FAMILY ALGEBRAS AND THE ISOTYPIC COMPONENTS OF $\mathfrak{g} \otimes \mathfrak{g}$

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ABSTRACT

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Matthew Tai

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Given a complex simple Lie algebra $\mathfrak{g}$ with adjoint group G, the space $S(\mathfrak{g})$ of polynomials on $\mathfrak{g}$ is isomorphic as a graded $\mathfrak{g}$-module to $(I(\mathfrak{g}) \otimes \mathcal{H}(\mathfrak{g})$ where $I(\mathfrak{g}) = (S(\mathfrak{g}))^G$ is the space of G-invariant polynomials and $\mathcal{H}(\mathfrak{g})$ is the space of G-harmonic polynomials. For a representation V of $\mathfrak{g}$, the generalized exponents of V are given by $\sum_{k \geq 0} dim(Hom_{\mathfrak{g}}(V, H_k(\mathfrak{g}))q^k$. We define an algebra $C_V(\mathfrak{g}) = Hom_{\mathfrak{g}}(End(V), S(\mathfrak{g}))$ and for the case of $V = \mathfrak{g}$ we determine the structure of $\mathfrak{g}$ using a combination of diagrammatic methods and information about representations of the Weyl-group of $\mathfrak{g}$. We find an almost uniform description of $C_{\mathfrak{g}}(\mathfrak{g})$ as an $I(\mathfrak{g})$-algebra and as an $I(\mathfrak{g})$-module and from there determine the generalized exponents of the irreducible components of $End(\mathfrak{g})$. The results support conjectures about $(T(\mathfrak{g}))^G$, the G-invariant part of the tensor algebra, and about a relation between generalized exponents and Lusztig's fake degrees.

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Chapter 1

Introduction

1.1 Simple Lie Algebras and Exponents

We consider the simple Lie algebras, these being the four classical series A_r, B_r, C_r, D_r and the five exceptional algebras G_2, F_4, E_6, E_7, E_8. Associated to each of these algebras is a list of numbers called their exponents, which appear in a number of ways. The name comes from the exponents of the hyperplane arrangement corresponding to the simple reflection planes of the Weyl Group of the Lie algebra. The exponents can also be considered topologically: for the compact group G associated to $\mathfrak{g}$, the Poincare polynomial of G is

$$P_G(q) = \sum_{k=0} rk(H^k(G,\mathbb{Z}))q^k = \prod_{k=1}^{r}(1+q^{2e_k+1})$$

The simple Lie algebras are summarized in the following table, where the descriptions of the exceptional Lie groups are based on the Rosenfeld projective planes [Ro97].

Table 1.1: Exponents for the simple Lie algebras

$\mathfrak{g}$	$\dim \mathfrak{g}$	Exponents	Notes
A_r	r^2+2r	$1,2,3\ldots,r$	$sl(r+1)$
B_r	$r(2r+1)$	$1,3,5,\ldots,2r-1$	$so(2r+1)$
C_r	$r(2r+1)$	$1,3,5,\ldots,2r-1$	$sp(r)$
D_r	$r(2r-1)$	$1,3,5,\ldots,2r-1,r-1$	$so(2r)$
G_2	14	1, 5	"$sl(1,\mathbb{O})$"
F_4	52	1, 5, 7, 11	"$sl(2,\mathbb{O})$"
E_6	78	1, 4, 5, 7, 8, 11	"$sl(2,\mathbb{C}\otimes\mathbb{O})$"
E_7	133	1, 5, 7, 9, 11, 13, 17	"$sl(2,\mathbb{H}\otimes\mathbb{O})$"
E_8	248	1, 7, 11, 13, 17, 19, 23, 29	"$sl(2,\mathbb{O}\otimes\mathbb{O})$"

1.2 Casimir Invariants

The exponents of $\mathfrak{g}$ also have representation-theoretic interpretations. G acts on $\mathfrak{g}$ via the conjugation action, and hence on $S(\mathfrak{g})$, the symmetric algebra on $\mathfrak{g}$ considered as a vector space. We denote by $I(\mathfrak{g})$ the G-invariant subspace $(S(\mathfrak{g}))^G$. In 1963, Kostant showed that $I(\mathfrak{g})$ for simple $\mathfrak{g}$ is a polynomial algebra where the number of generators is equal to the rank of $\mathfrak{g}$, and that furthermore the degrees of these generators are each one more than an exponent of $\mathfrak{g}$. For later use we establish a particular choice of generators of $I(\mathfrak{g})$, which we will call primitive Casimir operators.

For each Lie algebra, we pick a representation (V,π) with which to define the primitive Casimir operators. For $sl(r+1)$ we pick one of the two $r+1$-dimensional representations. For $so(n)$ we pick the n-dimensional representation, and for $sp(r)$ we pick the $2r$-dimensional representation. For G_2, F_4, E_6, E_7, and E_8 we pick the $7-, 26-, 27-, 56-$, and $248-$dimensional representations respectively. These are often also called the "standard" representations, and with a few exceptions are the nontrivial representations of minimal dimension. From here on out, unless otherwise specified, V and π refer to this defining representation.

Letting $\{x_\alpha\}$ be a basis of $\mathfrak{g}$ and K be the Killing form, define

$$M_d = \pi(x_\alpha) \otimes K^{\alpha\beta} x_\beta$$

regarded as an $S(\mathfrak{g})$-valued square matrix of dimension $\dim V$.

Except in the case of D_r, we can order the exponents of $\mathfrak{g}$ in increasing order, so that we can denote the exponents by

$$e_1 < e_2 < \ldots < e_r$$

We define the Casimir operator

$$c_k = tr(M_d^{e_k+1})$$

where the trace is taken in the minimal representation V. The set $\{c_k\}$ are the primitive Casimir operators for $\mathfrak{g}$.

In the case of D_r, the exponents are $1, 3, \ldots, 2r-3, r-1$. We order the exponents in increasing order, getting that $e_{\lceil \frac{r}{2} \rceil} = r-1$. For $k \neq \lceil \frac{r}{2} \rceil$, we write

$$c_k = tr(M_d^{e_k+1})$$

and for $k = \lceil \frac{r}{2} \rceil$ we define

$$c_k = Pf = \sqrt{\det(M_d)}$$

picking the sign of Pf arbitrarily.

For $sl(r+1)$, the exponents are $e_i = i$ for $1 \leq i \leq r$. For $k > r+1$, we have the reduction,

$$0 = \sum_{n_j m_j = k} \frac{1}{m_j!} \left(-\frac{tr(M_d^{n_j})}{n_j} \right)^{m_j}$$

where the notation $n_j m_j = k$ indicates a partition of k where n_j appears with multiplicity m_j. In particular, the relation for $k = r+2$ is the trace of the Cayley-Hamilton identity for M_d.

