i=1}^{256} = X^G$, and note that the inertia stack is of the form:

$$I_{\mathfrak{X}} = \mathfrak{X} \sqcup \left[\bigsqcup_{i=1}^{256} [x_i/G] \right]$$

We now choose our square roots of $\det(E_0^i)$, where we use a superscript i to denote the fiber of E at x_i. As this representation is trivial, we may choose $\chi_i = 1$ be our square root representations for all $x_i \in X^G$.

(II) Next, we compute the vacuum energies for the fixed points. For each $x_i \in X^G$, the fibers of the vector bundles TX^i and E^i are 4 and 28 copies of the nontrivial representation of G, respectively. Thus $\eta_j^T = \frac{1}{2} = \eta_k^E$ for $j = 1, \ldots, 4$ and $k = 1, \ldots, 28$. Thus the vacuum energy at x_i is:

$$E_{L,x_i} = \Lambda_T^i - \Lambda_E^i = \frac{1}{8}[4 - 28] = -3$$

This number lies in $\mathbb{Z}[\frac{1}{2}]$ and thus we may define our cohomology group of interest. We identify $d_0^i = 0$ and $d_1^i = 24$ as the ranks of the eigenbundles of E^i under the g action. In light of lemma 3.2.5, we consider segregated partitions of $F_{x_i} =$

$-|g|E_{L,x_i} = 6$ under the restrictions that:

$\mathsf{m} \equiv 0 \implies$ (multiplicity of m in P) ≤ 0 and (multiplicity of m in Q) ≤ 0

$\mathsf{m} \equiv 1 \implies$ (multiplicity of m in P) ≤ 24 and (multiplicity of m in Q) ≤ 24

(The congruences above are modulo $t_g = 2$.) In other words, our partitions must consist purely of odd integers, none occuring with multiplicity greater than $2d_1^i = 48$. So consider the only such partitions:

$$5+1 \qquad 3+3 \qquad 3+1+1+1 \qquad 1+1+1+1+1+1$$

(III) For each such partition we must vary over all permitted segregated partitions as in lemma 3.2.5, splitting each of the above up into P and Q. Then for each such we compute the relevant bundle whose cohomology is of interest. Note that this list is identical for each x_i, so we omit the i superscript for each bundle for ease of notation:

$\Upsilon(P\,\vert\,Q)$	$(P\,\vert\,Q)$
$E_1^* \otimes E_1^*$	$(1,5\,\vert\,)$
$E_1^* \otimes E_1$	$(5\,\vert\,1)$
$E_1^* \otimes E_1$	$(1\,\vert\,5)$
$E_1 \otimes E_1$	$(\,\vert\,1,5)$

$\Upsilon(P\,\vert\,Q)$	$(P\,\vert\,Q)$
$\wedge^2 E_1^*$	$(3,3\,\vert\,)$
$E_1^* \otimes E_1$	$(3\,\vert\,3)$
$\wedge^2 E_1$	$(\,\vert\,3,3)$

$\Upsilon(P\,|\,Q)$	$(P\,|\,Q)$
$\wedge^3 E_1^* \otimes E_1^*$	$(1,1,1,3\,|\,)$
$\wedge^2 E_1^* \otimes E_1^* \otimes E_1$	$(1,1,3\,|\,1)$
$E_1^* \otimes E_1^* \otimes \wedge^2 E_1$	$(1,3\,|\,1,1)$
$E_1^* \otimes \wedge^3 E_1$	$(3\,|\,1,1,1)$
$\wedge^3 E_1^* \otimes E_1$	$(1,1,1\,|\,3)$
$\wedge^2 E_1^* \otimes E_1 \otimes E_1$	$(1,1\,|\,1,3)$
$E_1^* \otimes \wedge^2 E_1 \otimes E_1$	$(1\,|\,1,1,3)$
$\wedge^3 E_1 \otimes E_1$	$(\,|\,1,1,1,3)$

$\Upsilon(P\,|\,Q)$	$(P\,|\,Q)$
$\wedge^6 E_1^*$	$(1,1,1,1,1,1\,|\,)$
$\wedge^5 E_1^* \otimes \wedge^1 E_1$	$(1,1,1,1,1\,|\,1)$
$\wedge^4 E_1^* \otimes \wedge^2 E_1$	$(1,1,1,1\,|\,1,1)$
$\wedge^3 E_1^* \otimes \wedge^3 E_1$	$(1,1,1\,|\,1,1,1)$
$\wedge^2 E_1^* \otimes \wedge^4 E_1$	$(1,1\,|\,1,1,1,1)$
$\wedge^1 E_1^* \otimes \wedge^5 E_1$	$(1\,|\,1,1,1,1,1)$
$\wedge^6 E_1$	$(\,|\,1,1,1,1,1,1)$

(IV) We now can compute the HCR cohomology using equation (3.3.4), giving us:

$$H^{\bullet}_{HCR} = H^{\bullet}(\mathfrak{X}, \wedge^{\bullet}\mathscr{E}^*) \oplus \left[\bigoplus_{i=1}^{256} \left[\Upsilon_{x_i} \otimes \wedge^{\bullet}(E_0^i)^*\right]^{\mathbb{Z}/2\mathbb{Z}}\right]$$

Where Υ_{x_i} is the direct sum of the sheaves/representations from the above tables. Note that:

$$E_0^i = 0$$

$$(E_1^i)^* = (\oplus_{i=1}^{28}\mathbb{C}^{\chi})^* \cong (\oplus_{i=1}^{28}\mathbb{C}^{\bar{\chi}}) = \oplus_{i=1}^{28}\mathbb{C}^{\chi} = E_1^i$$

Thus since every summand of Υ_{x_i} is the same tensor product of an even number of such representations, they are all trivial representations and thus $\mathbb{Z}/2\mathbb{Z}$-invariant. So we have:

$$H^{\bullet}_{HCR} = H^{\bullet}(\mathfrak{X}, \wedge^{\bullet}\mathscr{E}^*) \oplus [\Upsilon_{x_1}]^{\oplus 256}$$

To compute $H^\bullet(\mathfrak{X}, \wedge^\bullet \mathscr{E}^*) \cong H^\bullet(X, \wedge^\bullet \mathscr{E}^*)^G$, note that:

$$\begin{aligned}
\wedge^\bullet \mathscr{E}^* &= \wedge^\bullet (\mathscr{O}_X^\chi)^{\oplus 28} \\
&\cong \left[\bigoplus_{\substack{i=0 \\ i \text{ even}}}^{28} \mathscr{O}_X^{\oplus \binom{28}{i}} \right] \oplus \left[\bigoplus_{\substack{i=0 \\ i \text{ odd}}}^{28} (\mathscr{O}_X^\chi)^{\oplus \binom{28}{i}} \right]
\end{aligned}$$

Thus the contribution from the first of the two above summands is:

$$\begin{aligned}
\bigoplus_{k=0}^{4} H^k \left(X, \bigoplus_{\substack{i=0 \\ i \text{ even}}}^{28} \mathscr{O}_X^{\oplus \binom{28}{i}} \right)^G &= \bigoplus_{k=0}^{4} \bigoplus_{\substack{i=0 \\ i \text{ even}}}^{28} \left[H^k(X, \mathscr{O}_X)^G \right]^{\oplus \binom{28}{i}} \\
&\cong \bigoplus_{\substack{k=0 \\ k \text{ even}}}^{4} \bigoplus_{\substack{i=0 \\ i \text{ even}}}^{28} H^k(X, \mathscr{O}_X)^{\oplus \binom{28}{i}}
\end{aligned}$$

Here we are using that $H^k(X, \mathscr{O}_X) \cong H^{0,k}(X)$ which is spanned by k-fold wedges of $\{\mathrm{d}\bar{z}_i\}_{i=1}^4$, which is clearly G-invariant if and only if k is even. Likewise, for the second summand of the above we have only odd forms coming out due to twisting by χ:

$$\begin{aligned}
\bigoplus_{k=0}^{4} H^k \left(X, \bigoplus_{\substack{i=0 \\ i \text{ odd}}}^{28} (\mathscr{O}_X^\chi)^{\oplus \binom{28}{i}} \right)^G &= \bigoplus_{k=0}^{4} \bigoplus_{\substack{i=0 \\ i \text{ odd}}}^{28} \left[\left(H^k(X, \mathscr{O}_X) \otimes \chi \right)^G \right]^{\oplus \binom{28}{i}} \\
&\cong \bigoplus_{\substack{k=0 \\ k \text{ odd}}}^{4} \bigoplus_{\substack{i=0 \\ i \text{ odd}}}^{28} H^k(X, \mathscr{O}_X)^{\oplus \binom{28}{i}}
\end{aligned}$$

In sum, we have the following complete description of the HCR cohomology:

$$\boxed{H^\bullet_{HCR} \cong \left[\bigoplus_{\substack{k=0 \\ k \text{ even}}}^{4} \bigoplus_{\substack{i=0 \\ i \text{ even}}}^{28} H^k(X, \mathscr{O}_X)^{\oplus \binom{28}{i}} \right] \oplus \left[\bigoplus_{\substack{k=0 \\ k \text{ odd}}}^{4} \bigoplus_{\substack{i=0 \\ i \text{ odd}}}^{28} H^k(X, \mathscr{O}_X)^{\oplus \binom{28}{i}} \right] \oplus [\Upsilon_{x_1}]^{\oplus 256}}$$

Moreover, as the stabilizer groups are all 2-torsion, Serre duality interchanges the HCR cohomology over the same component of the inertia stack, and can be seen by simply swapping the P and Q portions of the corresponding partition.

