Pseudo-Retract Functors For Local Lattices And Bifinite L-Domains

MS-CIS-89-37
LOGIC & COMPUTATION 08

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June 1989

Acknowledgements:
This research was supported in part by ONR grant N00014-88-K-0633, DARPA grant N00014-85-K-0018, NSF grnats MCS-8219196-CER, IRI84-10413-A02 and U.S. Army grants DAA29-84-K-0061, DAA29-84-9-0027

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Abstract

Recently, a new category of domains used for the mathematical foundations of denotational semantics, that of L-domains, have been under study. In this paper we consider a related category of posets, that of *local lattices*. First, a completion operator taking posets to local lattices is developed, and then this operator is extended to a functor from posets with embedding-projection pairs to local lattices with embedding-projection pairs. The result of applying this functor to a local lattice yields a local lattice isomorphic to the first; this functor is a pseudo-retract.

Using the functor into local lattices, a continuous pseudo-retraction functor from ω-bifinite posets to ω-bifinite L-domains can be constructed. Such a functor takes a universal domain for the ω-bifinite posets to a universal domain for the ω-bifinite L-domains. Moreover, the existence of such a functor implies that, from the existence of a saturated universal domain for the ω-algebraic bifinites, we can conclude the existence of a saturated universal domain for the ω-bifinite L-domains.

1 Introduction

In the search for structures for use as mathematical foundations of the denotational semantics of programming languages, for several years attention has been focused on the category of bounded-complete (ω-algebraic) directed-complete posets. In particular, the requirement was made that every bounded subset of the poset should have a least upper bound. If one thinks of the points in the poset as being partial information, and one point is greater than another if it contains more information, then we are insisting that every demonstrably consistent set of information should have a join in the whole poset of information.

However, recently a new class of posets have been studied as an alternative, namely the (ω-algebraic) L-domains. An L-domain is a directed-complete poset (a cpo) with the property that the set of elements below each element forms a complete lattice.

So how do L-domains differ from bounded-complete cpo's? In an L-domain, bounded sets need not have joins in the poset as a whole. We only require that such a bounded set have a unique minimal upper bound below any given upper bound. Thinking in terms of information, we require a demonstrably consistent collection of information have a join relative to a given witness of this consistency, relative to each particular way we have more complete information. The idea of L-domains was introduced by Achim Jung in his investigations into extensions of Smyth's Theorem [7, 6]. They were independently discovered by Thierry Coquand as a special instance of his category of embeddings [1].

For the first part of this paper, we will focus our attention on a class of posets more general than the L-domains. The class we shall focus attention on is that of *local lattices.* A local lattice is a poset with the requirement that each principal ideal (the set of points below a given one) is a complete lattice. From the point of view of computation this is a crazy class to be looking at because we are not making the requirement that our posets be directed-complete. However, the construction we give here does not use the condition of being directed-complete, and would not simplify it. In the end we will restrict to finite posets (and then go to the ω-bifinites), and finite local lattices are the same thing as finite L-domains.

2 From Posets to Local Lattices

A classic example of a completion operator on posets is given by the following:

2.1 Definition. Let P be a poset. Given $X \subseteq P$, let

$$\underline{X} = \{y \mid y \leq x \text{ for all } x \in X\} = \{y \mid y \leq X\},$$

the set of all lower bounds for X in P. Let $M(P) = \{\underline{X} \mid X \subseteq P\}$ ordered by subset inclusion (*i.e.* $\underline{X} \leq \underline{Y}$ iff $\underline{X} \subseteq \underline{Y}$). Let $j : P \rightarrow M(P)$ by $j(x) = \underline{\{x\}} = \downarrow x$ the principal ideal generated by x. ∎

This construction is known as the MacNeille completion. Some properties which this construction has are:

1. $M(P)$ is a complete lattice.

2. $M(P) \cong P$ iff P is a complete lattice.

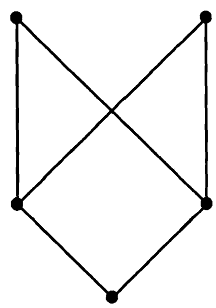
3. j is an injective function which preserves and reflects order, and which preserves all existing meets and joins.

(See Theorem III.3.11 in [5], for example.) In this section, we shall develop a similar construction for local lattices.

2.2 Definition. A poset L is called an *local lattice* if for every $x \in L$ the principal ideal generated by x (given by $\downarrow x = \underline{x}$) forms a complete lattice under the restricted ordering. ▮

Thus we have that an L-domain is a cpo which is also a local lattice. A fairly typical example of a local lattice is the following:

2.3 Example.



Notice that, while it is true that any bounded subset of a local lattice must have a meet, as the previous example shows, it need not be the case that a bounded subset have a join (*i.e.* a local lattice need not be bounded complete). However, what will be the case is that below any upper bound for a given bounded subset there will exist a unique minimal upper bound. That is, while we do not have absolute joins for bounded subsets, we do have "joins" for bounded subsets relative to given upper bounds.

2.4 Definition. Let P be a poset, X a bounded subset of P, and u an upper bound for X. Then v is a *join for X relative to u* if v is the join of X in the poset $\downarrow u$ with the restricted order relation. ▮

Note that an element v is the join of a bounded set X iff v is the join of X relative to every upper bound for X.

The following gives us a useful way of thinking about local lattices.

2.5 Lemma. *For any poset P the following are equivalent:*

1. *Every non-empty bounded subset has a meet.*
2. *For each bounded subset and each upper bound of it, there exists a join for the bounded subset relative to the upper bound.*
3. *P is a local lattice.* ∎

A complete lattice is a poset in which every subset has a meet. One way of thinking about the MacNeille completion is that it parsimoniously adds meets for all subsets of the original poset. (For any subset $X \subseteq P$, the new set $\underline{X}$ will be the meet of $j^*(X) = \{j(x) \mid x \in X\}$ in $M(P)$.) Given the above lemma, a first attempt at turning a poset into a local lattice might be to add, in a similar fashion, meets for all bounded subsets. This construction does not quite work. One of the difficulties is that, in adding meets for all bounded subsets of the original poset, you may create new bounded subsets without meets. While some sense can be made out of iterating the process (by injecting each partial result into the MacNeille completion of the original poset), there are other difficulties which arise. (Ultimately, we are aiming for a functor from posets to local lattices and this process just does not mesh well with functions.) A more fruitful approach lies in attempting to add relative joins for all bounded subsets in a reasonably parsimonious fashion, and this is the approach we shall pursue.