For the other simple Lie algebras, there are similar formulas for reducing $tr(M_d^k)$ for

k not an exponent plus 1, with varying complexity. For the exceptional Lie algebras, $\dim(V)$ is much larger than $e_r + 1$, so the Cayley-Hamilton identity doesn't yield much information about the how traces of low powers of M_d reduce. See [RSV99] for details.

1.3 Generalized Exponents

In 1963, Kostant [Ko63] proved that for a representation V of $\mathfrak{g}$ and hence of G, $(V \otimes S(\mathfrak{g}))^G = Hom_G(V^\vee, S(\mathfrak{g}))$ is a free $I(\mathfrak{g})$ module. Thus we can find a basis for $(V \otimes S(\mathfrak{g}))^G$ over $I(\mathfrak{g})$; Kostant calls the degrees of the polynomial components of this basis the generalized exponents of V (with multiplicity), usually expressed as a polynomial $P_V(q)$ for a variable q. Note that for $V = \mathfrak{g}$, the generalized exponents for $\mathfrak{g}$ match the classical notion of the exponents of $\mathfrak{g}$.

There is another description of the generalized exponents in terms of a space $\mathcal{H}(\mathfrak{g})$ of G-harmonic polynomials. Let $D(\mathfrak{g})$ be the space of G-invariant differential operators on $S(\mathfrak{g})$ with constant coefficients, and let $D_+(\mathfrak{g})$ be the subspace of $D(\mathfrak{g})$ with vanishing constant term. Then $\mathcal{H}(\mathfrak{g})$ is defined by

$$\mathcal{H}(\mathfrak{g}) = \{f \in S(\mathfrak{g}) | d(f) = 0 \ \forall d \in D_+(\mathfrak{g})\}$$

The condition $d(f) = 0$ generalizes the usual harmonic condition of $\Delta(f) = 0$, and thus the G-harmonic polynomials allow for studying functions defined on Lie groups using methods from harmonic analysis.

Kostant showed that $S(\mathfrak{g}) \cong I(\mathfrak{g}) \otimes \mathcal{H}(\mathfrak{g})$ as graded $\mathfrak{g}$-modules. Since the elements of

$I(\mathfrak{g})$ are invariant, all of the interesting behavior is contained in $\mathcal{H}(\mathfrak{g})$. Thus we can write the generalized exponents of a representation V as

$$P_V(q) = \sum_k \dim(Hom_{\mathfrak{g}}(V, \mathcal{H}_k(\mathfrak{g})))q^k$$

Hesselink [He80] gives a formula for computing the generalized exponents of an irreducible representation of a simple Lie algebra using a q-analogue of Kostant's multiplicity formula, but using this formula is computationally infeasible, involving computing the q-analogue of the partition function, which unlike the normal partition function doesn't vanish for negative weights, and then summing the partition function over the associated Weyl orbit. There are also combinatorial approaches such as the Kostka-Foulkes polynomials for $sl(n)$ [DLT94]. For representations where none of the weights are twice a root (called small representations), Broer [Br95] showed that the generalized exponents of V are equal to what Lusztig calls the fake degrees [Lu77] of V^T as a representation of W.

In general, however, there are no known closed-form expressions for the generalized exponents of arbitrary representations that don't require summation over the Weyl group.

Chapter 2

Introduction to Family Algebras

2.1 Definition of Family Algebras

In [Ki00], Kirillov introduced what he calls Family Algebras in the hopes of providing a new method for determining generalized exponents that doesn't involve summing over the Weyl group.

We fix a representation V of $\mathfrak{g}$ and consider $End(V)$, with the conjugation action on it induced from the action on V. We define the classical family algebra

$$C_V(\mathfrak{g}) = (End(V) \otimes S(\mathfrak{g}))^G$$

where G is the adjoint group of $\mathfrak{g}$ and acts by the action induced from $\mathfrak{g}$. This is an algebra with multiplication $\circ \otimes m$ inherited from

$$\circ : End(V) \otimes End(V) \to End(V)$$

via composition and

$$m : S(\mathfrak{g}) \otimes S(\mathfrak{g}) \to S(\mathfrak{g})$$

via polynomial multiplication.

If we pick a basis $\{v_a\}$ for V and let E_a^b be defined by $E_a^b v_b = v_a$, then we can write an element of the family algebra as

$$E_a^b \otimes P_b^a$$

where $P_b^a \in S(\mathfrak{g})$. We call the set $\{P_b^a\}$ the polynomial component of $E_a^b \otimes P_b^a$. Note that for two elements $E_a^b \otimes P_b^a$ and $E_a^b \otimes Q_b^a$, the multiplication looks like

$$(E_a^b \otimes P_b^a) \times (E_a^b \otimes Q_b^a) = E_a^b \otimes P_c^a Q_b^c$$

So the multiplication respects the natural grading on the polynomial components, and hence we say that an element $E_a^b \otimes P_b^a$ is homogeneous of degree k if all of the $\{P_b^a\}$ are homogeneous of degree k.

The phrase "family algebra" comes from the decomposition of

$$End(V) = \bigoplus_i V_i$$

for irreducible V_i, which Kirillov calls the children of V. The family algebra $C_V(\mathfrak{g})$ decomposes similarly into

$$C_V(\mathfrak{g}) = \bigoplus_i (V_i \otimes S(\mathfrak{g}))^G$$

Thus a family algebra gives us an $I(\mathfrak{g})$ module that is closed under multiplication and is built from a finite set of isotypic components. Note that if V_i is a component of

$End(V)$, then so is $V_i^\vee$; hence

$$C_V(\mathfrak{g}) = \bigoplus_i (V_i \otimes S(\mathfrak{g}))^G \cong \bigoplus_i Hom_{\mathfrak{g}}(V_i, S(\mathfrak{g}))$$

There is a natural quantization to what Kirillov calls the quantum family algebra,

$$Q_V(\mathfrak{g}) = (End(V) \otimes U(\mathfrak{g}))^G$$

$U(\mathfrak{g})$ is isomorphic to $S(\mathfrak{g})$ as G-modules, so there is a map that sends $Q_V(\mathfrak{g})$ to $C_V(\mathfrak{g})$, but the classical and quantum family algebras for a given V differ in their multiplicative structures. This dissertation will only consider classical family algebras.