We now consider examples over weighted projective stacks. Consider the weighted projective stack ($k_i \in \mathbb{N}$):

$$\mathfrak{X} = \mathbb{P}[k_0, k_1, \ldots, k_r] = [\mathbb{C}^{r+1} - \{0\}/\mathbb{C}^*]$$

With $\mathbb{C}^*$ action given by:

$$\lambda \cdot (x_0, x_1, \ldots, x_r) = (\lambda^{k_0} x_0, \lambda^{k_1} x_1, \ldots, \lambda^{k_r} x_r)$$

One may identify vector bundles on $\mathfrak{X}$ with $\mathbb{C}^*$-equivariant bundles over the punctured $(r+1)$-plane. Denote by $\mathscr{O}_{\mathfrak{X}}(n)$ the trivial bundle over $\mathbb{C}^{r+1} - \{0\}$ with the equivariant structure tensored by χ^n where χ is the tautological character of $\mathbb{C}^*$. Alternatively, the total space of $\mathscr{O}_{\mathfrak{X}}(n)$ is given by the quotient stack:

$$[\big(\mathbb{C}^{r+1} - \{0\}\big) \times \mathbb{C}/\mathbb{C}^*]$$

$$\lambda \cdot (x_0, \ldots, x_r, z) = (\lambda^{k_0} x_0, \ldots, \lambda^{k_r} x_r, \lambda^n z)$$

We consider examples where E fits into the following exact sequence:

$$0 \longrightarrow \mathscr{E} \longrightarrow \oplus_i \mathscr{O}_{\mathfrak{X}}(n_i) \longrightarrow \mathscr{O}_{\mathfrak{X}}(m) \longrightarrow 0$$

First we check for anomaly cancellation. Note that we have the following weighted euler sequence for the tangent bundle:

$$0 \longrightarrow \mathscr{O}_{\mathfrak{X}} \longrightarrow \oplus_{i=0}^r \mathscr{O}_{\mathfrak{X}}(k_i) \longrightarrow T\mathfrak{X} \longrightarrow 0$$

If we define $\alpha := \mathrm{c}_1^{\mathbb{C}^*}(\mathscr{O}_{\mathfrak{X}}(1))$ then we find that:

$$\mathrm{ch}_2^{\mathbb{C}^*}(\mathscr{E}) = \frac{1}{2}(\Sigma n_i^2 - m^2)\alpha^2 \qquad \det(E^*) \cong \mathscr{O}_{\mathfrak{X}}(m - \Sigma n_i)$$

$$\mathrm{ch}_2^{\mathbb{C}^*}(T\mathfrak{X}) = \frac{1}{2}(\Sigma k_i^2)\alpha^2 \qquad K_{\mathfrak{X}} \cong \mathscr{O}_{\mathfrak{X}}(-\Sigma k_i)$$

So anomaly cancellation is equivalent to:

$$\Sigma n_i - m = \Sigma k_i \qquad \Sigma n_i^2 - m^2 = \Sigma k_i^2 \tag{3.4.3}$$

Example 3.4.2. Consider the weighted projective stack $\mathbb{P}[1,2,3]$. Let E be the following kernel:

$$0 \longrightarrow \mathscr{E} \longrightarrow \mathscr{O}_{\mathfrak{X}}(-5) \oplus \mathscr{O}_{\mathfrak{X}}(-2) \oplus \mathscr{O}_{\mathfrak{X}}(2) \oplus \mathscr{O}_{\mathfrak{X}}(7)^{\oplus 2} \oplus \mathscr{O}_{\mathfrak{X}}(18) \xrightarrow{\phi} \mathscr{O}_{\mathfrak{X}}(21) \longrightarrow 0$$

$$\phi(s_1, s_2, s_3, s_4, s_5, s_6) = p_1 s_1 + p_2 s_2 + p_3 s_3 + p_4 s_4 + p_5 s_5 + p_6 s_6$$

Where the $p_i(x_0, x_1, x_2)$ are polynomials of the requisite degree, chosen generically so that they are never all zero and thus ϕ is surjective. One checks that they satisfy (3.4.3) so that anomaly cancellation holds. The inertia stack takes the form:

$$I_{\mathfrak{X}} = \mathfrak{X} \sqcup [p/(\mathbb{Z}/2\mathbb{Z})]_g \sqcup [q/(\mathbb{Z}/3\mathbb{Z})]_h \sqcup [\mathrm{pt}/(\mathbb{Z}/3\mathbb{Z})]_{h^2}$$

Here $p = [0:1:0]$ and $q = [0:0:1]$ are the points with nontrivial stabilizer, and g, h, and h^2 are used to label the two components of the inertia stack corresponding to the $\langle h \rangle \cong \mathbb{Z}/3\mathbb{Z}$ stabilizer subgroup of q and the $\langle g \rangle \cong \mathbb{Z}/2\mathbb{Z}$ stabilizer subgroup of p.

(I) The fibers $E|_{[p/(\mathbb{Z}/2\mathbb{Z})]}$ and $E|_{[q/(\mathbb{Z}/3\mathbb{Z})]}$ are representations of cyclic groups in which every non-identity element generates the group, thus the trivial eigenspaces of any such group element are trivial as representations as well. Thus the representations $\det(E_0^g)$, $\det(E_0^h)$, and $\det(E_0^{h^2})$ are all trivial and we may choose the trivial representation for their square roots.

(II) We compute the vacuum energies. Restricting the sequence defining E to the fiber above p, we obtain a short exact sequence of $(\mathbb{Z}/2\mathbb{Z})$-representations, which necessarily splits. We accordingly have that the fiber $E|_{[p/(\mathbb{Z}/2\mathbb{Z})]_g}$ decomposes:

$$E|_{[p/(\mathbb{Z}/2\mathbb{Z})]_g} = \mathbb{C}^{\oplus 3} \oplus (\mathbb{C}^{\chi})^{\oplus 2}$$

Here χ denotes the nontrivial $(\mathbb{Z}/2\mathbb{Z})$ character. Similarly we have the decomposition of the fiber at p for the tangent bundle:

$$T\mathfrak{X}|_{[p/(\mathbb{Z}/2\mathbb{Z})]_g} = (\mathbb{C}^{\chi})^{\oplus 2}$$

From here we see that the vacuum energy at p is simply the difference in the ranks of the nontrivial eigenspaces of the $(\mathbb{Z}/2\mathbb{Z})$-action, or:

$$E_{L,p,g} = 0$$

Restricting E to q, we obtain:

$$E|_{[q/(\mathbb{Z}/3\mathbb{Z})]_h} = (\mathbb{C}^{\eta})^{\oplus 4} \oplus \mathbb{C}^{\eta^2}$$

$$E|_{[q/(\mathbb{Z}/3\mathbb{Z})]_{h^2}} = \mathbb{C}^{\eta^2} \oplus (\mathbb{C}^{\eta})^{\oplus 4}$$

Where η is the tautolical representation when viewing $(\mathbb{Z}/3\mathbb{Z}) \subset \mathbb{C}^*$. For the tangent bundle we have:

$$T\mathfrak{X}|_{[q/(\mathbb{Z}/3\mathbb{Z})]_h} = \mathbb{C}^{\eta} \oplus \mathbb{C}^{\eta^2}$$

$$T\mathfrak{X}|_{[q/(\mathbb{Z}/3\mathbb{Z})]_{h^2}} = \mathbb{C}^{\eta^2} \oplus \mathbb{C}^{\eta}$$