But how frugal is frugal? If we only include one minimal upper bound for each bounded subset of our original poset, then we will turn it into a bounded-complete poset, and not just a local lattice. On the other hand, if we try to add one minimal upper bound for each existing upper bound of a given bounded subset we will obviously just get a big mess. The solution is to amalgamate all the upper bounds of a given bounded subset which must end up having the same minimal upper bound in the local lattice we are trying to build, and then for each such amalgamation add one relative join.

2.6 Definition. Let P be a poset. Given a bounded subset X of P, define

$$\overline{X} = \{y \mid y \geq x \text{ for all } x \in X\} = \{y \mid y \geq X\}.$$

That is, $\overline{X}$ is the set of upper bounds for X in P, or the *ceiling* of X in P. For each bounded subset $X \subseteq P$ define an equivalence relation $\sim$ on $\overline{X}$ by the following:

1. If a and b are elements of $\overline{X}$ and $\{a, b\}$ is bounded in P, then $a \sim b$.

2. If $a \sim b$ and $b \sim c$, then $a \sim c$.

That is, we take the transitive closure of the relation of consistency. We will say that a and b are *linked in* $\overline{X}$ if $a \sim b$. Let us write $[a]_X$ for the equivalence class of a in $\overline{X}$ under this equivalence relation. ∎

If we view a poset as a graph where there is an edge between two points precisely when one is greater than the other, then $[a]_X$ is just the connected component of $\overline{X}$ which contains a. Two elements a and b of $\overline{X}$ are linked iff there exists a chain $a_0, \ldots, a_n$ in $\overline{X}$ such that $a = a_0$, $b = a_n$, and for all i, $0 \leq i < n$ either $a_i \leq a_{i+1}$ or $a_i \geq a_{i+1}$. As a consequence the notion of being linked is preserved by monotone functions.

2.7 Lemma. *Let P and Q be posets, and let $f : P \to Q$ be a monotonic function. If X is a bounded subset of P and u and v are two upper bounds which are linked in $\overline{X}$, then $f(u)$ and $f(v)$ are linked in $\overline{Y}$ for all subsets Y of Q such that $f^*(\overline{X}) \subseteq \overline{Y}$ (where $f^*(\overline{X}) = \{f(w) \mid w \in \overline{X}\}$).* ∎

As the next lemma shows us, these equivalence classes do unite upper bounds which must have the same relative join.

2.8 Lemma. *Let X be a bounded subset of a local lattice L, and let y and z be two upper bounds of X which are linked in $\overline{X}$. Then a join for X relative to y is a join for X relative to z.*

Proof. Note that since L is a local lattice, there exists a relative join for each of y and z. Now, since y and z are linked in $\overline{X}$, there exists a sequence $y_1, \ldots, y_n$ in $\overline{X}$ such that $y = y_0$, $z = y_n$ and $\{y_i, y_{i+1}\}$ is related for $i = 0, \ldots, n-1$. Let us proceed by induction on n. If $y = z$, then obviously X has the same relative join with respect to y and z. By the inductive hypothesis, it is no loss to assume that y and z are related, say $y \leq z$ Since L is a local lattice, $\downarrow z$ is a complete lattice. Therefore, there exists a unique minimal upper bound w for X which is below z. However, y is below z and above X and hence is above this unique minimal upper bound w. Thus w must also be the unique minimal upper bound of X which is below y. ∎

Using equivalence classes of minimal upper bounds, we are now in a position to define our construction.

2.9 Definition. For any poset P, define

$$L(P) = \{A \subseteq P \mid A = [a]_X, \text{ for some } X \subseteq P \text{ bounded, some } a \in \overline{X}\}.$$

Define a partial ordering $\sqsubseteq$ on $L(P)$ by superset containment; that is, say that $A \sqsubseteq B$ if $A \supseteq B$. ▮

The ordering on $L(P)$ is really just the same as the ordering on the upper or Smyth power domain. Note that every element of $L(P)$ is an upwards-closed subset of P.

As things stand, $L(P)$ is a collection of sets, each of which is defined in terms of the existence of some subset. In order to be able to compute with the elements of $L(P)$ it will be helpful to have more structural information about these sets in their own right. The next three lemmas help us in this regard.

2.10 Lemma. *Let A be an element of $L(P)$ and let a be an element of A. Then $A = [a]_{\underline{A}}$.*

Proof. Since $A \subseteq \overline{(\underline{A})}$ and any two elements of A are linked in A, any two elements of A are linked as elements of $\overline{(\underline{A})}$. Therefore, $A \subseteq [a]_{\underline{A}}$. Since we have that A is an element of $L(P)$, there exists $X \subseteq P$ such that $A = [a]_X$. Note that X is contained in $\underline{A}$, and hence $\overline{(\underline{A})}$ is contained in $\overline{X}$. Therefore, since $a \in \overline{(\underline{A})}$, we have that $[a]_{\underline{A}} \subseteq [a]_X = A$. ▮

2.11 Lemma. *Let A and B be elements of $L(P)$ with $A \cap B$ non-empty. Then, for every subset $X \subseteq P$ with $A = [a]_X$ (where $a \in A$), we have $A \sqsubseteq B$ iff $X \subseteq \underline{B}$.*

Proof. Note that for any X with $A = [a]_X$ we have $X \subseteq \underline{A}$. If $A \sqsubseteq B$, then $A \supseteq B$ so $\underline{A} \subseteq \underline{B}$, and hence $X \subseteq \underline{B}$. So suppose $A = [a]_X$ for some $X \subseteq \underline{B}$. Then $B \subseteq \overline{(\underline{B})} \subseteq \overline{X}$. And since $A \cap B$ is non-empty, it is no loss to assume that $a \in A \cap B$. We want to show that $B \subseteq A$. That is, we want to show that for all $b \in B$, we have b is linked to a in $\overline{X}$. Let $b \in B$. Now, both a and b are elements of B, and as such they are linked in $\overline{(\underline{B})}$. However, $\overline{(\underline{B})} \subseteq \overline{X}$ and hence b is linked to a in $\overline{X}$. ▮

2.12 Lemma. *Let $A \in L(P)$. Then A is closed under meets when viewed as a subset of P.*

Proof. Suppose that W is a subset of A, and that $y = \bigwedge W$. If W is the empty set, then $y = \top$, and since A is an upwards closed set in P, we must have that $\top \in A$. So, let us suppose that W is non-empty. To show that $y \in A$, it suffices to show that $A \sqsubseteq [y]_{\{y\}}$. Since $y = \bigwedge W$, we have that $W \subseteq [y]_{\{y\}} \cap A = \uparrow y \cap A$ and hence $[y]_{\{y\}} \cap A$ is non-empty. Therefore, by Lemma 2.11, to show that $A \sqsubseteq [y]_{\{y\}}$, it suffices to show that $\underline{A} \subseteq \underline{\{y\}}$. However, we have this since $\underline{W} = \underline{\{y\}}$ since $y = \bigwedge W$, and $\underline{A} \subseteq \underline{W}$ since $W \subseteq A$. ∎

At last we are in a position to see that our operator L does as it is supposed to do and turns a poset into a local lattice.

