Clifford Algebras, Clifford Groups, and a Generalization of the Quaternions: The **Pin** and **Spin** Groups

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Abstract: One of the main goals of these notes is to explain how rotations in $\mathbb{R}^n$ are induced by the action of a certain group, $\mathbf{Spin}(n)$, on $\mathbb{R}^n$, in a way that generalizes the action of the unit complex numbers, $\mathbf{U}(1)$, on $\mathbb{R}^2$, and the action of the unit quaternions, $\mathbf{SU}(2)$, on $\mathbb{R}^3$ (*i.e.*, the action is defined in terms of multiplication in a larger algebra containing both the group $\mathbf{Spin}(n)$ and $\mathbb{R}^n$). The group $\mathbf{Spin}(n)$, called a *spinor group*, is defined as a certain subgroup of units of an algebra, Cl_n, the *Clifford algebra* associated with $\mathbb{R}^n$.

Since the spinor groups are certain well chosen subgroups of units of Clifford algebras, it is necessary to investigate Clifford algebras to get a firm understanding of spinor groups. These notes provide a tutorial on Clifford algebra and the groups **Spin** and **Pin**, including a study of the structure of the Clifford algebra $\mathrm{Cl}_{p,q}$ associated with a nondegenerate symmetric bilinear form of signature (p,q) and culminating in the beautiful "8-periodicity theorem" of Elie Cartan and Raoul Bott (with proofs).

Contents

Chapter 1

Clifford Algebras, Clifford Groups, and the Groups $\mathbf{Pin}(n)$ and $\mathbf{Spin}(n)$

1.1 Introduction: Rotations As Group Actions

One of the main goals of these notes is to explain how rotations in $\mathbb{R}^n$ are induced by the action of a certain group, $\mathbf{Spin}(n)$, on $\mathbb{R}^n$, in a way that generalizes the action of the unit complex numbers, $\mathbf{U}(1)$, on $\mathbb{R}^2$, and the action of the unit quaternions, $\mathbf{SU}(2)$, on $\mathbb{R}^3$ (*i.e.*, the action is defined in terms of multiplication in a larger algebra containing both the group $\mathbf{Spin}(n)$ and $\mathbb{R}^n$). The group $\mathbf{Spin}(n)$, called a *spinor group*, is defined as a certain subgroup of units of an algebra, Cl_n, the *Clifford algebra* associated with $\mathbb{R}^n$. Furthermore, for $n \geq 3$, we are lucky, because the group $\mathbf{Spin}(n)$ is topologically simpler than the group $\mathbf{SO}(n)$. Indeed, for $n \geq 3$, the group $\mathbf{Spin}(n)$ is simply connected (a fact that it not so easy to prove without some machinery), whereas $\mathbf{SO}(n)$ is not simply connected. Intuitively speaking, $\mathbf{SO}(n)$ is more twisted than $\mathbf{Spin}(n)$. In fact, we will see that $\mathbf{Spin}(n)$ is a double cover of $\mathbf{SO}(n)$.

Since the spinor groups are certain well chosen subroups of units of Clifford algebras, it is necessary to investigate Clifford algebras to get a firm understanding of spinor groups. These notes provide a tutorial on Clifford algebra and the groups **Spin** and **Pin**, including a study of the structure of the Clifford algebra $\mathrm{Cl}_{p,q}$ associated with a nondegenerate symmetric bilinear form of signature (p, q) and culminating in the beautiful "8-periodicity theorem" of Elie Cartan and Raoul Bott (with proofs). We also explain when $\mathbf{Spin}(p, q)$ is a double-cover of $\mathbf{SO}(p, q)$. The reader should be warned that a certain amount of algebraic (and topological) background is expected, and that these notes are not meant for a novice. This being said, perseverant readers will be rewarded by being exposed to some beautiful and nontrivial concepts and results, including Elie Cartan and Raoul Bott "8-periodicity theorem."

Going back to rotations as transformations induced by group actions, recall that if V is a vector space, a *linear action (on the left) of a group G on V* is a map, $\alpha \colon G \times V \to V$, satisfying the following conditions, where, for simplicity of notation, we denote $\alpha(g, v)$ by

$g \cdot v$:

(1) $g \cdot (h \cdot v) = (gh) \cdot v$, for all $g, h \in G$ and $v \in V$;

(2) $1 \cdot v = v$, for all $v \in V$, where 1 is the identity of the group G;

(3) The map $v \mapsto g \cdot v$ is a linear isomorphism of V for every $g \in G$.

For example, the (multiplicative) group, $\mathbf{U}(1)$, of unit complex numbers acts on $\mathbb{R}^2$ (by identifying $\mathbb{R}^2$ and $\mathbb{C}$) *via* complex multiplication: For every $z = a + ib$ (with $a^2 + b^2 = 1$), for every $(x, y) \in \mathbb{R}^2$ (viewing (x, y) as the complex number $x + iy$),

$$z \cdot (x, y) = (ax - by, ay + bx).$$

Now, every unit complex number is of the form $\cos\theta + i\sin\theta$, and thus, the above action of $z = \cos\theta + i\sin\theta$ on $\mathbb{R}^2$ corresponds to the rotation of angle θ around the origin. In the case $n = 2$, the groups $\mathbf{U}(1)$ and $\mathbf{SO}(2)$ are isomorphic, but this is an exception.

To represent rotations in $\mathbb{R}^3$ and $\mathbb{R}^4$, we need the quaternions. For our purposes, it is convenient to define the quaternions as certain 2×2 complex matrices. Let $\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}$ be the matrices

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathbf{i} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad \mathbf{j} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \mathbf{k} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

and let $\mathbb{H}$ be the set of all matrices of the form

$$X = a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d, \quad a, b, c, d \in \mathbb{R}.$$

Thus, every matrix in $\mathbb{H}$ is of the form

$$X = \begin{pmatrix} a + ib & c + id \\ -(c - id) & a - ib \end{pmatrix}, \quad a, b, c, d \in \mathbb{R}.$$

The quaternions $\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}$ satisfy the famous identities discovered by Hamilton:

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -\mathbf{1},$$
$$\mathbf{ij} = -\mathbf{ji} = \mathbf{k},$$
$$\mathbf{jk} = -\mathbf{kj} = \mathbf{i},$$
$$\mathbf{ki} = -\mathbf{ik} = \mathbf{j}.$$

As a consequence, it can be verified that $\mathbb{H}$ is a skew field (a noncommutative field) called the *quaternions*. It is also a real vector space of dimension 4 with basis $(\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k})$; thus, as a vector space, $\mathbb{H}$ is isomorphic to $\mathbb{R}^4$. The *unit quaternions* are the quaternions such that

$$\det(X) = a^2 + b^2 + c^2 + d^2 = 1.$$

It is easy to check that the matrices associated with the unit quaternions are exactly the matrices in $\mathbf{SU}(2)$. Thus, we call $\mathbf{SU}(2)$ the group of unit quaternions.

Now we can define an action of the group of unit quaternions, $\mathbf{SU}(2)$, on $\mathbb{R}^3$. For this, we use the fact that $\mathbb{R}^3$ can be identified with the pure quaternions in $\mathbb{H}$, namely, the quaternions of the form $x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$, where $(x_1, x_2, x_3) \in \mathbb{R}^3$. Then, we define the action of $\mathbf{SU}(2)$ over $\mathbb{R}^3$ by

$$Z \cdot X = ZXZ^{-1} = ZX\overline{Z},$$

where $Z \in \mathbf{SU}(2)$ and X is any pure quaternion. Now, it turns out that the map ρ_Z (where $\rho_Z(X) = ZX\overline{Z}$) is indeed a rotation, and that the map $\rho\colon Z \mapsto \rho_Z$ is a surjective homomorphism, $\rho\colon \mathbf{SU}(2) \to \mathbf{SO}(3)$, whose kernel is $\{-\mathbf{1}, \mathbf{1}\}$, where $\mathbf{1}$ denotes the multiplicative unit quaternion. (For details, see Gallier [16], Chapter 8).

We can also define an action of the group $\mathbf{SU}(2) \times \mathbf{SU}(2)$ over $\mathbb{R}^4$, by identifying $\mathbb{R}^4$ with the quaternions. In this case,

$$(Y, Z) \cdot X = YX\overline{Z},$$

where $(Y, Z) \in \mathbf{SU}(2) \times \mathbf{SU}(2)$ and $X \in \mathbb{H}$ is any quaternion. Then, the map $\rho_{Y,\overline{Z}}$ is a rotation (where $\rho_{Y,\overline{Z}}(X) = YX\overline{Z}$), and the map $\rho\colon (Y, Z) \mapsto \rho_{Y,\overline{Z}}$ is a surjective homomorphism, $\rho\colon \mathbf{SU}(2) \times \mathbf{SU}(2) \to \mathbf{SO}(4)$, whose kernel is $\{(\mathbf{1}, \mathbf{1}), (-\mathbf{1}, -\mathbf{1})\}$. (For details, see Gallier [16], Chapter 8).

Thus, we observe that for $n = 2, 3, 4$, the rotations in $\mathbf{SO}(n)$ can be realized *via* the linear action of some group (the case $n = 1$ is trivial, since $\mathbf{SO}(1) = \{1, -1\}$). It is also the case that the action of each group can be somehow be described in terms of multiplication in some larger algebra "containing" the original vector space $\mathbb{R}^n$ ($\mathbb{C}$ for $n = 2$, $\mathbb{H}$ for $n = 3, 4$). However, these groups appear to have been discovered in an ad hoc fashion, and there does not appear to be any universal way to define the action of these groups on $\mathbb{R}^n$. It would certainly be nice if the action was always of the form

$$Z \cdot X = ZXZ^{-1}(= ZX\overline{Z}).$$

A systematic way of constructing groups realizing rotations in terms of linear action, using a uniform notion of action, does exist. Such groups are the spinor groups, to be described in the following sections.

1.2 Clifford Algebras

We explained in Section 1.1 how the rotations in $\mathbf{SO}(3)$ can be realized by the linear action of the group of unit quaternions, $\mathbf{SU}(2)$, on $\mathbb{R}^3$, and how the rotations in $\mathbf{SO}(4)$ can be realized by the linear action of the group $\mathbf{SU}(2) \times \mathbf{SU}(2)$ on $\mathbb{R}^4$.

The main reasons why the rotations in $\mathbf{SO}(3)$ can be represented by unit quaternions are the following:

(1) For every nonzero vector $u \in \mathbb{R}^3$, the reflection s_u about the hyperplane perpendicular to u is represented by the map

$$v \mapsto -uvu^{-1},$$

where u and v are viewed as pure quaternions in $\mathbb{H}$ (i.e., if $u = (u_1, u_2, u_2)$, then view u as $u_1 \mathbf{i} + u_2 \mathbf{j} + u_3 \mathbf{k}$, and similarly for v).

(2) The group $\mathbf{SO}(3)$ is generated by the reflections.

As one can imagine, a successful generalization of the quaternions, i.e., the discovery of a group, G inducing the rotations in $\mathbf{SO}(n)$ *via* a linear action, depends on the ability to generalize properties (1) and (2) above. Fortunately, it is true that the group $\mathbf{SO}(n)$ is generated by the hyperplane reflections. In fact, this is also true for the orthogonal group, $\mathbf{O}(n)$, and more generally, for the group of direct isometries, $\mathbf{O}(\Phi)$, of any nondegenerate quadratic form, Φ, by the *Cartan-Dieudonné theorem* (for instance, see Bourbaki [6], or Gallier [16], Chapter 7, Theorem 7.2.1). In order to generalize (2), we need to understand how the group G acts on $\mathbb{R}^n$. Now, the case $n = 3$ is special, because the underlying space, $\mathbb{R}^3$, on which the rotations act, can be embedded as the pure quaternions in $\mathbb{H}$. The case $n = 4$ is also special, because $\mathbb{R}^4$ is the underlying space of $\mathbb{H}$. The generalization to $n \geq 5$ requires more machinery, namely, the notions of Clifford groups and Clifford algebras. As we will see, for every $n \geq 2$, there is a compact, connected (and simply connected when $n \geq 3$) group, $\mathbf{Spin}(n)$, the "spinor group," and a surjective homomorphism, $\rho \colon \mathbf{Spin}(n) \to \mathbf{SO}(n)$, whose kernel is $\{-1, 1\}$. This time, $\mathbf{Spin}(n)$ acts directly on $\mathbb{R}^n$, because $\mathbf{Spin}(n)$ is a certain subgroup of the group of units of the *Clifford algebra*, Cl_n, and $\mathbb{R}^n$ is naturally a subspace of Cl_n.

The group of unit quaternions $\mathbf{SU}(2)$ turns out to be isomorphic to the spinor group $\mathbf{Spin}(3)$. Because $\mathbf{Spin}(3)$ acts directly on $\mathbb{R}^3$, the representation of rotations in $\mathbf{SO}(3)$ by elements of $\mathbf{Spin}(3)$ may be viewed as more natural than the representation by unit quaternions. The group $\mathbf{SU}(2) \times \mathbf{SU}(2)$ turns out to be isomorphic to the spinor group $\mathbf{Spin}(4)$, but this isomorphism is less obvious.

In summary, we are going to define a group $\mathbf{Spin}(n)$ representing the rotations in $\mathbf{SO}(n)$, for any $n \geq 1$, in the sense that there is a linear action of $\mathbf{Spin}(n)$ on $\mathbb{R}^n$ which induces a surjective homomorphism, $\rho \colon \mathbf{Spin}(n) \to \mathbf{SO}(n)$, whose kernel is $\{-1, 1\}$. Furthermore, the action of $\mathbf{Spin}(n)$ on $\mathbb{R}^n$ is given in terms of multiplication in an algebra, Cl_n, containing $\mathbf{Spin}(n)$, and in which $\mathbb{R}^n$ is also embedded. It turns out that as a bonus, for $n \geq 3$, the group $\mathbf{Spin}(n)$ is topologically simpler than $\mathbf{SO}(n)$, since $\mathbf{Spin}(n)$ is simply connected, but $\mathbf{SO}(n)$ is not. By being astute, we can also construct a group, $\mathbf{Pin}(n)$, and a linear action of $\mathbf{Pin}(n)$ on $\mathbb{R}^n$ that induces a surjective homomorphism, $\rho \colon \mathbf{Pin}(n) \to \mathbf{O}(n)$, whose kernel is $\{-1, 1\}$. The difficulty here is the presence of the negative sign in (2). We will see how Atiyah, Bott and Shapiro circumvent this problem by using a "twisted adjoint action," as opposed to the usual adjoint action (where $v \mapsto uvu^{-1}$).

These notes are heavily influenced by Bröcker and tom Dieck [7], Chapter 1, Section 6, where most details can be found. This Chapter is almost entirely taken from the first 11

pages of the beautiful and seminal paper by Atiyah, Bott and Shapiro [3], Clifford Modules, and we highly recommend it. Another excellent (but concise) exposition can be found in Kirillov [18]. A very thorough exposition can be found in two places:

1. Lawson and Michelsohn [20], where the material on $\mathbf{Pin}(p, q)$ and $\mathbf{Spin}(p, q)$ can be found in Chapter I.

2. Lounesto's excellent book [21].

One may also want to consult Baker [4], Curtis [12], Porteous [24], Fulton and Harris (Lecture 20) [15], Choquet-Bruhat [11], Bourbaki [6], or Chevalley [10], a classic. The original source is Elie Cartan's book (1937) whose translation in English appears in [8].

