On the Role of Fractional Calculus in Electromagnetic Theory

Nader Engheta

Moore School of Electrical Engineering
University of Pennsylvania
Philadelphia, PA 19104, U.S.A
Tel: (215) 898-9777
Fax: (215) 573-2068
E-mail: engheta@pender.ee.upenn.edu

Keywords: Fractional derivative; fractional integral; electromagnetism; fractional multipole

In recent years, we have been interested in exploring the roles and potential applications of fractional calculus in electromagnetic theory, and in finding physical implications and possible utilities of these operators in certain electromagnetic problems [1-5]. Fractional calculus is an area of mathematics that addresses generalization of the mathematical operations of differentiation and integration to arbitrary, general, non-integer orders–orders that can be fractional or even complex (see, e.g., [6-21]).

The idea of non-integer-order differentiation and integration in mathematics, which dates back to as early as the late part of the seventeenth century, has been a subject of interest for many mathematicians in pure and applied mathematics over the years (see, e.g., [6-23]). Fractional derivatives and fractional integrals have interesting mathematical properties that may be utilized in treating certain mathematical problems (see, e.g., [6-23]).

We have applied the concept of fractional derivatives/integrals in several specific electromagnetic problems, and have obtained promising results and ideas that demonstrate that these mathematical operators can be interesting and useful tools in electromagnetic theory [1-5]. In this feature article, first we give a brief review of the general principles, definitions, and several features of fractional derivatives/integrals, and then we review some of our ideas and findings in exploring potential applications of fractional calculus in some electromagnetic problems.

1. What is fractional calculus?

Fractional derivatives, fractional integrals, and their properties are the subject of study in the field of fractional calculus. As is well known, in conventional differentiation and integration the symbols $\frac{d^n f(x)}{dx^n}$ and $\frac{d^{-m} f(x)}{dx^{-m}}$ are used to denote the nth-order derivative and m-fold integration of the function $f(x)$. In this sense, n and m are non-negative integers. What would happen if the order n or m were allowed to accept non-integer real (or even complex) values? How are such fractional derivatives and fractional integrals defined? This is a topic that is dealt with in the field of fractional calculus, and has been investigated by many mathematicians for years [6-22]. (Although the name "fractional calculus" has been used for this field in mathematics, the orders of integration and differentiation may, in general, also be irrational numbers). Fractional derivatives and integrals, which are also called "fractional differintegrals" by Oldham and Spanier [8], are shown symbolically by ${}_aD_x^\alpha f(x)$ (after Davis [12]), or $\frac{d^\alpha f(x)}{d(x-a)^\alpha}$ (after Oldham and Spanier [8]), where α is the general order of the operator (not necessarily a positive or negative integer), and a is the lower limit of the integrals used to define these operators. Some of these definitions and properties of fractional differintegrals are briefly reviewed below. The details of definitions of such operators can be found in several references, e.g., [8-12].

One of the definitions of fractional integrals is the one known as the Riemann-Liouville integral [8, p. 49]. It is written as

$${}_aD_x^\alpha f(x) \equiv \frac{1}{\Gamma(-\alpha)} \int_a^x (x-u)^{-\alpha-1} f(u)du \qquad (1)$$

$$\text{for } \alpha < 0 \text{ and } x > a,$$

where ${}_aD_x^\alpha$ denotes the (fractional) αth-order integration of the function $f(x)$, with the lower limit of integration being a, and $\Gamma(\bullet)$ is the Gamma function. This definition is a generalization of Cauchy's repeated-integration formula [8, p. 38]. That is, for $\alpha = -n$, the above definition results in

$${}_aD_x^{-n} f(x) = \frac{1}{(n-1)!} \int_a^x (x-u)^{n-1} f(u)du$$

which, according to Cauchy's repeated-integration formula, is equivalent to

$$\begin{aligned} {}_aD_x^{-n} f(x) &= \frac{1}{(n-1)!} \int_a^x (x-u)^{n-1} f(u)du \\ &= \int_a^x dx_{n-1} \int_a^{x_{n-1}} dx_{n-2} \dots \int_a^{x_1} f(x_0)dx_0 . \end{aligned} \qquad (2)$$

For fractional derivatives with $\alpha > 0$, the above Riemann-Liouville fractional-integration definition can still be applied, if used in conjunction with the following additional step: ${}_aD_x^\alpha f(x) = \frac{d^m}{dx^m}\, {}_aD_x^{\alpha-m} f(x)$ for $\alpha > 0$, where m is chosen so that $(\alpha - m) < 0$, and thus the Riemann-Liouville integration can be applied for ${}_aD_x^{\alpha-m} f(x)$ [8, p. 50]. Then, $\frac{d^m}{dx^m}$ is the ordinary mth-order differential operator [8, p. 50].

Another definition is given for the power series with fractional powers. Considers a function of one variable, $h(x)$, defined over a closed interval $[a,b]$. Let it be represented as the following power series: $h(x)=\sum_{i=0}^{\infty} A_i (x-a)^{v+i/n}$ for $v>-1$ and $a \le x \le b$, where n is a positive integer [8, p. 46]. The condition $v>-1$ is needed in order to have $h(x)$ be in the class of functions that can be differintegrated [8, p. 46]. Each term in this series may have a non-integer power. Now if one considers one of these terms, e.g., $(x-a)^p$, with the non-integer exponent denoted by p, where $p>-1$, according to Riemann ([8, p. 53], [7, p. 366]) a fractional αth-order derivative for such a term can be written as

$$ {}_aD_x^{\alpha}(x-a)^p \equiv \frac{d^{\alpha}(x-a)^p}{d(x-a)^{\alpha}} = \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)}(x-a)^{p-\alpha} \quad \text{for } x>\alpha, \tag{3} $$

where α may be a non-integer number ([7, 8]). (Riemann considered the case of $a=0$ [8, p. 53], [7, p. 366]). It can be easily observed that if α is a positive integer m, one gets the conventional definition for the mth-order derivative of $(x-a)^p$.

Fractional differintegration was also defined by Liouville for functions that can be expanded in a series of exponentials. When a function, $g(x)$, is expressible as $g(x)=\sum_{i=0}^{\infty} c_i e^{u_i x}$, according to Liouville ([6], [8, p. 53]) the αth-order differintegration (with lower limit $a=-\infty$) can be given as

$$ {}_{-\infty}D_x^{\alpha} g(x) \equiv \frac{d^{\alpha}g(x)}{dx^{\alpha}} \equiv \sum_{i=0}^{\infty} c_i u_i^{\alpha} e^{u_i x}. \tag{4} $$

There are several other equivalent definitions for fractional derivatives and integrals which can be found in various references, see, e.g., [8, Ch. 3].

Some of the features of fractional derivatives and integrals are briefly reviewed below.

1.1 Fractional operators are linear

Fractional operators are linear operators (like their integer-order counterparts), and thus they follow the usual properties of linearity, homogeneity, and scaling [8, pages 69 and 75]. That is,

$$ \frac{d^{\alpha}[f_1(x)+f_2(x)]}{dx^{\alpha}} = \frac{d^{\alpha}f_1(x)}{dx^{\alpha}} + \frac{d^{\alpha}f_2(x)}{dx^{\alpha}}, \tag{5} $$

$$ \frac{d^{\alpha}Af(x)}{dx^{\alpha}} = A\frac{d^{\alpha}f(x)}{dx^{\alpha}}, \tag{6} $$

$$ \frac{d^{\alpha}f(Cx)}{dx^{\alpha}} = C^{\alpha}\frac{d^{\alpha}f(Cx)}{d(Cx)^{\alpha}} \quad \text{for lower limit of } a=0. \tag{7} $$

If $a \ne 0$, Equation (7) would require a more complicated translation process [8, p. 75-76].

1.2 Dependence of the fractional derivative

As we mentioned in [1], one of the interesting features of the fractional derivative, as can be seen from Equation (1), is the fact that not only does the fractional derivative of a function evaluated at a point x depend on the value of the function $f(x)$ at the point x and its neighboring points, but it also depends on values of the function between x and the lower-limit point, a. For fractional and conventional integration, this is also true. However, this is not the case for the conventional integer-order derivatives, when α is a positive integer, e.g., $\alpha=m$, for which $\frac{d^m f(x)}{dx^m}$ would only depend on values of the function at x and its neighboring points around x in the limit, as regular derivatives should. What this implies is that in fractional derivatives, properties of differentiation and integration are effectively "mixed." Putting it another way, one can intuitively suggest that a fractional derivative is effectively "influenced" by some integration property. Similarly, the fractional integration has some "mixed" properties of differentiation and integration.

1.3 Fractional derivatives and integrals of a unit step function

In order to gain some insight, and to get a "feel" about fractional differentiation and integration, it is helpful to consider the case of the fractional derivatives and fractional integrals of a unit step function. For the unit step function, $U(x-d)$, where the value of U is unity for $x>d$, and is zero for $x<d$, the fractional differintegration for $\alpha<0$ gives rise [8, p. 105] to

$$ {}_aD_x^{\alpha}U(x-d) = \begin{cases} \frac{1}{\Gamma(1-\alpha)}(x-d)^{-\alpha} & \text{for } a<d<x \\ 0 & \text{for } x<d \end{cases}. \tag{8} $$

For $\alpha>o$, one should use the step ${}_aD_x^{\alpha}f(x)=\frac{d^m}{dx^m}\,{}_aD_x^{\alpha-m}f(x)$, where m is an integer chosen so that $(\alpha-m)<0$. Figure 1 shows a family of plots for fractional derivatives of the unit step function $U(x)$ (when $a=d=0$), for fractional orders α = 0, 0.2, 0.5, 0.8, and 1, and the fractional integral of $U(x)$ for orders α = –0.2, -0.5, –0.8, and –1. From Figure 1, we notice that for α = 0, one gets the original step function; for α = 1, we get the Dirac delta function, $\delta(x)$; and for α = –1, a ramp function is obtained for $x>0$, as expected. However, for fractional orders $0<\alpha<1$, and $-1<\alpha<0$, we obtain the non-zero function $x^{-\alpha}/\Gamma(1-\alpha)$ for $x>0$. So, unlike the conventional derivatives of a constant function being zero, the fractional αth-order derivative of a unit step function for $x>d$ is not zero [8, p. 62]! It is also useful to write the fractional differintegration of the Dirac delta function, $\delta(x-d)$. It is shown in [8, p. 106] that the fractional differintegration of $f(x)=\delta(x-d)$ for $\alpha<0$ can be written as

$$ {}_aD_x^{\alpha}\delta(x-d) = \begin{cases} \frac{1}{\Gamma(-\alpha)}(x-d)^{-\alpha-1} & \text{for } a<d<x \\ 0 & \text{for } x<d \end{cases}. \tag{9} $$

Once again, for $\alpha>0$, one can follow ${}_aD_x^{\alpha}=\frac{d^m}{dx^m}\,{}_aD_x^{\alpha-m}$.