2.13 Theorem. *For any poset P, the poset $L(P)$ is a local lattice.*

Proof. To show that $L(P)$ is a local lattice, it suffices to show that every non-empty bounded subset of $L(P)$ has a meet in $L(P)$. Note that if a subset of $L(P)$ is bounded by $B \in L(P)$, then it is bounded by $[b]_{\downarrow b}$ for every $b \in B$. Let $\{A_i \mid i \in I\}$ be a bounded subset of $L(P)$. Without loss of generality we may assume that it is bounded by $[b]_{\downarrow b}$ for some $b \in P$. Let $X = \bigcap\{\underline{A_i} \mid i \in I\}$. Now, $b \in A_i$ for each $i \in I$ since $\{A_i \mid i \in I\}$ is bounded by $[b]_{\downarrow b}$. Since $X \leq y$ for all $y \in A_i$ for all $i \in I$, we have $X \leq b$. Let $A = [b]_X$. Now, $b \in A_i$ and $X \subseteq \underline{A_i}$ for all $i \in I$. Therefore, by Lemma 2.11 we have that $A \sqsubseteq A_i$ for all $i \in I$. Suppose that $C \sqsubseteq A_i$ for every $i \in I$. Then $\underline{C} \subseteq \underline{A_i}$ for every $i \in I$, so $\underline{C} \subseteq X \subseteq \underline{[b]_X} = \underline{A}$. Moreover, $C \sqsubseteq B = [b]_{\downarrow b}$ so $b \in C$. Thus, again by Lemma 2.11, we have that $C \sqsubseteq A$. Therefore, A is that meet of $\{A_i \mid i \in I\}$. ∎

If $L(P)$ is to be thought of as the completion of P with respect to being a local lattice, we need to be able to view P as living in $L(P)$. That is, we need an injective function $\iota_P : P \to L(P)$ such that $\iota_P(p) \leq \iota_P(q)$ iff $p \leq q$.

2.14 Definition. For any poset P define $\iota_P : P \to L(P)$ by $\iota_P(p) = [p]_{\{p\}} = \uparrow p$, the principal filter generated by p. ∎

2.15 Theorem. *The function $\iota_P : P \to L(P)$ is an injective function which preserves and reflects order, (i.e. $\iota_P(p) \leq \iota_P(q) \Leftrightarrow p \leq q$), and preserves all existing meets and joins.*

Proof. That ι_P is an injective function that preserves and reflects order is immediate from the definition of ι_P and the definition of the order relation on $L(P)$.

If the meet of the empty set exists in P, then P has a top element, $\top$. In this case, $\{\top\} = \iota_P(\top)$ is clearly the top element of $L(P)$, and hence the meet of the empty set in $L(P)$. Suppose that $p = \bigwedge U$ and that $A \sqsubseteq \iota_P(u) = \uparrow u$ for every $u \in U$. Since $A \sqsubseteq \uparrow u$ for each $u \in U$, we have that $u \in A$ for every $u \in U$; we have $U \subseteq A$. But then $p \in A$, since, by Lemma 2.12, A is closed under meets. Therefore, $A \sqsubseteq \uparrow p = \iota_P(p)$. Hence, $\iota_P(p)$ is the meet in $L(P)$ of $\{\iota_P(u) \mid u \in U\}$.

If the join of the empty set exists in P, then P has a bottom element, $\perp$. In this case $P = \iota_P(\perp)$ is the bottom element of $L(P)$, and hence is the join of the empty set in $L(P)$. So suppose that $p = \bigvee U$ and $A \sqsupseteq \iota_P(u)$ for every $u \in U$. Then for every $a \in A$ and every $u \in U$ we have $a \geq u$. But then $a \geq p$ for every $a \in A$, and hence $A \sqsupseteq \iota_P(p)$. Therefore, $\iota_P(p) = \bigvee\{\iota_P(u) \mid u \in U\}$. ∎

If $L(P)$ is to be thought of as a completion of P with respect to being a local lattice, in addition to having ι_P, we need to know that if P is already a local lattice, then L doesn't change it (up to isomorphism).

2.16 Theorem. *For any poset P we have $P \cong L(P)$ iff P is a local lattice.*

Proof. By Theorem 2.13, $L(P)$ is always a local lattice. Therefore, if $P \cong L(P)$, then P must also be a local lattice.

Let P be a local lattice. Since, by Theorem 2.15, we have that $\iota_P : P \to L(P)$ is an injection which both preserves and reflects order, to show that $P \cong L(P)$ it suffices to show that ι_P is a surjection. Let $A \in L(P)$ and let $a \in A$. Now, $\underline{A}$ is a subset of P which is bounded by a. Therefore, since P is a local lattice, there exists $p \in P$ which is a join for $\underline{A}$ relative to a. Our goal is to show that $A = \iota_P(p)$.

Note that $a \geq p$, and hence that $a \in \uparrow p = \iota_P(p)$. Therefore, since $\underline{A} \subseteq \underline{\{p\}}$ and $a \in A \cap \iota_P(p)$, by Lemma 2.11 we have that $A \sqsubseteq \iota_P(p)$. On the other hand, every $x \in A$ is linked to a in $\overline{(\underline{A})}$. Therefore, by Lemma 2.8, p is the join of $\underline{A}$ relative to x for every $x \in A$. In particular, $p \leq x$ for every $x \in A$, and thus $A \subseteq \uparrow p = \iota_P(p)$. That is, $A \sqsupseteq \iota_P(p)$, and hence $A = \iota_P(p)$. ∎

It might be nice if we had that L were a minimal completion operator. That is, we might like to have that if P is a poset, M is local lattice, and $j : P \to M$ is an injection preserving and reflecting order, and preserving all existing meets and joins, then there exists a monotonic injection $\eta : L(P) \to M$ such that $j = \eta \circ \iota_P$. (Actually, we would want that η should be an embedding (see the next section)). Unfortunately, this is too much to hope for. However, we do get the following factorization result.