We begin by recalling what is an algebra over a field. Let K denote any (commutative) field, although for our purposes, we may assume that $K = \mathbb{R}$ (and occasionally, $K = \mathbb{C}$). Since we will only be dealing with associative algebras with a multiplicative unit, we only define algebras of this kind.

Definition 1.1 Given a field, K, a *K-algebra* is a K-vector space, A, together with a bilinear operation, $\cdot\colon A \times A \to A$, called *multiplication*, which makes A into a ring with unity, 1 (or 1_A, when we want to be very precise). This means that $\cdot$ is associative and that there is a multiplicative identity element, 1, so that $1 \cdot a = a \cdot 1 = a$, for all $a \in A$. Given two K-algebras A and B, a *K-algebra homomorphism, $h\colon A \to B$*, is a linear map that is also a ring homomorphism, with $h(1_A) = 1_B$.

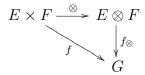For example, the ring, $M_n(K)$, of all $n \times n$ matrices over a field, K, is a K-algebra.

There is an obvious notion of *ideal* of a K-algebra: An ideal, $\mathfrak{A} \subseteq A$, is a linear subspace of A that is also a two-sided ideal with respect to multiplication in A. If the field K is understood, we usually simply say an algebra instead of a K-algebra.

We also need a quick review of tensor products. The basic idea is that tensor products allow us to view multilinear maps as linear maps. The maps become simpler, but the spaces (product spaces) become more complicated (tensor products). For more details, see Atiyah and Macdonald [2].

Definition 1.2 Given two K-vector spaces, E and F, a *tensor product of E and F* is a pair, $(E \otimes F, \otimes)$, where $E \otimes F$ is a K-vector space and $\otimes\colon E \times F \to E \otimes F$ is a bilinear map, so that for every K-vector space, G, and every bilinear map, $f\colon E \times F \to G$, there is a unique linear map, $f_\otimes\colon E \otimes F \to G$, with

$$f(u, v) = f_\otimes(u \otimes v) \quad \text{for all } u \in E \text{ and all } v \in V,$$

as in the diagram below:

$$
\begin{array}{ccc}
E \times F & \xrightarrow{\;\otimes\;} & E \otimes F \\
& {\scriptstyle f} \searrow & \downarrow {\scriptstyle f_\otimes} \\
& & G
\end{array}
$$

The vector space $E \otimes F$ is defined up to isomorphism. The vectors $u \otimes v$, where $u \in E$ and $v \in F$, generate $E \otimes F$.

Remark: We should really denote the tensor product of E and F by $E \otimes_K F$, since it depends on the field K. Since we usually deal with a fixed field K, we use the simpler notation $E \otimes F$.

We have natural isomorphisms

$$(E \otimes F) \otimes G \approx E \otimes (F \otimes G) \quad \text{and} \quad E \otimes F \approx F \otimes E.$$

Given two linear maps $f \colon E \to F$ and $g \colon E' \to F'$, we have a unique bilinear map $f \times g \colon E \times E' \to F \times F'$ so that

$$(f \times g)(a, a') = (f(a), g(a')) \quad \text{for all } a \in E \text{ and all } a' \in E'.$$

Thus, we have the bilinear map $\otimes \circ (f \times g) \colon E \times E' \to F \otimes F'$, and so, there is a unique linear map $f \otimes g \colon E \otimes E' \to F \otimes F'$, so that

$$(f \otimes g)(a \otimes a') = f(a) \otimes g(a') \quad \text{for all } a \in E \text{ and all } a' \in E'.$$

Let us now assume that E and F are K-algebras. We want to make $E \otimes F$ into a K-algebra. Since the multiplication operations $m_E \colon E \times E \to E$ and $m_F \colon F \times F \to F$ are bilinear, we get linear maps $m'_E \colon E \otimes E \to E$ and $m'_F \colon F \otimes F \to F$, and thus, the linear map

$$m'_E \otimes m'_F \colon (E \otimes E) \otimes (F \otimes F) \to E \otimes F.$$

Using the isomorphism $\tau \colon (E \otimes E) \otimes (F \otimes F) \to (E \otimes F) \otimes (E \otimes F)$, we get a linear map

$$m_{E \otimes F} \colon (E \otimes F) \otimes (E \otimes F) \to E \otimes F,$$

which defines a multiplication m on $E \otimes F$ (namely, $m(u, v) = m_{E \otimes F}(u \otimes v)$). It is easily checked that $E \otimes F$ is indeed a K-algebra under the multiplication m. Using the simpler notation $\cdot$ for m, we have
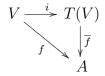
$$(a \otimes a') \cdot (b \otimes b') = (ab) \otimes (a'b')$$

for all $a, b \in E$ and all $a', b' \in F$.

Given any vector space, V, over a field, K, there is a special K-algebra, $T(V)$, together with a linear map, $i \colon V \to T(V)$, with the following universal mapping property: Given any K-algebra, A, for any linear map, $f \colon V \to A$, there is a unique K-algebra homomorphism, $\overline{f} \colon T(V) \to A$, so that

$$f = \overline{f} \circ i,$$

as in the diagram below:

$$V \xrightarrow{\;i\;} T(V)$$
$$f \searrow \quad \downarrow \overline{f}$$
$$A$$

The algebra, $T(V)$, is the *tensor algebra of V*. It may be constructed as the direct sum

$$T(V) = \bigoplus_{i \geq 0} V^{\otimes i},$$

where $V^0 = K$, and $V^{\otimes i}$ is the i-fold tensor product of V with itself. For every $i \geq 0$, there is a natural injection $\iota_n \colon V^{\otimes n} \to T(V)$, and in particular, an injection $\iota_0 \colon K \to T(V)$. The multiplicative unit, **1**, of $T(V)$ is the image, $\iota_0(1)$, in $T(V)$ of the unit, 1, of the field K. Since every $v \in T(V)$ can be expressed as a finite sum

$$v = v_1 + \cdots + v_k,$$

where $v_i \in V^{\otimes n_i}$ and the n_i are natural numbers with $n_i \neq n_j$ if $i \neq j$, to define multiplication in $T(V)$, using bilinearity, it is enough to define the multiplication $V^{\otimes m} \times V^{\otimes n} \longrightarrow V^{\otimes(m+n)}$. Of course, this is defined by

$$(v_1 \otimes \cdots \otimes v_m) \cdot (w_1 \otimes \cdots \otimes w_n) = v_1 \otimes \cdots \otimes v_m \otimes w_1 \otimes \cdots \otimes w_n.$$

(This has to be made rigorous by using isomorphisms involving the associativity of tensor products, for details, see see Atiyah and Macdonald [2].) The algebra $T(V)$ is an example of a *graded algebra*, where the *homogeneous elements of rank n* are the elements in $V^{\otimes n}$.

Remark: It is important to note that multiplication in $T(V)$ is **not** commutative. Also, in all rigor, the unit, **1**, of $T(V)$ is **not equal** to 1, the unit of the field K. However, in view of the injection $\iota_0 \colon K \to T(V)$, for the sake of notational simplicity, we will denote **1** by 1. More generally, in view of the injections $\iota_n \colon V^{\otimes n} \to T(V)$, we identify elements of $V^{\otimes n}$ with their images in $T(V)$.

Most algebras of interest arise as well-chosen quotients of the tensor algebra $T(V)$. This is true for the *exterior algebra,* $\bigwedge^\bullet V$ (also called *Grassmann algebra*), where we take the quotient of $T(V)$ modulo the ideal generated by all elements of the form $v \otimes v$, where $v \in V$, and for the *symmetric algebra,* $\mathrm{Sym}\, V$, where we take the quotient of $T(V)$ modulo the ideal generated by all elements of the form $v \otimes w - w \otimes v$, where $v, w \in V$. A Clifford algebra may be viewed as a refinement of the exterior algebra, in which we take the quotient of $T(V)$ modulo the ideal generated by all elements of the form $v \otimes v - \Phi(v) \cdot 1$, where Φ is the quadratic form associated with a symmetric bilinear form, $\varphi \colon V \times V \to K$, and $\cdot \colon K \times T(V) \to T(V)$ denotes the scalar product of the algebra $T(V)$. For simplicity, let us assume that we are now dealing with real algebras.

Definition 1.3 Let V be a real finite-dimensional vector space together with a symmetric bilinear form, $\varphi \colon V \times V \to \mathbb{R}$, and associated quadratic form, $\Phi(v) = \varphi(v, v)$. A *Clifford algebra associated with V and Φ* is a real algebra, $\mathrm{Cl}(V, \Phi)$, together with a linear map, $i_\Phi \colon V \to \mathrm{Cl}(V, \Phi)$, satisfying the condition $(i(v))^2 = \Phi(v) \cdot 1$ for all $v \in V$ and so that for every real algebra, A, and every linear map, $f \colon V \to A$, with

$$(f(v))^2 = \Phi(v) \cdot 1 \quad \text{for all } v \in V,$$

there is a unique algebra homomorphism, $\overline{f} \colon \mathrm{Cl}(V, \Phi) \to A$, so that

$$f = \overline{f} \circ i_\Phi,$$

as in the diagram below:

$$V \xrightarrow{\ i_\Phi\ } \mathrm{Cl}(V, \Phi)$$
$$f \searrow \quad \downarrow \overline{f}$$
$$A$$

We use the notation, $\lambda \cdot u$, for the product of a scalar, $\lambda \in \mathbb{R}$, and of an element, u, in the algebra $\mathrm{Cl}(V, \Phi)$ and juxtaposition, uv, for the multiplication of two elements, u and v, in the algebra $\mathrm{Cl}(V, \Phi)$.

By a familiar argument, any two Clifford algebras associated with V and Φ are isomorphic. We often denote i_Φ by i.

To show the existence of $\mathrm{Cl}(V, \Phi)$, observe that $T(V)/\mathfrak{A}$ does the job, where $\mathfrak{A}$ is the ideal of $T(V)$ generated by all elements of the form $v \otimes v - \Phi(v) \cdot 1$, where $v \in V$. The map $i_\Phi \colon V \to \mathrm{Cl}(V, \Phi)$ is the composition

$$V \xrightarrow{\ \iota_1\ } T(V) \xrightarrow{\ \pi\ } T(V)/\mathfrak{A},$$

where π is the natural quotient map. We often denote the Clifford algebra $\mathrm{Cl}(V, \Phi)$ simply by $\mathrm{Cl}(\Phi)$.

Remark: Observe that Definition 1.3 does not assert that i_Φ is injective or that there is an injection of $\mathbb{R}$ into $\mathrm{Cl}(V, \Phi)$, but we will prove later that both facts are true when V is finite-dimensional. Also, as in the case of the tensor algebra, the unit of the algebra $\mathrm{Cl}(V, \Phi)$ and the unit of the field $\mathbb{R}$ are **not equal**.

Since

$$\Phi(u + v) - \Phi(u) - \Phi(v) = 2\varphi(u, v)$$

and

$$(i(u + v))^2 = (i(u))^2 + (i(v))^2 + i(u)i(v) + i(v)i(u),$$

using the fact that

$$i(u)^2 = \Phi(u) \cdot 1,$$

we get

$$i(u)i(v) + i(v)i(u) = 2\varphi(u, v) \cdot 1.$$

As a consequence, if $(u_1, \ldots, u_n)$ is an orthogonal basis w.r.t. φ (which means that $\varphi(u_j, u_k) = 0$ for all $j \neq k$), we have

$$i(u_j)i(u_k) + i(u_k)i(u_j) = 0 \quad \text{for all } j \neq k.$$

Remark: Certain authors drop the unit, 1, of the Clifford algebra $\mathrm{Cl}(V, \Phi)$ when writing the identities

$$i(u)^2 = \Phi(u) \cdot 1$$

and

$$2\varphi(u, v) \cdot 1 = i(u)i(v) + i(v)i(u),$$

where the second identity is often written as

$$\varphi(u, v) = \frac{1}{2}(i(u)i(v) + i(v)i(u)).$$

This is very confusing and technically wrong, because we only have an injection of $\mathbb{R}$ into $\mathrm{Cl}(V, \Phi)$, but $\mathbb{R}$ is **not** a subset of $\mathrm{Cl}(V, \Phi)$.

We warn the readers that Lawson and Michelsohn [20] adopt the opposite of our sign convention in defining Clifford algebras, i.e., they use the condition

$$(f(v))^2 = -\Phi(v) \cdot 1 \quad \text{for all } v \in V.$$

The most confusing consequence of this is that their $\mathrm{Cl}(p, q)$ is our $\mathrm{Cl}(q, p)$.

Observe that when $\Phi \equiv 0$ is the quadratic form identically zero everywhere, then the Clifford algebra $\mathrm{Cl}(V, 0)$ is just the exterior algebra, $\bigwedge^\bullet V$.

Example 1.1 Let $V = \mathbb{R}$, $e_1 = 1$, and assume that $\Phi(x_1 e_1) = -x_1^2$. Then, $\mathrm{Cl}(\Phi)$ is spanned by the basis $(1, e_1)$. We have

$$e_1^2 = -1.$$

Under the bijection

$$e_1 \mapsto i,$$

the Clifford algebra, $\mathrm{Cl}(\Phi)$, also denoted by Cl_1, is isomorphic to the algebra of complex numbers, $\mathbb{C}$.

Now, let $V = \mathbb{R}^2$, (e_1, e_2) be the canonical basis, and assume that $\Phi(x_1 e_1 + x_2 e_2) = -(x_1^2 + x_2^2)$. Then, $\mathrm{Cl}(\Phi)$ is spanned by the basis by $(1, e_1, e_2, e_1 e_2)$. Furthermore, we have

$$e_2 e_1 = -e_1 e_2, \quad e_1^2 = -1, \quad e_2^2 = -1, \quad (e_1 e_2)^2 = -1.$$

Under the bijection

$$e_1 \mapsto \mathbf{i}, \quad e_2 \mapsto \mathbf{j}, \quad e_1 e_2 \mapsto \mathbf{k},$$

it is easily checked that the quaternion identities

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -\mathbf{1},$$
$$\mathbf{ij} = -\mathbf{ji} = \mathbf{k},$$
$$\mathbf{jk} = -\mathbf{kj} = \mathbf{i},$$
$$\mathbf{ki} = -\mathbf{ik} = \mathbf{j},$$

hold, and thus, the Clifford algebra $\mathrm{Cl}(\Phi)$, also denoted by Cl_2, is isomorphic to the algebra of quaternions, $\mathbb{H}$.
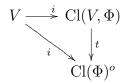
Our prime goal is to define an action of $\mathrm{Cl}(\Phi)$ on V in such a way that by restricting this action to some suitably chosen multiplicative subgroups of $\mathrm{Cl}(\Phi)$, we get surjective homomorphisms onto $\mathbf{O}(\Phi)$ and $\mathbf{SO}(\Phi)$, respectively. The key point is that a reflection in V about a hyperplane H orthogonal to a vector w can be defined by such an action, but some negative sign shows up. A correct handling of signs is a bit subtle and requires the introduction of a canonical anti-automorphism, t, and of a canonical automorphism, α, defined as follows:

Proposition 1.1 *Every Clifford algebra,* $\mathrm{Cl}(\Phi)$, *has a unique canonical anti-automorphism,* $t\colon \mathrm{Cl}(\Phi) \to \mathrm{Cl}(\Phi)$, *satisfying the properties*

$$t(xy) = t(y)t(x), \quad t \circ t = \mathrm{id}, \quad \text{and} \quad t(i(v)) = i(v),$$

for all $x, y \in \mathrm{Cl}(\Phi)$ *and all* $v \in V$.