Then the vacuum energy for both components is:

$$E_{L,q,h} = E_{L,q,h^2} = \frac{1}{2}\left[\frac{1}{3}\frac{2}{3} + \frac{2}{3}\frac{1}{3}\right] - \frac{1}{2}\left[4 \cdot \frac{1}{3}\frac{2}{3} + \frac{2}{3}\frac{1}{3}\right] = -\frac{1}{3}$$

(III) By Lemma 3.3.6, since the vacuum energies at the q components of $I_{\mathfrak{X}}$ are not in $\mathbb{Z}[\frac{1}{2}]$, the do not contribute to the HCR cohomology. The component corresponding to p contributes simply:

$$\left[\wedge^{\bullet}(E_0^g)^* \otimes \sqrt{\det(E_0^g)}\right]^{\mathbb{Z}/2\mathbb{Z}} = \wedge^{\bullet}(E_0^g)^* \cong \wedge^{\bullet}\mathbb{C}^{\oplus 3}$$

(IV) The total HCR cohomology is thus:

$$\boxed{H^{\bullet}_{HCR} \cong H^{\bullet}(\mathfrak{X}, \wedge^{\bullet}\mathscr{E}^*) \oplus \wedge^{\bullet}\mathbb{C}^{\oplus 3}}$$

Example 3.4.3. Let us consider a general weighted projective stack with 0-dimensional inertia, and let E be a vector bundle defined by a short exact sequence:

$$0 \longrightarrow \mathscr{E} \longrightarrow \bigoplus_{i=1}^{a} \mathscr{O}_{\mathfrak{X}}(n_i) \longrightarrow \mathscr{O}_{\mathfrak{X}}(m) \longrightarrow 0 \tag{3.4.4}$$

In this case we will find the vacuum energies for each component of the inertia stack in complete generality. Since these spaces produce a wealth of nontrivial examples, these computations should be of use.

Let $\mathfrak{X} = \mathbb{P}[k_0, k_1, \ldots, k_r]$. Then $\mathfrak{X}$ having 0-dimensional inertia is equivalent to $(k_i, k_j) = 1$ for $i \neq j$. In such a case, the only orbits of $\mathbb{C}^{r+1} - \{0\}$ under the $\mathbb{C}^*$ action defining the global quotient stack with nontrivial stabilizers are the coordinate axes, each with a cyclic stabilizer subgroup. These correspond to the following stacky points of $\mathfrak{X}$:

$$p_0 = [1 : 0 : 0 : \ldots : 0] = [\mathrm{pt}/(\mathbb{Z}/k_0\mathbb{Z})]$$

$$p_1 = [0 : 1 : 0 : \ldots : 0] = [\mathrm{pt}/(\mathbb{Z}/k_1\mathbb{Z})]$$

$$\vdots$$

$$p_j = [0 : \ldots 0 : 0 : 1] = [\mathrm{pt}/(\mathbb{Z}/k_j\mathbb{Z})]$$

Accordingly, the inertia stack decomposes:

$$I_{\mathfrak{X}} = \mathfrak{X} \sqcup \left[\bigsqcup_{i=0}^{r} \bigsqcup_{\substack{g \in (\mathbb{Z}/k_i\mathbb{Z}) \\ g \neq 1}} [\mathrm{pt}/(\mathbb{Z}/k_i\mathbb{Z})]_g \right]$$

We will use the letter g to denote the generator for the cyclic group of each component, and thus label the components by the pair (p_j, g^k), where $k = 1, \ldots, k_j - 1$. Let us compute the vacuum energy for an arbitrary component corresponding to (p_j, g^k). Restricting the short exact sequence defining E to the point p_j, we can consider it as a short exact sequence of $(\mathbb{Z}/k_j\mathbb{Z})$-representations:

$$0 \longrightarrow E_{p_j} \longrightarrow \oplus_i \mathbb{C}^{\chi_j^{n_i}} \longrightarrow \mathbb{C}^{\chi_j^m} \longrightarrow 0 \qquad (3.4.5)$$

Here χ_j is the tautological character of $(\mathbb{Z}/k_j\mathbb{Z}) \subset \mathbb{C}^*$. Let $\zeta_j = e^{\frac{2\pi i}{k_j}}$. Then g acts with eigenvalue $\zeta_j^{n_i}$ on $\mathbb{C}^{\chi_j^{n_i}}$, and thus g^k acts via $\zeta_j^{kn_i}$ on the same eigenspace. Since the above sequence splits, one of the eigenspaces in the middle space must be isomorphic to $\mathbb{C}^{\chi_j^m}$ as a representation. To compute the vacuum energy, we must take the logarithms of these eigenvalues. More specifically, we compute the unique $\exp^{-1}(\lambda) \in [0,1)$ for each eigenvalue where $\exp(x) = e^{2\pi i x}$. Let $\{x\} := x - \lfloor x \rfloor$ denote the fractional part of a real number. Then:

$$\exp^{-1}(\zeta_j^{kn_i}) = \exp^{-1}\exp\left(\frac{kn_i}{k_j}\right) = \left\{\frac{kn_i}{k_j}\right\}$$

These are the logarithms that we use to compute one half of the vacuum energy:

$$\Lambda_E^{p_j,g^k} = \frac{1}{2}\sum_{i=1}^{a}\left\{\frac{kn_i}{k_j}\right\}\left(1-\left\{\frac{kn_i}{k_j}\right\}\right) - \frac{1}{2}\left\{\frac{km}{k_j}\right\}\left(1-\left\{\frac{km}{k_j}\right\}\right) \tag{3.4.6}$$

Note that the subtracted term serves to cancel out the contribution from whichever representation is mapped isomorphically onto $\mathbb{C}^{\chi_j^m}$ in equation (3.4.5). Similarly, we restrict the euler sequence to compute the contribution coming from $T\mathfrak{X}$:

$$0 \longrightarrow \mathbb{C} \longrightarrow \oplus_i \mathbb{C}^{\chi_j^{k_i}} \longrightarrow T\mathfrak{X}_{p_j} \longrightarrow 0$$

From this we obtain that g^k acts with eigenvalues $\zeta_j^{kk_i}$, and thus the second half of the vacuum energy is:

$$\Lambda_T^{p_j,g^k} = \frac{1}{2}\sum_{i=0}^{r}\left\{\frac{kk_i}{k_j}\right\}\left(1-\left\{\frac{kk_i}{k_j}\right\}\right) \tag{3.4.7}$$

Note that when $i = j$ that term gives 0 contribution to the above sum, as it should, since that eigenspace is actually the kernel of the above short exact sequence and is thus not a summand of $T\mathfrak{X}_{p_j}$. Thus we have shown:

Theorem 3.4.4. *Let E be a vector bundle over $\mathfrak{X} = \mathbb{P}[k_0, \ldots, k_r]$ where $(k_i, k_j) = 1$ for $i \neq j$ which fits into the exact sequence* (3.4.4). *Then the vacuum energy* *corresponding to the component of the inertia stack* (p_j, g^k) *is:*

$$E_{L,(p_j,g^k)} = \frac{1}{2}\sum_{i=0}^{r}\left\{\frac{kk_i}{k_j}\right\}\left(1-\left\{\frac{kk_i}{k_j}\right\}\right) - \frac{1}{2}\sum_{i=1}^{a}\left\{\frac{kn_i}{k_j}\right\}\left(1-\left\{\frac{kn_i}{k_j}\right\}\right)\dots$$
$$+\frac{1}{2}\left\{\frac{km}{k_j}\right\}\left(1-\left\{\frac{km}{k_j}\right\}\right)$$

Proof. Combine equations (3.4.6) and (3.4.7) along with the equation:

$$E_{L,(p_j,g^k)} = \Lambda_T^{p_j,g^k} - \Lambda_E^{p_j,g^k}$$

□

Remark 3.4.5. In light of Lemma 3.3.6, if one desires examples for which the HCR cohomology contribution from the stacky points to be nonzero, one is interested when the above sum is in $\mathbb{Z}\left[\frac{1}{2}\right]$.

Chapter 4

Elliptic Fibrations and Fourier-Mukai Transforms

In this chapter we review and fix notation of derived categories and Fourier-Mukai transforms in preparation for the following chapter, where we will need this language for investigating the twisted anomaly cancellation conditions.