2.17 Proposition. *Given a poset P, a local lattice M, and a monotonic function $f : P \to M$, there exists monotonic function $\eta : L(P) \to M$ such that $f = \eta \circ \iota_P$. Moreover, if $\nu : L(P) \to M$ is another monotonic function such that $f = \nu \circ \iota_P$, then $\eta(A) \leq \nu(A)$ for all $A \in L(P)$*

Proof. Define η by $\eta([a]_{\underline{A}}) = \bigwedge\{y \in M \mid y \leq f(a) \text{ and } y \geq f(x) \text{ for all } x \in \underline{A}\}$. That is, we map A to the join of $\underline{A}$ relative a in M. That the function η is well-defined is given to us by Lemmas 2.7 and 2.8. It is immediate that η is monotonic. To show is that $f = \eta \circ \iota_P$, compute.

$$\begin{aligned}\eta(\iota_P(p)) &= \eta([p]_{\underline{\uparrow p}}) \\ &= \bigwedge\{y \in M \mid y \leq f(p) \text{ and } y \geq f(x) \text{ for all } x \in \underline{\uparrow p}\} \\ &= f(p)\end{aligned}$$

since $p \in \underline{\uparrow p}$. Finally, suppose that $\nu : L(P) \to M$ with $\nu \circ \iota_P = f$. Then, for all $x \in \underline{A}$ we have $\nu(A) \geq \nu(\iota_P(x)) = f(x)$ and $\nu(A) \leq \nu(\iota_P(a)) = f(a)$ for any $a \in A$. Therefore, $\nu(A) \geq \eta(A)$. ▮

3 Dealing with Functions

So far, we have built an operator L which takes a poset and turns it into an local lattice, and if the poset is already a local lattice, it leaves it alone (up to isomorphism). Moreover, we have a way of viewing P as living in $L(P)$ via our monotone injective function ι_P. But what we are after is more, namely that L be (extended to) a functor from an appropriate category of posets to a correspondingly appropriate category of local lattices. The morphisms we will be interested in are embedding-projection pairs.

3.1 Definition. Given posets P and Q, a monotone function $\epsilon : P \to Q$ is an *embedding* if there exists a monotone function $\rho : Q \to P$ such that $\rho \circ \epsilon = \mathrm{id}_P$ and $\epsilon \circ \rho(q) \leq q$ for all $q \in Q$. The function ρ is called a *projection* from Q to P ▮

It should be noted that an embedding is an injection and a projection is a surjection. It is a fairly straight-forward and well-known fact (see Proposition 0.3.2 in [2], for example) that an embedding uniquely determines its corresponding projection, and likewise a projection uniquely determines its corresponding embedding. Therefore, instead of referring to the category of posets with embedding-projection pairs, we could equally well refer to the category of posets with embeddings or to the category of posets with projections.

In order to extend L to a functor from the category of posets with embedding-projection pairs ($\mathbf{PO}^{ep}$) to the category of local lattices with embedding-projection pairs ($\mathbf{LL}^{ep}$), we first need to extend L to act on embedding-projection pairs.

For sake of convenience, if $f : X \to Y$, let $f^* : 2^X \to 2^Y$ denote the function $f^*(U) = \{f(u) \mid u \in U\}$.

3.2 Definition. Let P and Q be posets, and suppose that $(\epsilon : P \to Q, \rho : Q \to P)$ is an embedding-projection pair. Define $(L(\epsilon) : L(P) \to L(Q), L(\rho) : L(Q) \to L(P))$ by

$$L(\epsilon)(A) = [\epsilon(a)]_{\epsilon^*(\underline{A})} \qquad \text{some } a \in A$$

and

$$L(\rho)(B) = [\rho(b)]_{\rho^*(\underline{B})} \qquad \text{some } b \in B.$$

∎

That $L(\epsilon)$ and $L(\rho)$ are well-defined functions, *i.e.* that their definition is independent of the choices of a and b is given to us by Lemma 2.7. The next thing we need to know is that if (ϵ, ρ) is an embedding-projection pair, then so is $(L(\epsilon), L(\rho))$. For this the following lemma will be useful.

3.3 Lemma. *Let $\rho : Q \to P$ be a projection, and let B be a subset. Then $\rho^*(\underline{B}) = \underline{\rho^*(B)}$.*

Proof. Since ρ is monotone, $\rho^*(\underline{B}) \subseteq \underline{\rho^*(B)}$. If $y \in \underline{\rho^*(B)}$, then $y \leq \rho(b)$ for every $b \in B$. Let $\epsilon : P \to Q$ be the embedding associated with ρ. Then $\epsilon(y) \leq \epsilon(\rho(b)) \leq b$ for every $b \in B$. Therefore, $\epsilon(y) \in \underline{B}$. But then $y = \rho(\epsilon(y)) \in \rho(\underline{B})$. ∎

3.4 Lemma. *If P and Q are posets and $(\epsilon : P \to Q, \rho : Q \to P)$ is an embedding-projection pair, then $(L(\epsilon), L(\rho))$ is also an embedding-projection pair.*

Proof. Let R and S be arbitrary posets, let $A \sqsubseteq B \in L(R)$ and let $f : R \to S$ be a monotone function. Let $c \in B$, and hence $c \in A$. Now, $\underline{A} \subseteq \underline{B}$, and thus, since f is monotone,

$$f^*(\underline{A}) \subseteq f^*(\underline{B}) \subseteq \underline{[f(c)]_{f(\underline{B})}}.$$

Therefore, by Lemma 2.11 we have $[f(c)]_{f(\underline{A})} \sqsubseteq [f(c)]_{f^*(\underline{B})}$. In particular, both $L(\epsilon)$ and $L(\rho)$ are monotone functions, given that both ϵ and ρ are.