Proof. Consider the opposite algebra $\mathrm{Cl}(\Phi)^o$, in which the product of x and y is given by yx. It has the universal mapping property. Thus, we get a unique isomorphism, t, as in the diagram below:

$$V \xrightarrow{\ i\ } \mathrm{Cl}(V, \Phi)$$
$$\quad\quad {\scriptstyle i} \searrow \quad \downarrow {\scriptstyle t}$$
$$\mathrm{Cl}(\Phi)^o$$

$\square$

We also denote $t(x)$ by x^t. When V is finite-dimensional, for a more palatable description of t in terms of a basis of V, see the paragraph following Theorem 1.4.
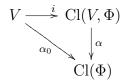
The canonical automorphism, α, is defined using the proposition

Proposition 1.2 *Every Clifford algebra,* $\mathrm{Cl}(\Phi)$, *has a unique canonical automorphism,* $\alpha\colon \mathrm{Cl}(\Phi) \to \mathrm{Cl}(\Phi)$, *satisfying the properties*

$$\alpha \circ \alpha = \mathrm{id}, \quad \text{and} \quad \alpha(i(v)) = -i(v),$$

for all $v \in V$.

Proof. Consider the linear map $\alpha_0\colon V \to \mathrm{Cl}(\Phi)$ defined by $\alpha_0(v) = -i(v)$, for all $v \in V$. We get a unique homomorphism, α, as in the diagram below:

$$V \xrightarrow{\ i\ } \mathrm{Cl}(V, \Phi)$$
$$\quad\quad {\scriptstyle \alpha_0} \searrow \quad \downarrow {\scriptstyle \alpha}$$
$$\mathrm{Cl}(\Phi)$$

Furthermore, every $x \in \mathrm{Cl}(\Phi)$ can be written as

$$x = x_1 \cdots x_m,$$

with $x_j \in i(V)$, and since $\alpha(x_j) = -x_j$, we get $\alpha \circ \alpha = \mathrm{id}$. It is clear that α is bijective. $\square$

Again, when V is finite-dimensional, a more palatable description of α in terms of a basis of V can be given. If $(e_1, \ldots, e_n)$ is a basis of V, then the Clifford algebra $\mathrm{Cl}(\Phi)$ consists of certain kinds of "polynomials," linear combinations of monomials of the form $\sum_J \lambda_J e_J$, where $J = \{i_1, i_2, \ldots, i_k\}$ is any subset (possibly empty) of $\{1, \ldots, n\}$, with $1 \leq i_1 < i_2 \cdots < i_k \leq n$, and the monomial e_J is the "product" $e_{i_1} e_{i_2} \cdots e_{i_k}$. The map α is the linear map defined on monomials by

$$\alpha(e_{i_1} e_{i_2} \cdots e_{i_k}) = (-1)^k e_{i_1} e_{i_2} \cdots e_{i_k}.$$

For a more rigorous explanation, see the paragraph following Theorem 1.4.

We now show that if V has dimension n, then i is injective and $\mathrm{Cl}(\Phi)$ has dimension 2^n. A clever way of doing this is to introduce a graded tensor product.

First, observe that

$$\mathrm{Cl}(\Phi) = \mathrm{Cl}^0(\Phi) \oplus \mathrm{Cl}^1(\Phi),$$

where

$$\mathrm{Cl}^i(\Phi) = \{x \in \mathrm{Cl}(\Phi) \mid \alpha(x) = (-1)^i x\}, \quad \text{where } i = 0, 1.$$

We say that we have a $\mathbb{Z}/2$-*grading*, which means that if $x \in \mathrm{Cl}^i(\Phi)$ and $y \in \mathrm{Cl}^j(\Phi)$, then $xy \in \mathrm{Cl}^{i+j \,(\mathrm{mod}\, 2)}(\Phi)$.

When V is finite-dimensional, since every element of $\mathrm{Cl}(\Phi)$ is a linear combination of the form $\sum_J \lambda_J e_J$, as explained earlier, in view of the description of α given above, we see that the elements of $\mathrm{Cl}^0(\Phi)$ are those for which the monomials e_J are products of an even number of factors, and the elements of $\mathrm{Cl}^1(\Phi)$ are those for which the monomials e_J are products of an odd number of factors.

Remark: Observe that $\mathrm{Cl}^0(\Phi)$ is a subalgebra of $\mathrm{Cl}(\Phi)$, whereas $\mathrm{Cl}^1(\Phi)$ is not.

Given two $\mathbb{Z}/2$-graded algebras $A = A^0 \oplus A^1$ and $B = B^0 \oplus B^1$, their *graded tensor product* $A \,\widehat{\otimes}\, B$ is defined by

$$
\begin{aligned}
(A \,\widehat{\otimes}\, B)^0 &= (A^0 \oplus B^0) \otimes (A^1 \oplus B^1), \\
(A \,\widehat{\otimes}\, B)^1 &= (A^0 \oplus B^1) \otimes (A^1 \oplus B^0),
\end{aligned}
$$

with multiplication

$$(a' \otimes b)(a \otimes b') = (-1)^{ij}(a'a) \otimes (bb'),$$

for $a \in A^i$ and $b \in B^j$. The reader should check that $A \,\widehat{\otimes}\, B$ is indeed $\mathbb{Z}/2$-graded.

Proposition 1.3 *Let V and W be finite dimensional vector spaces with quadratic forms Φ and Ψ. Then, there is a quadratic form, $\Phi \oplus \Psi$, on $V \oplus W$ defined by*

$$(\Phi + \Psi)(v, w) = \Phi(v) + \Psi(w).$$

If we write $i \colon V \to \mathrm{Cl}(\Phi)$ and $j \colon W \to \mathrm{Cl}(\Psi)$, we can define a linear map,

$$f \colon V \oplus W \to \mathrm{Cl}(\Phi) \,\widehat{\otimes}\, \mathrm{Cl}(\Psi),$$

by

$$f(v, w) = i(v) \otimes 1 + 1 \otimes j(w).$$

Furthermore, the map f induces an isomorphism (also denoted by f)

$$f \colon \mathrm{Cl}(V \oplus W) \to \mathrm{Cl}(\Phi) \,\widehat{\otimes}\, \mathrm{Cl}(\Psi).$$

Proof. See Bröcker and tom Dieck [7], Chapter 1, Section 6, page 57. $\square$

As a corollary, we obtain the following result:

Theorem 1.4 *For every vector space, V, of finite dimension n, the map $i \colon V \to \mathrm{Cl}(\Phi)$ is injective. Given a basis $(e_1, \ldots, e_n)$ of V, the $2^n - 1$ products*

$$i(e_{i_1})i(e_{i_2}) \cdots i(e_{i_k}), \quad 1 \le i_1 < i_2 \cdots < i_k \le n,$$

and 1 form a basis of $\mathrm{Cl}(\Phi)$. Thus, $\mathrm{Cl}(\Phi)$ has dimension 2^n.

Proof. The proof is by induction on $n = \dim(V)$. For $n = 1$, the tensor algebra $T(V)$ is just the polynomial ring $\mathbb{R}[X]$, where $i(e_1) = X$. Thus, $\mathrm{Cl}(\Phi) = \mathbb{R}[X]/(X^2 - \Phi(e_1))$, and the result is obvious. Since

$$i(e_j)i(e_k) + i(e_k)i(e_j) = 2\varphi(e_i, e_j) \cdot 1,$$

it is clear that the products

$$i(e_{i_1})i(e_{i_2}) \cdots i(e_{i_k}), \quad 1 \le i_1 < i_2 \cdots < i_k \le n,$$

and 1 generate $\mathrm{Cl}(\Phi)$. Now, there is always a basis that is orthogonal with respect to φ (for example, see Artin [1], Chapter 7, or Gallier [16], Chapter 6, Problem 6.14), and thus, we have a splitting

$$(V, \Phi) = \bigoplus_{k=1}^{n}(V_k, \Phi_k),$$

where V_k has dimension 1. Choosing a basis so that $e_k \in V_k$, the theorem follows by induction from Proposition 1.3. $\square$

Since i is injective, for simplicity of notation, from now on, we write u for $i(u)$. Theorem 1.4 implies that if $(e_1, \ldots, e_n)$ is an orthogonal basis of V, then $\mathrm{Cl}(\Phi)$ is the algebra presented by the generators $(e_1, \ldots, e_n)$ and the relations

$$
\begin{aligned}
e_j^2 &= \Phi(e_j) \cdot 1, \quad 1 \le j \le n, \quad \text{and} \\
e_j e_k &= -e_k e_j, \quad 1 \le j, k \le n, \ j \ne k.
\end{aligned}
$$

If V has finite dimension n and $(e_1, \ldots, e_n)$ is a basis of V, by Theorem 1.4, the maps t and α are completely determined by their action on the basis elements. Namely, t is defined by

$$
\begin{aligned}
t(e_i) &= e_i \\
t(e_{i_1} e_{i_2} \cdots e_{i_k}) &= e_{i_k} e_{i_{k-1}} \cdots e_{i_1},
\end{aligned}
$$

where $1 \le i_1 < i_2 \cdots < i_k \le n$, and, of course, $t(1) = 1$. The map α is defined by

$$
\begin{aligned}
\alpha(e_i) &= -e_i \\
\alpha(e_{i_1} e_{i_2} \cdots e_{i_k}) &= (-1)^k e_{i_1} e_{i_2} \cdots e_{i_k}
\end{aligned}
$$

where $1 \le i_1 < i_2 \cdots < i_k \le n$, and, of course, $\alpha(1) = 1$. Furthermore, the even-graded elements (the elements of $\mathrm{Cl}^0(\Phi)$) are those generated by 1 and the basis elements consisting of an even number of factors, $e_{i_1} e_{i_2} \cdots e_{i_{2k}}$, and the odd-graded elements (the elements of $\mathrm{Cl}^1(\Phi)$) are those generated by the basis elements consisting of an odd number of factors, $e_{i_1} e_{i_2} \cdots e_{i_{2k+1}}$.

We are now ready to define the Clifford group and investigate some of its properties.

1.3   Clifford Groups

First, we define *conjugation* on a Clifford algebra, $\mathrm{Cl}(\Phi)$, as the map

$$
x \mapsto \overline{x} = t(\alpha(x)) \quad \text{for all } x \in \mathrm{Cl}(\Phi).
$$

Observe that

$$
t \circ \alpha = \alpha \circ t.
$$

If V has finite dimension n and $(e_1, \ldots, e_n)$ is a basis of V, in view of previous remarks, conjugation is defined by

$$
\begin{aligned}
\overline{e_i} &= -e_i \\
\overline{e_{i_1} e_{i_2} \cdots e_{i_k}} &= (-1)^k e_{i_k} e_{i_{k-1}} \cdots e_{i_1}
\end{aligned}
$$

where $1 \le i_1 < i_2 \cdots < i_k \le n$, and, of course, $\overline{1} = 1$. Conjugation is an anti-automorphism.

The multiplicative group of invertible elements of $\mathrm{Cl}(\Phi)$ is denoted by $\mathrm{Cl}(\Phi)^*$. Observe that for any $x \in V$, if $\Phi(x) \ne 0$, then x is invertible because $x^2 = \Phi(x)$; that is, $x \in \mathrm{Cl}(\Phi)^*$.

Definition 1.4 Given a finite dimensional vector space, V, and a quadratic form, Φ, on V, the *Clifford group of* Φ is the group

$$\Gamma(\Phi) = \{x \in \mathrm{Cl}(\Phi)^* \mid \alpha(x)vx^{-1} \in V \quad \text{for all } v \in V\}.$$

The map $N \colon \mathrm{Cl}(Q) \to \mathrm{Cl}(Q)$ given by

$$N(x) = x\overline{x}$$

is called the *norm* of $\mathrm{Cl}(\Phi)$.

We see that the group $\Gamma(\Phi)$ acts on V *via*

$$x \cdot v = \alpha(x)vx^{-1},$$

where $x \in \Gamma(\Phi)$ and $v \in V$. Actually, it is not entirely obvious why the action $\Gamma(\Phi) \times V \longrightarrow V$ is a linear action, and for that matter, why $\Gamma(\Phi)$ is a group.

This is because V is finite-dimensional and α is an automorphism. As a consequence, for any $x \in \Gamma(\Phi)$, the map ρ_x from V to V defined by

$$v \mapsto \alpha(x)vx^{-1}$$

is linear and injective, and thus bijective, since V has finite dimension. It follows that $x^{-1} \in \Gamma(\Phi)$ (the reader should fill in the details).

We also define the group $\Gamma^+(\Phi)$, called the *special Clifford group*, by

$$\Gamma^+(\Phi) = \Gamma(\Phi) \cap \mathrm{Cl}^0(\Phi).$$

Observe that $N(v) = -\Phi(v) \cdot 1$ for all $v \in V$. Also, if $(e_1, \dots, e_n)$ is a basis of V, we leave it as an exercise to check that

$$N(e_{i_1}e_{i_2} \cdots e_{i_k}) = (-1)^k \Phi(e_{i_1})\Phi(e_{i_2}) \cdots \Phi(e_{i_k}) \cdot 1.$$

Remark: The map $\rho \colon \Gamma(\Phi) \to \mathbf{GL}(V)$ given by $x \mapsto \rho_x$ is called the *twisted adjoint representation*. It was introduced by Atiyah, Bott and Shapiro [3]. It has the advantage of not introducing a spurious negative sign, i.e., when $v \in V$ and $\Phi(v) \neq 0$, the map ρ_v is the reflection s_v about the hyperplane orthogonal to v (see Proposition 1.6). Furthermore, when Φ is nondegenerate, the kernel $\mathrm{Ker}\,(\rho)$ of the representation ρ is given by $\mathrm{Ker}\,(\rho) = \mathbb{R}^* \cdot 1$, where $\mathbb{R}^* = \mathbb{R} - \{0\}$. The earlier *adjoint representation* (used by Chevalley [10] and others) is given by

$$v \mapsto xvx^{-1}.$$

Unfortunately, in this case, ρ_x represents $-s_v$, where s_v is the reflection about the hyperplane orthogonal to v. Furthermore, the kernel of the representation ρ is generally bigger than $\mathbb{R}^* \cdot 1$. This is the reason why the twisted adjoint representation is preferred (and must be used for a proper treatment of the **Pin** group).

Proposition 1.5 *The maps α and t induce an automorphism and an anti-automorphism of the Clifford group, $\Gamma(\Phi)$.*

Proof. It is not very instructive, see Bröcker and tom Dieck [7], Chapter 1, Section 6, page 58.

The following proposition shows why Clifford groups generalize the quaternions.

Proposition 1.6 *Let V be a finite dimensional vector space and Φ a quadratic form on V. For every element x, of the Clifford group $\Gamma(\Phi)$, if $x \in V$ and $\Phi(x) \neq 0$, then the map $\rho_x \colon V \to V$ given by*

$$v \mapsto \alpha(x)vx^{-1} \quad \text{for all } v \in V$$

is the reflection about the hyperplane H orthogonal to the vector x.