4.1 Fourier-Mukai Transforms

First we have the following definitions:

Definition 4.1.1. *Let X be a projective variety. We define its* **bounded derived category** $D^b(X)$ *to be the bounded derived category of the associated abelian cate-*

gory of coherent sheaves on X:

$$D^b(X) := D^b(Coh(X))$$

We frequently identify $D^b(X)$ as the full triangulated subcategory of $D^b(\mathrm{Qcoh}(X))$ with coherent cohomology objects - see [17] for a detailed introduction. Objects are complexes $\mathscr{E}^\bullet$ a differential of degree $+1$, and we define the shift functor $T(\mathscr{E}^\bullet) = \mathscr{E}^\bullet[1]$, where $(\mathscr{E}^\bullet[1])^n = \mathscr{E}^{n+1}$. Given any morphism $f : X \to Y$, one can define the usual derived functors:

$$\mathbb{R}f_* : D^+(X) \to D^+(Y)$$

$$\mathbb{L}f^* : D^-(Y) \to D^-(X)$$

Further for an object $\mathscr{E}^\bullet \in D^b(X)$ one can define the derived tensor product:

$$\mathscr{E}^\bullet \otimes^{\mathbb{L}} \_ : D^-(X) \to D^-(X)$$

Definition 4.1.2. *Given any two projective varieties X and Y along with an object $\mathscr{E}^\bullet \in D^b(X \times Y)$, we define the* **integral transform** $\Phi^{\mathscr{E}^\bullet}_{X\to Y}$ *to be the composition of derived functors:*

$$\Phi^{\mathscr{E}^\bullet}_{X\to Y}(\mathscr{F}^\bullet) := \mathbb{R}\pi_{Y*}(\pi_X^*\mathscr{F}^\bullet \otimes^{\mathbb{L}} \mathscr{E}^\bullet)$$

For some such functors they induce equivalences of triangulated categories:

Definition 4.1.3. *Let $\mathscr{E}^\bullet \in D^b(X \times Y)$ be such that $\Phi^{\mathscr{E}^\bullet}_{X\to Y}$ defines an equivalence of triangulated categories. Then we call the functor $\Phi^{\mathscr{E}^\bullet}_{X\to Y}$ a* **Fourier-Mukai transform**.

4.2 Elliptic Fibrations & Relative Fourier-Mukai

Let $f : X \to B$ be a smooth morphism of projective varieties such that the fibers are smooth genus one curves. We call such a morphism a smooth genus one fibration. If we further impose that the fibration has a section $\sigma : B \to X$, then we call f a smooth elliptic fibration. If f is elliptic, one can globally perform a Fourier-Mukai transform over the fibers by using the section σ to globally identify each elliptic fiber E_b (for $b \in B$) with its dual $\mathrm{Pic}^0(E_b)$. More precisely if we define the sheaf:

$$\mathscr{P} := \mathscr{O}_{X \times_B X}(\Delta - [\sigma \times_B X] - [X \times_B \sigma] - \xi^*(D))$$

Here $\xi : X \times_B X \to B$ is the projection. Then $\mathscr{P}|_{E_b \times E_b}$ is the usual Poincaré sheaf on each fiber, inducing a fiberwise Fourier-Mukai transform. A bundle satisfying such a condition fiberwise is determined only up to a pull back from B, in which case the divisor D is chosen to impose the normalization conditions $\mathscr{P}|_{\sigma \times_B X} \cong \mathscr{O}$ and $\mathscr{P}|_{X \times_B \sigma} \cong \mathscr{O}$.

When no such section exists, one can only do the above construction locally (assuming local sections exist). So one can construct local Poincaré sheaves $\mathscr{P}_i$ over $f^{-1}(U_i)$ for some $U_i \subset B$. The obstruction to gluing them, namely the bundles $\mathscr{N}_{ij} = \mathscr{P}_i|_{U_{ij}} \otimes \mathscr{P}_j^*|_{U_{ij}}$, can be considered as a geometric representation of an $\mathscr{O}_X^*$ gerbe $\mathfrak{X}$ over X - see section 1.1 of [8]. What this is really telling us is that the moduli problem of interest (relative degree zero line bundles) is not parametrized by a space but by the stack $\mathfrak{X}$, so thus our induced Fourier-Mukai transform should be thought of as a map from $D^b(X)$ to $D^b(\mathfrak{X})$. The stacks $\mathfrak{X}$ over X which arise in the above dualities are $\mathscr{O}_X^*$-gerbes and are parametrized by the cohomology class $H^2(X, \mathscr{O}_X^*)$.

4.2.1 Cohomological Fourier-Mukai

Since we are interested in the induced map on characteristic classes, we make the following definition:

Definition 4.2.1. *Let $\Phi_{X\to Y}^{\mathscr{E}^\bullet}$ be a Fourier-Mukai transform. Then the cohomological Fourier-Mukai transform $\mathbf{fm}_{\mathscr{E}^\bullet}$ is defined on the subspace of $V_X \subseteq H^\bullet(X, \mathbb{Q})$ lying in the image of the Chern character. It is uniquely defined by making the following diagram commute:*

$$\begin{tikzcd} D^b(X) \arrow[r, "\Phi_{X\to Y}^{\mathscr{E}^\bullet}"] \arrow[d, "ch"] & D^b(Y) \arrow[d, "ch"] \\ V_X \arrow[r, "\mathbf{fm}_{\mathscr{E}^\bullet}"] & V_Y \end{tikzcd}$$

Remark 4.2.2. Typically one adjusts the morphism to the cohomology ring by the (invertible) square root of the Todd class (this adjustment is known as the Mukai vector) in order to make the cohomological Fourer-Mukai morphism more directly mirror that of the Fourier-Mukai morphism itself. We do not use this convention.

4.2.2 Elliptic Fibrations Without Section

Given a smooth elliptic fibration $f : X \to B$ with section σ, one has the sheaf of sections $\mathcal{X}$ of f taking values in abelian groups. One can thus define the **Tate-**

Shafarevich group:

$$Ш_B(X) = H^1(B, \mathcal{X})$$

This sheaf cohomology group describes twisted versions of the original fibration, some of which may not have sections. The above group exists in more general situations - for a detailed reference see Dolgachev and Gross [11]. It suffices for our purposes to note that in this situation we have an exact sequence:

$$0 \longrightarrow \mathrm{Br}(B) \longrightarrow \mathrm{Br}(X) \xrightarrow{\pi} Ш_B(X) \longrightarrow 0$$

Here $\mathrm{Br}(X)$ denotes the **Brauer group**, which is a subgroup of $H^2(X, \mathscr{O}_X^*)$. Such classes correspond to equivalence classes of gerbes. A theorem of Căldăraru (Theorem 4.4.1 of [8]) shows that the above morphism π directly relates gerbes arising as obstructions to the existence of global Poincaré sheaves to twisted copies of the fibration $f : X \to B$. More precisely, if $g : Y \to B$ is a twisted version of f, then if one takes the obstruction $\alpha \in \mathrm{Br}(X)$ to the existence of a Poincaré sheaf on $Y \times_B X$, then $\pi(\alpha) = [g] \in Ш_B(X)$. In particular, if $\mathrm{Br}(B) = 0$ we have that the orders of α and $[g]$ are equal. The order of α corresponds to the smallest k such that we have an α^k-twisted line bundle, while the order of $[g]$ corresponds to the minimal degree of a multisection of the fibration $[g]$. Further, if $\alpha^k = 1$ then we can represent the gerbe by a cocycle $\tilde{\alpha} \in H^2(X, \mu_k)$, and thus represent the gerbe as a Deligne-Mumford stack. Here we are using the long exact sequence coming from:

$$0 \longrightarrow \mu_k \longrightarrow \mathscr{O}_X^* \xrightarrow{(\_)^k} \mathscr{O}_X^* \longrightarrow 0$$

Thus the minimal degree of a multisection of a twisted fibration will also tell us the minimal order of the cyclic group G such that we can describe the image of the transform as lying on a G-gerbe. This is important since we are interested in applying Riemann-Roch, thus we want to have an inertia stack with finitely many connected components. We will be interested in a fibration which admits no section but admits a multisection of degree 2. By the above arguments, such a transform can take values in a μ_2-gerbe over an elliptic fibration.

Chapter 5

Twisted Anomaly Cancellation for Gerbes

We now set our sights on comparing the anomaly cancellation conditions at the level of Chern-reps. Let us give an outline of how we will proceed. We begin by describing a smooth K3 surface X which contains two distinct genus one fibrations which we will call $\rho_i : X \to \mathbb{P}^1$ for $i = 1, 2$. Of these, only one will have a section, say ρ_1. We use the fibration with a section to transform an omalous bundle V over X to spectral data on X, and then discuss how one could transform this spectral data along ρ_2 to obtain a vector bundle over some associated gerbe solving the relative moduli problem along the fibers of ρ_2.