1.4 Special functions

It is important to note that the fractional calculus has also been applied in describing several special functions in mathematical physics in terms of fractional differintegrals of more elementary functions. For instance, the Bessel function $J_v(x)$ of order v

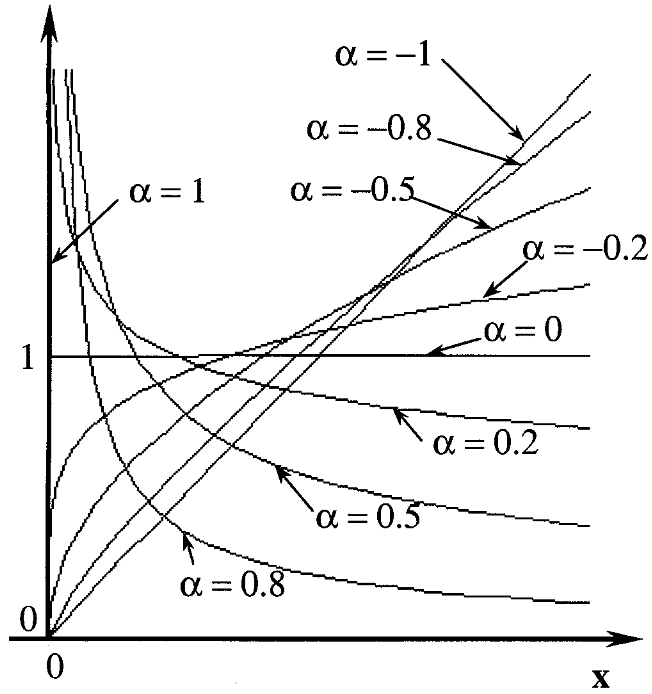


Figure 1. Sketches of fractional derivatives of the unit step function, for fractional orders of α = 0, 0.2, 0.5, 0.8, and 1, and fractional integrals of the unit step function for orders of α = − 0.2, − 0.5, − 0.8, and − 1. As can be seen from this Figure, for α = 0, the original step function, $U(x)$; for α = 1, the Dirac delta function, $\delta(x)$; and for α = − 1, a ramp function, $R(x)$; are obtained for $x > 0$, as expected. However, as is evident from Equation (8), for fractional orders $0 < \alpha < 1$ and $-1 < \alpha < 0$, one obtains the non-zero function $x^{-\alpha}/\Gamma(1-\alpha)$ for $x > 0$. For $x < 0$, the values of the fractional derivatives/integrals of $U(x)$ are zero.

(ν being non-integer, in general) can be written [17, vol. 1, p. 121] in terms of the fractional differintegral of $\cos(x)/x$ as

$$J_\nu(x) = \frac{1}{\sqrt{\pi}(2x)^{-\nu}} \, {}_0D_{x^2}^{-\nu-1/2} \frac{\cos x}{x}, \tag{10}$$

where the subscript x^2 indicates fractional differintegration with respect to the variable x^2. Representation of special functions in this way might, in some cases, introduce novel physical interpretations in the study of physical problems involving such functions.

Fractional calculus has had applications in various topics, such as differential equations, complex analysis, Mellin transforms, and generalized functions, to name a few (see, e.g., [8-19]). Among applications of fractional calculus, one of the earliest appears to be that given by Abel, in 1823, in his study of the tautochrone problem (see the bibliography prepared by Ross, reprinted on pp. 3-15 in [8], also [8, pp. 183-186], and [20, pp. 11-19]). Furthermore, Lützen, in his article [21], has mentioned that Liouville, who was one of the major pioneers of fractional calculus, was probably inspired by the problem of the fundamental force law in Ampère's electrodynamics, and treated a fractional differential equation in that problem. One should also mention Heaviside, who introduced operational calculus, and used fractional derivatives in his work [22, vol. 2, Chs. 6-8]. Scott Blair also used this tool in his work in rheology [23]. For a historical review of the field of fractional calculus, the reader is referred to the excellent bibliography prepared by B. Ross that is reprinted on pp. 3-15 of the monograph by Oldham and Spanier [8], and also to the historical outline given in [9]. In recent years, researchers interested in the field of fractals and related subjects have also begun using the concept of fractional calculus in some of their investigations. Among those, one should mention the work of LeMéhauté, Héliodore and their co-workers ([24, 25]), and Nigmatulin ([26, 27]). Some of their work and the related work of others were presented in a recent summer school/workshop on fractal geometry and fractional derivatives and fractals [28]. Furthermore, researchers in some fields of applied science have used some aspects of fractional derivatives/integrals in their work (see, e.g., [29-33]). It is also important to note that there has been much work conducted on fractionalization of another common operator in mathematics, namely, fractionalization of the Fourier transform, by several researchers, such as Namias [34], Lohmann (see. e.g., [35, 36]), Mendlovic and Ozaktas (see. e.g., [36, 37]), and Shamir and Cohen [38], to name a few. These works, although not directly to the area of fractional derivatives/integrals, show the richness and power of fractionalization of operators.

Our interest in fractional calculus has been particularly focused on finding out what possible applications and/or physical roles these mathematical operators can have in electromagnetic theory. Needless to say, electromagnetics is a field in which the use of conventional (integer-order) calculus plays a major role, and it is of interest to see how fractional calculus may offer useful mathematical tools in this field. It is also of interest to find physical interpretations or intuitive roles that may be attributed to the non-integer order of such differential or integral operators in electromagnetic problems. We have applied the concept of fractional derivatives/integrals to certain electromagnetic problems, and have obtained interesting results and ideas showing that these mathematical operators can be interesting and useful mathematical tools in electromagnetic theory ([1-5]). Some of these ideas include the novel concept of "fractional" multipoles in electromagnetism, electrostatic "fractional" image methods for perfectly conducting wedges and cones, "fractional" solutions for the standard scalar Helmholtz equation, and the mathematical link between the electrostatic image methods for the conducting sphere and the dielectric sphere. Below, a brief review of these problems is given.

2. Fractional multipoles in electromagnetism

Multipole expansion of sources, potentials, and fields is a very well known subject in electromagnetism, and has been studied extensively (e.g., [39-43]). Point multipoles, such as monopoles, dipoles, quadrupoles, octopoles, etc., are well defined sources with specific potential and field distributions. The spatial distributions of these point multipoles can be expressed in terms of the Dirac delta function and its spatial derivatives (see. e.g., [40, Ch. 2], [41-43]). For instance, the charge distribution of a point monopole can be written as $\rho(r) = q\delta(r)$, and that of a point dipole can be expressed as $\rho(\boldsymbol{r}) = -\boldsymbol{p}\cdot\nabla\delta(\boldsymbol{r})$. So, for a point monopole, the Dirac delta function is used, whereas for a point-dipole, the first derivatives of the Dirac delta function are involved. We asked the following questions: since for the point dipole, the first derivatives of $\delta(r)$ are involved, and for a monopole, $\delta(r)$, itself, is used, what will we get if we take the fractional αth-order spatial derivative (with respect to one of the spatial variables, e.g., z) of the Dirac delta function? If α is a fractional value between zero and unity, what form of charge distribution will we obtain? We have shown in [1] that if we start with a charge distribution of a simple monopole, $\rho_1(\boldsymbol{r}) = q\delta(\boldsymbol{r})$, and we take the fractional αth-order derivative (say, with respect to z) of this function, we will obtain the following charge distribution:

$$\rho_{2^\alpha,z}(\boldsymbol{r}) \equiv q\, l^\alpha\, {}_{-\infty}D_z^\alpha \delta(\boldsymbol{r})$$

$$= q\, l^\alpha \frac{\partial^2}{\partial z^2}\begin{bmatrix} \delta(x)\delta(y)\dfrac{1}{\Gamma(2-\alpha)} z^{1-\alpha} & \text{for } z > 0 \\ 0 & \text{for } z < 0 \end{bmatrix}$$

$$= q\, l^\alpha \frac{\partial^2}{\partial z^2}\left[\delta(x)\delta(y)U(z)\frac{1}{\Gamma(2-\alpha)} z^{1-\alpha}\right]. \quad (11)$$