Proof. Recall that the reflection s about the hyperplane H orthogonal to the vector x is given by

$$s(u) = u - 2\,\frac{\varphi(u, x)}{\Phi(x)} \cdot x.$$

However, we have

$$x^2 = \Phi(x) \cdot 1 \quad \text{and} \quad ux + xu = 2\varphi(u, x) \cdot 1.$$

Thus, we have

$$
\begin{aligned}
s(u) &= u - 2\,\frac{\varphi(u, x)}{\Phi(x)} \cdot x \\
&= u - 2\varphi(u, x) \cdot \left(\frac{1}{\Phi(x)} \cdot x \right) \\
&= u - 2\varphi(u, x) \cdot x^{-1} \\
&= u - 2\varphi(u, x) \cdot (1x^{-1}) \\
&= u - (2\varphi(u, x) \cdot 1)x^{-1} \\
&= u - (ux + xu)x^{-1} \\
&= -xux^{-1} \\
&= \alpha(x)ux^{-1},
\end{aligned}
$$

since $\alpha(x) = -x$, for $x \in V$. $\square$

In general, we have a map

$$\rho \colon \Gamma(\Phi) \to \mathbf{GL}(V)$$

defined by

$$\rho(x)(v) = \alpha(x)vx^{-1},$$

for all $x \in \Gamma(\Phi)$ and all $v \in V$. We would like to show that ρ is a surjective homomorphism from $\Gamma(\Phi)$ onto $\mathbf{O}(\varphi)$ and a surjective homomorphism from $\Gamma^+(\Phi)$ onto $\mathbf{SO}(\varphi)$. For this,

we will need to assume that φ is nondegenerate, which means that for every $v \in V$, if $\varphi(v, w) = 0$ for all $w \in V$, then $v = 0$. For simplicity of exposition, we first assume that Φ is the quadratic form on $\mathbb{R}^n$ defined by

$$\Phi(x_1, \ldots, x_n) = -(x_1^2 + \cdots + x_n^2).$$

Let Cl_n denote the Clifford algebra $\mathrm{Cl}(\Phi)$ and Γ_n denote the Clifford group $\Gamma(\Phi)$. The following lemma plays a crucial role:

Lemma 1.7 *The kernel of the map* $\rho\colon \Gamma_n \to \mathbf{GL}(n)$ *is* $\mathbb{R}^* \cdot 1$, *the multiplicative group of nonzero scalar multiples of* $1 \in \mathrm{Cl}_n$.

Proof. If $\rho(x) = \mathrm{id}$, then

$$\alpha(x)v = vx \quad \text{for all } v \in \mathbb{R}^n. \tag{1}$$

Since $\mathrm{Cl}_n = \mathrm{Cl}_n^0 \oplus \mathrm{Cl}_n^1$, we can write $x = x^0 + x^1$, with $x^i \in \mathrm{Cl}_n^i$ for $i = 1, 2$. Then, equation (1) becomes

$$x^0 v = v x^0 \quad \text{and} \quad -x^1 v = v x^1 \quad \text{for all } v \in \mathbb{R}^n. \tag{2}$$

Using Theorem 1.4, we can express x^0 as a linear combination of monomials in the canonical basis $(e_1, \ldots, e_n)$, so that

$$x^0 = a^0 + e_1 b^1, \quad \text{with } a^0 \in \mathrm{Cl}_n^0, \ b^1 \in \mathrm{Cl}_n^1,$$

where neither a^0 nor b^1 contains a summand with a factor e_1. Applying the first relation in (2) to $v = e_1$, we get

$$e_1 a^0 + e_1^2 b^1 = a^0 e_1 + e_1 b^1 e_1. \tag{3}$$

Now, the basis $(e_1, \ldots, e_n)$ is orthogonal w.r.t. Φ, which implies that

$$e_j e_k = -e_k e_j \quad \text{for all } j \neq k.$$

Since each monomial in a^0 is of even degre and contains no factor e_1, we get

$$a^0 e_1 = e_1 a^0.$$

Similarly, since b^1 is of odd degree and contains no factor e_1, we get

$$e_1 b^1 e_1 = -e_1^2 b^1.$$

But then, from (3), we get

$$e_1 a^0 + e_1^2 b^1 = a^0 e_1 + e_1 b^1 e_1 = e_1 a^0 - e_1^2 b^1,$$

and so, $e_1^2 b^1 = 0$. However, $e_1^2 = -1$, and so, $b_1 = 0$. Therefore, x_0 contains no monomial with a factor e_1. We can apply the same argument to the other basis elements $e_2, \ldots, e_n$, and thus, we just proved that $x^0 \in \mathbb{R} \cdot 1$.

A similar argument applying to the second equation in (2), with $x^1 = a^1 + e_1 b^0$ and $v = e_1$ shows that $b^0 = 0$. We also conclude that $x^1 \in \mathbb{R} \cdot 1$. However, $\mathbb{R} \cdot 1 \subseteq \mathrm{Cl}_n^0$, and so, $x^1 = 0$. Finally, $x = x^0 \in (\mathbb{R} \cdot 1) \cap \Gamma_n = \mathbb{R}^* \cdot 1$. $\square$

Remark: If Φ is any nondegenerate quadratic form, we know (for instance, see Artin [1], Chapter 7, or Gallier [16], Chapter 6, Problem 6.14) that there is an orthogonal basis $(e_1, \ldots, e_n)$ with respect to φ (i.e. $\varphi(e_j, e_k) = 0$ for all $j \neq k$). Thus, the commutation relations

$$\begin{aligned} e_j^2 &= \Phi(e_j) \cdot 1, \quad \text{with } \Phi(e_j) \neq 0, \quad 1 \leq j \leq n, \quad \text{and} \\ e_j e_k &= -e_k e_j, \quad 1 \leq j, k \leq n, \ j \neq k \end{aligned}$$

hold, and since the proof only rests on these facts, Lemma 1.7 holds for any nondegenerate quadratic form.

However, Lemma 1.7 may fail for degenerate quadratic forms. For example, if $\Phi \equiv 0$, then $\mathrm{Cl}(V, 0) = \bigwedge^\bullet V$. Consider the element $x = 1 + e_1 e_2$. Clearly, $x^{-1} = 1 - e_1 e_2$. But now, for any $v \in V$, we have

$$\alpha(1 + e_1 e_2)v(1 + e_1 e_2)^{-1} = (1 + e_1 e_2)v(1 - e_1 e_2) = v.$$

Yet, $1 + e_1 e_2$ is not a scalar multiple of 1.

The following proposition shows that the notion of norm is well-behaved.

Proposition 1.8 *If $x \in \Gamma_n$, then $N(x) \in \mathbb{R}^* \cdot 1$.*

Proof. The trick is to show that $N(x)$ is in the kernel of ρ. To say that $x \in \Gamma_n$ means that

$$\alpha(x)vx^{-1} \in \mathbb{R}^n \quad \text{for all } v \in \mathbb{R}^n.$$

Applying t, we get

$$t(x)^{-1}vt(\alpha(x)) = \alpha(x)vx^{-1},$$

since t is the identity on $\mathbb{R}^n$. Thus, we have

$$v = t(x)\alpha(x)v(t(\alpha(x))x)^{-1} = \alpha(\overline{x}x)v(\overline{x}x)^{-1},$$

so $\overline{x}x \in \mathrm{Ker}\,(\rho)$. By Proposition 1.5, we have $\overline{x} \in \Gamma_n$, and so, $x\overline{x} = \overline{\overline{x}}\,\overline{x} \in \mathrm{Ker}\,(\rho)$. $\square$

Remark: Again, the proof also holds for the Clifford group $\Gamma(\Phi)$ associated with any nondegenerate quadratic form Φ. When $\Phi(v) = -\|v\|^2$, where $\|v\|$ is the standard Euclidean norm of v, we have $N(v) = \|v\|^2 \cdot 1$ for all $v \in V$. However, for other quadratic forms, it is possible that $N(x) = \lambda \cdot 1$ where $\lambda < 0$, and this is a difficulty that needs to be overcome.

Proposition 1.9 *The restriction of the norm, N, to Γ_n is a homomorphism, $N \colon \Gamma_n \to \mathbb{R}^* \cdot 1$, and $N(\alpha(x)) = N(x)$ for all $x \in \Gamma_n$.*

Proof. We have

$$N(xy) = xy\overline{y}\,\overline{x} = xN(y)\overline{x} = x\overline{x}N(y) = N(x)N(y),$$

where the third equality holds because $N(x) \in \mathbb{R}^* \cdot 1$. We also have

$$N(\alpha(x)) = \alpha(x)\alpha(\overline{x}) = \alpha(x\overline{x}) = \alpha(N(x)) = N(x).$$

$\square$

Remark: The proof also holds for the Clifford group $\Gamma(\Phi)$ associated with any nondegenerate quadratic form Φ.

Proposition 1.10 *We have $\mathbb{R}^n - \{0\} \subseteq \Gamma_n$ and $\rho(\Gamma_n) \subseteq \mathbf{O}(n)$.*

Proof. Let $x \in \Gamma_n$ and $v \in \mathbb{R}^n$, with $v \neq 0$. We have

$$N(\rho(x)(v)) = N(\alpha(x)vx^{-1}) = N(\alpha(x))N(v)N(x^{-1}) = N(x)N(v)N(x)^{-1} = N(v),$$

since $N \colon \Gamma_n \to \mathbb{R}^* \cdot 1$. However, for $v \in \mathbb{R}^n$, we know that

$$N(v) = -\Phi(v) \cdot 1.$$

Thus, $\rho(x)$ is norm-preserving, and so, $\rho(x) \in \mathbf{O}(n)$. $\square$

Remark: The proof that $\rho(\Gamma(\Phi)) \subseteq \mathbf{O}(\Phi)$ also holds for the Clifford group $\Gamma(\Phi)$ associated with any nondegenerate quadratic form Φ. The first statement needs to be replaced by the fact that every non-isotropic vector in $\mathbb{R}^n$ (a vector is non-isotropic if $\Phi(x) \neq 0$) belongs to $\Gamma(\Phi)$. Indeed, $x^2 = \Phi(x) \cdot 1$, which implies that x is invertible.

   We are finally ready for the introduction of the groups **Pin**(n) and **Spin**(n).

1.4   The Groups Pin(n) and Spin(n)

Definition 1.5 We define the *pinor group*, **Pin**(n), as the kernel Ker (N) of the homomorphism $N \colon \Gamma_n \to \mathbb{R}^* \cdot 1$, and the *spinor group*, **Spin**(n), as **Pin**$(n) \cap \Gamma_n^+$.

   Observe that if $N(x) = 1$, then x is invertible and $x^{-1} = \overline{x}$, since $x\overline{x} = N(x) = 1$. Thus, we can write

$$\mathbf{Pin}(n) = \{x \in \mathrm{Cl}_n \mid xvx^{-1} \in \mathbb{R}^n \quad \text{for all } v \in \mathbb{R}^n, \quad N(x) = 1\},$$

and

$$\mathbf{Spin}(n) = \{x \in \mathrm{Cl}_n^0 \mid xvx^{-1} \in \mathbb{R}^n \quad \text{for all } v \in \mathbb{R}^n, \quad N(x) = 1\}.$$

Remark: According to Atiyah, Bott and Shapiro, the use of the name **Pin**(k) is a joke due to Jean-Pierre Serre (Atiyah, Bott and Shapiro [3], page 1).

Theorem 1.11 *The restriction of ρ to the pinor group,* **Pin**(n)*, is a surjective homomorphism, ρ:* **Pin**$(n) \to$ **O**(n)*, whose kernel is* $\{-1, 1\}$*, and the restriction of ρ to the spinor group,* **Spin**(n)*, is a surjective homomorphism, ρ:* **Spin**$(n) \to$ **SO**(n)*, whose kernel is* $\{-1, 1\}$*.*

Proof. By Proposition 1.10, we have a map ρ: **Pin**$(n) \to$ **O**(n). The reader can easily check that ρ is a homomorphism. By the Cartan-Dieudonné theorem (see Bourbaki [6], or Gallier [16], Chapter 7, Theorem 7.2.1), every isometry $f \in$ **SO**(n) is the composition $f = s_1 \circ \cdots \circ s_k$ of hyperplane reflections s_j. If we assume that s_j is a reflection about the hyperplane H_j orthogonal to the nonzero vector w_j, by Proposition 1.6, $\rho(w_j) = s_j$. Since $N(w_j) = \|w_j\|^2 \cdot 1$, we can replace w_j by $w_j / \|w_j\|$, so that $N(w_1 \cdots w_k) = 1$, and then

$$f = \rho(w_1 \cdots w_k),$$

and ρ is surjective. Note that

$$\mathrm{Ker}\,(\rho \mid \mathbf{Pin}(n)) = \mathrm{Ker}\,(\rho) \cap \ker(N) = \{t \in \mathbb{R}^* \cdot 1 \mid N(t) = 1\} = \{-1, 1\}.$$

As to **Spin**(n), we just need to show that the restriction of ρ to **Spin**(n) maps Γ_n into **SO**(n). If this was not the case, there would be some improper isometry $f \in$ **O**(n) so that $\rho(x) = f$, where $x \in \Gamma_n \cap \mathrm{Cl}_n^0$. However, we can express f as the composition of an odd number of reflections, say

$$f = \rho(w_1 \cdots w_{2k+1}).$$

Since

$$\rho(w_1 \cdots w_{2k+1}) = \rho(x),$$

we have $x^{-1} w_1 \cdots w_{2k+1} \in \mathrm{Ker}\,(\rho)$. By Lemma 1.7, we must have

$$x^{-1} w_1 \cdots w_{2k+1} = \lambda \cdot 1$$

for some $\lambda \in \mathbb{R}^*$, and thus,

$$w_1 \cdots w_{2k+1} = \lambda \cdot x,$$

where x has even degree and $w_1 \cdots w_{2k+1}$ has odd degree, which is impossible. $\square$

Let us denote the set of elements $v \in \mathbb{R}^n$ with $N(v) = 1$ (with norm 1) by S^{n-1}. We have the following corollary of Theorem 1.11:

Corollary 1.12 *The group* **Pin**(n) *is generated by S^{n-1} and every element of* **Spin**(n) *can be written as the product of an even number of elements of S^{n-1}.*

Example 1.2 The reader should verify that

$$\mathbf{Pin}(1) \approx \mathbb{Z}/4\mathbb{Z}, \quad \mathbf{Spin}(1) = \{-1, 1\} \approx \mathbb{Z}/2\mathbb{Z},$$

and also that

$$\mathbf{Pin}(2) \approx \{ae_1 + be_2 \mid a^2 + b^2 = 1\} \cup \{c1 + de_1e_2 \mid c^2 + d^2 = 1\}, \quad \mathbf{Spin}(2) = \mathbf{U}(1).$$

We may also write $\mathbf{Pin}(2) = \mathbf{U}(1) + \mathbf{U}(1)$, where $\mathbf{U}(1)$ is the group of complex numbers of modulus 1 (the unit circle in $\mathbb{R}^2$). It can also be shown that $\mathbf{Spin}(3) \approx \mathbf{SU}(2)$ and $\mathbf{Spin}(4) \approx \mathbf{SU}(2) \times \mathbf{SU}(2)$. The group $\mathbf{Spin}(5)$ is isomorphic to the symplectic group $\mathbf{Sp}(2)$, and $\mathbf{Spin}(6)$ is isomorphic to $\mathbf{SU}(4)$ (see Curtis [12] or Porteous [24]).