2.2 Relation to the Generalized Exponents

For an irreducible representation V_i, there is an $I(\mathfrak{g})$-linear basis of $Hom_{\mathfrak{g}}(V_i, S(\mathfrak{g}))$ where there is a bijection between generalized exponents e_{ij} (with multiplicity) and basis elements A_{ij} such that

$$A_{ij} \in Hom_{\mathfrak{g}}(V_i, \mathcal{H}_{e_{ij}}(\mathfrak{g}))$$

Hence, using the decomposition of $End(V)$ into irreducible representations, there is an $I(\mathfrak{g})$-linear basis of $C_V(\mathfrak{g})$ where each basis element is in $Hom_{\mathfrak{g}}(V_i, \mathcal{H}_{e_{ij}}(\mathfrak{g}))$ for some V_i in the decomposition of $End(V)$ and some e_{ij} a generalized exponent of V_i.

Thus the general strategy of family algebras is to determine the algebraic structure of a given family algebra, use that to determine an $I(\mathfrak{g})$-linear basis, turn that basis into a harmonic basis and from there compute the generalized exponents.

2.3 Restriction to the Cartan subalgebra

Given a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ with corresponding torus $T \subset G$, we can look at $S(\mathfrak{h})$, and in particular the restriction $res : S(\mathfrak{g}) \to S(\mathfrak{h})$ given by viewing the two algebras as $Pol[\mathfrak{g}^\vee]$ and $Pol[\mathfrak{h}^\vee]$ respectively, and sending an element $f \in S(\mathfrak{g})$ to $f|_{\mathfrak{h}^\vee}$. While this map is generally not an injection, there are some useful aspects. Chevalley's restriction theorem [Br95] says that

$$res|_{I(\mathfrak{g})} : I(\mathfrak{g}) \to I(\mathfrak{h}) = S(\mathfrak{h})^W$$

is an isomorphism. We get a map

$$Res : (V \otimes S(\mathfrak{g}))^G \to (V^T \otimes S(\mathfrak{h}))^W$$

induced by restricting from V to V^T and from $S(\mathfrak{g})$ to $S(\mathfrak{h})$, which Kostant shows is an injection. The result by Broer mentioned in the first chapter is a necessary and sufficient condition for Res to be an isomorphism.

Writing $B_V(\mathfrak{h})$ for $End(V)^T \otimes S(\mathfrak{h})$ we get that

$$Res : C_V(\mathfrak{g}) \to B_V(\mathfrak{h})^W$$

is an injection. We can make Res into a surjection by localizing with respect to the non-zero part of $I(\mathfrak{g})$. In particular, let K_0 be the fraction field of $I(\mathfrak{g}) \cong S(\mathfrak{h})^W$. Then by [Ki01] we have that

$$C_V(\mathfrak{g}) \otimes_{I(\mathfrak{g})} K_0 \cong B_V(\mathfrak{h})^W \otimes_{I(\mathfrak{h})} K_0$$

This then tells us that the dimension of $C_V(\mathfrak{g})$ over $I(\mathfrak{g})$ is equal to the dimension of $B_V(\mathfrak{h})^W$ over $I(\mathfrak{h})$.

$$End(V)^T = \bigoplus_{\mu \in Wt(V)} Mat_{m_V(\mu)}(\mathbb{C})$$

and so $B_V(\mathfrak{h})^W$ is the W-invariant subalgebra of the sum of matrix algebras

$$\bigoplus_{\mu \in Wt(V)} Mat_{m_V(\mu)}(S(\mathfrak{h}))$$

Since the multiplicity over $I(\mathfrak{h})$ of a representation ϕ of W in $S(\mathfrak{h})$ is $\dim(\phi)$, the dimension of $B_V(\mathfrak{h})^W$ over $I(\mathfrak{h})$, i.e. the dimension of $C_V(\mathfrak{g})$, is given by the sum of the dimensions of the matrix algebras:

$$\sum_{\mu \in Wt(V)} m_V(\mu)^2$$

Given a weight $\lambda \in Wt(V)$, we can consider an element of $B_V(\mathfrak{h})^W$ that is the identity on the matrix algebras corresponding to weights in $W.\lambda$ and vanish elsewhere. Such an element lifts to an element of $C_V(\mathfrak{g})$ which, for some $P \in I(\mathfrak{h})$, restricts to P times the identity on the matrix algebras corresponding to weights in $W.\lambda$ and vanish elsewhere. We can use these elements of $C_V(\mathfrak{g})$ as analogues to projection operators.

Chapter 3

Results for $V = \mathfrak{g}$

This dissertation will focus on the particular case of the adjoint representation, i.e. setting $V = \mathfrak{g}$. The weights in question are then the roots of $\mathfrak{g}$ as well as 0 with multiplicity r, where r is the rank of $\mathfrak{g}$. We denote the image of $(End(\mathfrak{g})^T \otimes S(\mathfrak{h}))^W$ in $Mat_r(S(\mathfrak{h}))$ by the torus part of the algebra, and everything else by the vector part, as it is composed of 1-dimensional and hence scalar algebras. Note that the vector part is commutative, since $S(\mathfrak{h})$ is commutative, so any non-commutativity in the family algebra appears only in the torus part.

3.1 Algebraic Structure

The decomposition of $End(\mathfrak{g})$ into irreducible components depends on $\mathfrak{g}$, but is uniform for all of the A_n, uniform for B_n, C_n and D_n, and is uniform for the five exceptional Lie algebras [CV08]. The main result of this dissertation is that though the form of $C_{\mathfrak{g}}(\mathfrak{g})$ as a $\mathfrak{g}$-module ends up quite different, the algebraic structures of $C_{\mathfrak{g}}(\mathfrak{g})$ are very similar for all of the simple Lie algebras. There are two generators common to all of family algebras in question, denoted M and S, and then a set of r other generators, labelled R_1 through R_r, that depend on the structure of $\mathfrak{g}$. A set of $I(\mathfrak{g})$-linearly independent basis elements of $C_{\mathfrak{g}}(\mathfrak{g})$ is then

$$M^m R_k \text{ for } 0 \le m \le e_r + 1, 1 \le k \le r$$

$$R_m S R_n + R_n S R_m \text{ for } 1 \le m \le n \le r-1$$

$$R_m S R_n - R_n S R_m \text{ for } 1 \le m < n \le r$$

Here R_1 is a scalar, left in for uniformity of expression.

The R_k can themselves be generated by either M, S and R_2 in the cases of A_r, B_r, C_r and G_2, by M, S, R_2 and R_r for D_r, or by M, S, R_2 and R_3.in the cases of F_4, E_6, E_7 and E_8.

There are several relations common to all of the cases. The terms $M^m R_k$ for $m \ge 1$ vanish on the torus part, and any term involving S vanishes on the vector part. Hence M is central and $MS = SM = 0$. The R_k commute with each other and with M, but not with S. $SR_kS = P_kS$ for some $P_k \in I(\mathfrak{g})$, although the form of P_k depends on $\mathfrak{g}$. The relations describing the products of the R_k also depend on $\mathfrak{g}$, in particular the existence or absence of primitive Casimir elements in particular degrees.