Here, $U(z)$ is the unit step function. The multiplicative constant l^α, where l is an arbitrary constant with physical dimension of length, is introduced to keep the dimensions of $\rho_{2^\alpha,z}(\boldsymbol{r})$ equal to Coulomb/m^3. The subscript z in ${}_{-\infty}D_z^\alpha$ denotes fractional differ-integration with respect to z, with the lower limit $a = -\infty$. The subscript 2^α in $\rho_{2^\alpha,z}(\boldsymbol{r})$ indicates the multipole fractional order of this new charge distribution. This subscript is chosen such that for $\alpha = 0$, we get the original point-monopole $\rho_1(r)$, and for $\alpha = 1$, we obtain the first z derivative of $\delta(r)$. This shows a charge distribution of dipole $\rho_r(r)$ along the z axis. At $z = 0$, the value of $\rho_{2^\alpha,z}(\boldsymbol{r})$ has a singularity. Therefore, strictly speaking, this should be treated as a generalized function. For $0 < \alpha < 1$, as we mentioned earlier in the definition of fractional derivatives, the operator ${}_{-\infty}D_z^\alpha \delta(z)$ can be treated as ${}_{-\infty}D_z^\alpha \delta(z) = \dfrac{\partial^m}{\partial z^m}\, {}_{-\infty}D_z^{\alpha-m}\delta(z)$, where $\alpha - m < 0$. In the above equation, $m = 2$. Then, when evaluating $\dfrac{\partial^m}{\partial z^m}$, care must be exercised at $z = 0$. Figure 2 shows a sketch of the charge distribution of Equation (11). To gain some insights, the two limiting cases of $\alpha = 0$ (i.e., a point monopole $\delta(r)$) and $\alpha = 1$ (i.e., a point dipole $\_(\delta(\mathbf{r}))/\_z$ are also sketched. This 2^α-pole charge distribution is a line charge along the positive z axis, with a volume charge density given in Equation (11). The volume charge densities of these charge distributions are also sketched in Figure 2 as a function of z. This charge distribution, which we have named fractional 2^α-order pole [1], is effectively an "intermediate" charge distribution, between the cases of the point monopole and the point dipole. It approaches these two limiting cases of point monopole (for $\alpha = 0$) and point dipole (for $\alpha = 1$). Several interesting features of this fractional-order pole are given in our work reported in [1]. For the electrostatic case, we analyzed and studied the scalar potential of this charge distribution in [1]. The expression for this scalar electric potential is given by

$$\Phi_{2^\alpha,z}(x,y,z) = \frac{q\, l^\alpha \Gamma(\alpha+1)}{4\pi\varepsilon R^{1+\alpha}} P_\alpha(-\cos\theta), \quad (12)$$

where $P_\alpha(-\cos\theta)$ is the Legendre function of the first kind and the (non-integer) degree α. Figure 3 presents a series of contour plots for this potential in the x-z plane, in the region $-2 \le z \le 2$ and $0 < x \le 2$, for several values of α in the interval $0 \le \alpha \le 1$. As can be seen from Equation (12), this potential drops as $R^{-1-\alpha}$ (with $0 < \alpha < 1$) as R increases. Thus, effectively, this is a potential distribution that can be regarded as the "intermediate" case between the potential of a monopole (which drops as R^{-1}) and the potential of a point dipole (which drops as R^{-2}). Although this electrostatic scalar potential drops as $R^{-1-\alpha}$, it does certainly satisfy Gauss' law, as we have shown in [1]. As described in the caption of Figure 3, this potential of the "intermediate" fractional-order multipole approaches the well-known limiting cases of point monopole and point dipole when α approaches zero and unity, respectively. Some of the interesting features of this potential distribution are detailed in [1].

We have also developed the concept of fractional multipoles in the two-dimensional case. The "intermediate" fractional mul-

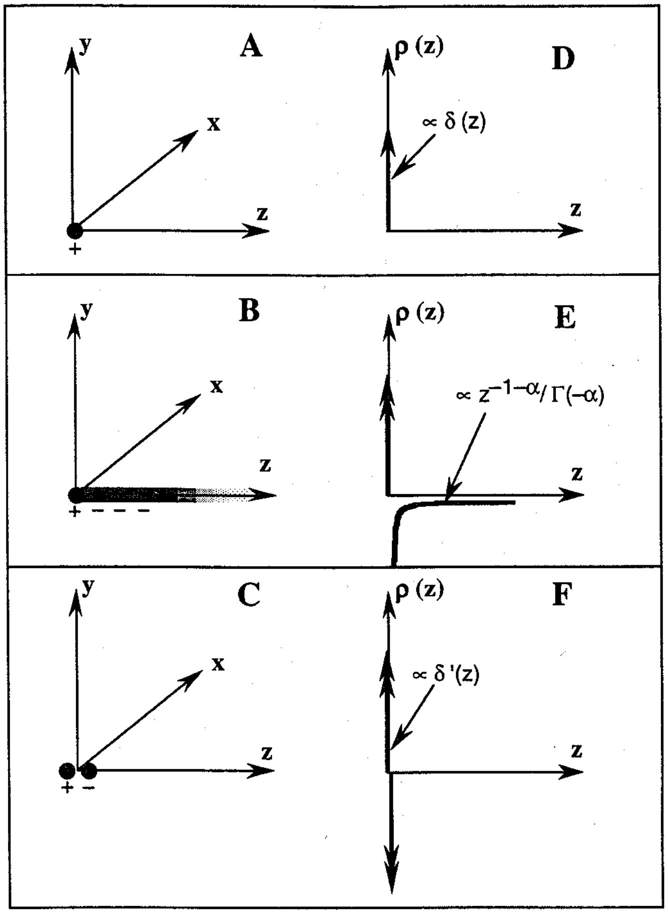


Figure 2 [1]. Sketches of fractional 2^α-pole charge distributions (with $0 \le \alpha \le 1$) are shown in panels A, B, and C. The Cartesian coordinate system (x,y,z) is used. In A and C, an electric point-monopole ($\alpha = 0$) and an electric point-dipole ($\alpha = 1$) are depicted, respectively. In B, a case of a fractional-order 2^α-pole is shown, with $0 < \alpha < 1$. (The αth-order fractional derivative is taken with respect to z). In panels D, E, and F, the forms of the volume charge densities for the charge distributions shown in A, B, and C are illustrated, respectively. These volume charge densities have the form of $\rho(z)\delta(x)\delta(y)$, where the form of the function $\rho(z)$ as a function of z is only shown. In D, $\delta(z)\delta(x)\delta(y)$ corresponds to the point-monopole (in A). In F, $\delta'(z)\delta(x)\delta(y)$ corresponds to the point-dipole (in C). Finally, E shows the form of $\rho(z)$ for the fractional 2^α-pole (shown in B) for a fractional value for α. (Here, for the purpose of plotting $\rho(z)$ for $z > 0$, α is taken to be $\alpha = 1/2$.) In E, at $z = 0$, the expression for Equation (11) has a singularity in the form resulting from $\dfrac{\partial^2}{\partial z^2}\left[U(z)z^{1-\alpha}/\Gamma(2-\alpha)\right]$ at $z = 0$.

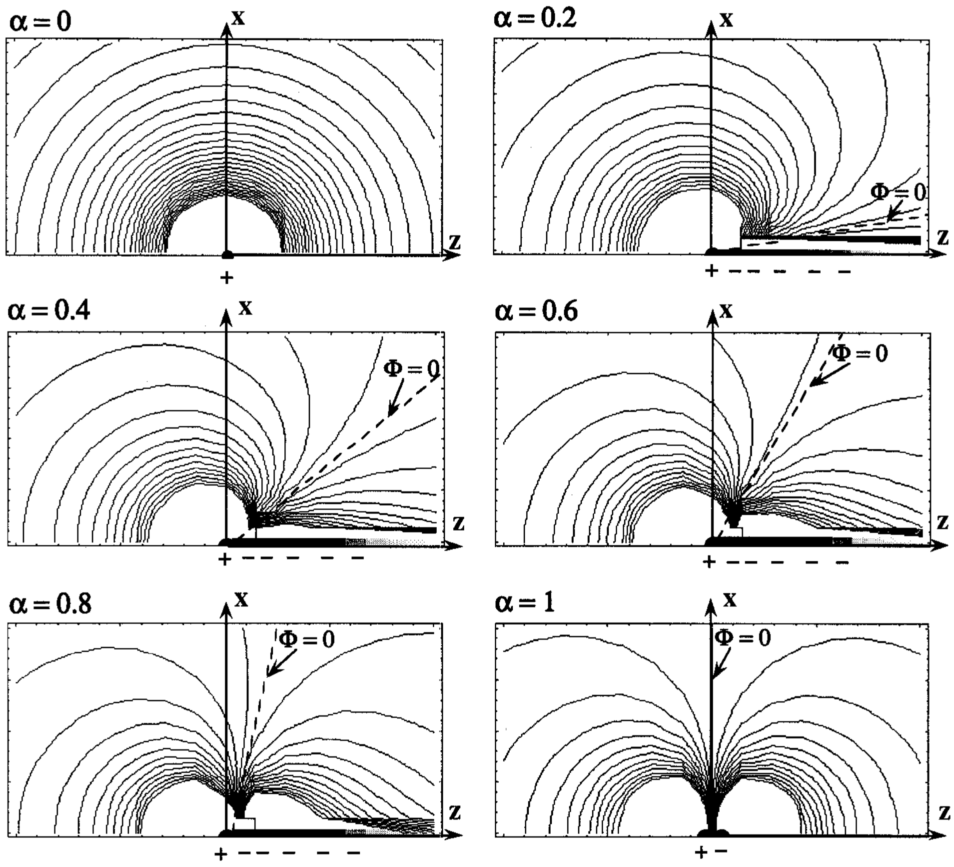


Figure 3 [1]. Contour plots for the electrostatic scalar potential of the fractional 2^{α}-pole charge distribution of Equation (11) for $0 \le \alpha \le 1$. The expression for the potential is given in Equation (12). Since this scalar potential is azimuthally symmetric, the intersection of equipotential surfaces with the *x-z* plane is only shown here. The contours are shown for the region $-2 \le z \le +2$ and $0 < x \le 2$, and for six different values of α. For $\alpha = 0$ (top-left panel), the source is a single electric point-monopole (shown as +), located at $x = y = z = 0$, and the contours are independent of θ. For $\alpha = 1$ (bottom-right panel), the source is an electric point-dipole (shown as + –) located at the origin, and the angular dependence of the potential is $-\cos(\theta)$. For fractional values of α between zero and unity, we can see from the above contour plots that the scalar potentials of fractional 2^{α}-poles are "intermediate" cases between those of the point-monopole and point-dipole, and in a way they "evolve" from one case to the other. The dashed lines, which were added later on top of the contour plots, show the approximate location of zero potential (root of $P_{\alpha}[-\cos(\theta)] = 0$.) In the region of the *x-z* plane shown above, to the left of this dashed line, the potential is positive, and to the right, it is negative. Along the positive *z* axis, the charge distribution is also sketched. The plus and minus signs below the *z* axis indicate the sign of the charge distribution along the *z* axis.

tipoles and their corresponding electrostatic scalar-potential distributions in the two-dimensional case have been analyzed and reported in [1].