Let us take a closer look at $\mathbf{Spin}(2)$. The Clifford algebra Cl_2 is generated by the four elements

$$1, \; e_1, \; e_2, \; , e_1e_2,$$

and they satisfy the relations

$$e_1^2 = -1, \quad e_2^2 = -1, \quad e_1e_2 = -e_2e_1.$$

The group $\mathbf{Spin}(2)$ consists of all products

$$\prod_{i=1}^{2k} (a_ie_1 + b_ie_2)$$

consisting of an even number of factors and such that $a_i^2 + b_i^2 = 1$. In view of the above relations, every such element can be written as

$$x = a1 + be_1e_2,$$

where x satisfies the conditions that $xvx^{-1} \in \mathbb{R}^2$ for all $v \in \mathbb{R}^2$, and $N(x) = 1$. Since

$$\overline{X} = a1 - be_1e_2,$$

we get

$$N(x) = a^2 + b^2,$$

and the condition $N(x) = 1$ is simply $a^2 + b^2 = 1$. We claim that $xvx^{-1} \in \mathbb{R}^2$ if $x \in \mathrm{Cl}_2^0$. Indeed, since $x \in \mathrm{Cl}_2^0$ and $v \in \mathrm{Cl}_2^1$, we have $xvx^{-1} \in \mathrm{Cl}_2^1$, which implies that $xvx^{-1} \in \mathbb{R}^2$, since the only elements of Cl_2^1 are those in $\mathbb{R}^2$. Then, $\mathbf{Spin}(2)$ consists of those elements $x = a1 + be_1e_2$ so that $a^2 + b^2 = 1$. If we let $\mathbf{i} = e_1e_2$, we observe that

$$\begin{aligned} \mathbf{i}^2 &= -1, \\ e_1\mathbf{i} &= -\mathbf{i}e_1 = -e_2, \\ e_2\mathbf{i} &= -\mathbf{i}e_2 = e_1. \end{aligned}$$

Thus, $\mathbf{Spin}(2)$ is isomorphic to $\mathbf{U}(1)$. Also note that

$$e_1(a1 + b\mathbf{i}) = (a1 - b\mathbf{i})e_1.$$

Let us find out explicitly what is the action of **Spin**(2) on $\mathbb{R}^2$. Given $X = a1 + b\mathbf{i}$, with $a^2 + b^2 = 1$, for any $v = v_1 e_1 + v_2 e_2$, we have

$$
\begin{aligned}
\alpha(X)vX^{-1} &= X(v_1 e_1 + v_2 e_2)X^{-1} \\
&= X(v_1 e_1 + v_2 e_2)(-e_1 e_1)\overline{X} \\
&= X(v_1 e_1 + v_2 e_2)(-e_1)(e_1 \overline{X}) \\
&= X(v_1 1 + v_2 \mathbf{i})X e_1 \\
&= X^2(v_1 1 + v_2 \mathbf{i})e_1 \\
&= (((a^2 - b^2)v_1 - 2abv_2)1 + (a^2 - b^2)v_2 + 2abv_1)\mathbf{i})e_1 \\
&= ((a^2 - b^2)v_1 - 2abv_2)e_1 + (a^2 - b^2)v_2 + 2abv_1)e_2.
\end{aligned}
$$

Since $a^2 + b^2 = 1$, we can write $X = a1 + b\mathbf{i} = (\cos\theta)1 + (\sin\theta)\mathbf{i}$, and the above derivation shows that

$$
\alpha(X)vX^{-1} = (\cos 2\theta v_1 - \sin 2\theta v_2)e_1 + (\cos 2\theta v_2 + \sin 2\theta v_1)e_2.
$$

This means that the rotation ρ_X induced by $X \in \mathbf{Spin}(2)$ is the rotation of angle 2θ around the origin. Observe that the maps

$$
v \mapsto v(-e_1), \quad X \mapsto Xe_1
$$

establish bijections between $\mathbb{R}^2$ and $\mathbf{Spin}(2) \simeq \mathbf{U}(1)$. Also, note that the action of $X = \cos\theta + i\sin\theta$ viewed as a complex number yields the rotation of angle θ, whereas the action of $X = (\cos\theta)1 + (\sin\theta)\mathbf{i}$ viewed as a member of $\mathbf{Spin}(2)$ yields the rotation of angle 2θ. There is nothing wrong. In general, $\mathbf{Spin}(n)$ is a two–to–one cover of $\mathbf{SO}(n)$.

Next, let us take a closer look at $\mathbf{Spin}(3)$. The Clifford algebra Cl$_3$ is generated by the eight elements

$$
1, \ e_1, \ e_2, \ , e_3, \ , e_1 e_2, \ e_2 e_3, \ e_3 e_1, \ e_1 e_2 e_3,
$$

and they satisfy the relations

$$
e_i^2 = -1, \quad e_j e_j = -e_j e_i, \quad 1 \leq i, j \leq 3, \ i \neq j.
$$

The group $\mathbf{Spin}(3)$ consists of all products

$$
\prod_{i=1}^{2k}(a_i e_1 + b_i e_2 + c_i e_3)
$$

consisting of an even number of factors and such that $a_i^2 + b_i^2 + c_i^2 = 1$. In view of the above relations, every such element can be written as

$$
x = a1 + be_2 e_3 + ce_3 e_1 + de_1 e_2,
$$

where x satisfies the conditions that $xvx^{-1} \in \mathbb{R}^3$ for all $v \in \mathbb{R}^3$, and $N(x) = 1$. Since

$$\overline{X} = a1 - be_2e_3 - ce_3e_1 - de_1e_2,$$

we get

$$N(x) = a^2 + b^2 + c^2 + d^2,$$

and the condition $N(x) = 1$ is simply $a^2 + b^2 + c^2 + d^2 = 1$.

It turns out that the conditions $x \in Cl_3^0$ and $N(x) = 1$ imply that $xvx^{-1} \in \mathbb{R}^3$ for all $v \in \mathbb{R}^3$. To prove this, first observe that $N(x) = 1$ implies that $x^{-1} = \pm\overline{x}$, and that $\overline{v} = -v$ for any $v \in \mathbb{R}^3$, and so,

$$\overline{xvx^{-1}} = -xvx^{-1}.$$

Also, since $x \in Cl_3^0$ and $v \in Cl_3^1$, we have $xvx^{-1} \in Cl_3^1$. Thus, we can write

$$xvx^{-1} = u + \lambda e_1e_2e_3, \quad \text{for some } u \in \mathbb{R}^3 \text{ and some } \lambda \in \mathbb{R}.$$

But

$$\overline{e_1e_2e_3} = -e_3e_2e_1 = e_1e_2e_3,$$

and so,

$$\overline{xvx^{-1}} = -u + \lambda e_1e_2e_3 = -xvx^{-1} = -u - \lambda e_1e_2e_3,$$

which implies that $\lambda = 0$. Thus, $xvx^{-1} \in \mathbb{R}^3$, as claimed. Then, **Spin**(3) consists of those elements $x = a1 + be_2e_3 + ce_3e_1 + de_1e_2$ so that $a^2 + b^2 + c^2 + d^2 = 1$. Under the bijection

$$\mathbf{i} \mapsto e_2e_3, \ \mathbf{j} \mapsto e_3e_1, \ \mathbf{k} \mapsto e_1e_2,$$

we can check that we have an isomorphism between the group **SU**(2) of unit quaternions and **Spin**(3). If $X = a1 + be_2e_3 + ce_3e_1 + de_1e_2 \in$ **Spin**(3), observe that

$$X^{-1} = \overline{X} = a1 - be_2e_3 - ce_3e_1 - de_1e_2.$$

Now, using the identification

$$\mathbf{i} \mapsto e_2e_3, \ \mathbf{j} \mapsto e_3e_1, \ \mathbf{k} \mapsto e_1e_2,$$

we can easily check that

$$\begin{aligned}
(e_1e_2e_3)^2 &= 1, \\
(e_1e_2e_3)\mathbf{i} &= \mathbf{i}(e_1e_2e_3) = -e_1, \\
(e_1e_2e_3)\mathbf{j} &= \mathbf{j}(e_1e_2e_3) = -e_2, \\
(e_1e_2e_3)\mathbf{k} &= \mathbf{k}(e_1e_2e_3) = -e_3, \\
(e_1e_2e_3)e_1 &= -\mathbf{i}, \\
(e_1e_2e_3)e_2 &= -\mathbf{j}, \\
(e_1e_2e_3)e_3 &= -\mathbf{k}.
\end{aligned}$$

Then, if $X = a1 + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \in \mathbf{Spin}(3)$, for every $v = v_1 e_1 + v_2 e_2 + v_3 e_3$, we have

$$
\begin{aligned}
\alpha(X)vX^{-1} &= X(v_1 e_1 + v_2 e_2 + v_3 e_3)X^{-1} \\
&= X(e_1 e_2 e_3)^2(v_1 e_1 + v_2 e_2 + v_3 e_3)X^{-1} \\
&= (e_1 e_2 e_3)X(e_1 e_2 e_3)(v_1 e_1 + v_2 e_2 + v_3 e_3)X^{-1} \\
&= -(e_1 e_2 e_3)X(v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k})X^{-1}.
\end{aligned}
$$

This shows that the rotation $\rho_X \in \mathbf{SO}(3)$ induced by $X \in \mathbf{Spin}(3)$ can be viewed as the rotation induced by the quaternion $a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ on the pure quaternions, using the maps

$$
v \mapsto -(e_1 e_2 e_3)v, \quad X \mapsto -(e_1 e_2 e_3)X
$$

to go from a vector $v = v_1 e_1 + v_2 e_2 + v_3 e_3$ to the pure quaternion $v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k}$, and back.

We close this section by taking a closer look at $\mathbf{Spin}(4)$. The group $\mathbf{Spin}(4)$ consists of all products

$$
\prod_{i=1}^{2k}(a_i e_1 + b_i e_2 + c_i e_3 + d_i e_4)
$$

consisting of an even number of factors and such that $a_i^2 + b_i^2 + c_i^2 + d_i^2 = 1$. Using the relations

$$
e_i^2 = -1, \quad e_j e_j = -e_j e_i, \quad 1 \le i, j \le 4, \ i \ne j,
$$

every element of $\mathbf{Spin}(4)$ can be written as

$$
x = a_1 1 + a_2 e_1 e_2 + a_3 e_2 e_3 + a_4 e_3 e_1 + a_5 e_4 e_3 + a_6 e_4 e_1 + a_7 e_4 e_2 + a_8 e_1 e_2 e_3 e_4,
$$

where x satisfies the conditions that $xvx^{-1} \in \mathbb{R}^4$ for all $v \in \mathbb{R}^4$, and $N(x) = 1$. Let

$$
\mathbf{i} = e_1 e_2, \ \mathbf{j} = e_2 e_3, \ \mathbf{k} = 3_3 e_1, \ \mathbf{i}' = e_4 e_3, \ \mathbf{j}' = e_4 e_1, \ \mathbf{k}' = e_4 e_2,
$$

and $\mathbb{I} = e_1 e_2 e_3 e_4$. The reader will easily verify that

$$
\begin{aligned}
\mathbf{ij} &= \mathbf{k} \\
\mathbf{jk} &= \mathbf{i} \\
\mathbf{ki} &= \mathbf{j} \\
\mathbf{i}^2 &= -1, \quad \mathbf{j}^2 = -1, \quad \mathbf{k}^2 = -1 \\
\mathbf{i}\mathbb{I} &= \mathbb{I}\mathbf{i} = \mathbf{i}' \\
\mathbf{j}\mathbb{I} &= \mathbb{I}\mathbf{j} = \mathbf{j}' \\
\mathbf{k}\mathbb{I} &= \mathbb{I}\mathbf{k} = \mathbf{k}' \\
\mathbb{I}^2 &= 1, \quad \bar{\mathbb{I}} = \mathbb{I}.
\end{aligned}
$$

Then, every $x \in \mathbf{Spin}(4)$ can be written as

$$
x = u + \mathbb{I}v, \quad \text{with} \quad u = a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \quad \text{and} \quad v = a'\mathbf{1} + b'\mathbf{i} + c'\mathbf{j} + d'\mathbf{k},
$$

with the extra conditions stated above. Using the above identities, we have

$$(u + \mathbb{I}v)(u' + \mathbb{I}v') = uu' + vv' + \mathbb{I}(uv' + vu').$$

As a consequence,

$$N(u + \mathbb{I}v) = (u + \mathbb{I}v)(\overline{u} + \mathbb{I}\overline{v}) = u\overline{u} + v\overline{v} + \mathbb{I}(u\overline{v} + v\overline{u}),$$

and thus, $N(u + \mathbb{I}v) = 1$ is equivalent to

$$u\overline{u} + v\overline{v} = 1 \quad \text{and} \quad u\overline{v} + v\overline{u} = 0.$$

As in the case $n = 3$, it turns out that the conditions $x \in \mathrm{Cl}_4^0$ and $N(x) = 1$ imply that $xvx^{-1} \in \mathbb{R}^4$ for all $v \in \mathbb{R}^4$. The only change to the proof is that $xvx^{-1} \in \mathrm{Cl}_4^1$ can be written as

$$xvx^{-1} = u + \sum_{i,j,k} \lambda_{i,j,k} e_i e_j e_k, \quad \text{for some } u \in \mathbb{R}^4, \quad \text{with } \{i, j, k\} \subseteq \{1, 2, 3, 4\}.$$

As in the previous proof, we get $\lambda_{i,j,k} = 0$. Then, **Spin**(4) consists of those elements $u + \mathbb{I}v$ so that

$$u\overline{u} + v\overline{v} = 1 \quad \text{and} \quad u\overline{v} + v\overline{u} = 0,$$

with u and v of the form $a1 + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$. Finally, we see that **Spin**(4) is isomorphic to **Spin**(2) × **Spin**(2) under the isomorphism

$$u + v\mathbb{I} \mapsto (u + v, u - v).$$

Indeed, we have

$$N(u + v) = (u + v)(\overline{u} + \overline{v}) = 1,$$

and

$$N(u - v) = (u - v)(\overline{u} - \overline{v}) = 1,$$

since

$$u\overline{u} + v\overline{v} = 1 \quad \text{and} \quad u\overline{v} + v\overline{u} = 0,$$

and

$$(u + v, u - v)(u' + v', u' - v') = (uu' + vv' + uv' + vu', uu' + vv' - (uv' + vu')).$$

Remark: It can be shown that the assertion if $x \in \mathrm{Cl}_n^0$ and $N(x) = 1$, then $xvx^{-1} \in \mathbb{R}^n$ for all $v \in \mathbb{R}^n$, is true up to $n = 5$ (see Porteous [24], Chapter 13, Proposition 13.58). However, this is already false for $n = 6$. For example, if $X = 1/\sqrt{2}(1 + e_1e_2e_3e_4e_5e_6)$, it is easy to see that $N(X) = 1$, and yet, $Xe_1X^{-1} \notin \mathbb{R}^6$.

1.5   The Groups $\mathbf{Pin}(p, q)$ and $\mathbf{Spin}(p, q)$

For every nondegenerate quadratic form Φ over $\mathbb{R}$, there is an orthogonal basis with respect to which Φ is given by

$$\Phi(x_1, \ldots, x_{p+q}) = x_1^2 + \cdots + x_p^2 - (x_{p+1}^2 + \cdots + x_{p+q}^2),$$

where p and q only depend on Φ. The quadratic form corresponding to (p, q) is denoted $\Phi_{p,q}$ and we call (p, q) the *signature of* $\Phi_{p,q}$. Let $n = p + q$. We define the group $\mathbf{O}(p, q)$ as the group of isometries w.r.t. $\Phi_{p,q}$, i.e., the group of linear maps f so that

$$\Phi_{p,q}(f(v)) = \Phi_{p,q}(v) \quad \text{for all } v \in \mathbb{R}^n$$

and the group $\mathbf{SO}(p, q)$ as the subgroup of $\mathbf{O}(p, q)$ consisting of the isometries, $f \in \mathbf{O}(p, q)$, with $\det(f) = 1$. We denote the Clifford algebra $\mathrm{Cl}(\Phi_{p,q})$ where $\Phi_{p,q}$ has signature (p, q) by $\mathrm{Cl}_{p,q}$, the corresponding Clifford group by $\Gamma_{p,q}$, and the special Clifford group $\Gamma_{p,q} \cap \mathrm{Cl}_{p,q}^0$ by $\Gamma_{p,q}^+$. Note that with this new notation, $\mathrm{Cl}_n = \mathrm{Cl}_{0,n}$.