5.1 Construction of X

First, we recall some properties of the relevant space X constructed in section 4.3 of [7]. The space X is K3 with two genus one fibrations ρ_1 and ρ_2, with only ρ_1 having a section. The fibration ρ_2 has two I_0^* fibers. The double component of each I_0^* is a section for the ρ_1 fibration. The ρ_1 fibration has 8 singular fibers of type I_2. The two sections of the ρ_1 fibration intersect two different components of each of the I_2 fibers.

5.2 Construction of the Spectral Data

Now given X as constructed above, we seek to find a curve $\iota : C \hookrightarrow X$ and a line bundle $\mathscr{N} \in \mathrm{Pic}(C)$ such that the sheaf $\iota_* \mathscr{N}$ is Fourier-Mukai dual (with respect to the elliptic fibration ρ_1) to a vector bundle V over X satisfying the anomaly cancellation conditions:

$$\mathrm{c}_1(V) = \mathrm{c}_1(TX) \qquad \mathrm{ch}_2(V) = \mathrm{ch}_2(TX)$$

As X is K3, we have that $\mathrm{c}_1(TX) = 0$ and $\mathrm{ch}_2(TX) = -\mathrm{c}_2(TX) = -24$ (where we identify $H^4(X, \mathbb{Z}) \cong \mathbb{Z}$ via the fundamental class of X). Denote by $\mathbf{FM}_{\rho_1} := \Phi^{\mathscr{P}}_{X \to X}$ the induced Fourer-Mukai transform. We will work in the following steps:

1. Compute $\mathrm{ch}(\iota_* \mathscr{N})$ for a large family of possible choices of spectral data depending on some parameters.

2. Compute the cohomological Fourier-Mukai transform $\mathbf{fm}_{\rho_1}(\eta)$ for a set of cohomology classes $\eta \in H^\bullet(X, \mathbb{Q})$ spanning the image of $\mathrm{ch}(\iota_* \mathscr{N})$.

3. Use the information from step 2 to compute the transform $\mathbf{fm}_{\rho_1}(\mathrm{ch}(\iota_* \mathscr{N})) = \mathrm{ch}(\mathbf{FM}_{\rho_1}(\iota_* \mathscr{N})) = \mathrm{ch}(\mathbf{V})$ and determine for which values of the parameters in step 1 do we have that the bundle V satisfies anomaly cancellation.

5.2.1 Chern Character of Spectral Data (Step 1)

Before we compute the characteristic classes of $\iota_* \mathscr{N}$ with the Grothendieck-Riemann-Roch theorem, we fix some notation.

Notation: Let $NS(X)$ denote the Néron-Severi group of X. Let f denote the class of a fiber of ρ_1 and let σ_1 and σ_2 be the two rational curves making up the two sections of ρ_1 inside the I_0^* fibers of ρ_2. We use the notation $\eta_\alpha \in H^\bullet(X, \mathbb{Q})$ to denote the Poincaré dual of a homology class α of X. In particular, $\eta_f, \eta_{\sigma_1}, \eta_{\sigma_2} \in H^2(X, \mathbb{Q})$ are duals of the homology classes of the respective curves in X, and $\eta_{pt} \in H^4(X, \mathbb{Z})$ is the class of the volume form, and $\eta_X = 1 \in H^0(X, \mathbb{Z})$. On the spectral curve C we denote by $\omega \in H^2(C, \mathbb{Z})$ the volume form of C.

Lemma 5.2.1. *We will choose our spectral curve C to be in the linear system:*

$$C \in |a\sigma_1 + b\sigma_2 + cf| \textit{ For some } a, b, c \in \mathbb{Z}$$

The line bundle $\mathscr{N}$ will be in $Pic^d(C)$ for some d. Then we have that:

$$ch(\iota_*\mathscr{N}) = a\eta_{\sigma_1} + b\eta_{\sigma_2} + c\eta_f + [d + \frac{1}{2}\chi(C)]\eta_{pt}$$

Proof. The Grothendieck-Riemann-Roch theorem applied to the sheaf $\mathscr{N}$ and the morphism $\iota : C \hookrightarrow X$ yields:

$$\iota_*\left(\mathrm{ch}(\mathscr{N}) \cdot \mathrm{td}(C)\right) = \mathrm{ch}\left(\iota_!\mathscr{N}\right) \cdot \mathrm{td}(X) \tag{5.2.1}$$

Now we have that:

$$\mathrm{td}(X) = 1 + \frac{1}{2}\mathrm{c}_1(TX) + \frac{1}{12}\left[\mathrm{c}_1^2(TX) + \mathrm{c}_2(TX)\right] = 1 + 2\eta_{pt}$$

And therefore:

$$\mathrm{td}(X)^{-1} = 1 - 2\eta_{pt}$$

Also, we have:

$$\mathrm{td}(C) = 1 + \frac{1}{2}\mathrm{c}_1(TC) = 1 + \frac{1}{2}\chi(C)\omega \ \text{ and } \ \mathrm{ch}(\mathscr{N}) = 1 + d\omega$$

Since ι is an affine morphism we have that:

$$\iota_!(\mathscr{N}) = \Sigma_{i\geq 0}(-1)^i\mathbb{R}^i\iota_*\mathscr{N} = \iota_*\mathscr{N}$$

Substituting these equalities in to 5.2.1 and solving for $\mathrm{ch}(\iota_*\mathscr{N})$ gives us:

$$\begin{aligned}\mathrm{ch}(\iota_*\mathscr{N}) &= \iota_*\left([1+d\omega]\cdot[1+\frac{1}{2}\chi(C)\omega]\right)\cdot(1-2\eta_{pt})\\ &= \iota_*\left(1+[d+\frac{1}{2}\chi(C)]\omega\right)\cdot(1-2\eta_{pt})\\ &= \left(\eta_C+[d+\frac{1}{2}\chi(C)]\eta_{pt}\right)\cdot(1-2\eta_{pt})\\ &= \eta_C+[d+\frac{1}{2}\chi(C)]\eta_{pt}\\ &= a\eta_{\sigma_1}+b\eta_{\sigma_2}+c\eta_f+[d+\frac{1}{2}\chi(C)]\eta_{pt}\end{aligned}$$

5.2.2 Cohomological Fourier-Mukai Transform (Step 2)

We now compute the cohomological Fourier-Mukai for the classes of interest. Let σ_1 and σ_2 also denote the morphisms $\sigma_i : \mathbb{P}^1 \to X$. We begin with the following:

Lemma 5.2.2. *The Chern characters of the following classes are:*

$$ch(\sigma_{i*}\mathscr{O}_{\mathbb{P}^1}) = \eta_{\sigma_i}+\eta_{pt} \ (for\ i=1,2)$$

$$ch(\sigma_{1*}\mathscr{O}_t)=\eta_{pt} \qquad ch(\rho_1^*\mathscr{O}_t)=\eta_f \qquad ch(\mathscr{O}_X)=\eta_X$$

Proof. The first follows from the short exact sequence:

$$0 \longrightarrow \mathscr{O}_X(-\sigma_i) \longrightarrow \mathscr{O}_X \longrightarrow \sigma_{i*}\mathscr{O}_{\mathbb{P}^1} \longrightarrow 0$$

Then we have that:

$$\begin{aligned}
\mathrm{ch}(\sigma_{i*}\mathscr{O}_{\mathbb{P}^1}) &= \mathrm{ch}(\mathscr{O}_X) - \mathrm{ch}(\mathscr{O}_X(-\sigma_i)) \\
&= 1 - (1 - \eta_{\sigma_i} + \frac{1}{2}\eta_{\sigma_i}^2) \\
&= \eta_{\sigma_i} + \eta_{pt}
\end{aligned}$$

The last equality is justified as follows. Since the cup product is dual to the intersection product and then using the adjunction formula we have: $\eta_{\sigma_i}^2 = (\sigma_i \,.\, \sigma_i)\eta_{pt} = -\chi(\sigma_i)\eta_{pt} = -2\eta_{pt}$.