Now, for $(L(\epsilon), L(\rho))$ to be an embedding-projection pair, we need $L(\rho) \circ L(\epsilon) = \mathrm{id}_{L(P)}$ and $L(\epsilon) \circ L(\rho)(B) \sqsubseteq B$ for all $B \in L(Q)$. To see that $L(\rho) \circ L(\epsilon) = \mathrm{id}_{L(P)}$, fix $A \in L(P)$ and $a \in A$. Then

$$\begin{aligned}
L(\rho)(L(\epsilon)(A)) &= L(\rho)([\epsilon(a)]_{\epsilon^*(\underline{A})}) && \\
&= [\rho(\epsilon(a))]_{\rho^*(\underline{\epsilon^*(\underline{A})})} && \\
&= [\rho(\epsilon(a))]_{\underline{\rho^*(\epsilon^*(\underline{A}))}} && \text{by Lemma 3.3} \\
&= [a]_{\underline{A}} = A && \text{since } \rho \circ \epsilon = \text{id.}
\end{aligned}$$

To see that $L(\epsilon) \circ L(\rho)(B) \sqsubseteq B$ for all $B \in L(Q)$, fix $B \in L(Q)$ and $b \in B$. Then

$$\begin{aligned}
L(\epsilon)(L(\rho)(B)) &= [\epsilon(\rho(b))]_{\epsilon^*(\underline{\rho^*(\underline{B})})} && \\
&= [b]_{\epsilon^*(\underline{\rho^*(\underline{B})})} && \text{since } b \geq \epsilon(\rho(b)) \\
&= [b]_{\epsilon^*(\rho^*(\underline{B}))} && \text{by Lemma 3.3} \\
&\sqsubseteq [b]_{\underline{B}} = B &&
\end{aligned}$$

by Lemma 2.11 since $\epsilon^*(\rho^*(\underline{B})) \subseteq \underline{B}$. ∎

3.5 Theorem. *Let $\mathbf{PO}^{ep}$ be the category of posets and embeddings-projection pairs (where the arrow points in the direction of the embedding). Let $\mathbf{LL}^{ep}$ be the category of local lattices with embeddings-projection pairs. Then $L : \mathbf{PO}^{ep} \to \mathbf{LL}^{ep}$ is a functor.*

Proof. By Theorem 2.13 we have that L takes posets to local lattices. By the previous lemma we have that $L : \mathrm{Hom}_{\mathbf{PO}^{ep}}(P, Q) \to \mathrm{Hom}_{\mathbf{LL}^{ep}}(L(P), L(Q))$ for all P and Q in $\mathbf{PO}^{ep}$. It is immediate from the definition of L that $L(\mathrm{id}_P) = \mathrm{id}_{L(P)}$. To show that L is a functor it remains to show that L preserves composition of embedding-projection pairs. To this end it suffices to show that L preserves composition of projections. Let P, Q and R be posets with projections $\rho_1 : P \to Q$ and $\rho_2 : Q \to R$. Fix $B \in P$ and

$b \in B$. Then

$$\begin{aligned} L(\rho_2)(L(\rho_1(B))) &= [\rho_2(\rho_1(b))]_{\rho_2^*(\underline{\rho_1^*(\underline{B})})} \\ &= [\rho_2(\rho_1(b))]_{\rho_2^*(\rho_1^*(\underline{B}))} && \text{by Lemma 3.3} \\ &= [\rho_2 \circ \rho_1(b)]_{(\rho_2 \circ \rho_1)^*(\underline{B})} \\ &= L(\rho_2 \circ \rho_1)(B). \end{aligned}$$

Therefore, L preserves composition, and hence is a functor. ∎

We have more from L, namely that it commutes with the action of the ι_P's.

3.6 Lemma. *For every P and $Q \in \mathbf{PO}^{ep}$, and $(\epsilon, \rho) \in \mathrm{Hom}_{\mathbf{PO}^{ep}}(P, Q)$ we have*

$$L(\epsilon) \circ \iota_P = \iota_Q \circ \epsilon \qquad \textit{and} \qquad \iota_P \circ L(\rho) = \rho \circ \iota_Q. \ ∎$$

3.7 Definition. Let $\mathbf{C}$ be a category and let $\mathbf{B}$ be a full subcategory. A functor $F : \mathbf{C} \to \mathbf{B}$ is a *pseudo-retraction* if there exists a natural isomorphism from the identity functor on $\mathbf{B}$ to F restricted to $\mathbf{B}$, *i.e.* if there exists a family of isomorphisms $i_B : B \cong F(B)$, one for each object in $\mathbf{B}$, such that for all arrows $f : B_1 \to B_2$ in $\mathbf{B}$, we have $F(f) \circ i_{B_1} = i_{B_2} \circ f$. ∎

3.8 Corollary. *The functor $L : \mathbf{PO}^{ep} \to \mathbf{LL}^{ep}$ is a pseudo-retraction.*

Proof. By the proof of 2.16, we have that for all L in $\mathbf{LL}^{ep}$ ι_L is an isomorphism. Therefore, Lemma 3.6 tells us that the family $\{\iota_L \mid L \in \mathbf{LL}^{ep}\}$ provides the desired natural isomorphism. ∎

4 Restricting to the Bifinites

So far, we have been looking at the categories of posets and of local lattices. However, for a class of posets to be used as a mathematical foundation for the denotational semantics of programming languages, (in order to sensibly model computation) one usually restricts to the class of directed-complete posets (cpo's). Unfortunately, the functor L which we have built does not take cpo's to cpo's; it does not in general preserve directed-completeness. However, it does preserve finiteness.

Recall that an *ω-bifinite domain* is a directed-complete poset which is an ω-bilimit (directed colimit) of an ω-chain in the category $\mathbf{DCPO}_\perp^{ep}$ of directed-complete posets with continuous embedding-projection pairs of finite posets with least element. (An ω-chain in a category $\mathbf{C}$ is a functor $F : \omega \to \mathbf{C}$ from the ordinal ω viewed as a category.) In [4] it is observed that the subcategories of ω-bifinites ($\omega\mathbf{B}^{ep}$) and ω-bifinite L-domains ($\omega\mathbf{BL}^{ep}$) are closed under the formation of bilimits of ω-chains in the category of directed complete posets with embedding-projection pairs. If we restrict our functor L to the finite posets, we may use it to construct a functor L^* from the ω-bifinite posets to the ω-bifinite L-domains which has much the same behaviour as L.

4.1 Definition. Let $\mathbf{C}$ be a category with colimits of ω-chains and let $\mathbf{B}$ be a non-empty full subcategory of $\mathbf{C}$. We shall say that $\mathbf{B}$ is an *ω-closed* subcategory of $\mathbf{C}$ if $\mathbf{B}$ is closed under the formation of colimits of ω-chains and if for every object A of $\mathbf{C}$ such that there exists an object B of $\mathbf{B}$ and an arrow $f : A \to B$ in $\mathbf{C}$ we have that A is an object of $\mathbf{B}$. ∎

The following lemma is certainly a previously-known fact, although the author could find no direct reference to it. The proof shall be omitted here, for it is moderately long but fairly straight-forward.