After we obtained the electrostatic-potential distributions of the fractional multipoles in both the three- and two-dimensional cases, we noticed the resemblance of these potential distributions to certain sets of solutions of the Laplace equation, for static potential distributions in front of a perfectly conducting cone (compared with the three-dimensional fractional multipoles), and to those in front of a perfectly conducting wedge (compared with the two-dimensional fractional multipoles). This theoretical observation led us to develop the electrostatic "fractional" image methods for perfectly conducting wedges and cones [2]. This will be briefly reviewed in the next section. The interested reader is referred to [2] for further details.

3. Fractional calculus and theory of images in electrostatics

As mentioned in the previous paragraph, using fractional calculus and the knowledge of electrostatic potentials of fractional multipoles, we have shown that a theory for electrostatic "fractional" image methods can be developed for perfectly conducting wedges and cones with arbitrary wedge and cone angles. What is an electrostatic "fractional" image for a two-dimensional perfectly conducting wedge, or for a three-dimensional perfectly conducting cone? Here, we briefly describe this issue for the case of a two-dimensional perfectly conducting wedge.

Consider a two-dimensional perfectly conducting wedge with outer angle β. It is well known that if an infinitely long uniform electrostatic line charge is located in front of this wedge and parallel with the edge of the wedge, the standard method of images provides us with a set of discrete image charges only when the outer angle β takes the specific value of π/s, where s is a positive integer s = 1, 2, 3,.... In this case, there are $2s-1$ discrete image line charges [44, p. 70]. Then one may ask, what if $\beta \neq \pi/s$? What kind of image, if any, would one expect to obtain when $\beta \neq \pi/s$? One form of solution to this problem was obtained by Nikoskinen and Lindell, who have elegantly analyzed the problem of image solutions for the Poisson equation for the dielectric-wedge geometry [45]. In their solutions, they found that for the perfectly conducting wedge with an arbitrary angle, their image solutions have a distributed portion in the imaginary angular domain [45]. We have found another way to approach this problem, using the concept of fractional calculus and fractional multipoles [2]. In our method, it is possible to describe the equivalent charge distributions ("images") that effectively behave as distributed "intermediate" cases between those discrete images obtained for the specific wedge angles. As will be reviewed below, we have shown that the fractional orders of these equivalent charges, which we have called "fractional" images, depend on the wedge's angle.

Consider a Cartesian coordinate system (x,y,z), with the z axis along the edge of the wedge, and the x-z plane being the symmetry plane of the wedge. The wedge outer angle is β (Figure 4). We also use a cylindrical coordinate system (ρ,φ,z), with $x=\rho\cos\varphi$ and $y=\rho\sin\varphi$. An infinitely long uniform static line charge, with a charge density per unit length of Q_l (Coulomb/m), is located parallel with the z axis at an arbitrary point with coordinates (ρ_0, φ_0). The problem, therefore, is a two-dimensional problem, and the electrostatic potential, $\Phi(\rho,\varphi)$, is independent of the z coordinate. The electrostatic potential, $\Phi(\rho,\varphi)$, due to this line charge in front of the perfectly conducting wedge (i.e., in the angular range of $\pi-\beta/2<\varphi<\pi+\beta/2$) can be written as

$$\Phi(\rho,\varphi) = \sum_{m=1}^{\infty} \frac{Q_l}{m\pi\varepsilon}\left(\frac{\rho_<}{\rho_>}\right)^{\frac{m\pi}{\beta}} \sin\left[\frac{m\pi}{2}-\frac{m\pi}{\beta}(\pi-\varphi)\right]\sin\left[\frac{m\pi}{2}-\frac{m\pi}{\beta}(\pi-\varphi_o)\right] \qquad (13)$$

where $\rho_<$ ($\rho_>$) is the smaller (larger) of ρ_o and ρ, respectively, and ε is the permittivity of the homogeneous isotropic medium in front of the wedge [46, p. 76]. We first consider the region where $\rho>\rho_0$ and $\pi-\beta/2<\varphi<\pi+\beta/2$. If one removes the wedge and the original line charge in front of it, what form of equivalent charge distribution should be inserted in order to obtain the same potential $\Phi(\rho,\varphi)$ in this region? As can be seen from Equation (13), in this region, the ρ dependencies of various terms of the potential involve the powers $-m\pi/\beta$, which are, in general, non-integers. For any two-dimensional electrostatic charge density, with its cross section in the x-y plane being bounded in a finite region (which is not the case under study here), the conventional multipole expansion of the potential involves terms the ρ dependencies of which have negative-integer powers (except the zeroth-order line-monopole, the potential of which has logarithmic dependence on ρ). Thus, one can conclude that for the wedge problem with $\beta \neq \pi/s$, shown in Figure 4, an equivalent representation of charge distribution, which would provide a potential distribution similar to that of the wedge in the region $\rho>\rho_0$ and $\pi-\beta/2<\varphi<\pi+\beta/2$, will not be limited to a certain bounded region in the two-dimensional x-y plane (i.e., it would not be discrete–localized–in the x-y plane). To clarify this matter, consider the dominant term in Equation (13) for $\rho \gg \rho_0$. The ρ dependence of this term has the form $\rho^{-\pi/\beta}$. As $\rho\to\infty$, this decays faster than the $-\ln(\rho)$ term (of a line-monopole), and it decays more slowly than ρ^{-n} (of a line 2^n-pole), where n is the first positive integer larger than π/β. Therefore, for $\beta>\pi$, it appears that in the absence of the wedge (and the original line charge), an equivalent charge distribution, to produce a potential with $\rho^{-\pi/\beta}$ dependence will have a form which is neither a line-monopole nor a line-dipole at the origin. Instead, it should effectively be an "intermediate" case, between a two-dimensional line-monopole and a two-dimensional line-dipole. Thus, a two-dimensional fractional-order multipole can be of use here. Following detailed mathematical steps described in [2], we have shown that if the wedge and its original line charge are replaced with the following equivalent charge distribution,

$$Q_{equiv.}(x,y) = \sum_{n=0}^{\infty}\left\{\frac{2\pi\rho_o^{\frac{(2n+1)\pi}{\beta}}\cos\left[\frac{(2n+1)\pi}{\beta}(\pi-\varphi_o)\right]}{\beta\Gamma\left[\frac{(2n+1)\pi}{\beta}+1\right]}\,{}_{-\infty}D_x^{\frac{(2n+1)\pi}{\beta}}\left[Q_l\delta(x)\delta(y)\right]\right\}$$
$$+\sum_{n=0}^{\infty}\left\{\frac{2\pi\rho_o^{\frac{(2n+2)\pi}{\beta}}\sin\left[\frac{(2n+2)\pi}{\beta}(\pi-\varphi_o)\right]}{\beta\Gamma\left[\frac{(2n+2)\pi}{\beta}+1\right]}\,{}_{-\infty}D_x^{\frac{(2n+2)\pi}{\beta}-1}\left[-Q_l\delta(x)\delta^{(1)}(y)\right]\right\} \qquad (14)$$

the same electrostatic potential in the region $\rho>\rho_0$ and $\pi-\beta/2<\varphi<\pi+\beta/2$ is obtained. This equivalent charge distribution, which involves fractional derivatives of the line charge $Q_l\delta(x)\delta(y)$ and of the line dipole $-Q_l\delta(x)\delta^{(1)}(y)$, has several notable features that are addressed in detail in [2]. To have a better "feel" for the physical meaning of the equivalent (or substituting) charge given

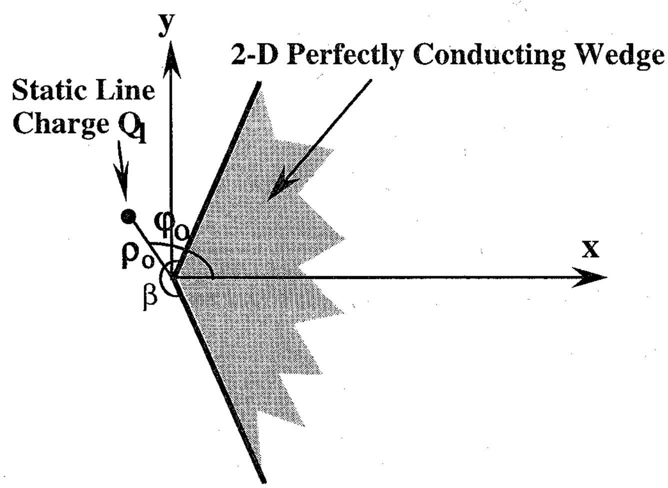


Figure 4. A two-dimensional perfectly conducting wedge with the outer angle β. An infinitely long static line charge, with uniform line-charge density of Q_l (Coulomb/m) is located at (ρ_0, φ_0). The Cartesian (x,y,z) and cylindrical (ρ,φ,z) coordinate systems are used. The geometry of the problem is independent of z (adapted from [2]).