Since we have $(t \in \mathbb{P}^1)$ $\rho_1^*\mathscr{O}_t = \mathscr{O}_f$, the short exact sequence of the divisor f yields:

$$\begin{aligned}
\mathrm{ch}(\rho_1^*\mathscr{O}_t) = \mathrm{ch}(\mathscr{O}_f) &= \mathrm{ch}(\mathscr{O}_X) - \mathrm{ch}(\mathscr{O}_X(-f)) \\
&= 1 - \left(1 - \eta_f + \frac{1}{2}\eta_f^2\right) \\
&= \eta_f
\end{aligned}$$

To compute $\mathrm{ch}(\sigma_{1*}\mathscr{O}_t)$ we push forward the exact sequence via σ_{1*}:

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^1}(-1) \longrightarrow \mathscr{O}_{\mathbb{P}^1} \longrightarrow \mathscr{O}_t \longrightarrow 0$$

As the morphism σ_1 is affine, we preserve exactness after pushing forward and deduce that:

$$\mathrm{ch}(\sigma_{1*}\mathscr{O}_t) = \mathrm{ch}(\sigma_{1*}\mathscr{O}_{\mathbb{P}^1}) - \mathrm{ch}(\sigma_{1*}\mathscr{O}_{\mathbb{P}^1}(-1))$$

Since $\sigma_{1*}\mathscr{O}_{\mathbb{P}^1}(-1) \cong \sigma_{1*}{\sigma_1}^*\mathscr{O}_X(-f) \cong \sigma_{1*}\mathscr{O}_{\mathbb{P}^1} \otimes \mathscr{O}_X(-f)$ from the projection formula, we thus have:

$$\begin{aligned}
\operatorname{ch}(\sigma_{1*}\mathscr{O}_t) &= \operatorname{ch}(\sigma_{1*}\mathscr{O}_{\mathbb{P}^1})\,(1 - \operatorname{ch}(\mathscr{O}_X(-f))) \\
&= (\eta_{\sigma_1} + \eta_{pt})\left(1 - [1 - \eta_f - \frac{1}{2}\eta_f^2]\right) \\
&= (\sigma_1.f)\eta_{pt} \\
&= \eta_{pt}
\end{aligned}$$

$\square$

Now we have a series of sheaves on X which can generate any of the possible Chern characters of our spectral data. Notice that all of these sheaves are either a pushforward of a sheaf from a section of the fibration ρ_1 or the pullback of a sheaf via ρ_1. Our next objective is to see how the Fourier-Mukai transform maps sheaves of these two types. Let $\mathscr{P} := \mathscr{O}_{X\times_{\mathbb{P}^1}X}(\Delta - [\sigma_1 \times X] - [X \times \sigma_1] - 2F)$ denote the normalized Poincaré sheaf relative to the ρ_1 fibration and using σ_1 as the identity section, where $F := \xi^{-1}(pt)$ where $\xi : X \times_{\mathbb{P}^1} X \to \mathbb{P}^1$ is the projection. This factor of $2F$ is the normalization factor which imposes the conditions that $\mathscr{P}|_{X\times_{\mathbb{P}^1}\sigma} \cong \mathscr{O}$ and $\mathscr{P}|_{\sigma\times_{\mathbb{P}^1}X} \cong \mathscr{O}$, in turn making the isomorphism class of such a $\mathscr{P}$ unique.

Theorem 5.2.3. *Let $\mathscr{F}$ be a sheaf on X. Then we have that:*

$$\mathbf{FM}_{\rho_1}(\rho_1^*\mathscr{F}) = \sigma_{1*}(\mathscr{F} \otimes \mathscr{O}_{\mathbb{P}^1}(-2))[-1]$$

Proof. By definition we have that:

$$\begin{aligned}
\mathbf{FM}_{\rho_1}(\rho_1^*\mathscr{F}) &= \mathbb{R}\pi_{2*}(\pi_1^*\rho_1^*\mathscr{F} \otimes \mathscr{P}) \\
&\cong \mathbb{R}\pi_{2*}(\pi_2^*\rho_1^*\mathscr{F} \otimes \mathscr{P}) \\
&\cong \rho_1^*\mathscr{F} \otimes^{\mathbb{L}} \mathbb{R}\pi_{2*}\mathscr{P}
\end{aligned}$$

Here we used the projection formula and the commutativity of:

$$\begin{tikzcd}
 & X \times_{\mathbb{P}^1} X \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
X \arrow[dr, "\rho_1"'] & & X \arrow[dl, "\rho_1"] \\
 & \mathbb{P}^1 &
\end{tikzcd}$$

Now for any $p \in X$ consider the base change morphism induced from the following cartesian diagram:

$$\begin{tikzcd}
X \times_{\mathbb{P}^1} \{p\} \arrow[r, hook, "\iota_{X_p}"] \arrow[d, "\rho_1"] & X \times_{\mathbb{P}^1} X \arrow[d, "\pi_2"] \\
\{p\} \arrow[r, hook, "\iota_p"] & X
\end{tikzcd}$$

For each i we have the morphism:

$$\mathbb{R}^i\pi_{2*}\mathscr{P} \otimes \kappa(p) \to H^i(X_p, (\iota_{X_p})^*\mathscr{P})$$

We have $(\iota_{X_p})^*\mathscr{P} \cong \mathscr{O}_{X_p}((\sigma_1 \circ \rho_1)(p) - p)$ which has nonzero cohomology if and only if $p \in \sigma_1$. Thus, via [16] theorem 12.11 followed by Nakayama's lemma we have that the sheaves $\mathbb{R}^i\pi_{2*}\mathscr{P}$ vanish when restricted to $X - \sigma_1$. In other words, they are supported on σ_1.

Since $\mathscr{P}$ is a torsion-free sheaf, so also is its pushforward $\mathbb{R}^0\pi_{2*}\mathscr{P} = \pi_{2*}\mathscr{P}$. Since $\mathrm{supp}(\mathscr{P}) \subseteq \sigma_1$, it is also torsion, so that $\pi_{2*}\mathscr{P} = 0$. Moreover, because the fibers of π_2 are curves, $\mathbb{R}^i\pi_{2*}\mathscr{P} = 0$ for $i \geq 2$. Thus we have $\mathbb{R}\pi_{2*}\mathscr{P} = \mathbb{R}^1\pi_{2*}\mathscr{P}[-1]$. We can now apply another base change theorem to compute this.

Recall [5] proposition A.85 which states:

Theorem 5.2.4 (Base change in the derived category)**.** *Consider a cartesian diagram of algebraic varieties:*

$$\begin{tikzcd} X \times_Z Y \arrow[r, "v"] \arrow[d, "g"] & X \arrow[d, "f"] \\ Y \arrow[r, "u"] & Z \end{tikzcd}$$

Then for any complex $\mathscr{M}^\bullet$ of $\mathscr{O}_X$-modules the is a natural morphism:

$$\mathbb{L}u^*\mathbb{R}f_*\mathscr{M}^\bullet \to \mathbb{R}g_*\mathbb{L}v^*\mathscr{M}^\bullet$$

Moreover, if $\mathscr{M}^\bullet$ has quasi-coherent cohomology and either f or u is flat, then this is an isomorphism.

Applying this to our current situation and the following cartesian diagram:

$$\begin{tikzcd} X \cong X \times_{\mathbb{P}^1} \sigma_1 \arrow[r, "g"] \arrow[d, "\rho_1"] & X \times_{\mathbb{P}^1} X \arrow[d, "\pi_2"] \\ \mathbb{P}^1 \arrow[r, "\sigma_1"] & X \end{tikzcd}$$

Since π_2 is flat, the induced map:

$$\mathbb{L}\sigma_1^*\mathbb{R}\pi_{2*}\mathscr{P} \to \mathbb{R}\rho_{1*}\mathbb{L}g^*\mathscr{P} \tag{5.2.2}$$

is an isomorphism. The lefthand side is:

$$\mathbb{L}\sigma_1^*(\mathbb{R}^1\pi_{2*}\mathscr{P})[-1]$$

Here we have used the vanishing of the other derived functors discussed above. Because of the normalization condition on $\mathscr{P}$, the righthand side of 5.2.2 becomes:

$$\mathbb{R}\rho_{1*}\mathbb{L}g^*\mathscr{P} \cong \mathbb{R}\rho_{1*}\mathscr{O}_X$$

Further from relative duality we have that:

$$\mathbb{R}^1\rho_{1*}\mathscr{O}_X \cong (\rho_{1*}\omega_{X/\mathbb{P}^1})^\vee \cong (\rho_{1*}[\omega_X \otimes \rho_1^*\omega_{\mathbb{P}^1}^\vee])^\vee \cong (\rho_{1*}\rho_1^*\mathscr{O}_{\mathbb{P}^1}(2))^\vee \cong \mathscr{O}_{\mathbb{P}^1}(-2)$$