> As we mentioned earlier, since Lawson and Michelsohn [20] adopt the opposite of our sign convention in defining Clifford algebras, their $\mathrm{Cl}(p, q)$ is our $\mathrm{Cl}(q, p)$.

As we mentioned in Section 1.3, we have the problem that $N(v) = -\Phi(v) \cdot 1$ but $-\Phi(v)$ is not necessarily positive (where $v \in \mathbb{R}^n$). The fix is simple: Allow elements $x \in \Gamma_{p,q}$ with $N(x) = \pm 1$.

Definition 1.6 We define the *pinor group,* $\mathbf{Pin}(p, q)$, as the group

$$\mathbf{Pin}(p, q) = \{x \in \Gamma_{p,q} \mid N(x) = \pm 1\},$$

and the *spinor group,* $\mathbf{Spin}(p, q)$, as $\mathbf{Pin}(p, q) \cap \Gamma_{p,q}^+$.

Remarks:

(1) It is easily checked that the group $\mathbf{Spin}(p, q)$ is also given by

$$\mathbf{Spin}(p, q) = \{x \in \mathrm{Cl}_{p,q}^0 \mid xv\overline{x} \in \mathbb{R}^n \quad \text{for all } v \in \mathbb{R}^n, \quad N(x) = 1\}.$$

This is because $\mathbf{Spin}(p, q)$ consists of elements of even degree.

(2) One can check that if $N(x) \neq 0$, then

$$\alpha(x)vx^{-1} = xvt(x)/N(x).$$

Thus, we have

$$\mathbf{Pin}(p, q) = \{x \in \mathrm{Cl}_{p,q} \mid xvt(x)N(x) \in \mathbb{R}^n \quad \text{for all } v \in \mathbb{R}^n, \quad N(x) = \pm 1\}.$$

When $\Phi(x) = -\|x\|^2$, we have $N(x) = \|x\|^2$, and

$$\mathbf{Pin}(n) = \{x \in \mathrm{Cl}_n \mid xvt(x) \in \mathbb{R}^n \quad \text{for all } v \in \mathbb{R}^n, \quad N(x) = 1\}.$$

Theorem 1.11 generalizes as follows:

Theorem 1.13 *The restriction of ρ to the pinor group,* **Pin**(p,q), *is a surjective homomorphism,* $\rho\colon$ **Pin**$(p,q) \to$ **O**(p,q), *whose kernel is* $\{-1,1\}$, *and the restriction of ρ to the spinor group,* **Spin**(p,q), *is a surjective homomorphism,* $\rho\colon$ **Spin**$(p,q) \to$ **SO**(p,q), *whose kernel is* $\{-1,1\}$.

Proof. The Cartan-Dieudonné also holds for any nondegenerate quadratic form Φ, in the sense that every isometry in **O**(Φ) is the composition of reflections defined by hyperplanes orthogonal to non-isotropic vectors (see Dieudonné [13], Chevalley [10], Bourbaki [6], or Gallier [16], Chapter 7, Problem 7.14). Thus, Theorem 1.11 also holds for any nondegenerate quadratic form Φ. The only change to the proof is the following: Since $N(w_j) = -\Phi(w_j) \cdot 1$, we can replace w_j by $w_j/\sqrt{|\Phi(w_j)|}$, so that $N(w_1 \cdots w_k) = \pm 1$, and then

$$f = \rho(w_1 \cdots w_k),$$

and ρ is surjective. $\square$

If we consider $\mathbb{R}^n$ equipped with the quadratic form $\Phi_{p,q}$ (with $n = p+q$), we denote the set of elements $v \in \mathbb{R}^n$ with $N(v) = 1$ by $S_{p,q}^{n-1}$. We have the following corollary of Theorem 1.13 (generalizing Corollary 1.14):

Corollary 1.14 *The group* **Pin**(p,q) *is generated by $S_{p,q}^{n-1}$ and every element of* **Spin**(p,q) *can be written as the product of an even number of elements of $S_{p,q}^{n-1}$.*

Example 1.3 The reader should check that

$$\mathrm{Cl}_{0,1} \approx \mathbb{C}, \quad \mathrm{Cl}_{1,0} \approx \mathbb{R} \oplus \mathbb{R}.$$

We also have

$$\mathbf{Pin}(0,1) \approx \mathbb{Z}/4\mathbb{Z}, \quad \mathbf{Pin}(1,0) \approx \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z},$$

from which we get **Spin**$(0,1) =$ **Spin**$(1,0) \approx \mathbb{Z}/2\mathbb{Z}$. Also, show that

$$\mathrm{Cl}_{0,2} \approx \mathbb{H}, \quad \mathrm{Cl}_{1,1} \approx M_2(\mathbb{R}), \quad \mathrm{Cl}_{2,0} \approx M_2(\mathbb{R}),$$

where $M_n(\mathbb{R})$ denotes the algebra of $n \times n$ matrices. One can also work out what are **Pin**$(2,0)$, **Pin**$(1,1)$, and **Pin**$(0,2)$; see Choquet-Bruhat [11], Chapter I, Section 7, page 26. Show that

$$\mathbf{Spin}(0,2) = \mathbf{Spin}(2,0) \approx \mathbf{U}(1),$$

and

$$\mathbf{Spin}(1,1) = \{a1 + be_1e_2 \mid a^2 - b^2 = 1\}.$$

Observe that **Spin**$(1,1)$ is not connected.

More generally, it can be shown that $\mathrm{Cl}_{p,q}^0$ and $\mathrm{Cl}_{q,p}^0$ are isomorphic, from which it follows that **Spin**(p,q) and **Spin**(q,p) are isomorphic, but **Pin**(p,q) and **Pin**(q,p) are not isomorphic in general, and in particular, **Pin**$(p,0)$ and **Pin**$(0,p)$ are not isomorphic in general (see Choquet-Bruhat [11], Chapter I, Section 7). However, due to the "8-periodicity" of the Clifford algebras (to be discussed in the next section), it follows that $\mathrm{Cl}_{p,q}$ and $\mathrm{Cl}_{q,p}$ are isomorphic when $|p - q| = 0 \bmod 4$.

1.6   Periodicity of the Clifford Algebras $\mathrm{Cl}_{p,q}$

It turns out that the real algebras $\mathrm{Cl}_{p,q}$ can be build up as tensor products of the basic algebras $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$. As pointed out by Lounesto (Section 23.16 [21]), the description of the real algebras $\mathrm{Cl}_{p,q}$ as matrix algebras and the 8-periodicity was first observed by Elie Cartan in 1908; see Cartan's article, *Nombres Complexes*, based on the original article in German by E. Study, in Molk [23], article I-5 (fasc. 3), pages 329-468. These algebras are defined in Section 36 under the name "'Systems of Clifford and Lipschitz," page 463-466. Of course, Cartan used a very different notation; see page 464 in the article cited above. These facts were rediscovered independently by Raoul Bott in the 1960's (see Raoul Bott's comments in Volume 2 of his Collected papers.).

We will use the notation $\mathbb{R}(n)$ (resp. $\mathbb{C}(n)$) for the algebra $M_n(\mathbb{R})$ of all $n \times n$ real matrices (resp. the algebra $M_n(\mathbb{C})$ of all $n \times n$ complex matrices). As mentioned in Example 1.3, it is not hard to show that

$$\begin{aligned}
\mathrm{Cl}_{0,1} &= \mathbb{C} & \mathrm{Cl}_{1,0} &= \mathbb{R} \oplus \mathbb{R} \\
\mathrm{Cl}_{0,2} &= \mathbb{H} & \mathrm{Cl}_{2,0} &= \mathbb{R}(2)
\end{aligned}$$

and

$$\mathrm{Cl}_{1,1} = \mathbb{R}(2).$$

The key to the classification is the following lemma:

Lemma 1.15 *We have the isomorphisms*

$$\begin{aligned}
\mathrm{Cl}_{0,n+2} &\approx \mathrm{Cl}_{n,0} \otimes \mathrm{Cl}_{0,2} \\
\mathrm{Cl}_{n+2,0} &\approx \mathrm{Cl}_{0,n} \otimes \mathrm{Cl}_{2,0} \\
\mathrm{Cl}_{p+1,q+1} &\approx \mathrm{Cl}_{p,q} \otimes \mathrm{Cl}_{1,1},
\end{aligned}$$

for all $n, p, q \geq 0$.

Proof. Let $\Phi_{0,n}(x) = -\|x\|^2$, where $\|x\|$ is the standard Euclidean norm on $\mathbb{R}^{n+2}$, and let $(e_1, \ldots, e_{n+2})$ be an orthonormal basis for $\mathbb{R}^{n+2}$ under the standard Euclidean inner product. We also let $(e_1', \ldots, e_n')$ be a set of generators for $\mathrm{Cl}_{n,0}$ and (e_1'', e_2'') be a set of generators for $\mathrm{Cl}_{0,2}$. We can define a linear map $f\colon \mathbb{R}^{n+2} \to \mathrm{Cl}_{n,0} \otimes \mathrm{Cl}_{0,2}$ by its action on the basis $(e_1, \ldots, e_{n+2})$ as follows:

$$f(e_i) = \begin{cases} e_i' \otimes e_1'' e_2'' & \text{for } 1 \leq i \leq n \\ 1 \otimes e_{i-n}'' & \text{for } n+1 \leq i \leq n+2. \end{cases}$$

Observe that for $1 \leq i, j \leq n$, we have

$$f(e_i)f(e_j) + f(e_j)f(e_i) = (e_i' e_j' + e_j' e_i') \otimes (e_1'' e_e'')^2 = -2\delta_{ij} 1 \otimes 1,$$

since $e_1'' e_2'' = -e_2'' e_1''$, $(e_1'')^2 = -1$, and $(e_2'')^2 = -1$, and $e_i' e_j' = -e_j' e_i'$, for all $i \neq j$, and $(e_i')^2 = 1$, for all i with $1 \leq i \leq n$. Also, for $n + 1 \leq i, j \leq n + 2$, we have

$$f(e_i)f(e_j) + f(e_j)f(e_i) = 1 \otimes (e_{i-n}'' e_{j-n}'' + e_{j-n}'' e_{i-n}'') = -2\delta_{ij} 1 \otimes 1,$$

and

$$f(e_i)f(e_k) + f(e_k)f(e_i) = 2e_i' \otimes (e_1'' e_2'' e_{n-k}'' + e_{n-k}'' e_1'' e_2'') = 0,$$

for $1 \leq i, j \leq n$ and $n + 1 \leq k \leq n + 2$ (since $e_{n-k}'' = e_1''$ or $e_{n-k}'' = e_2''$). Thus, we have

$$f(x)^2 = -\|x\|^2 \cdot 1 \otimes 1 \quad \text{for all } x \in \mathbb{R}^{n+2},$$

and by the universal mapping property of $\mathrm{Cl}_{0,n+2}$, we get an algebra map

$$\widetilde{f} \colon \mathrm{Cl}_{0,n+2} \to \mathrm{Cl}_{n,0} \otimes \mathrm{Cl}_{0,2}.$$

Since $\widetilde{f}$ maps onto a set of generators, it is surjective. However

$$\dim(\mathrm{Cl}_{0,n+2}) = 2^{n+2} = 2^n \cdot 2 = \dim(\mathrm{Cl}_{n,0})\dim(\mathrm{Cl}_{0,2}) = \dim(\mathrm{Cl}_{n,0} \otimes \mathrm{Cl}_{0,2}),$$

and $\widetilde{f}$ is an isomorphism.

The proof of the second identity is analogous. For the third identity, we have

$$\Phi_{p,q}(x_1, \ldots, x_{p+q}) = x_1^2 + \cdots + x_p^2 - (x_{p+1}^2 + \cdots + x_{p+q}^2),$$

and let $(e_1, \ldots, e_{p+1}, \epsilon_1, \ldots, \epsilon_{q+1})$ be an orthogonal basis for $\mathbb{R}^{p+q+2}$ so that $\Phi_{p+1,q+1}(e_i) = +1$ and $\Phi_{p+1,q+1}(\epsilon_j) = -1$ for $i = 1, \ldots, p+1$ and $j = 1, \ldots, q+1$. Also, let $(e_1', \ldots, e_p', \epsilon_1', \ldots, \epsilon_q')$ be a set of generators for $\mathrm{Cl}_{p,q}$ and (e_1'', ϵ_1'') be a set of generators for $\mathrm{Cl}_{1,1}$. We define a linear map $f \colon \mathbb{R}^{p+q+2} \to \mathrm{Cl}_{p,q} \otimes \mathrm{Cl}_{1,1}$ by its action on the basis as follows:

$$f(e_i) = \begin{cases} e_i' \otimes e_1'' \epsilon_1'' & \text{for } 1 \leq i \leq p \\ 1 \otimes e_1'' & \text{for } i = p+1, \end{cases}$$

and

$$f(\epsilon_j) = \begin{cases} \epsilon_j' \otimes e_1'' \epsilon_1'' & \text{for } 1 \leq j \leq q \\ 1 \otimes \epsilon_1'' & \text{for } j = q+1. \end{cases}$$

We can check that

$$f(x)^2 = \Phi_{p+1,q+1}(x) \cdot 1 \otimes 1 \quad \text{for all } x \in \mathbb{R}^{p+q+2},$$

and we finish the proof as in the first case. $\square$

To apply this lemma, we need some further isomorphisms among various matrix algebras.

Proposition 1.16 *The following isomorphisms hold:*

$$\begin{aligned}
\mathbb{R}(m) \otimes \mathbb{R}(n) &\approx \mathbb{R}(mn) && \text{for all } m, n \geq 0 \\
\mathbb{R}(n) \otimes_{\mathbb{R}} K &\approx K(n) && \text{for } K = \mathbb{C} \text{ or } K = \mathbb{H} \text{ and all } n \geq 0 \\
\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} &\approx \mathbb{C} \oplus \mathbb{C} \\
\mathbb{C} \otimes_{\mathbb{R}} \mathbb{H} &\approx \mathbb{C}(2) \\
\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} &\approx \mathbb{R}(4).
\end{aligned}$$

Proof. Details can be found in Lawson and Michelsohn [20]. The first two isomorphisms are quite obvious. The third isomorphism $\mathbb{C} \oplus \mathbb{C} \to \mathbb{C} \otimes \mathbb{C}$ is obtained by sending

$$(1,0) \mapsto \frac{1}{2}(1 \otimes 1 + i \otimes i), \quad (0,1) \mapsto \frac{1}{2}(1 \otimes 1 - i \otimes i).$$

The field $\mathbb{C}$ is isomorphic to the subring of $\mathbb{H}$ generated by $\mathbf{i}$. Thus, we can view $\mathbb{H}$ as a $\mathbb{C}$-vector space under left scalar multiplication. Consider the $\mathbb{R}$-bilinear map $\pi \colon \mathbb{C} \times \mathbb{H} \to \mathrm{Hom}_{\mathbb{C}}(\mathbb{H}, \mathbb{H})$ given by

$$\pi_{y,z}(x) = yx\overline{z},$$

where $y \in \mathbb{C}$ and $x, z \in \mathbb{H}$. Thus, we get an $\mathbb{R}$-linear map $\pi \colon \mathbb{C} \otimes_{\mathbb{R}} \mathbb{H} \to \mathrm{Hom}_{\mathbb{C}}(\mathbb{H}, \mathbb{H})$. However, we have $\mathrm{Hom}_{\mathbb{C}}(\mathbb{H}, \mathbb{H}) \approx \mathbb{C}(2)$. Furthermore, since

$$\pi_{y,z} \circ \pi_{y',z'} = \pi_{yy',zz'},$$

the map π is an algebra homomorphism

$$\pi \colon \mathbb{C} \times \mathbb{H} \to \mathbb{C}(2).$$

We can check on a basis that π is injective, and since

$$\dim_{\mathbb{R}}(\mathbb{C} \times \mathbb{H}) = \dim_{\mathbb{R}}(\mathbb{C}(2)) = 8,$$

the map π is an isomorphism. The last isomorphism is proved in a similar fashion. $\square$

We now have the main periodicity theorem.