3.2 Fake Degrees

For the Weyl group W acting on the Cartan subalgebra $\mathfrak{h}$, there is a notion called "fake degrees" analogous to that of the generalized exponents, in that there is a polynomial $P_U(q)$ describing the maps from a W-representation U into a space $\mathfrak{M}(\mathfrak{h})$ of W-harmonic polynomials. For the representations relevant to $\mathfrak{g}\otimes\mathfrak{g}$, we have the following statement: if $V^T = \oplus_i U_i$ as W-modules, then

$$P_V(q) = \sum_i q^{k_i} P_{U_i}(q)$$

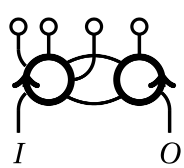
for some set of exponents k_i, although the k_i are not uniquely determined.

For each classical families there are uniform expressions for the q_i in terms of r, as well as for the E_r family.

Chapter 4

Diagrams

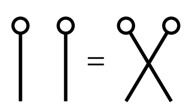
We can write elements of the family algebra diagrammatically using the Feynman-Penrose-Cvitanović “birdtrack” notation [Cv08]. We consider graphs with two types of edges, called reference and adjoint edges. An adjoint edge is marked here by a thin line, a reference edge by a thick line with an arrow on it. All edges that end in a univalent vertex must be adjoint edges, and for every diagram one of these univalent vertices is labelled with an “I”, one with an “O”, and the rest with a white dot. A diagram with k dotted vertices is considered to have degree k. The other types of allowed vertices depend on the Lie algebra in question. For example,



In the usual particle interpretation of Feynman diagrams, the adjoint edges are bosons and the reference edges are fermions carrying the corresponding charge, and vertices of valence higher than 1 being interactions, with G being the gauge group of the interactions. Momentum constraints are ignored here.

In the Lie algebra interpretation, the reference edges correspond to copies of the reference representation, the adjoint edges are copies of the adjoint representation, and vertices of valence higher than 1 are invariants. In particular, the vertices with one reference edge pointing in, one reference edge pointing out and one adjoint edge attached are Clebsches for $V \otimes V^{\vee} \to Adj$. In the standard index notation for tensors, each vertex is an invariant tensor with an upper reference index for each arrow going in, a lower reference index for each arrow going out, and an adjoint index for each adjoint edge attached; two indices are contracted if they are connected by an edge. The ability to turn upper adjoint indices into lower adjoint indices via the Killing form allows us to not require arrows on the adjoint edges.

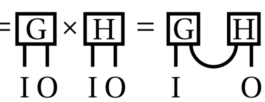
We consider the dotted vertices as indistinguishable, so that if two diagrams differ only by which of a pair of adjoint edges connect to which of a pair of dotted vertices, we consider the diagrams equivalent.



The dotted vertices correspond to our polynomial part $\{P^{\alpha}_{\beta}\}$. The I and O vertices correspond to our coordinate indices E^{β}_{α}. A component that is not connected to either of the I or O vertices is contained entirely within $S(\mathfrak{g})$, and hence in $I(\mathfrak{g})$, so a component with only dotted vertices acts as a coefficient. A diagram is considered as the tensor product of its connected components.

These diagrams are all naturally G-invariant, being built out of G-invariant objects, and hence all diagrams are naturally in $(End(\mathfrak{g}) \otimes S(\mathfrak{g}))^G$. Thus any diagram as defined above automatically gives a family algebra element, as opposed to the initial setup of defining $End(\mathfrak{g}) \otimes S(\mathfrak{g})$ and then imposing $\mathfrak{g}$-invariance as an additional property. By Cvitanović, all elements of the family algebra are formal $\mathbb{C}$-linear combinations of such diagrams, so we can consider the algebra in terms of these diagrams.

The family algebra product of two diagrams is the diagram created by removing the I vertex of one diagram and the O vertex of the other and identifying the adjoint edges those vertices were attached to, which is the equivalent of contracting the adjoint indices that the two edges corresponded to.

$G \times H=$ 

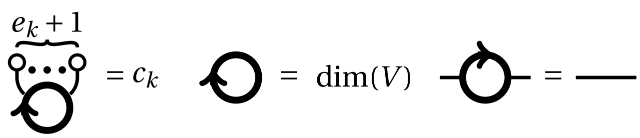
We can also define the trace of a family algebra element similarly, by removing both the I and the O vertices of a family algebra element and identifying the adjoint edges those vertices were attached to.

4.1 Casimir Operators and Structure Constants

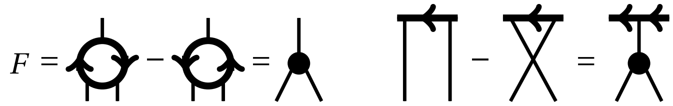
Given a reference loop going through n Clebsche vertices, we have n adjoint edges coming off of the loop, and the loop corresponds to

$$tr(\pi(X_1)\pi(X_2)\cdots\pi(X_n))$$

where the X_i are the adjoint edges, i.e. elements of $\mathfrak{g}$. As such, a loop of reference edges going through k Clebsche vertices will be called a "trace" of order k from now on. A trace of degree e_k+1 whose adjoint edges all end in dotted vertices evaluates to the primitive Casimir operator c_k, except in the case of the $e_r = r-1$ exponent of D_r, which will be handled in the section on D_r. A trace of order 0 evaluates to $\dim(V)$. We normalize so that traces of degree 2 are equivalent to just adjoint lines.



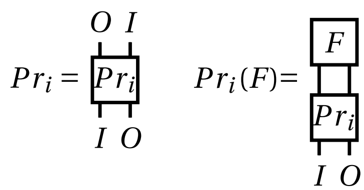
The structure constant $f^{\alpha}_{\beta\gamma}$ can be written as a diagram F as the difference of two traces each with three adjoint edges coming off, differing only in the direction of the reference edges. We abbreviate it using Cvitanović's notation of a big black dot. Given two Clebsche vertices connected to single a reference edge, swapping the ends of the adjoint edges can be written using an F node. This is just the Lie algebra relation $\pi(X)\pi(Y) - \pi(Y)\pi(X) = \pi([X,Y])$ applied to the reference representation:



We say a diagram is simple if the connected components containing the I and O vertices are each a primitive Casimir operator attached to some number of trees built out of structure constants.