From here, taking cohomology objects at degree 1 in 5.2.2 we obtain:

$$\mathcal{H}^1(\mathbb{L}\sigma_1^*\mathbb{R}^1\pi_{1*}\mathscr{P}[-1]) \cong \mathcal{H}^1(\mathbb{R}\rho_{1*}\mathscr{O}_X)$$

$$\mathcal{H}^0(\mathbb{L}\sigma_1^*\mathbb{R}^1\pi_{1*}\mathscr{P}) \cong \mathbb{R}^1\rho_{1*}\mathscr{O}_X$$

$$\sigma_1^*\mathbb{R}^1\pi_{1*}\mathscr{P} \cong \mathscr{O}_{\mathbb{P}^1}(-2)$$

Now we can deduce the following after pushing the above forward via σ_1:

$$\begin{aligned}\mathbf{FM}_{\rho_1}(\rho_1^*\mathscr{F}) &\cong \left(\rho_1^*\mathscr{F} \otimes^{\mathbb{L}} \sigma_{1*}\mathscr{O}_{\mathbb{P}^1}(-2)\right)[-1] \\ &\cong \mathbb{R}\sigma_{1*}\left(\mathbb{L}\sigma_1^*\rho_1^*\mathscr{F} \otimes^{\mathbb{L}} \mathscr{O}_{\mathbb{P}^1}(-2)\right)[-1] \\ &\cong \sigma_{1*}\left(\mathscr{F} \otimes \mathscr{O}_{\mathbb{P}^1}(-2)\right)[-1]\end{aligned}$$

Where here we have used that σ_1 is an affine morphism, $\rho_1 \circ \sigma_1 = \mathrm{id}_{\mathbb{P}^1}$, and that $\mathscr{O}_{\mathbb{P}^1}(-2)$ is locally free to remove the derived functors. $\square$

Now that we have this result, let us consider sheaves which are pushed forward to X via some section of ρ_1.

Theorem 5.2.5. *Let $s : \mathbb{P}^1 \to X$ be a section of ρ_1. Then we have that:*

$$\mathbf{FM}_{\rho_1}(s_*\mathscr{F}) = \rho_1^*\mathscr{F} \otimes \mathscr{O}_X(s - \sigma_1) \otimes \rho_1^*\mathscr{O}_{\mathbb{P}^1}(-s.\sigma_1 - 2)$$

Proof. By definition we have that $\mathbf{FM}_{\rho_1}(s_*\mathscr{F}) = \mathbb{R}\pi_{2*}\left(\pi_1^* s_*\mathscr{F} \otimes \mathscr{P}\right)$. Since $\mathrm{supp}(s_*\mathscr{F}) \subseteq s$, we have that $\mathrm{supp}(\pi_1^* s_*\mathscr{F}) \subseteq s \times_{\mathbb{P}^1} X$. Tensoring with $\mathscr{P}$ will not change the support, so we have $\mathrm{supp}(\pi_1^* s_*\mathscr{F} \otimes \mathscr{P}) \subseteq s \times_{\mathbb{P}^1} X$. Consider now the following diagram:

$$X \cong s \times_{\mathbb{P}^1} X \xhookrightarrow{\iota} X \times_{\mathbb{P}^1} X \xrightarrow{\pi_1} X$$

For any sheaf $\mathscr{G}$ supported on $s \times_{\mathbb{P}^1} X$, we have an isomorphism $\mathscr{G} \to \iota_*\iota^*\mathscr{G}$, so we may apply this to the above sheaf. So we have:

$$\begin{aligned}
\mathbf{FM}_{\rho_1}(\mathscr{F}) &\cong \mathbb{R}\pi_{2*}\iota_*\iota^*\left(\pi_1^* s_*\mathscr{F} \otimes \mathscr{P}\right) \\
&\cong \underbrace{\mathbb{R}\pi_{2*}\iota_*}_{\mathrm{id}_*}(\iota^*\pi_1^* s_*\mathscr{F}) \otimes \iota^*\mathscr{P} \\
&\cong s_*\mathscr{F} \otimes \iota^*\mathscr{P}
\end{aligned}$$

Here we have used that $\pi_1 \circ \iota$ is the identity map when restricted to s, thus the pullback of $s_*\mathscr{F}$ via this morphism is itself. Now it suffices to notice that:

$$\begin{aligned}
\iota^*\mathscr{P} &\cong \mathscr{O}_{X \times_{\mathbb{P}^1} X}(\Delta - [\sigma_1 \times X] - [X \times \sigma_1] - 2F)|_{s \times_{\mathbb{P}^1} X} \\
&\cong \mathscr{O}_X(s - (s.\sigma_1)f - \sigma_1 - 2f) \\
&\cong \mathscr{O}_X(s - \sigma_1) \otimes \rho_1^*\mathscr{O}_{\mathbb{P}^1}(-(s.\sigma_1) - 2)
\end{aligned}$$

$\square$

We can now compute the cohomological Fourier-Mukai transform for the classes of interest:

Theorem 5.2.6. *Let X be constructed as above. Let $\{1, \eta_{\sigma_1}, \eta_{\sigma_2}, \eta_f, \eta_{pt}\}$ be an ordered basis for the subspace $V \subset H^{\bullet}(X, \mathbb{Q})$ of interest. Then relative to this basis, the cohomological Fourier-Mukai transform with respect to the ρ_1 fibration takes the form:*

$$\mathbf{fm}_{\rho_1}\begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & -3 & 0 & 1 \\ -1 & 0 & 2 & -1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix}$$

Proof.

$$\begin{aligned}
\mathbf{fm}_{\rho_1}(1) &= \mathbf{fm}_{\rho_1}(\mathrm{ch}(\mathscr{O}_X)) \\
&= \mathrm{ch}(\mathbf{FM}_{\rho_1}(\rho_1^*\mathscr{O}_{\mathbb{P}^1})) \\
&= \mathrm{ch}(\sigma_{1*}(\mathscr{O}_{\mathbb{P}^1} \otimes \mathscr{O}_{\mathbb{P}^1}(-2))[-1]) \qquad \text{(Theorem 5.2.3)} \\
&= \mathrm{ch}(\sigma_{1*}\mathscr{O}_{\mathbb{P}^1}(-2)[-1]) \\
&= -\mathrm{ch}(\sigma_{1*}\mathscr{O}_{\mathbb{P}^1}(-2)) \\
&= -\mathrm{ch}(\sigma_{1*}\sigma_1^*\mathscr{O}_X(-2f)) \qquad \text{(Since } \sigma_1 \,.\, f = 1\text{)} \\
&= -\mathrm{ch}(\mathscr{O}_X(-2f) \otimes \sigma_{1*}\mathscr{O}_{\mathbb{P}^1}) \qquad \text{(Projection formula)} \\
&= -\mathrm{ch}(\mathscr{O}_X(-2f))\mathrm{ch}(\sigma_{1*}\mathscr{O}_{\mathbb{P}^1}) \\
&= -(1-\eta_f)(1-[1-\eta_{\sigma_1}+\frac{1}{2}\eta_{\sigma_1}^2]) \\
&= -\eta_{\sigma_1} - \eta_{pt}
\end{aligned}$$

$$\begin{aligned}
\mathbf{fm}_{\rho_1}(\eta_{pt}) &= \mathbf{fm}_{\rho_1}(\mathrm{ch}(\sigma_{1*}\mathscr{O}_t)) \qquad \text{(Lemma 5.2.2)} \\
&= \mathrm{ch}(\mathbf{FM}_{\rho_1}(\sigma_{1*}\mathscr{O}_t)) \\
&= \mathrm{ch}(\rho_1^*\mathscr{O}_t \otimes \mathscr{O}_X(\sigma_1 - \sigma_1) \otimes \rho_1^*\mathscr{O}_{\mathbb{P}^1}(-\sigma_1^2 - 2)) \qquad \text{(Theorem 5.2.5)} \\
&= \mathrm{ch}(\rho_1^*\mathscr{O}_t) \\
&= \eta_f \qquad \text{(Lemma 5.2.2)}
\end{aligned}$$

$$
\begin{aligned}
\mathbf{fm}_{\rho_1}(\eta_{\sigma_1} + \eta_{pt}) &= \mathbf{fm}_{\rho_1}(\mathrm{ch}(\sigma_{1*}\mathscr{O}_{\mathbb{P}^1})) \qquad \text{(Lemma 5.2.2)} \\
&= \mathrm{ch}(\mathbf{FM}_{\rho_1}(\sigma_{1*}\mathscr{O}_{\mathbb{P}^1})) \\
&= \mathrm{ch}(\rho_1^*\mathscr{O}_{\mathbb{P}^1} \otimes \mathscr{O}_X(\sigma_1 - \sigma_1) \otimes \rho_1^*\mathscr{O}_{\mathbb{P}^1}(-\sigma_1^2 - 2)) \qquad \text{(Theorem 5.2.5)} \\
&= \mathrm{ch}(\rho_1^*\mathscr{O}_{\mathbb{P}^1}) \\
&= \mathrm{ch}(\mathscr{O}_X) \\
&= 1
\end{aligned}
$$