in Equation (14), let us consider observation points in the far region, i.e., where $\rho >> \rho_0$. For this case, the dominant term in the wedge potential, given in Equation (13), is the term with $\rho^{-\pi/\beta}$ dependence. If we remove the perfectly conducting wedge and the line charge in its proximity, and replace them with the "dominant" term of the charge distribution in Equation (14), which is

$$Q_{equiv.}(x,y)$$

$$\approx \frac{2\pi\rho_o^{\frac{\pi}{\beta}}\cos\left[\frac{\pi}{\beta}(\pi-\varphi_o)\right]}{\beta\Gamma\left[\frac{\pi}{\beta}+1\right]}\, {}_{-\infty}D_x^{\frac{\pi}{\beta}}\left[Q_l\delta(x)\delta(y)\right] \tag{15}$$

$$= \frac{2\pi Q_l\rho_o^{\frac{\pi}{\beta}}\cos\left[\frac{\pi}{\beta}(\pi-\varphi_o)\right]}{\beta\Gamma\left[\frac{\pi}{\beta}+1\right]}\frac{\partial^p}{\partial x^p}\begin{cases}\delta(y)\dfrac{1}{\Gamma\left[p-\frac{\pi}{\beta}\right]}x^{-\frac{\pi}{\beta}+p-1} & \text{for } x>0\\ 0 & \text{for } x<0\end{cases}$$

we will obtain the same potential as the dominant term of Equation (13). Here, integer p is chosen such that $(\pi/\beta - p)$ becomes negative, and thus the Riemann-Liouville Integral for the definition of a fractional integral can be used. At $x = 0$, care must be exercised when evaluating $\frac{\partial^p}{\partial x^p}$. The integer p can be chosen such that the function $x^{-\frac{\pi}{\beta}+p-1}$ becomes continuous at $x = 0$. However, the pth-order derivative of this function at $x = 0$ is singular and discontinuous. For non-integer values of π/β, this two-dimensional charge density distribution is a (non-localized) surface charge distribution on the x-z plane for $x > 0$, with volume charge density given in the above equation. Figure 5 shows the sketch of this charge distribution, which replaces the wedge and its neighboring line charge. What would happen if π/β were an integer? As said earlier, it is well known from the conventional image method that for these values, one can find a finite number of discrete (i.e., localized) image charges [44, p. 70]. When π/β is an integer, the orders of derivatives involved in the definition of the equivalent charge distribution become integer orders, thus yielding conventional differentiation of the Dirac delta function. For example, if $\beta = \pi$, the wedge becomes a two-dimensional perfectly conducting half space, with flat interface on the y-z plane. From Equation (13), one gets

$$\Phi(\rho,\varphi)_{\beta=\pi} \approx Q_l\rho_o\cos(\varphi_o)\frac{\cos(\varphi)}{\pi\varepsilon\rho} \quad \text{for } \rho >> \rho_o \text{ and } \frac{\pi}{2}<\varphi<\frac{3\pi}{2}. \quad \frac{\pi}{2}<\varphi_o<\frac{3\pi}{2} \tag{16}$$

This is effectively the potential of a two-dimensional line-dipole, with lines parallel with the z-axis, located at the origin, and with the vector of the dipole $\boldsymbol{p} = 2Q_l\rho_o\cos(\varphi_o)\hat{\boldsymbol{a}}_x$ ($\hat{\boldsymbol{a}}_x$ being the unit vector along the x-axis). Equation (15) for this case ($\beta = \pi$) can be written as

$$Q_{equiv.}(x,y)_{\beta=\pi} \approx -2\rho_o\cos(\varphi_o)\ {}_{-\infty}D_x^1\left[Q_l\delta(x)\delta(y)\right]$$

$$= 2Q_l\rho_o\cos(\varphi_o)\left[-\delta^{(1)}(x)\delta(y)\right] \tag{17}$$

$$\text{for } \rho >> \rho_o \text{ and } \pi/2 < \varphi_o < 3\pi/2.$$

This is, indeed, the volume charge density of a line-dipole located at the origin, with the same dipole vector $\boldsymbol{p} = 2Q_l\rho_o\cos(\varphi_o)\hat{\boldsymbol{a}}_x$ as mentioned above. This is expected, because when the original line charge at (ρ_0, φ_0) is parallel with the flat perfectly conducting interface at the y-z plane, there is an image line charge with charge density per unit length $-Q_l$ located at $(\rho_0, \pi-\varphi_0)$ and parallel with the original line charge. This image line charge ($-Q_l$) and the original line charge (Q_l) together form a line-dipole, which provides the dominant term of the potential in the far region, as given in Equation (16). Similar steps can be taken for the case of $\beta = \pi/2$, which leads to an equivalent line quadrupole (see [2] for details). Thus, when π/β is an integer value, the conventional integer-order derivatives of the Dirac delta function in Equation (14) lead to a charge distribution that effectively behaves as discrete (localized) image line charges. When π/β is not an integer, then fractional derivatives of the Dirac delta function are employed, which yield distributed (non-localized) charge densities for the equivalent charges. So, effectively, as the angle of the

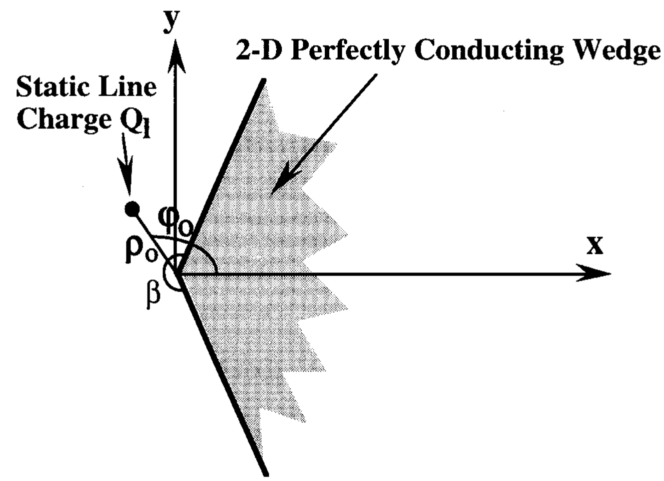


Figure 5a [2]. This shows the conducting wedge in Figure 4. For the electrostatic potential in the far region (where $\rho >> \rho_0$) and in $\pi - \beta/2 < \varphi < \pi + \beta/2$, the wedge and the original line charge can be replaced with the dominant term of the equivalent charge given in Equation (15), which is proportional to the fractional π/β-derivative of the line-monopole.

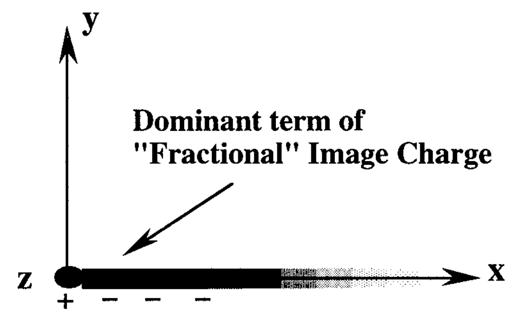


Figure 5b [2]. The wedge and the original line charge of Figure 5a have been replaced with the dominant term of the equivalent charge given in Equation (15), which is proportional to the fractional π/β-derivative of the line-monopole. This fractional $2^{\pi/\beta}$-pole gives rise to a potential that is similar to the potential in front of the wedge in Figure 5a for observers in the far region (where $\rho >> \rho_0$) in $\pi - \beta/2 < \varphi < \pi + \beta/2$.

wedge varies, the equivalent charge (Equation (14) or (15)) "evolves" between discrete (localized) and distributed (non-localized) charge densities, continually "filling the gap" between the discrete cases of localized image line charges as line-dipole, line-quadrupole, line-octopole, etc. We have named such "intermediate" charge distributions the "fractional images."

It can be shown that when $\beta > \pi$, for observation points close to the edge of the wedge where $\rho << \rho_0$, the equivalent charge distribution is Equation (15), with π/β replaced with $-\pi/\beta$ [2]. We have also extended the fractional-image method, described above, to the case of a perfectly conducting circular cone with arbitrary cone angle. This can be found in [2].

4. Mathematical link between the electrostatic images for a dielectric sphere and the images for a perfectly conducting sphere

We have developed another interesting application of fractional calculus in electrostatic-image theory, namely, the mathematical link between the electrostatic images for a dielectric sphere and a perfectly conducting sphere. The solution by the image method for the potential distribution of an electrostatic charge in front of a perfectly conducting sphere has been attributed to Lord Kelvin in 1848 [47, p. 245]. The dielectric counterpart of this problem was elegantly treated by Lindell in 1992 [48]. Later, as pointed out by Lindell et al. in [49], it was discovered that a similar solution for the dielectric sphere was given by Neumann in 1883. However, clearly Neumann's idea was not widely known, until independent rediscovery by Lindell.

Inspired by the tool of fractional calculus, we have shown [3], as reviewed below, that the distributions of image charges for the dielectric sphere and image charges for the perfectly conducting sphere can be related via fractional differintegration. We have also shown that the order of such fractional operators depends on the dielectric property of the sphere, i.e., its dielectric constant.