4.2 Projections

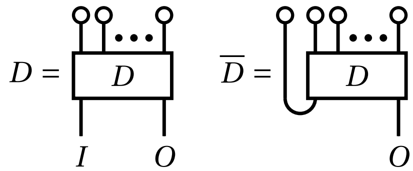
When looking for generalized exponents, we want objects not in $(\mathfrak{g} \otimes \mathfrak{g} \otimes S(\mathfrak{g}))^G$, where these diagrams naturally live, but in $(V_i \otimes S(\mathfrak{g}))^G$ for a given irreducible component V_i. We denote by Pr_i the projection operator that sends $\mathfrak{g}\otimes\mathfrak{g}$ into the subspace isomorphic to V_i. Diagrammatically, such a projector looks like a diagram with two I vertices, two O vertices and no dotted vertices. Similar to multiplication, an O vertex of the projector connects to the I vertex of the diagram being projected, but now also an I vertex of the projector connects to the O vertex of the diagram being projected, yielding a new diagram with a single I and a single O vertex:

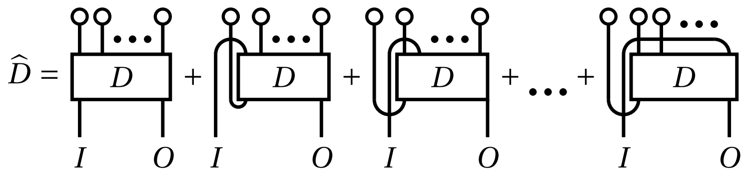


The projection operators, like the diagrams themselves, can be expressed entirely in terms of traces of reference edges connected to adjoint edges, so the adjoint edges between the projector and the diagram being projected can be expanded out, allowing for diagrammatic evaluation of the projected diagram. See [Cv08] for details.

4.3 Symmetrization

For a generic diagram D, we consider the diagram $\overline{D}$ created by replacing the I vertex in D by a dotted vertex. We also consider the half-symmetrization $\widehat{D}$, which is the sum over all diagrams derived from D by swapping the I and one of the dotted vertices, plus D itself. $\widehat{D}$ is the sum over all diagrams created by replacing one of the dotted vertices in $\overline{D}$ by the I vertex.





$\overline{D}$ belongs to $(\mathfrak{g} \otimes S(\mathfrak{g}))^G$, and thus decomposes into $\sum a_k D_k$ where $a_k \in I(\mathfrak{g})$ and D_k is the diagram created by taking the diagram corresponding to the primitive Casimir element c_k and replacing one of the dotted vertices with the O vertex. D_k is a simple diagram, and the a_k is not connected to anything in D_k, so $\overline{D}$ can be written in terms of a finite set of simple diagrams, and thus $\widehat{D}$ can also be written in terms of a finite set of simple diagrams. Thus if the other terms in $\widehat{D}$ can be written in terms of simple diagrams, so can D.

Chapter 5

Invariant Tensors

5.1 General Statement for most Classical Lie Algebras

For A_r, B_r and C_r there is a particularly elegant expression for all the elements of the invariant tensors $(T(\mathfrak{g}))^G$ coming from the reference representations.

Theorem 5.1.1 (Invariant Tensors for A_r, B_r and C_r)**.** *For $\mathfrak{g} = A_r, B_r$ or C_r with the corresponding reference representation (V, π), the elements of $(T(\mathfrak{g}))^G$ can be expressed as tensor products of*

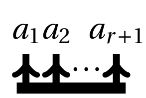
$$tr_V(\pi(X_{\alpha_1})\pi(X_{\alpha_2})\cdots\pi(X_{\alpha_k}))X^{\alpha_1} \otimes X^{\alpha_2} \otimes \cdots \otimes X^{\alpha_k} \in T(\mathfrak{g})$$

along with permutations of the indices.

Diagrammatically, this corresponds to the statement that all diagrams with only adjoint edges leaving the diagram are expressible as loops over the reference representation with adjoint edges attached, where no two loops are connected by an adjoint edge.

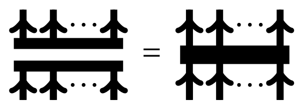
5.2 A_r

Any invariant tensor in $\otimes A_r$ can be written in terms of representations of A_r, invariants of those representations, and Clebsches between representations. In turn, any representation of A_r can be written in terms of V and $V^\vee$, symmetrized and antisymmetrized. Thus we can write any tensor in $\otimes A_r$ in terms of V and the adjoint representation. Diagrammatically, this corresponds to diagrams with only reference and adjoint edges, with all the internal edges written as reference edges and all of the edges leading out of the diagrams being adjoint edges. By the first fundamental theorem of the invariant theory of $SL(r+1)$ acting on the $r+1$-dimensional representation [FH04], the possible vertices are the Clebsches converting between the adjoint representation and $V \otimes V^\vee$, and the two forms of the Levi-Civita tensor, one with $r+1$ reference edges in, the other with $r+1$ reference edges out, corresponding to tensor that takes $r+1$ vectors and returns a scalar, and the dual of that tensor. We write the Levi-Civita tensor not as a vertex but as a black bar, following [Cv08]:

$\epsilon_{a_1,a_2,\ldots,a_{r+1}} =$ 

$\epsilon^\vee =$ 

Since the only edges that can lead out of the diagram have to be adjoint edges, corresponding to the fact that all of our tensors are in $T(A_r)$, any instance of the Levi-Civita tensor in the tensor must be matched by an instance of the dual of the Levi-Civita tensor, as those are the only possible sources and sinks for reference edges. Furthermore, given a Levi-Civita tensor and a dual of the Levi-Civita tensor, we can combine them to yield reference edges without source or sink:

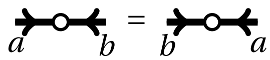


where the black bar across the reference edges on the right side of the previous equation means a full antisymmetrization of the corresponding vectors.

Hence since every Levi-Civita tensor is matched by a dual of the Levi-Civita tensor, we can expand them into reference edges without Levi-Civita tensors. Since these reference edges cannot lead out of the diagram, they must close up. Hence we end up with loops of reference edges with Clebsche vertices attaching these loops to adjoint edges that lead out of the diagram. These are all traces of powers of the adjoint representation over V, as claimed.

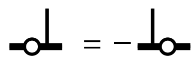
5.3 B_r

For B_r we use the $2r+1$-dimensional representation as the reference representation; we have a symmetric form generally called the metric, which we denote by a white circle:

$g_{ab}=$ 

$\delta_a^b =$ 

The invariance of the metric is given by



Although B_r has spinor representations, the group that acts on $\mathfrak{g}$ is $SO(2r+1)$ rather than $Spin(2r+1)$ and hence the invariants must be expressible in terms of representations of $SO(2r+1)$, which in turn can be written in terms of the reference representation V. Hence we can write all tensors in $(T(B_r))^{SO(2r+1)}$ as graphs with reference and adjoint edges. By the first fundamental theorem of the invariant theory of $SO(2r+1)$ acting on the $2r+1$-dimensional representation [FH04], the relevant vertices are Clebsches between the adjoint and $V \otimes V^\vee$, as well as the bilinear form and Levi-Civita tensor for V. We will use the bilinear form on V to identify V with $V^\vee$ and remove the arrows from the reference edges.