Using the previous two calculations we have that:

$$
\mathbf{fm}_{\rho_1}(\eta_{\sigma_1}) = \mathbf{fm}_{\rho_1}(\eta_{\sigma_1} + \eta_{pt}) - \mathbf{fm}_{\rho_1}(\eta_{pt}) = 1 - \eta_f
$$

$$
\begin{aligned}
\mathbf{fm}_{\rho_1}(\eta_{\sigma_2} + \eta_{pt}) &= \mathbf{fm}_{\rho_1}(\mathrm{ch}(\sigma_{2*}\mathscr{O}_{\mathbb{P}^1})) \qquad \text{(Lemma 5.2.2)} \\
&= \mathrm{ch}(\mathbf{FM}_{\rho_1}(\sigma_{2*}\mathscr{O}_{\mathbb{P}^1})) \\
&= \mathrm{ch}(\rho_1^*\mathscr{O}_{\mathbb{P}^1} \otimes \mathscr{O}_X(\sigma_2 - \sigma_1) \otimes \rho_1^*\mathscr{O}_{\mathbb{P}^1}(-\sigma_2 \,.\, \sigma_1 - 2)) \qquad \text{(Theorem 5.2.5)} \\
&= \mathrm{ch}(\mathscr{O}_X(\sigma_2) \otimes \mathscr{O}_X(-\sigma_1) \otimes \mathscr{O}_X(-2f)) \\
&= (1 + \eta_{\sigma_2} + \frac{1}{2}\eta_{\sigma_2}^2)(1 - \eta_{\sigma_1} + \frac{1}{2}\eta_{\sigma_1}^2)(1 - 2\eta_f) \\
&= 1 - \eta_{\sigma_1} - \eta_{\sigma_2} - 2\eta_f + 2\eta_{pt}
\end{aligned}
$$

Using the above calculation we have that:

$$
\mathbf{fm}_{\rho_1}(\eta_{\sigma_2}) = \mathbf{fm}_{\rho_1}(\eta_{\sigma_2} + \eta_{pt}) - \mathbf{fm}_{\rho_1}(\eta_{pt}) = 1 - \eta_{\sigma_1} - \eta_{\sigma_2} - 3\eta_f + 2\eta_{pt}
$$

$$
\begin{aligned}
\mathbf{fm}_{\rho_1}(\eta_f) &= \mathbf{fm}_{\rho_1}(\mathrm{ch}(\rho_1^*\mathscr{O}_t)) \\
&= \mathrm{ch}(\mathbf{FM}_{\rho_1}(\rho_1^*\mathscr{O}_t)) \\
&= \mathrm{ch}(\sigma_{1*}(\mathscr{O}_t \otimes \mathscr{O}_{\mathbb{P}^1}(-2))[-1]) \qquad \text{(Theorem 5.2.3)} \\
&= -\mathrm{ch}(\sigma_{1*}\mathscr{O}_t) \\
&= -\eta_{pt} \qquad \text{(Lemma 5.2.2)}
\end{aligned}
$$

□

5.2.3 Describing the Parameters (Step 3)

Using the computations of the cohomological Fourier-Mukai transform in theorem 5.2.6, we deduce the following simple corollary:

Corollary 5.2.7. *Let $\iota : C \to X$ be any curve in the linear system $|r\sigma_1 + 24f|$ for $r \geq 0$ and choose any $\mathscr{N} \in Pic^{25r-r^2}(C)$. Then the Fourier-Mukai transform $\mathbf{FM}_{\rho_1}(\iota_*\mathscr{N}) = \mathscr{E}$ is a vector bundle of rank r on X satisfying the anomaly cancellation conditions.*

Proof. Suppose more generally that C is a curve in the linear system $|a\sigma_1+b\sigma_2+cf|$, and $\mathscr{N} \in \mathrm{Pic}^d(C)$. By theorem 5.2.1, in the ordered basis of theorem 5.2.6 we have that $\mathrm{ch}(\iota_*\mathscr{N}) = (0, a, b, c, d + \frac{1}{2}\chi(C))$. We apply the cohomological Fourier-Mukai transform by using the matrix in 5.2.6, and set this equal to the vector $(r, 0, 0, 0, -24)$ (as X is K3, so $\mathrm{c}_1(X) = 0$, $\mathrm{ch}_2(X) = -24\eta_{pt}$, and the rank can be arbitrary):

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & -3 & 0 & 1 \\ -1 & 0 & 2 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ a \\ b \\ c \\ d + \frac{1}{2}\chi(C) \end{pmatrix} = \begin{pmatrix} r \\ 0 \\ 0 \\ 0 \\ -24 \end{pmatrix}$$

From this system of equations, one deduces that $a = r$, $b = 0$, $c = 24$, and

$$-r + d + \frac{1}{2}\chi(C) = 0$$

For this last equation, consider:

$$\begin{aligned} \chi(C) &= -C \,.\, C \qquad (X \text{ is K3}) \\ &= -(r\sigma_1 + 24f)^2 \\ &= -(r^2\sigma_1^2 + 48r\sigma_1 \,.\, f) \\ &= -(-2r^2 + 48r) \\ &= 2r^2 - 48r \end{aligned}$$

Plugging this in to the equation for d above, we obtain:

$$d = 25r - r^2$$

$\square$

5.3 Twisted Transform

Now theoretically one can take the spectral data constructed in the previous section and transform relative to the genus one fibration $\rho_2 : X \to \mathbb{P}^1$ without a section to obtain a vector bundle over the dual gerbe. To do so, we would want to follow the construction of [13]. We recall the following theorem:

Theorem 5.3.1 (Donagi-Pantev)**.** *Suppose $f : J \to B \cong \mathbb{P}^1$ is a non-isotrivial elliptic fibration with a section on a smooth complex surface J. Assume that f has I_1 fibers at worst. Let $\alpha, \beta \in Ш_B(J)$ be two elements such that β is torsion. Then there is an equivalence*

$$\mathbf{FM} : D^b_1({}_\alpha J_\beta) \to D^b_1({}_{-\beta} J_\alpha)$$

of the derived category of weight 1 coherent sheaves on the gerbe ${}_\alpha J_\beta$ over J_β and the derived category of weight 1 coherent sheaves on the gerbe ${}_{-\beta} J_\alpha$ over J_α.

For the precise construction of the gerbes see [13]. It suffices to note that given a class $\delta \in Ш_B(J)$ that is n-torsion, then any gerbe of the form ${}_\delta J_\alpha$ will correspond to an n-torsion class in $H^2(J_\alpha, \mathscr{O}^*_{J_\alpha})$, and therefore some gerbe over J_α. Thus, the gerbe comes from a class in $H^2(J_\alpha, \mu_n)$ under the inclusion of $\mu_n \hookrightarrow \mathscr{O}^*_{J_\alpha}$.

In our situation we have slightly worse singular fibers, so the construction of the above gerbes would have to be extended. Our fibration $\rho_2 : X \to \mathbb{P}^1$ corresponds to a 2-torsion class $\beta \in Ш_{\mathbb{P}^1}(J)$. If we assume the above construction can be extended to our situation, then we could apply it to the case where $\alpha = 0$ and β defines ρ_2.

What we would obtain is a Fourier-Mukai equivalence between weight one sheaves on a trivial gerbe ${}_0J_\beta$ over $J_\beta \cong X$ and a weight one sheaves on the nontrivial gerbe ${}_\beta J_0$ over $J_0 \cong J \to \mathbb{P}^1$ (note $\beta = -\beta$). Since the gerbe ${}_0J_\beta$ is trivial, we have an equivalence:

$$D_0^b({}_0J_\beta) \cong D_1^b({}_0J_\beta)$$

From here we deduce the following composition of equivalences will map our spectral data to the desired bundle over a gerbe ${}_\beta J_0$:

$$D^b(X) = D^b(J_\beta) \longrightarrow D_0^b({}_0J_\beta) \longrightarrow D_1^b({}_0J_\beta) \xrightarrow{\mathbf{FM}} D_1^b({}_\beta J_0)$$

Since the target gerbe is induced by a μ_2-gerbe, one can consider the sheaves as lying over a μ_2-gerbe. This allows one to compute the Chern reps, noting that the corresponding inertia stack will just be two copies of the μ_2-gerbe itself. This must be left for future work.

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