Consider a point charge, Q, in front of a perfectly conducting sphere of radius a, at a distance d $(\geq a)$ from its center. According to Lord Kelvin's solution, the image charges, as seen by the observer outside the sphere, consist of an image point source $-aQ/d$, located at distance a^2/d away from the sphere's center on the line connecting the real point charge Q and the center, and, if the sphere is ungrounded with a zero net charge, another image point source aQ/d, located at the center of the sphere. Mathematically, the volume charge density of these image charges can be written as

$$\rho_c(z) = -\frac{aQ}{d}\delta_-\left(z-\frac{a^2}{d}\right)+\frac{aQ}{d}\delta_+(z). \tag{18}$$

z is the coordinate along the line connecting the center to the real point charge, with $z=0$ denoting the center. Here, we use the notations $\delta_+(\cdot)$ and $\delta_-(\cdot)$ for "right-handed" and "left-handed" Dirac delta functions, respectively, as used and described by Lindell in his work [48]. As described in [48], $\delta_+(u)=\delta(u-c)$ when $c \to 0$, and $\delta_-(u)=\delta(u+c)$ when $c \to 0$ (c approaches zero through positive values in both cases). When the sphere is made of a dielectric material with permittivity $\varepsilon = \varepsilon_r\varepsilon_0$, where ε_0 is the permittivity of the outside region (free space), according to Lindell [48], the image charges as seen by the observer outside the sphere can be expressed as

$$\rho_d(z) = -\frac{\varepsilon_r-1}{\varepsilon_r+1}\frac{Qa}{d}\delta_-(z-\frac{a^2}{d})+\frac{\varepsilon_r-1}{(\varepsilon_r+1)^2}\frac{Q}{a}\left(\frac{zd}{a^2}\right)^{-\frac{\varepsilon_r}{\varepsilon_r+1}}. \tag{19}$$

This indicates that the image charges have two parts: an image point source, located at distance a^2/d away from the sphere's center (similar to the perfectly conducting case but with different charge intensity), and a line charge distributed along this connecting line between the center and the image point source.

Comparing the two image charges described in Equations (18) and (19), we noticed that the first terms in both equations are similar (aside from different multiplicative constants), and they represent image point sources at $z=a^2/d$. The second terms, however, are quite different: for the perfectly conducting sphere. This term is again an image point source, whereas in the dielectric case this image source is distributed as $z^{-\frac{\varepsilon_r}{\varepsilon_r+1}}$ along the connecting line with a singularity at the origin, where the order of such a singularity depends on ε_r. It seems as though in going from the perfectly conducting case to the dielectric case, the second image point source at $z=0$ is effectively "stretched" along the connecting line in the interval $0<z<a^2/d$. Inspired by the relationship used for the fractional integral of the Dirac delta function (see Equation (9)), we have shown [3] that the second terms in both Equations (18) and (19) can be related via fractional integration as follows:

$$\rho_d(z) = \frac{\varepsilon_r-1}{\varepsilon_r+1}\left[-\frac{aQ}{d}\delta_-\left(z-\frac{a^2}{d}\right)+\Gamma\left(\frac{\varepsilon_r+2}{\varepsilon_r+1}\right)\frac{\partial^{-\frac{1}{\varepsilon_r+1}}}{\partial(zd/a^2)^{-\frac{1}{\varepsilon_r+1}}}\left[\frac{aQ}{d}\delta_+(z)\right]\right] \tag{20}$$

for $0<z<a^2/d$.

As can be seen from the above equation, there is a relationship between the image charge distributions for the perfectly conducting sphere (insulated zero net charge case), and the dielectric sphere using fractional integration of order $(\varepsilon_r+1)^{-1}$. Thus, the dielectric property of the sphere appears in the expression for the order of fractional integration. This presents an interesting case in which the fractional order of the operator may be interpreted to contain some physical meaning. It must be noted that recently it has been shown by Lindell that Heaviside's operational calculus can also be used in the theory of images [50]. For the case of the dielectric sphere, such operational calculus leads to an expression for the image charges equivalent with Equation (20), given above.

5. What is an "intermediate wave," between a plane wave and a cylindrical wave?

Another case for study for the role of fractional calculus in electromagnetics that we have considered is the possibility of the use of fractional derivatives/integrals in finding certain "fractional" or "intermediate" solutions for the standard scalar Helmholtz equation [4]. In treating the scalar Helmholtz equation, it is well known that canonical solutions for the one-, two- and three-dimensional cases are identified as plane, cylindrical, and spherical waves, respectively. The corresponding sources of these canonical solutions can be one-, two- and three-dimensional Dirac delta functions, respectively. One may ask: Does there exist a solution for the scalar Helmholtz equation that effectively behaves as the "intermediate" case between these canonical cases, for example, an "intermediate" wave between the cases of a plane wave and a cylindrical wave? If yes, what kind of source distributions would effectively behave as the intermediate step between the integer-dimensional Dirac delta functions, e.g., between the one- and two-dimensional Dirac delta functions? Since fractional derivatives/integrals are operators with features that are "intermediate" between the conventional operators of integer-order differentiation and integration, one may not be surprised to find certain roles of these operators in exploring answers to the above questions. Our study has shown that fractional integrations can be utilized to find the "intermediate" sources and waves that can satisfy the conventional scalar Helmholtz equation [4]. Here we briefly review these results. The details can be found in [4].

Consider a Cartesian coordinate system (x,y,z) in a three-dimensional physical space. The Green's function for the scalar Helmholtz equation should satisfy

$$\nabla^2 G(\mathbf{r},\mathbf{r}_o;k) + k^2 G(\mathbf{r},\mathbf{r}_o;k) = -\delta(\mathbf{r}-\mathbf{r}_o) \tag{21}$$

where $\mathbf{r} = x\hat{\mathbf{a}}_x + y\hat{\mathbf{a}}_y + z\hat{\mathbf{a}}_z$ and $\mathbf{r}_0 = x_0\hat{\mathbf{a}}_x + y_0\hat{\mathbf{a}}_y + z_0\hat{\mathbf{a}}_z$ are the position vectors for the observation and source points, respectively, and $\hat{\mathbf{a}}_x$, $\hat{\mathbf{a}}_y$, and $\hat{\mathbf{a}}_z$ are the unit vectors in the coordinate system. ∇^2 is the Laplacian operator, which, in the three-dimensional Cartesian coordinate system, is expressed as $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$. k is a scalar constant, for a homogeneous isotropic space. It is well known that the solution to the scalar Helmholtz equation, for a one-dimensional Dirac-delta-function source, $\delta_1(\mathbf{r}) \equiv \delta(x)$, located at the y-z plane, is a plane wave [51, p. 811] given as

$$G_1(\mathbf{r},\mathbf{r}_0;k) = G_1(|x|;k) = i\frac{e^{ik|x|}}{2k}. \tag{22}$$

When the source is the two-dimensional Dirac delta function, $\delta_2(\mathbf{r}) \equiv \delta(x)\delta(y)$ (i.e., a uniform line source along the z axis), the solution to Equation (21) is a cylindrical wave [51, p. 811]:

$$G_2(\mathbf{r},\mathbf{r}_0;k) = G_2\left(\sqrt{x^2+y^2};k\right) = \frac{i}{4}H_0^{(1)}\left(k\sqrt{x^2+y^2}\right), \tag{23}$$

where $H_o^{(1)}(\cdot)$ is the zeroth-order Hankel function of the first kind [52, chapter 9]. Through certain mathematical steps, we have shown that a source that is described as follows,

$$ {}_eS_f(x,y) = (1/2)\left[{}_{-\infty}D_y^{f-2}\delta(x)\delta(y) + {}_{-\infty}D_{-y}^{f-2}\delta(x)\delta(y)\right] = \frac{\delta(x)|y|^{1-f}}{2\Gamma(2-f)} \quad \text{for } 1<f<2, \tag{24}$$

can be considered to be an "intermediate" source between the two cases of one- and two-dimensional Dirac-delta-function sources [4]. The subscript f is a parameter, with fractional values between zero and unity, and the pre-subscript e indicates the "even" symmetric nature of this source with respect to the y coordinate. This source, which involves fractional integrals (of order $f-2$) of the two-dimensional Dirac delta function $\delta(x)\delta(y)$, is a distributed source in the y-z plane. (In this fractional integration, the lower limit of integration in the definition of the Riemann-Liouville integral is taken to be $a=-\infty$.) When $f \to 2$, the above source approaches the two-dimensional Dirac delta function located at $x = y = 0$ (i.e., at the z axis). When $f \to 1$, the source in Equation (24) becomes $(1/2)\delta(x)$, which is a one-dimensional source located at $x = 0$ (i.e., in the y-z plane). With the above "intermediate" source as the source term in the right-hand side of the non-homogeneous Helmholtz equation, the solution to this equation, for the observation points on the symmetry plane (i.e. on the x-z plane) and for $x>0$, can be exactly obtained and written [4] as

$$\Psi_f(x,y=0;k) = \frac{-i\Gamma[(f-1)/2]\cos(f\pi/2)}{4\sqrt{\pi}}\left(\frac{x}{2k}\right)^{(2-f)/2} H_{(f-2)/2}^{(1)}(kx) \quad \text{for } x>0 \text{ and } 1<f<2. \tag{25}$$

$H_{(f-2)/2}^{(1)}$ is the Hankel function of order $(f-2)/2$ and of the first kind [52, p. 358]. For observation points outside the x-z plane, we have found the asymptotic solution suitable for the far-zone region [4]. Denoting $\rho \equiv \sqrt{x^2+y^2}$ and $\varphi \equiv \tan^{-1}(y/x)$, for $k\rho \gg 1$ and $\varphi > 0$ but not too small, the far-zone solution is given as

$$ {}_e\Psi_f(x,y;k) \underset{k\rho\gg 1}{\cong} \frac{-i}{4\pi}\cos\left(\frac{f\pi}{2}\right)(k\sin|\varphi|)^{f-2}\sqrt{\frac{2\pi}{k\rho}}e^{ik\rho - i\pi/4} + \frac{i}{4k^{2-f}\Gamma(2-f)}\frac{e^{ik|x|}}{(k|y|)^{f-1}} \quad \text{for } 1<f<2. \tag{26}$$

Several salient features of the solutions given in Equations (25) and (26) should be highlighted. As can be seen from Equation (26), this far-zone solution to the Helmholtz equation, with the "intermediate" source given in Equation (24), has two parts: a cylindrical wave, which drops as $\rho^{-1/2}$ in the far zone, and a non-uniform plane wave, which propagates in the x direction, but its amplitude drops with y as $|y|^{1-f}$ for $1<f<2$. It can be easily shown that when f approaches one of the limits of 1 or 2, the above solution becomes the asymptotic form for the one or two-dimensional case. Specifically, when $f=2$, the source given in Equation (24) represents a two-dimensional Dirac delta function at $x=y=0$. In this case, the second term in Equation (26), representing a "plane-wave" portion of the solution, disappears, due to the $\Gamma(2-f)$ in its denominator. The first term becomes ${}_e\Psi_{f=2}(x,y;k) \underset{k\rho\gg 1}{\cong} \frac{i}{4}\sqrt{\frac{2}{\pi k\rho}}e^{ik\rho - i\pi/4}$, which is the asymptotic form for $\frac{i}{4}H_o^{(1)}(k\rho)$, as expected. For $f=1$, the source in Equation (24) takes the form $(1/2)\delta(x)$. In this case, the cylindrical term disappears, and the plane-wave part of Equation (26) becomes $(i/4k)\exp(ik|x|)$, which is half of the value of $G_1(x;k)$ when the source is $\delta(x)$. So, indeed, the above solution effectively behaves as an "intermediate wave" between the cases of plane and cylindrical waves.