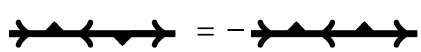
Since the tensors have no reference indices, every reference edge must either form a loop or end in a Levi-Civita tensor. Since the Levi-Civita tensors have odd degree, they must appear in pairs, and so again we can cancel them to leave only possible metric forms and dual metric forms. Since the metric form has two edges coming out and no edges going in, for each instance of the metric in the diagram there must be a copy of the dual of the metric form connected to it by a reference edge. The metric form can be moved past an attached adjoint edge at the cost of a sign change, so the metric form and its dual can be placed next to each other and thus cancelled. Hence all instances of the metric forms and its dual can be removed from a diagram in $(T(B_r))^{SO(2r+1)}$, leaving only loops in the reference representation attached to adjoint edges.

5.4 C_r

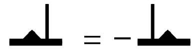
For C_r, all representations of $Sp(r)$ can be written in terms of the $2r$-dimensional representation. By the first fundamental theorem of the invariant theory of $SO(2r+1)$ acting on the $2r+1$-dimensional representation [FH04], the relevant vertices for the $2r$-dimensional representation are the Clebsches and the symplectic form and its dual. Here we denote the symplectic form by a triangle:

$\omega_{ab} =$ 

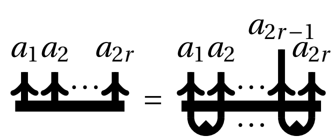
The inverse ω^{ab} is denoted by a triangle with the arrows pointing away, with the convention:

$\delta_a^b =$ 

The invariance of the symplectic form is given by



The Levi-Civita tensor can itself be replaced by a fully antisymmetrized multiple of $(\omega)^{\otimes r}$



where again the black bar on the right indicates full antisymmetrization. Similarly, the dual of the Levi-Civita tensor can be replaced by copies of the dual of the symplectic form. Hence the only relevant invariant is the symplectic form.

Since the symplectic form has two edges coming out and no edges going in, for each instance of the symplectic in the diagram there must be a copy of the dual of the symplectic form connected to it by a reference edge. The symplectic form can be moved past an attached adjoint edge at the cost of a sign change, so the symplectic form and its dual can be placed next to each other and thus cancelled. Hence all instances of the symplectic forms and its dual can be removed from a diagram in $(T(C_r))^{Sp(r)}$, leaving only loops in the reference representation attached to adjoint edges.

5.5 Other simple Lie algebras

For D_r, the Levi-Civita tensor does not need to appear in pairs since it has an even number of reference edges attached to it. Hence there are invariants of D_r that are not traces over the reference representation, including one of the primitive Casimir operators. While for $r = 2k+1$ the set of primitive Casimir operators for D_r can be expressed as traces in one of the spins representations, for r even there are two degree r primitive Casimir operators, and since there is up to scaling only one possible degree r fully symmetrized trace in any single representation, there cannot be a single representation for which all of the invariant tensors in $(T(D_r))^{SO(2r)}$ can be expressed via traces. The invariant tensors of D_r can be described similarly to those of the other classical

Lie algebras, but due to these complications will be handled in the chapter on D_r.

For the exceptional Lie algebras, the author conjectures that the statement given above does hold for them, but the above methods for showing such do not work due to the existence of higher-order invariants in their reference representations that do not vanish as simply as the ones for A_r, B_r and C_r.

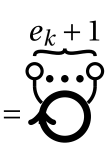
Chapter 6

The A_r case

As an example, we will use the case of A_3, as A_1 and A_2 have been fully worked out in [Ro01].

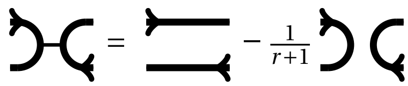
6.1 Diagrams for A_r

The reference representation V of A_r we take to be the $r+1$-dimensional representation. A_r has exponents $e_i = i$ and primitive Casimir operators all of the form $c_i = tr_V(M_d^{e_i+1})$:

$c_k =$ 

In our example, A_3 has exponents $1, 2$ and 3, and has primitive Casimir elements of degree $2, 3$ and 4.

The projection from $V \otimes V^\vee$ to the adjoint representation can be represented diagrammatically as

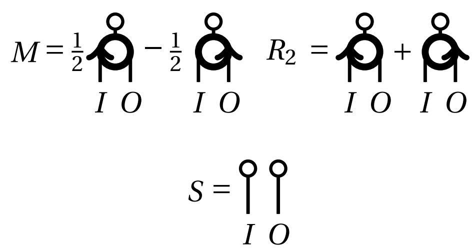


which corresponds to removing the trace from a tensor in $V \otimes V^\vee$.

6.2 Structure of the Family Algebra

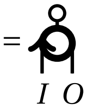
The family algebra is generated as an algebra over $I(\mathfrak{g})$ by the following pieces, written in diagrammatic notation:

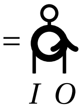
Theorem 6.2.1 (**Generators'**). *The family algebra $C_\mathfrak{g}(A_r)$ is generated over $I(A_r)$ by the following:*



However, the relations in terms of these generators are fairly ugly. In particular, the relations for powers of M and R_2 are complicated. Instead, we replace M and R_2 with the following:

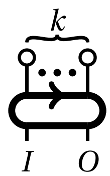
Theorem 6.2.2 (**Generators**). *The family algebra $C_\mathfrak{g}(A_r)$ is generated over $I(A_r)$ by the following:*

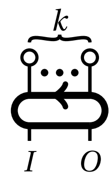
$K =$ 

$L =$ 

Note that $K = \frac{R_2}{2} + M$ and $L = \frac{R_2}{2} - M$, so the algebra generated by M, R_2 and S is isomorphic to the algebra generated by K, L and S.

For the relations, first we define the following elements:

$K_k =$ 

$L_k =$ 

As will be shown, K_k and L_k are expressible in terms of K, L and S in a uniform manner. Then for all r, we get the following relations:

Theorem 6.2.3 (**Relations**)**.** *The following relations are sufficient for defining the family algebra with the generators above*

$$KL = LK, \; KS = LS, \; SK = SL$$

$$SK_mL_nS = (c_{m+n+1} - \frac{1}{r+1} c_m c_n)S$$
$$K_{r+1} = \sum_{k=0}^{r-1} d_{r-k+1} K_k, \; L_{r+1} = \sum_{k=0}^{r-1} d_{r-k} L_k$$
$$\sum_{l=0}^{r} K^{r-l} L^l = \sum_{k=0}^{r-2} d_{r-k} \sum_{l=0}^{k} K^{k-l} L^l$$

Note that these relations are not independent. The SK_mL_nS relations become redundant when $m + n > r$. The K_{r+1} relation minus the L_{r+1} relation gives the K^kL^l relation times $K - L$, and hence the three relations are not independent. We present both the K_{r+1} and the L_{r+1} relation because each of them is easier to prove individually than any linear combination of them that isn't a multiple of the $K^k L^l$ relation.