4.2 Lemma. *Let $\mathbf{B}$ be an ω-closed subcategory of $\omega\mathbf{B}^{ep}$, and let $\mathbf{A}$ be the intersection of $\mathbf{B}$ with the finite posets (i.e. the full subcategory of $\mathbf{B}$ whose objects are exactly the finite posets of $\mathbf{B}$). Let $I : \mathbf{A} \to \mathbf{B}$ be the inclusion functor. Suppose that F is a functor from $\mathbf{A}$ to any ω-complete category $\mathbf{C}$. Then there exists a functor $\widehat{F} : \mathbf{B} \to \mathbf{C}$ which preserves ω-colimits and whose restriction to $\mathbf{A}$, i.e. $\widehat{F} \circ I$, is naturally isomorphic to F. Moreover, any other functor $G : \mathbf{B} \to \mathbf{C}$ which preserves ω-colimits and whose restriction to $\mathbf{A}$ is naturally isomorphic to F is naturally isomorphic to $\widehat{F}$.* ∎

4.3 Corollary. *Let $\mathbf{B}$ be an ω-closed subcategory of $\omega\mathbf{B}^{ep}$, and let $\mathbf{A}$ be the intersection of $\mathbf{B}$ with the finite posets, $\mathbf{PO}_{<\omega}^{ep}$. Let $I : \mathbf{A} \to \mathbf{B}$ be the inclusion functor of $\mathbf{A}$ into $\mathbf{B}$, and let $J : \mathbf{PO}_{<\omega}^{ep} \to \omega\mathbf{B}^{ep}$ be the inclusion functor of finite posets with embedding-projection pairs into the bifinites. Let $G : \mathbf{PO}_{<\omega}^{ep} \to \mathbf{A}$ be a pseudo-retraction functor. Then there exists a pseudo-retraction $\overline{G} : \omega\mathbf{B}^{ep} \to \mathbf{B}$ which preserves ω-bilimits and whose restriction to the finite posets, $\overline{G} \circ J$, is naturally isomorphic to $I \circ G$.* ∎

Proof. Let $K : \mathbf{A} \to \mathbf{PO}^{ep}_{<\omega}$ be the inclusion functor of $\mathbf{A}$ into $\mathbf{PO}^{ep}_{<\omega}$, the category of finte posets with embedding-projection pairs, and let $L : \mathbf{B} \to \omega\mathbf{B}^{ep}$ be the corresponding inclusion functor of $\mathbf{B}$ into $\omega\mathbf{B}^{ep}$. Then we have $L \circ I = J \circ K$. Let $\overline{G} = \widehat{I \circ G}$, the ω-colimit preserving extension of $I \circ G$ given to us by the preceding lemma. (We take $\mathbf{C}$ to be $\mathbf{B}$ and F to be $I \circ G$). Then we have that $\overline{G}$ preserves ω-bilimits (ω-colimits) and that $\overline{G} \circ J$ is naturally isomorphic to $I \circ G$. What remains to be seen is that $\overline{G}$ is a pseudo-retraction. For this, we need to show that $\overline{G} \circ K$ is naturally isomorphic to $\mathrm{Id}_{\mathbf{B}}$. However, apply the second half of the previous lemma, but with $I \circ \mathrm{Id}_{\mathbf{A}}$ for F, since $\mathrm{Id}_{\mathbf{B}}$ is an ω-colimit preserving extension of $I \circ \mathrm{Id}_{\mathbf{A}}$, it suffices to show that $(\overline{G} \circ L) \circ I$ is naturally isomorphic to $I \circ \mathrm{Id}_{\mathbf{A}}$. Now, $\overline{G} \circ L \circ I = \overline{G} \circ J \circ K$ which is naturally isomorphic to $I \circ G \circ K$. But $G \circ K$ is naturally isomorphic to $\mathrm{Id}_{\mathbf{A}}$, since G is a pseudo-retraction. Therefore, $(\overline{G} \circ L) \circ I$ is naturally isomorphic to $I \circ \mathrm{Id}_{\mathbf{A}}$. ▮

In [4], it is also shown that the category of ω-bifinite L-domains has a universal domain.

4.4 Definition. Let $\mathbf{C}$ be a category. An object $\mathcal{U}$ is *universal* in $\mathbf{C}$ if for every object A of $\mathbf{C}$, there is a (not necessarily unique) arrow $f : A \to \mathcal{U}$. ▮

Notice that it is immediate from the definitions, that if $\mathcal{U}$ is universal in a category $\mathbf{C}$, and $F : \mathbf{C} \to \mathbf{B}$ is a pseudo-retraction, then $F(\mathcal{U})$ is universal in $\mathbf{B}$. That a universal domain $\mathcal{U}$ exists for the ω-bifinites was shown by Gunter in [3]. Therefore, we have

4.5 Corollary. *Let $\mathcal{U}$ be a universal domain for the ω-bifinites, and let $\overline{L}$ be a continuous extension of the functor L restricted to the finite posets. Then $\overline{L}(\mathcal{U})$ is a universal domain in the category of ω-bifinite L-domains.* ▮

5 Functors and Universal Domains

As we have already seen, pseudo-retraction functors provide us with an easy means to conclude that a subcategory of ω-bifintes has a universal domain. In [4] the notion of a specific kind of universal domain, that of a *fully-saturated* universal domain, was introduced. As was shown in that paper, fully-saturated universal domains have the advantage that, when they exist, they are unique up to isomorphism. In the remainer of this paper, we shall show that pseudo-retraction functors also provide us with a means to conclude that a subcategory of ω-bifintes has a fully-saturated universal domain.

We begin with some preliminary definitions and facts.

5.1 Definition (Gunter and Jung). An arrow $f : A \to B$ in a category $\mathbf{C}$ is an *increment* if for every pair of arrows h, g in $\mathbf{C}$ with $f = h \circ g$ either h or g is an isomorphism. A category $\mathbf{C}$ is *incremental* if

1. $\mathbf{C}$ has an initial object,
2. $\mathbf{C}$ has colimits of ω-chains,
3. every object A of $\mathbf{C}$ is a colimt of an ω-chain (A_i, a_{ij}) where A_0 is initial, each A_i is finite (in the category $\mathbf{C}$), and each arrow $a_{i+1,i} : A_i \to A_{i+1}$ is an increment. ∎

5.2 Lemma. *Let* $\mathbf{C}$ *be an incremental category and let* $\mathbf{B}$ *be an* ω*-closed full subcategory of* $\mathbf{C}$*. Then* $\mathbf{B}$ *is an incremental category.*

Proof. See the proof of Corollary 10 in [4]. ∎

In particular, we have that $\omega\mathbf{BL}^{ep}$ is incremental, since $\omega\mathbf{B}^{ep}$ is.