For far-zone observation points along the x-z plane, the magnitude of the exact solution given in Equation (25) can be analyzed for $kx \gg 1$. This results in

$$|\Psi_f(x,y=0;k)| \underset{kx\gg 1}{=} \frac{\Gamma[(f-1)/2]|\cos(f\pi/2)|}{4\pi 2^{(1-f)/2}k^{(3-f)/2}}|x|^{(1-f)/2} \quad \text{for } 1<f<2. \tag{27}$$

So, on the symmetric x-z plane, the magnitude of the solution drops as $|x|^{(1-f)/2}$ in the far zone. The rate of drop of this magnitude along the x axis depends on the fractional order, f. For $f = 2$, Equation (27) provides $|x|^{-1/2}$ as the dependence on x. This is expected since, for $f = 2$, the cylindrical wave is the solution. For $f = 1$, it can be shown that Equation (25) will become $(i/4k)\exp(ikx)$, as expected for the plane-wave solution. So, effectively, the two- and one-dimensional Green's functions of Equation (21) have been "smoothly connected" by varying the order of fractional integration of the two-dimensional delta functions in the source term, Equation (24). Since ${}_e\Psi_f(x, y; k)$ provides a solution, the magnitude along the x axis in the far zone of which drops as $|x|^{(1-f)/2}$, and since this solution is the "intermediate" solution between the Green's functions of the Helmholtz equations for the one- and two-dimensional Dirac-delta-function sources, we have named it the "fractional" solution of the Helmholtz equation [4]. For observation points outside the x-z plane (and not too close to this plane), the far-zone solution given in Equation (26) has both cylindrical- and plane-wave portions. Furthermore, in each of the limits as $f \to 1$ and $f \to 2$, one of these parts disappears, and the other represents the far-zone solution for a limiting case.

We have extended the above analysis to the cases that are intermediate between n- and $(n-1)$-dimensional Dirac delta functions for the scalar Helmholtz equation in n dimensions (n being a positive integer). The interested reader is referred to [4]. We have shown that starting from the Green's function for the scalar Helmholtz operator with the n-dimensional-delta-function source, we can "smoothly collapse" to the $(n-1)$ dimensional case, using sources described as $(n-f)$th-order integrals of n-dimensional Dirac delta functions, where f is a real non-integer number between $(n-1)$ and n. One can then continue this process from $(n-1)$ to $(n-2)$, and from $(n-2)$ to $(n-3)$, and so on, until one reaches the Green's function for the one-dimensional-delta-function source.

6. Summary

In this feature article, we have briefly reviewed some of the roles and applications of fractional calculus in electromagnetics that we have recently introduced and explored. These cases, although limited and specific in nature, might reveal interesting features of fractional derivatives and integrals and their possible utilities in electromagnetic theory. Since fractional derivatives/integrals are effectively the intermediate case between the conventional integer-order differentiation/integration, one may speculate that use of these fractional operators in electromagnetics may provide interesting, novel, "intermediate" cases in electromagnetics. Cases such as fractional multipoles, fractional solutions for Helmholtz equations, and fractional-image methods are the ones that we have studied and briefly reviewed here. Some other cases, such as the fractionalization of the curl operator and its electromagnetic applications, are currently under study by the author. Preliminary results of this study will be presented in the upcoming IEEE AP-S International Symposium/URSI North American Radio Science Meeting in Montreal, Canada, in July, 1997.

7. References

1. N. Engheta, "On Fractional Calculus and Fractional Multipoles in Electromagnetism," *IEEE Transactions on Antennas and Propagation*, **AP-44**, 4, April 1996, pp. 554-566.

2. N. Engheta, "Electrostatic 'Fractional' Image Methods for Perfectly Conducting Wedges and Cones," *IEEE Transactions on Antennas and Propagation*, **AP-44**, 12, December 1996, pp. 1565-1574.

3. N. Engheta, "A Note on Fractional Calculus and the Image Method for Dielectric Spheres," *Journal of Electromagnetic Waves and Applications*, **9**, 9, September 1995, pp. 1179-1188.

4. N. Engheta, "Use of Fractional Integration to Propose Some 'Fractional' Solutions for the Scalar Helmholtz Equation," in Jin A. Kong (ed.), *Progress in Electromagnetics Research (PIER) Monograph Series Volume 12*, Cambridge, MA, EMW Pub., 1996, pp. 107-132.

5. N. Engheta, "Fractional Differintegrals and Electrostatic Fields and Potentials near Sharp Conducting Edges," presented in "Ecole d'ÉTÉ Internationale sur Geometrie Fractale et Hyperbolique, Derivation Fractionnaire et Fractal, Applications dans les Sciences de l'Ingenieur et en Economie," Bordeaux, France, July 3-8, 1994, organized by Alain Le Méhauté from Institut Supérieur des Matériaux du Mans (ISMANS), Le Mans, France, and A. Oustaloup from Equipe CRONE, Laboratoire d'Automatique et de Productique, ENSERB Université Bordeaux I, Bordeaux, France.

6. J. Liouville, "Mémoire Sur le Calcul des Différentielles à Indices Quelconques," *Journal de l'Ecole Polytechnique*, **13**, Sec. 21, 1832, pp. 71-162.

7. B. Riemann, "Versuch einer allgemeinen Auffasung der Integration und Differentiation," in H. Weber (ed.), *The Collected Works of Bernhard Riemann*, (second edition), New York, Dover, 1953, Ch. 19, pp. 353-366.

8. K. B. Oldham and J. Spanier, *The Fractional Calculus*, New York, Academic Press, 1974.

9. S. G. Samko, A. A. Kilbas, and O. I. Marchichev, *Fractional Integrals and Derivatives, Theory and Applications*, Langhorne, Pennsylvania, Gordon and Breach Science Publishers, 1993. (Originally published in Russian by Minsk, Nauka i Tekhnika, 1987).

10. A. C. McBride and G. F. Roach (eds.), *Fractional Calculus*, (Research Notes in Mathematics 138), Boston, Pitman Advanced Publishing Program, 1985.

11. V. Kiryakova, *Generalized Fractional Calculus and Its Applications*, (Pitman Research Notes in Mathematics Series 301), Essex, UK, Longman Scientific & Technical, Longman Group, 1994.

12. H. T. Davis, *The Theory of Linear Operators*, Bloomington, Indiana, The Principia Press, 1936, pp. 64-77 and pp. 276-292.

13. E. R. Love, "Fractional Derivatives of Imaginary Order," *Journal of the London Mathematical Society*, **3**, Part 2, Second Series, February 1971, pp. 241-259.

14. A. Erdélyi and I. N. Sneddon, "Fractional Integration and Dual Integral Equations," *Canadian Journal of Mathematics*, **XIV**, 4, 1962, pp. 685-693.

15. G. K. Kalisch, "On Fractional Integrals of Pure Imaginary Order in L_p," *Proceedings of the American Mathematical Society*, **18**, 1967, pp. 136-139.

16. T. J. Osler, "The Fractional Derivative of a Composite Function," *SIAM Journal on Mathematical Analysis*, **1**, 2, May 1970, pp. 288-293.

17. I. M. Gel'fand and G. E. Shilov, *Generalized Functions*, New York, Academic Press, (5 volumes) 1964-68.

18. J. A. de Durán and S. L. Kalla, "An Application of Fractional Calculus to the Solution of $(n+1)$th-order Ordinary and Partial Differential Equations," *Journal of Fractional Calculus*, **2**, November 1992, pp. 67-76.

19. B. Rubin, "Fractional Integrals and Weakly Singular Integral Equations of the First Kind in the n-Dimensional Ball," *Journal d'Analyse Mathématique*, **63**, 1994, pp. 55-102.

20. N. H. Abel, "Solution de Quelques Problèmes à l'Aide d'Intégrales Définies," in L. Sylow and S. Lie (eds.), *Œuvres Complètes de Niels Henrik Abel, Volume 1*, Christiana, Norway, Grondahl, 1881, Chapter 2, pp. 11-18. (As mentioned in this reference, the original article was published in *Magazin for Naturvidenskaberne*, Aargang 1, Bind 2, 1823.)

21. J. Lützen, "Liouville's Differential Calculus of Arbitrary Order and Its Electrodynamical Origin," in *Proceedings of the Nineteenth Nordic Congress of Mathematicians*, Reykjavik, 1984, published by the Icelandic Mathematical Society, 1985, pp. 149-160.

22. O. Heaviside, *Electromagnetic Theory, Volume II*, London, Ernest Benn Pub., (second printing), 1925.

23. G. W. Scott Blair, "The Role of Psychophysics in Rheology," *J. Colloid Sci.*, **2**, 1947, pp. 21-32.

24. A. Le Méhauté, F. Héliodore, and V. Dionnet, "Overview of Electrical Processes in Fractal Geometry: From Electrodynamic Relaxation to Superconductivity," *Proceedings of the IEEE*, **81**, 10, 1993, pp. 1500-1510.