In our example, the r-dependent relations become

$$SK_mRl_nS = (c_{m+n-1} - \frac{c_m c_n}{4})S$$

$$Lk_4 = d_2K_2 + d_3K + d_4, L_4 = d_2L_2 + d_3L + d_4$$

$$K^3 + K^2L + KL^2 + L^3 = d_2(K + L) + d_3$$

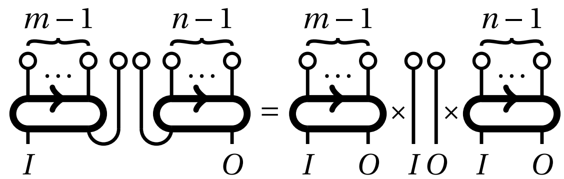
Here $d_2 = c_1/2$, $d_3 = c_2/3$ and $d_4 = \frac{c_3}{4} - \frac{c_1^2}{8}$.

6.3 Sufficiency of the Generators

As shown in the previous chapter, all of our elements of $(\otimes A_r)^{A_r}$ are tensor products of traces, so our family algebra elements are thus all tensor products of traces. We now consider our three types of univalent vertices, the I, O and dotted vertices. A trace with only dotted vertices on the ends of the attached adjoint edges is an element of $I(A_r)$, so we only have to generate the connected components of the diagram with the I or O vertices. But as we saw, the only diagrams we need are those whose connected components are traces. Thus we need to generate all diagrams where both the I and the O vertices are connected to the same trace, and all diagrams where they're connected to different traces.

First we show that, given K_k and L_k for all k, we can generate an element where the

I vertex is connected to a trace of degree m and the O vertex is connected to another trace of degree n, the two traces being distinct connected components:



$$= K_{m-1} S K_{n-1}$$

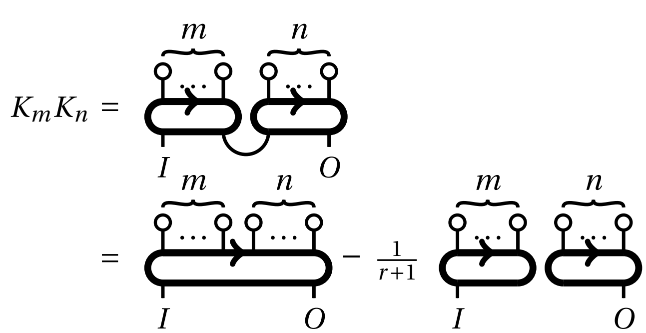
Note that for a trace connected to the I vertex but not the O vertex, the direction of the arrow is irrelevant, since all of the dotted vertices are symmetrized over. Similarly, for a trace connected to the O vertex but not the I vertex, the direction of the arrow is irrelevant. Hence $K_{m-1}S = L_{m-1}S$ and $SK_{n-1} = SL_{n-1}$.

Now we show that K_k and L_k can be generated via K, L and S. We shall show the derivation for K_k; the L_k case is analogous.

Lemma 6.3.1. *K_k can be generated over $I(A_r)$ by K and S*

We first note that $K_1 = K$, and then proceed by induction.

We assume that we can generate K_m, K_n, K_{m-1} and K_{n-1}, and now we show that we can generate K_{m+n}:



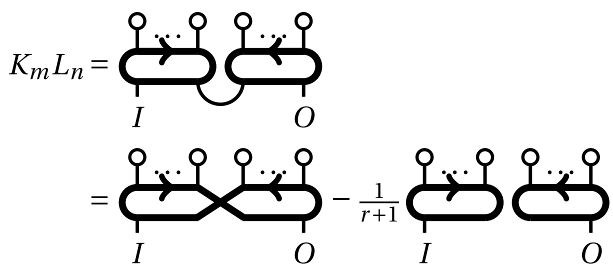
$$= K_{m+n} - \frac{1}{r+1} K_{m-1} S K_{n-1}$$

The second line uses the projection from $V \otimes V^\vee$ to the adjoint representation to remove the internal adjoint edge. Thus we can write

$$K_{m+n} = K_m K_n + \frac{1}{r+1} K_{m-1} S K_{n-1}$$

So thus we can generate K_k for all k.

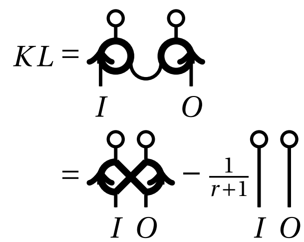
Finally, we just have to generate all of the other diagrams where both the I and O vertices are connected to the same trace. These are all traces where the reference edge attaches to the I vertex, then to m dotted vertices, then to the O vertex, and then to n vertices.



The first term in the last line is precisely what we want, and both $K_m L_n$ and the last term, $K_{m-1} S L_{n-1}$, can be generated from K, L and S by assumption. Hence our generators are sufficient to generate the whole family algebra.

6.4 Proof of the Relations

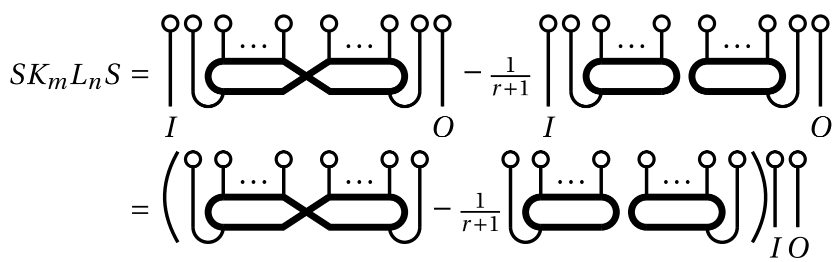
We have already seen that the relations $KS = LS$ and $SK = SL$ hold, as special cases of $K_m S = L_m S$ and $SK_n = SL_n$. Now we can prove the other relations. We start with



where the traces in the second term of the last expression have only two adjoint edges attached, and hence by our normalization become just adjoint edges.

LK differs only in the direction of the arrows on in the first term, so we end up with a trace attached to the I vertex, then a dotted vertex, then the O vertex, and then another dotted vertex. But the dotted vertices are interchangeable, so which of the two dotted vertices we pass through first doesn't matter. Hence the direction of the arrow doesn't matter and so $KL = LK$.

The SK_mL_nS relation follows from the expression for the product of K_mL_n computed above:



The contents of the parentheses, being unconnected to the I and O vertices, is an element of $I(A_r)$, and counting the dotted vertices coming off each trace gives a factor of $c_{m+n-1} - \frac{1}{r+1} c_m c_n$.