5.3 Definition (Gunter and Jung). Let $\mathbf{C}$ be an incremental category and let A be an object in $\mathbf{C}$. An object A^+ together with an arrow $s : A \to A^+$ is a *saturation* of A in $\mathbf{C}$ if, for every increment $f : B \to B'$ and arrow $g : B \to A$, there exists an arrow $h : B' \to A^+$ such that $h \circ f = s \circ g$. The category $\mathbf{C}$ is said to have *finite saturations* if for every finite object A in $\mathbf{C}$ there exists a saturation $s : A \to A^+$ such that A^+ is finite. ∎

The following fact about saturations follows immediately from the definition.

5.4 Lemma. *Let* A*,* A^+ *be objects in an incremental category* $\mathbf{C}$*, and let* $f : A \to A^+$ *be a saturation of* A*. Then, for all objects* C *and* D *in* $\mathbf{C}$*, if there exists arrowS* $g : C \to A$ *and* $h : A^+ \to D$*, then* $h \circ f : A \to D$ *is another saturation of* A *and* $f \circ g : C \to A^+$ *is a saturation of* C*.* ∎

In general, the image of a finite saturation under a functor wil not be a finite saturation. Even if the functor is a pseudo-retraction, the image of a finite saturation will not generally be a finite saturation. However, as the next lemma shows, a pseudo-retraction will carry some finite saturations to finite saturations.

5.5 Lemma. *Let* $\mathbf{C}$ *be an incremental category and let* $\mathbf{B}$ *be an* ω*-closed full subcategory of* $\mathbf{C}$*. Let* $F : \mathbf{C} \to \mathbf{B}$ *be a pseudo-retraction of* $\mathbf{C}$ *to* $\mathbf{B}$*. If* A *is an object in* $\mathbf{B}$ *and* $s : A \to A^+$ *is a saturation of* A *in* $\mathbf{C}$*, then* $F(s) : F(A) \to F(A^+)$ *is a saturation of* $F(A)$ *in* $\mathbf{B}$*.*

Proof. Since $F : \mathbf{C} \to \mathbf{B}$ is a pseudo-retraction, we may fix $\iota_B : B \to F(B)$ a family of isomorphisms in $\mathbf{B}$ which form a natural transformation from the identity functor on $\mathbf{B}$ to F restricited to $\mathbf{B}$. Then for all objects B, C and arrows $f : B \to C$ in $\mathbf{B}$, we have $\iota_C \circ f = F(f) \circ \iota_B$.

Now, suppose that we have objects B, B' and arrows $f : B \to B'$ and $g : B \to F(A)$ in $\mathbf{B}$. We want to show that there exists an arrow $h : B' \to F(A^+)$ such that $F(s) \circ g = h \circ f$. Since A is an object in $\mathbf{B}$, the arrow $\iota_A : A \to F(A)$ is an isomorphism. Therefore, we can form the arrow $\iota_A^{-1} \circ g : B \to A$. Since $s : A \to A^+$ is a saturation for A in $\mathbf{C}$, there exists an arrow $k : B' \to A^+$ in $\mathbf{C}$ such that $s \circ \iota_A^{-1} \circ g = k \circ f$. Applying the functor F, we then have $F(s) \circ F(\iota_A^{-1} \circ g) = F(k) \circ F(f)$. Let $h = F(k) \circ \iota_{B'}$. Then

$$\begin{aligned} h \circ f &= F(k) \circ \iota_{B'} \circ f \\ &= F(k) \circ F(f) \circ \iota_B \\ &= F(s) \circ F(\iota_A^{-1} \circ g) \circ \iota_B \\ &= F(s) \circ \iota_A \circ \iota_A^{-1} \circ g \\ &= F(s) \circ g \end{aligned}$$

as was to be shown. Pictorially, we have the following commutative diagram:

$$
\begin{tikzcd}
A \arrow[rrr, "s"] \arrow[dr, "\iota_A"] & & & A^{+} \\
& F(A) \arrow[r, "F(s)"] & F(A^{+}) & \\
& F(B) \arrow[r, "F(f)"] \arrow[u, "F(\iota_A^{-1} \circ g)"'] & F(B') \arrow[u, "F(k)"] & \\
B \arrow[uuu, "\iota_A^{-1} \circ g"] \arrow[uur, "g"] \arrow[ur, "\iota_B"'] \arrow[rrr, "f"] & & & B' \arrow[uuu, "k"'] \arrow[uul, "h"'] \arrow[ul, "\iota_{B'}"]
\end{tikzcd}
$$ ∎

5.6 Definition (Gunter and Jung). Let $\mathbf{B}$ be an ω-closed full sub-category of $\omega\mathbf{B}^{ep}$. An object U of $\mathbf{B}$ is *fully saturated* in $\mathbf{B}$ if for every pair of objects M, N and arrows $f : M \to U$ and $g : M \to N$ in $\mathbf{B}$ there exists an arrow $h : N \to U$ such that $f = h \circ g$. ∎

From the definition we can see that a fully saturated object is a universal domain. More then that, it was shown in [4] that any fully saturated ojbect is unigue up to isomorphism. Also in that paper a means for constucting a fully saturated ojbect was given.

5.7 Theorem (Gunter and Jung). *Suppose that $\mathbf{B}$ is a closed full sub-category of $\omega\mathbf{B}^{ep}$. Let (S_i, s_{ij}) be an ω-chain in $\mathbf{B}$ where S_0 is initial, each S_i is finite, and each $s_{i+1,i} : S_i \to S_{i+1}$ is a saturation in $\mathbf{B}$ and let $\mathcal{U}$ be the bilimit of this ω-chain. Then $\mathcal{U}$ is fully saturated in $\mathbf{B}$.*

Proof. See the proofs of Theorem 11 and Theorem 19 in [4]. ∎

Now, a pseudo-retraction functor does not preserve arbitrary finite saturations, so just taking the image of an ω-chain of finite saturations and then taking its bilimt need not get us a fully saturated domain. (In fact, in the case of $\overline{L}$ it will not.) However, the pseudo-retraction does give us a way of constructing an ω-chain of finite saurations out of the image of an ω-chain of finite saurations.