25. A. Le Méhauté, F. Héliodore, D. Cottevieille, and F. Latreille, "Introduction to Wave Phenomena and Uncertainty in a Fractal Space-I," *Chaos, Solitons & Fractals*, **4**, 3, 1994, pp. 389-402.

26. R. R. Nigmatulin, "On the Theory of Relaxation for Systems with "Remnant" Memory," *Physica Status Solidi (b)*, **124**, 1, July 1984, pp. 389-393.

27. R. R. Nigmatulin, "The Temporal Fractional Integral and Its Physical Sense," in "Ecole d'ÉTÉ Internationale sur Geometrie Fractale et Hyperbolique, Derivation Fractionnaire et Fractal, Applications dans les Sciences de l'Ingenieur et en Economie," Bordeaux, France, July 3-8, 1994 organized by Alain Le Méhauté from Institut Superieur des Materiaux du Mans (ISMANS), Le Mans, France and A. Oustaloup from Equipe CRONE, Laboratoire d'Automatique et de Productique. ENSERB Université Bordeaux I, Bordeaux, France.

28. "Ecole d'ÉTÉ Internationale sur Geometrie Fractale et Hyperbolique, Derivation Fractionnaire et Fractal, Applications dans les Sciences de l'Ingenieur et en Economie," Bordeaux, France, July 3-8, 1994 organized by Alain Le Méhauté from Institut Superieur des Materiaux du Mans (ISMANS), Le Mans, France and A. Oustaloup from Equipe CRONE, Laboratoire d'Automatique et de Productique. ENSERB Université Bordeaux I, Bordeaux, France.

29. M. Ochmann and S. Makarov, "Representation of the Absorption of Nonlinear Waves by Fractional Derivatives," *Journal of Acoustical Society of America*, **94**, 6, December 1993, pp. 3392-3399.

30. W. R. Schneider and W. Wyss, "Fractional Diffusion and Wave Equation," *Journal of Mathematical Physics*, **30**, 1, January 1989, pp. 134-144.

31. M. Caputo and F. Mainardi, "Linear Models of Dissipation in Anelastic Solids," *La Rivista del Nuovo Cimento (della società italiana di fisica)*, **1**, series 2, 2, April-June 1971, pp. 161-198.

32. C. F. Chen, Y. T. Tsay, and T. T. Wu, "Walsh Operational Matrices for Fractional Calculus and Their Application to Distributed Systems," *Journal of the Franklin Institute*, **303**, 3, March 1977, pp. 267-284.

33. Z. Altman, D. Renaud, and H. Baudrand, "On the Use of Differential Equations of Nonentire Order to Generate Entire Domain Basis Functions with Edge Singularity," *IEEE Transactions on Microwave Theory and Techniques*, **MTT-42**, 10, October 1994, pp. 1966-1972.

34. V. Namias, "The Fractional Order Fourier Transform and its Application to Quantum Mechanics," *J. Inst. Maths. Applics.*, **25**, 1980, pp. 241-265.

35. A. W. Lohmann, "Image Rotation, Wigner Rotation, and the Fractional Fourier Transform," *Journal of the Optical Society of America A*, **10**, 10, 1993, pp. 2181-2186.

36. D. Mendlovic, H. M. Ozaktas, and A. W. Lohmann, "Graded-Index Fibers, Wigner-Distribution Functions, and the Fractional Fourier Transform," *Applied Optics*, **33**, 26, 10 September 1994, pp. 6188-6193.

37. D. Mendlovic and H. M. Ozaktas, "Fractional Fourier Transform and Their Optical Implementation: I," *Journal of the Optical Society of America A*, **10**, 9, 1993, pp. 1875-1881.

38. J. Shamir and N. Cohen, "Root and Power Transformations in Optics," *Journal of the Optical Society of America A*, **12**, 11, 1995, pp. 2415-2425.

39. C. H. Papas, *Theory of Electromagnetic Wave Propagation*, New York, Dover Publications, 1988 (unabridged republication of work first published by New York, McGraw-Hill, 1965).

40. J. Van Bladel, *Singular Electromagnetic Fields and Sources*, New York, Oxford Science Publications, Oxford University Press, 1991.

41. J. Van Bladel, "The Multipole Expansion Revisited," *Archiv für Elektronik und Übertragungstechnik*, **31**, 1977, pp. 407-411.

42. V. Namias, "Application of the Dirac delta function to Electric Charge and Multipole Distributions," *American Journal of Physics*, **45**, 7, July 1977, p. 624-630.

43. C. A. Kocher, "Point-Multipole Expansion for Charge and Current Distributions," *American Journal of Physics*, **46**, 5, May 1978, pp. 578-579.

44. W. R. Smythe, *Static and Dynamic Electricity*, New York, McGraw-Hill, (3rd edition), 1968.

45. K. I. Nikoskinen and I. V. Lindell, "Image Solution for Poisson's Equation in Wedge Geometry," *IEEE Transactions on Antennas and Propagation*, **AP-43**, 2, 1995, pp. 179-187.

46. J. D. Jackson, *Classical Electrodynamics*, (2nd edition), New York, Wiley, 1975.

47. J. C. Maxwell, *Treatise on Electricity and Magnetism, Volume 1*, reprint by New York, Dover Publications, 1954.

48. I. V. Lindell, "Electrostatic Image Theory for the Dielectric Sphere," *Radio Science*, **27**, 1, January-February 1992, pp. 1-8.

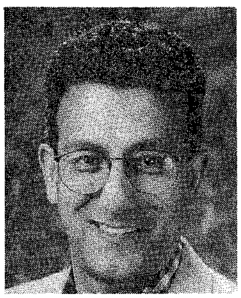
49. I. V. Lindell, J. C-E. Sten, and R. E. Kleinman, "Low-Frequency Image Theory for the Dielectric Sphere," *Journal of Electromagnetic Waves and Applications*, **8**, 3, 1994, pp. 295-313. In this work, the work of C. Neumann, *Hydrodynamische Untersuchungen nebst einem anhange über die Probleme der Elektrostatik und der Magnetischen Induction*, Leipzig, Teubner, 1883, is cited.

50. I. V. Lindell, "Heaviside Operational Calculus in Electromagnetic Image Theory," *Journal of Electromagnetic Waves and Applications*, **11**, 1, January 1997, pp. 119-132.

51. P. M. Morse and H. Feshbach, *Methods of Theoretical Physics*, New York, McGraw-Hill, (2 volumes), 1953.

52. M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, New York, Dover, 1970.

Introducing Feature Article Author



Nader Engheta received the BS degree (with honors) in electrical engineering from the University of Tehran in 1978, the MS degree in electrical engineering and the PhD degree in electrical engineering, with a minor in physics, from the California Institute of Technology in 1979 and 1982, respectively. From 1982 to 1983, he was a Postdoctoral Research Fellow at Caltech, and from 1983 to 1987, he was a Senior Research Scientist at Kaman Sciences Corporation, Dikewood Division, in Santa Monica, California. In July, 1987, he joined the faculty of the Moore School of Electrical Engineering, University of Pennsylvania, where he is currently Professor and Graduate Group Chair of Electrical Engineering. He is also a member of the David Mahoney Institute of Neurological Sciences at UPenn, and a member of the Bioengineering Graduate Group at UPenn. His current research interests are in the areas of electromagnetics, electromagnetic materials, wave interaction with complex and chiral media, fractional calculus in electromagnetics, biologically inspired polarization imaging, photoreceptor waveguides, and optics. He is a Fellow of the IEEE. He is a member of the Optical Society of America, the American Physical Society, the American Association for the Advancement of Science, and of Sigma Xi. He is an Associate Editor of the *IEEE Transactions on Antennas and Propagation*, and on the Editorial Board of *Journal of Electromagnetic Waves and Applications*. He was an Associate Editor of *Radio Science* (1991-1996). He is a member of Commissions B and D of USNC/URSI, and of the Electromagnetics Academy. He was a recipient of a 1989 NSF Presidential Young Investigator (PYI) Award, the 1993 S. Reid Warren, Jr. Award for distinguished teaching from UPenn's School of Engineering and Applied Science, the 1994 Christian F. and Mary R. Lindback Foundation Award for distinguished teaching from UPenn, and the W. M. Keck Foundation's 1995 Engineering Teaching Excellence Award. He is the recipient of the Fulbright Naples Chair award for March-May 1998.

Changes of Address

Information regarding subscription addresses is maintained by IEEE headquarters. It is *not* maintained, nor can it be changed, by any member of the *Magazine* Staff. If you are a member of the IEEE, your subscription is sent to the address in your IEEE member record. To record a change of address, contact IEEE headquarters: Member Address Records, IEEE Headquarters, 445 Hoes Lane, Piscataway, NJ 08855-1331; Tel: (908) 981-0060 or (800) 678-4333; Fax: (908) 981-9667; E-mail: address.change@ieee.org. If you are an institutional subscriber, contact IEEE Customer Service at the above address, telephone and fax numbers; E-mail: customer.service@ieee.org. Do *not* send requests to any member of the *Magazine* Staff.

Robert Smrek Named Production Director

Robert Smrek, who has been Production Manager for Magazines/Newsletters for the IEEE, has been promoted to Production Director for IEEE Periodicals. In this new position, he will be responsible for the production departments of both Transactions/Journals and Magazines/Newsletters at the IEEE. He will continue to manage the art department of the Magazines group. He will also take on management of CD-ROM production for all periodical-related products. Bob has been the production interface to the printer, Cadmus, for the *Magazine* since he joined the IEEE, five and one-half years ago.

According to Fran Zappulla, Staff Director for IEEE Periodicals, "These changes reflect our goal of uniting both the Transactions/Journals and Magazines/Newsletters departments even further. We hope it will bring more synergy to the group in working with our publication customers."