Theorem 1.17 *(Cartan/Bott) For all $n \geq 0$, we have the following isomorphisms:*

$$\begin{aligned}
\mathrm{Cl}_{0,n+8} &\approx \mathrm{Cl}_{0,n} \otimes \mathrm{Cl}_{0,8} \\
\mathrm{Cl}_{n+8,0} &\approx \mathrm{Cl}_{n,0} \otimes \mathrm{Cl}_{8,0}.
\end{aligned}$$

Furthermore,

$$\mathrm{Cl}_{0,8} = \mathrm{Cl}_{8,0} = \mathbb{R}(16).$$

Proof. By Lemma 1.15 we have the isomorphisms

$$\begin{aligned}
\mathrm{Cl}_{0,n+2} &\approx \mathrm{Cl}_{n,0} \otimes \mathrm{Cl}_{0,2} \\
\mathrm{Cl}_{n+2,0} &\approx \mathrm{Cl}_{0,n} \otimes \mathrm{Cl}_{2,0},
\end{aligned}$$

and thus,

$$\mathrm{Cl}_{0,n+8} \approx \mathrm{Cl}_{n+6,0} \otimes \mathrm{Cl}_{0,2} \approx \mathrm{Cl}_{0,n+4} \otimes \mathrm{Cl}_{2,0} \otimes \mathrm{Cl}_{0,2} \approx \cdots \approx \mathrm{Cl}_{0,n} \otimes \mathrm{Cl}_{2,0} \otimes \mathrm{Cl}_{0,2} \otimes \mathrm{Cl}_{2,0} \otimes \mathrm{Cl}_{0,2}.$$

Since $\mathrm{Cl}_{0,2} = \mathbb{H}$ and $\mathrm{Cl}_{2,0} = \mathbb{R}(2)$, by Proposition 1.16, we get

$$\mathrm{Cl}_{2,0} \otimes \mathrm{Cl}_{0,2} \otimes \mathrm{Cl}_{2,0} \otimes \mathrm{Cl}_{0,2} \approx \mathbb{H} \otimes \mathbb{H} \otimes \mathbb{R}(2) \otimes \mathbb{R}(2) \approx \mathbb{R}(4) \otimes \mathbb{R}(4) \approx \mathbb{R}(16).$$

The second isomorphism is proved in a similar fashion. $\square$

From all this, we can deduce the following table:

n	0	1	2	3	4	5	6	7	8
$\mathrm{Cl}_{0,n}$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H} \oplus \mathbb{H}$	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}(8) \oplus \mathbb{R}(8)$	$\mathbb{R}(16)$
$\mathrm{Cl}_{n,0}$	$\mathbb{R}$	$\mathbb{R} \oplus \mathbb{R}$	$\mathbb{R}(2)$	$\mathbb{C}(2)$	$\mathbb{H}(2)$	$\mathbb{H}(2) \oplus \mathbb{H}(2)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$	$\mathbb{R}(16)$

A table of the Clifford groups $\mathrm{Cl}_{p,q}$ for $0 \leq p, q \leq 7$ can be found in Kirillov [18], and for $0 \leq p, q \leq 8$, in Lawson and Michelsohn [20] (but beware that their $\mathrm{Cl}_{p,q}$ is our $\mathrm{Cl}_{q,p}$). It can also be shown that

$$\begin{aligned}
\mathrm{Cl}_{p+1,q} &\approx \mathrm{Cl}_{q+1,p} \\
\mathrm{Cl}_{p,q} &\approx \mathrm{Cl}_{p-4,q+4}
\end{aligned}$$

with $p \geq 4$ in the second identity (see Lounesto [21], Chapter 16, Sections 16.3 and 16.4). Using the second identity, if $|p-q| = 4k$, it is easily shown by induction on k that $\mathrm{Cl}_{p,q} \approx \mathrm{Cl}_{q,p}$, as claimed in the previous section.

We also have the isomorphisms

$$\mathrm{Cl}_{p,q} \approx \mathrm{Cl}^0_{p,q+1},$$

frow which it follows that

$$\mathbf{Spin}(p,q) \approx \mathbf{Spin}(q,p)$$

(see Choquet-Bruhat [11], Chapter I, Sections 4 and 7). However, in general, $\mathbf{Pin}(p,q)$ and $\mathbf{Pin}(q,p)$ are not isomorphic. In fact, $\mathbf{Pin}(0,n)$ and $\mathbf{Pin}(n,0)$ are not isomorphic if $n \neq 4k$, with $k \in \mathbb{N}$ (see Choquet-Bruhat [11], Chapter I, Section 7, page 27).

1.7 The Complex Clifford Algebras Cl(n, $\mathbb{C}$)

One can also consider Clifford algebras over the complex field $\mathbb{C}$. In this case, it is well-known that every nondegenerate quadratic form can be expressed by

$$\Phi_n^{\mathbb{C}}(x_1, \ldots, x_n) = x_1^2 + \cdots + x_n^2$$

in some orthonormal basis. Also, it is easily shown that the complexification $\mathbb{C} \otimes_{\mathbb{R}} \mathrm{Cl}_{p,q}$ of the real Clifford algebra $\mathrm{Cl}_{p,q}$ is isomorphic to $\mathrm{Cl}(\Phi_n^{\mathbb{C}})$. Thus, all these complex algebras are isomorphic for $p + q = n$, and we denote them by $\mathrm{Cl}(n, \mathbb{C})$. Theorem 1.15 yields the following periodicity theorem:

Theorem 1.18 *The following isomorphisms hold:*

$$\mathrm{Cl}(n + 2, \mathbb{C}) \approx \mathrm{Cl}(n, \mathbb{C}) \otimes_{\mathbb{C}} \mathrm{Cl}(2, \mathbb{C}),$$

with $\mathrm{Cl}(2, \mathbb{C}) = \mathbb{C}(2)$.

Proof. Since $\mathrm{Cl}(n, \mathbb{C}) = \mathbb{C} \otimes_{\mathbb{R}} \mathrm{Cl}_{0,n} = \mathbb{C} \otimes_{\mathbb{R}} \mathrm{Cl}_{n,0}$, by Lemma 1.15, we have

$$\mathrm{Cl}(n + 2, \mathbb{C}) = \mathbb{C} \otimes_{\mathbb{R}} \mathrm{Cl}_{0,n+2} \approx \mathbb{C} \otimes_{\mathbb{R}} (\mathrm{Cl}_{n,0} \otimes_{\mathbb{R}} \mathrm{Cl}_{0,2}) \approx (\mathbb{C} \otimes_{\mathbb{R}} \mathrm{Cl}_{n,0}) \otimes_{\mathbb{C}} (\mathbb{C} \otimes_{\mathbb{R}} \mathrm{Cl}_{0,2}).$$

However, $\mathrm{Cl}_{0,2} = \mathbb{H}$, $\mathrm{Cl}(n, \mathbb{C}) = \mathbb{C} \otimes_{\mathbb{R}} \mathrm{Cl}_{n,0}$, and $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{H} \approx \mathbb{C}(2)$, so we get $\mathrm{Cl}(2, \mathbb{C}) = \mathbb{C}(2)$ and

$$\mathrm{Cl}(n + 2, \mathbb{C}) \approx \mathrm{Cl}(n, \mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}(2),$$

and the theorem is proved. $\square$

As a corollary of Theorem 1.18, we obtain the fact that

$$\mathrm{Cl}(2k, \mathbb{C}) \approx \mathbb{C}(2^k) \quad \text{and} \quad \mathrm{Cl}(2k + 1, \mathbb{C}) \approx \mathbb{C}(2^k) \oplus \mathbb{C}(2^k).$$

The table of the previous section can also be completed as follows:

n	0	1	2	3	4	5	6	7	8
$\mathrm{Cl}_{0,n}$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H} \oplus \mathbb{H}$	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}(8) \oplus \mathbb{R}(8)$	$\mathbb{R}(16)$
$\mathrm{Cl}_{n,0}$	$\mathbb{R}$	$\mathbb{R} \oplus \mathbb{R}$	$\mathbb{R}(2)$	$\mathbb{C}(2)$	$\mathbb{H}(2)$	$\mathbb{H}(2) \oplus \mathbb{H}(2)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$	$\mathbb{R}(16)$
$\mathrm{Cl}(n, \mathbb{C})$	$\mathbb{C}$	$2\mathbb{C}$	$\mathbb{C}(2)$	$2\mathbb{C}(2)$	$\mathbb{C}(4)$	$2\mathbb{C}(4)$	$\mathbb{C}(8)$	$2\mathbb{C}(8)$	$\mathbb{C}(16)$.

where $2\mathbb{C}(k)$ is an abbrevation for $\mathbb{C}(k) \oplus \mathbb{C}(k)$.

1.8 The Groups **Pin**(p, q) and **Spin**(p, q) as double covers of **O**(p, q) and **SO**(p, q)

It turns out that the groups **Pin**(p, q) and **Spin**(p, q) have nice topological properties w.r.t. the groups **O**(p, q) and **SO**(p, q). To explain this, we review the definition of covering maps

and covering spaces (for details, see Fulton [14], Chapter 11). Another interesting source is Chevalley [9], where is is proved that $\mathbf{Spin}(n)$ is a universal double cover of $\mathbf{SO}(n)$ for all $n \geq 3$.

Since $C_{p,q}$ is an algebra of dimension 2^{p+q}, it is a topological space as a vector space isomorphic to $V = \mathbb{R}^{2^{p+q}}$. Now, the group $C_{p,q}^*$ of units of $C_{p,q}$ is open in $C_{p,q}$, because $x \in C_{p,q}$ is a unit if the linear map $y \mapsto xy$ is an isomorphism, and $\mathbf{GL}(V)$ is open in $\mathrm{End}(V)$, the space of endomorphisms of V. Thus, $C_{p,q}^*$ is a Lie group, and since $\mathbf{Pin}(p,q)$ and $\mathbf{Spin}(p,q)$ are clearly closed subgroups of $C_{p,q}^*$, they are also Lie groups.

Definition 1.7 Given two topological spaces X and Y, a *covering map* is a continuous surjective map, $p \colon Y \to X$, with the property that for every $x \in X$, there is some open subset, $U \subseteq X$, with $x \in U$, so that $p^{-1}(U)$ is the disjoint union of open subsets, $V_\alpha \subseteq Y$, and the restriction of p to each V_α is a homeomorphism onto U. We say that U is *evenly covered by p*. We also say that Y is a *covering space of X*. A covering map $p \colon Y \to X$ is called *trivial* if X itself is evenly covered by p (i.e., Y is the disjoint union of open subsets, Y_α, each homeomorphic to X), and *nontrivial*, otherwise. When each fiber, $p^{-1}(x)$, has the same finite cardinaly n for all $x \in X$, we say that p is an *n-covering* (or *n-sheeted covering*).

Note that a covering map, $p \colon Y \to X$, is not always trivial, but always *locally trivial* (i.e., for every $x \in X$, it is trivial in some open neighborhood of x). A covering is trivial iff Y is isomorphic to a product space of $X \times T$, where T is any set with the discrete topology. Also, if Y is connected, then the covering map is nontrivial.

Definition 1.8 An *isomorphism* φ between covering maps $p \colon Y \to X$ and $p' \colon Y' \to X$ is a homeomorphism, $\varphi \colon Y \to Y'$, so that $p = p' \circ \varphi$.

Typically, the space X is connected, in which case it can be shown that all the fibers $p^{-1}(x)$ have the same cardinality.

One of the most important properties of covering spaces is the path–lifting property, a property that we will use to show that $\mathbf{Spin}(n)$ is path-connected.

Proposition 1.19 *(Path lifting) Let $p \colon Y \to X$ be a covering map, and let $\gamma \colon [a,b] \to X$ be any continuous curve from $x_a = \gamma(a)$ to $x_b = \gamma(b)$ in X. If $y \in Y$ is any point so that $p(y) = x_a$, then there is a unique curve, $\widetilde{\gamma} \colon [a,b] \to Y$, so that $y = \widetilde{\gamma}(a)$ and*

$$p \circ \widetilde{\gamma}(t) = \gamma(t) \quad \text{for all } t \in [a,b].$$

Proof. See Fulton [15], Chapter 11, Lemma 11.6. $\square$

Many important covering maps arise from the action of a group G on a space Y. If Y is a topological space, an *action (on the left) of a group G on Y* is a map $\alpha \colon G \times Y \to Y$ satisfying the following conditions, where, for simplicity of notation, we denote $\alpha(g,y)$ by $g \cdot y$:

(1) $g \cdot (h \cdot y) = (gh) \cdot y$, for all $g, h \in G$ and $y \in Y$;

(2) $1 \cdot y = y$, for all $\in Y$, where 1 is the identity of the group G;

(3) The map $y \mapsto g \cdot y$ is a homeomorphism of Y for every $g \in G$.

We define an equivalence relation on Y as follows: $x \equiv y$ iff $y = g \cdot x$ for some $g \in G$ (check that this is an equivalence relation). The equivalence class $G \cdot x = \{g \cdot x \mid g \in G\}$ of any $x \in Y$ is called the *orbit of x*. We obtain the quotient space Y/G and the projection map $p \colon Y \to Y/G$ sending every $y \in Y$ to its orbit. The space Y/G is given the quotient topology (a subset U of Y/G is open iff $p^{-1}(U)$ is open in Y).

Given a subset V of Y and any $g \in G$, we let

$$g \cdot V = \{g \cdot y \mid y \in V\}.$$

We say that G *acts evenly on Y* if for every $y \in Y$ there is an open subset V containing y so that $g \cdot V$ and $h \cdot V$ are disjoint for any two distinct elements $g, h \in G$.

The importance of the notion a group acting evenly is that such actions induce a covering map.

Proposition 1.20 *If G is a group acting evenly on a space Y, then the projection map, $p \colon Y \to Y/G$, is a covering map.*

Proof. See Fulton [15], Chapter 11, Lemma 11.17. $\square$

The following proposition shows that **Pin**(*p, q*) and **Spin**(*p, q*) are nontrivial covering spaces unless $p = q = 1$.