5.8 Lemma. *Let (S_i, s_{ij}) be an ω-chain in $\omega\mathbf{B}^{ep}$ where S_0 is initial, each S_i is finite, and each $s_{i+1,i} : S_i \to S_{i+1}$ is a saturation. Let $\mathbf{B}$ be an ω-closed full subcategory of $\omega\mathbf{B}^{ep}$, and suppose there exists a pseudo-retraction $F : \omega\mathbf{B}^{ep} \to \mathbf{B}$ which takes finite objects of $\omega\mathbf{B}^{ep}$ to finite objects of $\mathbf{B}$. Then there exists a subchain $(S_{n_i}, s_{n_i n_j})$ (with $n_i > n_j$ for all $i > j$) such that $F(S_{n_0})$ is initial in $\mathbf{B}$, and a sequence of embeddings $e_i : F(S_{n_i}) \to S_{n_{i+1}}$ such that $e_i \circ F(s_{n_i n_{i-1}}) = s_{n_{i+1} n_i} \circ e_{i-1}$ and such that $F(e_i) : F^2(S_{n_i}) \to F(S_{n_{i+1}})$ is a saturation in $\mathbf{B}$.*

Proof. Let $(\mathcal{U}, (u_i)_{i\in\omega})$ be a bilimit of the ω-chain (S_i, s_{ij}). By Theorem 5.7 we have that $\mathcal{U}$ is a fully saturated object in $\omega\mathbf{B}^{ep}$.

We shall define the subchain $(S_{n_i}, s_{n_i n_j})$ and embeddings $e_i : F(S_{n_i}) \to S_{n_{i+1}}$ inductively. Let $S_{n_0} = S_0$. Since S_0 is initial in $\omega\mathbf{B}^{ep}$, it is an object of $\mathbf{B}$, and hence initial in $\mathbf{B}$. However, since S_0 in an object of $\mathbf{B}$, we have that $S_0 \cong F(S_0)$, so $F(S_0)$ is intial in $\mathbf{B}$. Let $S_{n_1} = S_1$ and let $e_o = s_{1,0} \circ \iota_{S_o}^{-1}$. By Lemma 5.4, e_0 is a saturation in $\mathbf{B}$.

Now, suppose S_{n_i} and e_{i-1} have been selected. Then we have the arrows $u_{n_i} \circ e_{i-1} : F(S_{n_{i-1}}) \to \mathcal{U}$ and $F(s_{n_i n_{i-1}}) : F(S_{n_{i-1}}) \to F(S_{n_i})$. Since $\mathcal{U}$ is fully saturated, there exists an arrow $g : F(S_{n_i}) \to \mathcal{U}$ such that $g \circ F(s_{n_i n_{i-1}}) = u_{n_i} \circ e_{i-1}$. However, $F(S_{n_i})$ is finite, being the image under F of a finite object. Therefore, there exists a k and an arrow $h : F(S_{n_i}) \to S_k$ such that $g = u_k \circ h$. Moreover, we may choose k such that $k \geq n_i$. Let $n_{i+1} = k + 1$ and let $e_i = s_{k+1,k} \circ h = s_{n_{i+1}k} \circ h$. It remains to show that $e_i \circ F(s_{n_i n_{i-1}}) = s_{n_{i+1} n_i} \circ e_{i-1}$ and that $F(e_i) : F^2(Sn_i) \to F(S_{n_{i+1}})$ is a saturation.

To see that $e_i \circ F(s_{n_i n_{i-1}}) = s_{n_{i+1} n_i} \circ e_{i-1}$, we'll compute the equation for the embeddings. If f is an arrow in $\omega\mathbf{B}^{ep}$, let f^e be the embedding and f^p be the corresponding projection. Then

$$
\begin{aligned}
s^e_{n_{i+1}n_i} \circ e^e_{i-1} &= s^e_{n_{i+1}k} \circ s^e_{kn_i} \circ e^e_{i-1} \\
&= s^e_{n_{i+1}k} \circ u^p_k \circ u^e_k \circ s^e_{kn_i} \circ e^e_{i-1} \\
&= s^e_{n_{i+1}k} \circ u^p_k \circ u^e_{n_i} \circ e^e_{i-1} \\
&= s^e_{n_{i+1}k} \circ u^p_k \circ g^e \circ F(s^e_{n_i n_{i-1}}) \\
&= s^e_{n_{i+1}k} \circ u^p_k \circ u^e_k \circ h^e \circ F(s^e_{n_i n_{i-1}}) \\
&= s^e_{n_{i+1}k} \circ h^e \circ F(s^e_{n_i n_{i-1}}) \\
&= e^e_i \circ F(s^e_{n_i n_{i-1}}).
\end{aligned}
$$

Since the equation holds for the embeddings, it holds for the embedding-projection pairs.

Now, $e_i = s_{k+1,k} \circ h$ and $s_{k+1,k}$ is a saturation of S_k. Therefore, by Lemma 5.4, we have that $e_i : F(S_{n_i}) \to S_{n_{i+1}}$ is a saturation in $\omega\mathbf{B}^{ep}$, and thus by Lemma 5.5 we have that $F(e_i) : F^2(S_{n_i}) \to F(S_{n_{i+1}})$ is a saturation in $\mathbf{B}$. ∎

5.9 Theorem. *Let* $\mathbf{B}$ *be an* ω*-closed full subcategory of* $\omega\mathbf{B}^{ep}$*, and suppose there exists a pseuod-retraction* $F : \omega\mathbf{B}^{ep} \to \mathbf{B}$ *which takes finite objects of* $\omega\mathbf{B}^{ep}$ *to finite objects of* $\mathbf{B}$*. Then there exists a fully saturated object in* $\mathbf{B}$*.*

Proof. Let (S_i, s_{ij}) be an ω-chain in $\omega\mathbf{B}^{ep}$ where S_0 is initial, each S_i is finite, and each $s_{i+1,i} : S_i \to S_{i+1}$ is a saturation. That such an ω-chain exists was shown by Gunter in [3]. Since $\mathbf{B}$ is an ω-closed full subcategory of $\omega\mathbf{B}^{ep}$, there exists a sequence of embeddings $e_i : F(S_{n_i}) \to S_{n_{i+1}}$ such that $F(e_i) : F^2(S_{n_i}) \to F(S_{n_{i+1}})$ is a saturation in $\mathbf{B}$. Let η be the natural isomorphism from the identity on $\mathbf{B}$ to F resticted to $\mathbf{B}$. Then $F(e_i) \circ \eta_{F(S_{n_i})}$ is also a saturation for each i. Then he bilimit of the ω-chain $(F(S_{n_i}), \eta_{F^2(S_{n_j})} \circ F(e_j))$ is fully saturated in $\mathbf{B}$. ∎

5.10 Corollary. *There exists a fully saturated object for the category* ω*-bifinite L-domains.* ∎

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