Proposition 1.21 *For all $p, q \geq 0$, the groups* **Pin**(*p, q*) *and* **Spin**(*p, q*) *are double covers of* **O**(*p, q*) *and* **SO**(*p, q*)*, respectively. Furthermore, they are nontrivial covers unless $p = q = 1$.*

Proof. We know that kernel of the homomorphism $\rho \colon$ **Pin**(*p, q*) $\to$ **O**(*p, q*) is $\mathbb{Z}_2 = \{-1, 1\}$. If we let $\mathbb{Z}_2$ act on **Pin**(*p, q*) in the natural way, then **O**(*p, q*) $\approx$ **Pin**(*p, q*)$/\mathbb{Z}_2$, and the reader can easily check that $\mathbb{Z}_2$ acts evenly. By Proposition 1.20, we get a double cover. The argument for $\rho \colon$ **Spin**(*p, q*) $\to$ **SO**(*p, q*) is similar.

Let us now assume that $p \neq 1$ and $q \neq 1$. In order to prove that we have nontrivial covers, it is enough to show that -1 and 1 are connected by a path in **Pin**(*p, q*) (If we had **Pin**(*p, q*) $= U_1 \cup U_2$ with U_1 and U_2 open, disjoint, and homeomorphic to **O**(*p, q*), then -1 and 1 would not be in the same U_i, and so, they would be in disjoint connected components. Thus, -1 and 1 can't be path–connected, and similarly with **Spin**(*p, q*) and **SO**(*p, q*).) Since $(p, q) \neq (1, 1)$, we can find two orthogonal vectors e_1 and e_2 so that $\Phi_{p,q}(e_1) = \Phi_{p,q}(e_2) = \pm 1$. Then,

$$\gamma(t) = \pm \cos(2t)\, 1 + \sin(2t)\, e_1 e_2 = (\cos t\, e_1 + \sin t\, e_2)(\sin t\, e_2 - \cos t\, e_1),$$

for $0 \leq t \leq \pi$, defines a path in $\mathbf{Spin}(p, q)$, since

$$(\pm \cos(2t)\, 1 + \sin(2t)\, e_1 e_2)^{-1} = \pm \cos(2t)\, 1 - \sin(2t)\, e_1 e_2,$$

as desired. $\square$

In particular, if $n \geq 2$, since the group $\mathbf{SO}(n)$ is path-connected, the group $\mathbf{Spin}(n)$ is also path-connected. Indeed, given any two points x_a and x_b in $\mathbf{Spin}(n)$, there is a path γ from $\rho(x_a)$ to $\rho(x_b)$ in $\mathbf{SO}(n)$ (where $\rho \colon \mathbf{Spin}(n) \to \mathbf{SO}(n)$ is the covering map). By Proposition 1.19, there is a path $\widetilde{\gamma}$ in $\mathbf{Spin}(n)$ with origin x_a and some origin $\widetilde{x_b}$ so that $\rho(\widetilde{x_b}) = \rho(x_b)$. However, $\rho^{-1}(\rho(x_b)) = \{-x_b, x_b\}$, and so, $\widetilde{x_b} = \pm x_b$. The argument used in the proof of Proposition 1.21 shows that x_b and $-x_b$ are path-connected, and so, there is a path from x_a to x_b, and $\mathbf{Spin}(n)$ is path-connected. In fact, for $n \geq 3$, it turns out that $\mathbf{Spin}(n)$ is simply connected. Such a covering space is called a *universal cover* (for instance, see Chevalley [9]).

This last fact requires more algebraic topology than we are willing to explain in detail, and we only sketch the proof. The notions of fibre bundle, fibration, and homotopy sequence associated with a fibration are needed in the proof. We refer the perseverant readers to Bott and Tu [5] (Chapter 1 and Chapter 3, Sections 16–17) or Rotman [25] (Chapter 11) for a detailed explanation of these concepts.

Recall that a topological space is *simply connected* if it is path connected and $\pi_1(X) = (0)$, which means that every closed path in X is homotopic to a point. Since we just proved that $\mathbf{Spin}(n)$ is path connected for $n \geq 2$, we just need to prove that $\pi_1(\mathbf{Spin}(n)) = (0)$ for all $n \geq 3$. The following facts are needed to prove the above assertion:

(1) The sphere S^{n-1} is simply connected for all $n \geq 3$.

(2) The group $\mathbf{Spin}(3) \simeq \mathbf{SU}(2)$ is homeomorphic to S^3, and thus, $\mathbf{Spin}(3)$ is simply connected.

(3) The group $\mathbf{Spin}(n)$ acts on S^{n-1} in such a way that we have a fibre bundle with fibre $\mathbf{Spin}(n-1)$:
$$\mathbf{Spin}(n-1) \longrightarrow \mathbf{Spin}(n) \longrightarrow S^{n-1}.$$

Fact (1) is a standard proposition of algebraic topology and a proof can found in many books. A particularly elegant and yet simple argument consists in showing that any closed curve on S^{n-1} is homotopic to a curve that omits some point. First, it is easy to see that in $\mathbb{R}^n$, any closed curve is homotopic to a piecewise linear curve (a polygonal curve), and the radial projection of such a curve on S^{n-1} provides the desired curve. Then, we use the stereographic projection of S^{n-1} from any point omitted by that curve to get another closed curve in $\mathbb{R}^{n-1}$. Since $\mathbb{R}^{n-1}$ is simply connected, that curve is homotopic to a point, and so is its preimage curve on S^{n-1}. Another simple proof uses a special version of the Seifert—van Kampen's theorem (see Gramain [17]).

Fact (2) is easy to establish directly, using (1).

To prove (3), we let **Spin**(n) act on S^{n-1} via the standard action: $x \cdot v = xvx^{-1}$. Because **SO**(n) acts transitively on S^{n-1} and there is a surjection **Spin**$(n) \longrightarrow$ **SO**(n), the group **Spin**(n) also acts transitively on S^{n-1}. Now, we have to show that the stabilizer of any element of S^{n-1} is **Spin**$(n-1)$. For example, we can do this for e_1. This amounts to some simple calculations taking into account the identities among basis elements. Details of this proof can be found in Mneimné and Testard [22], Chapter 4. It is still necessary to prove that **Spin**(n) is a fibre bundle over S^{n-1} with fibre **Spin**$(n-1)$. For this, we use the following results whose proof can be found in Mneimné and Testard [22], Chapter 4:

Lemma 1.22 *Given any topological group G, if H is a closed subgroup of G and the projection $\pi \colon G \to G/H$ has a local section at every point of G/H, then*

$$H \longrightarrow G \longrightarrow G/H$$

is a fibre bundle with fibre H.

Lemma 1.22 implies the following key proposition:

Proposition 1.23 *Given any linear Lie group G, if H is a closed subgroup of G, then*

$$H \longrightarrow G \longrightarrow G/H$$

is a fibre bundle with fibre H.

Now, a fibre bundle is a fibration (as defined in Bott and Tu [5], Chapter 3, Section 16, or in Rotman [25], Chapter 11). For a proof of this fact, see Rotman [25], Chapter 11, or Mneimné and Testard [22], Chapter 4. So, there is a homotopy sequence associated with the fibration (Bott and Tu [5], Chapter 3, Section 17, or Rotman [25], Chapter 11, Theorem 11.48), and in particular, we have the exact sequence

$$\pi_1(\mathbf{Spin}(n-1)) \longrightarrow \pi_1(\mathbf{Spin}(n)) \longrightarrow \pi_1(S^{n-1}).$$

Since $\pi_1(S^{n-1}) = (0)$ for $n \geq 3$, we get a surjection

$$\pi_1(\mathbf{Spin}(n-1)) \longrightarrow \pi_1(\mathbf{Spin}(n)),$$

and so, by induction and (2), we get

$$\pi_1(\mathbf{Spin}(n)) \approx \pi_1(\mathbf{Spin}(3)) = (0),$$

proving that **Spin**(n) is simply connected for $n \geq 3$.

We can also show that $\pi_1(\mathbf{SO}(n)) = \mathbb{Z}/2\mathbb{Z}$ for all $n \geq 3$. For this, we use Theorem 1.11 and Proposition 1.21, which imply that **Spin**(n) is a fibre bundle over **SO**(n) with fibre $\{-1, 1\}$, for $n \geq 2$:

$$\{-1, 1\} \longrightarrow \mathbf{Spin}(n) \longrightarrow \mathbf{SO}(n).$$

Again, the homotopy sequence of the fibration exists, and in particular, we get the exact sequence

$$\pi_1(\mathbf{Spin}(n)) \longrightarrow \pi_1(\mathbf{SO}(n)) \longrightarrow \pi_0(\{-1,+1\}) \longrightarrow \pi_0(\mathbf{SO}(n)).$$

Since $\pi_0(\{-1,+1\}) = \mathbb{Z}/2\mathbb{Z}$, $\pi_0(\mathbf{SO}(n)) = (0)$, and $\pi_1(\mathbf{Spin}(n)) = (0)$, when $n \geq 3$, we get the exact sequence

$$(0) \longrightarrow \pi_1(\mathbf{SO}(n)) \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow (0),$$

and so, $\pi_1(\mathbf{SO}(n)) = \mathbb{Z}/2\mathbb{Z}$. Therefore, $\mathbf{SO}(n)$ is not simply connected for $n \geq 3$.

Remark: Of course, we have been rather cavalier in our presentation. Given a topological space, X, the group $\pi_1(X)$ is the *fundamental group of X*, *i.e.*, the group of homotopy classes of closed paths in X (under composition of loops). But $\pi_0(X)$ is generally *not* a group! Instead, $\pi_0(X)$ is the set of path-connected components of X. However, when X is a Lie group, $\pi_0(X)$ is indeed a group. Also, we have to make sense of what it means for the sequence to be exact. All this can be made rigorous (see Bott and Tu [5], Chapter 3, Section 17, or Rotman [25], Chapter 11).

1.9   More on the Topology of $\mathbf{O}(p,q)$ and $\mathbf{SO}(p,q)$

It turns out that the topology of the group, $\mathbf{O}(p,q)$, is completely determined by the topology of $\mathbf{O}(p)$ and $\mathbf{O}(q)$. This result can be obtained as a simple consequence of some standard Lie group theory. The key notion is that of a pseudo-algebraic group.

Consider the group, $\mathbf{GL}(n,\mathbb{C})$, of invertible $n \times n$ matrices with complex coefficients. If $A = (a_{kl})$ is such a matrix, denote by x_{kl} the real part (resp. y_{kl}, the imaginary part) of a_{kl} (so, $a_{kl} = x_{kl} + iy_{kl}$).

Definition 1.9 A subgroup, G, of $\mathbf{GL}(n,\mathbb{C})$ is *pseudo-algebraic* iff there is a finite set of polynomials in $2n^2$ variables with real coefficients, $\{P_i(X_1,\ldots,X_{n^2},Y_1,\ldots,Y_{n^2})\}_{i=1}^{t}$, so that

$$A = (x_{kl} + iy_{kl}) \in G \quad \text{iff} \quad P_i(x_{11},\ldots,x_{nn},y_{11},\ldots,y_{nn}) = 0, \quad \text{for } i = 1,\ldots,t.$$

Recall that if A is a complex $n \times n$-matrix, its *adjoint*, A^*, is defined by $A^* = (\overline{A})^{\top}$. Also, $\mathbf{U}(n)$ denotes the group of unitary matrices, i.e., those matrices $A \in \mathbf{GL}(n,\mathbb{C})$ so that $AA^* = A^*A = I$, and $\mathbf{H}(n)$ denotes the vector space of Hermitian matrices, i.e., those matrices A so that $A^* = A$. Then, we have the following theorem which is essentially a refined version of the polar decomposition of matrices:

Theorem 1.24 *Let G be a pseudo-algebraic subgroup of $\mathbf{GL}(n,\mathbb{C})$ stable under adjunction (i.e., we have $A^* \in G$ whenever $A \in G$). Then, there is some integer, $d \in \mathbb{N}$, so that G is homeomorphic to $(G \cap \mathbf{U}(n)) \times \mathbb{R}^d$. Moreover, if $\mathfrak{g}$ is the Lie algebra of G, the map*

$$(\mathbf{U}(n) \cap G) \times (\mathbf{H}(n) \cap \mathfrak{g}) \longrightarrow G, \quad \text{given by} \quad (U,H) \mapsto Ue^{H},$$

is a homeomorphism onto G.

Proof. A proof can be found in Knapp [19], Chapter 1, or Mneimné and Testard [22], Chapter 3. $\square$

We now apply Theorem 1.24 to determine the structure of the space $\mathbf{O}(p,q)$. Let $J_{p,q}$ be the matrix

$$J_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}.$$

We know that $\mathbf{O}(p,q)$ consists of the matrices, A, in $\mathbf{GL}(p+q,\mathbb{R})$ such that

$$A^\top J_{p,q} A = J_{p,q},$$

and so, $\mathbf{O}(p,q)$ is clearly pseudo-algebraic. Using the above equation, it is easy to determine the Lie algebra, $\mathfrak{o}(p,q)$, of $\mathbf{O}(p,q)$. We find that $\mathfrak{o}(p,q)$ is given by

$$\mathfrak{o}(p,q) = \left\{ \begin{pmatrix} X_1 & X_2 \\ X_2^\top & X_3 \end{pmatrix} \;\middle|\; X_1^\top = -X_1, \; X_3^\top = -X_3, \; X_2 \text{ arbitrary} \right\}$$

where X_1 is a $p \times p$ matrix, X_3 is a $q \times q$ matrix and X_2 is a $p \times q$ matrix. Consequently, it immediately follows that

$$\mathfrak{o}(p,q) \cap \mathbf{H}(p+q) = \left\{ \begin{pmatrix} 0 & X_2 \\ X_2^\top & 0 \end{pmatrix} \;\middle|\; X_2 \text{ arbitrary} \right\},$$

a vector space of dimension pq.

Some simple calculations also show that

$$\mathbf{O}(p,q) \cap \mathbf{U}(p+q) = \left\{ \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} \;\middle|\; X_1 \in \mathbf{O}(p), \; X_2 \in \mathbf{O}(q) \right\} \cong \mathbf{O}(p) \times \mathbf{O}(q).$$

Therefore, we obtain the structure of $\mathbf{O}(p,q)$:

Proposition 1.25 *The topological space* $\mathbf{O}(p,q)$ *is homeomorphic to* $\mathbf{O}(p) \times \mathbf{O}(q) \times \mathbb{R}^{pq}$.

Since $\mathbf{O}(p)$ has two connected components when $p \geq 1$, we see that $\mathbf{O}(p,q)$ has four connected components when $p, q \geq 1$. It is also obvious that

$$\mathbf{SO}(p,q) \cap \mathbf{U}(p+q) = \left\{ \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} \;\middle|\; X_1 \in \mathbf{O}(p), \; X_2 \in \mathbf{O}(q), \; \det(X_1)\det(X_2) = 1 \right\}.$$

This is a subgroup of $\mathbf{O}(p) \times \mathbf{O}(q)$ that we denote $S(\mathbf{O}(p) \times \mathbf{O}(q))$. Furthermore, it is easy to show that $\mathfrak{so}(p,q) = \mathfrak{o}(p,q)$. Thus, we also have

Proposition 1.26 *The topological space* $\mathbf{SO}(p,q)$ *is homeomorphic to* $S(\mathbf{O}(p) \times \mathbf{O}(q)) \times \mathbb{R}^{pq}$.

Note that $\mathbf{SO}(p,q)$ has two connected components when $p,q \geq 1$. The connected component of I_{p+q} is a group denoted $\mathbf{SO}_0(p,q)$. This latter space is homeomorphic to $\mathbf{SO}(p) \times \mathbf{SO}(q) \times \mathbb{R}^{pq}$.

As a closing remark observe that the dimension of all these spaces depends only on $p+q$: It is $(p+q)(p+q-1)/2$.

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