The other relations mentioned follow from variations of the Cayley-Hamilton identity. Since the reference representation is $r+1$-dimensional, for any $X \in A_r$ we have a relation

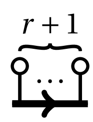
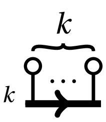
$$[\pi(X)]^{r+1} = \sum_{k=0}^{r} d_{r+1-k}(X)[\pi(X)]^k$$

where d_k is a degree k polynomial of the entries of $\pi(X)$. Thus for the matrix $M_d = \pi(X_\alpha) \otimes X^\alpha$, we get a relation

$$M_d^{r+1} = \sum_{k=0}^{r} d_{r+1-k} M_d^k$$

where now d_k is an element of $I(A_r)$.

Diagrammatically, this translates as

 $= \sum_{k=0}^{r} d_{r+1-k}$ 

Now we note that K_k contains a reference line attached to k dotted vectors, and so for K_{r+1} we can make the above replacement. This yields the relation

$$K_{r+1} = \sum_{k=0}^{r} d_{r+1-k} K_k$$

And similarly for L_{r+1}.

The final relation comes from the decomposition of $tr(M_d^{r+2})$ into primitive Casimir operators. We have the following relation, mentioned in the section on Casimir operators:

$$0 = \sum_{n_j m_j = r+2} \prod_j \frac{1}{m_j!} \left(-\frac{tr(M_d^{n_j})}{n_j} \right)^{m_j}$$

The coefficient of $tr(M_d^{r+2})$ on the right side is -1, so this gives an expression for $tr(M_d^{r+2})$ in terms of the primitive Casimir operators, recalling that $tr(M_d^{k+1}) = c_k$ for

$1 \leq k \leq r$.

We can translate this fact into one about the family algebra by writing all of the traces as diagrammatic traces, connected only to dotted vertices, and then for each diagram writing out all the ways to replace a dotted vertex by the I vertex and another dotted vertex by the O vertex. This is equivalent to taking derivatives with respect to the vectors corresponding to the edges connected to the I and O vertices.

Given a trace of degree d with only dotted vertices, there are d ways to replace one dotted vertex by the I vertex, and all the ways yield the same diagram. Given a product of traces with only dotted vertices, the number of ways to replace a dotted vertex by the I vertex is equal to the total degree of the product, with each trace of degree d_i yielding d_i identical diagrams.

Given a trace with the I vertex and $d-1$ dotted vertices, there are now $d-1$ ways to replace a dotted vertex by the O vertex. Given a product of traces with one vertex being the I vertex and the rest dotted, the number of ways to replace a dotted vertex by the O vertex is the degree of the product (which only counts the dotted vertices). Note that we have two possibilities here: the O vertex could be on the same or on a different trace as the I vertex.

Using the fact that

$$d_k = \sum_{m_i n_i = k} \prod_i \frac{1}{m_i!} \left(- \frac{tr(M_d)^{n_i}}{n_i} \right)^{m_i}$$

for $k \leq r+1$, we get that the sum is thus

$$0 = \sum_{i,j} d_{r-i-j-2} K_i S L_j + \sum_{i,j} d_{r-i-j} Q_{i,j}$$

where $Q_{i,j}$ is a trace connected to the I vertex, then i dotted vertices, then the O vertex, and then j dotted vertices, which we saw above can be written as

$$K_i L_j + \frac{1}{r+1} K_{i-1} S L_{j-1}$$

Writing out K_m and L_n in terms of K, L and S, we get that all of the terms involving S automatically cancel out, leaving the relation

$$\sum_{k=0}^{r} K^k L^{r-k} = \sum_{j=0}^{r-2} d_{r-j} \sum_{k=0}^{j} K^k L^{j-k}$$

6.5 The Sufficiency of the Relations

Here we show that the relations listed above are sufficient to determine the algebra, i.e. that any further relations on the algebra can be derived from the relations already given.

Lemma 6.5.1. *No monic polynomial in $K+L$ with coefficients in $I(\mathfrak{g})$ and degree less than r can vanish.*

Proof. If we consider $K^k L^{m-k}$, lower the raised coordinate using the Killing form, and then symmetrize over all of the indices, coordinate or otherwise, we end up with a polynomial in Casimir elements of degree $m+2$ including a term of c_{m+2} and all other terms products of Casimir elements of lower degree.

Now suppose that we have a monic polynomial in $K+L$ with coefficients in $I(\mathfrak{g})$ and degree m less than r. Then the leading terms, i.e. the terms involving no nontrivial

Casimir elements, has positive coefficients for all terms of the form $K^k L^{m-k}$ and thus the symmetrization of this polynomial then yields a polynomial in Casimir elements with nonvanishing c_{m+2} coefficient.

Since $m < r$, we have that $m+2 < r+2$ and hence c_{m+2} is algebraically independent of the Casimir elements of lower degree; hence the symmetrization cannot vanish, and hence the polynomial in $(L+R)$ cannot vanish. □

Consider now the $K^k L^l$ relation. The leading term is

$$\sum_k K^k L^{r-k}$$

and hence multiplying this leading term by $(K+L)^m$ yields a polynomial in K and L that only has positive coefficients. In particular, the term $K^r L^m$ has positive coefficient in this polynomial. For $m \leq r$, neither K^r nor L^m can be reduced by the K_{r+1} or L_{r+1} relations.

Hence we get that for $m < r$, $(K+L)^m$ times the $K^k L^l$ relation yields a relation in each degree greater than $r-1$ that cannot be deduced from the other relations. Since no polynomial in $K+L$ vanishes for degree less than r, we get that the relations of the form $(K+L)^m$ times the $K^k L^l$ relation are themselves linearly independent from one another over $I(\mathfrak{g})$. We also get a relation in degree $2r$ by squaring the $K^k L^l$ relation, and this one is also linearly independent from the other relations since the $K^k L^l$ relation is itself a polynomial in K and L that is linearly independent from all polynomials in $K+L$, just by comparing leading coefficients.

Now we show that there cannot be any other relations that are not in the ideal generated by the ones listed. We do so by counting the number of $I(\mathfrak{g})$-linearly independent monomials.

Note that since $KS = LS$, we can write $K^r S = \frac{1}{r+1} \sum_k K^k L^{r-k} S$. Hence, using the $K^k L^l$ relation, we can reduce $L^r S$ to terms involving nontrivial Casimir elements. Hence we get that $K^r SL^l$ is not linearly independent over $I(\mathfrak{g})$ from terms of lower degree. Similarly, $K^k SL^r$ cannot be linearly independent.

Thus we get that our linearly independent monomials are $K^k SL^l$ for $0 \le k, l \le r-1$ and $K^k L^l$ for $0 \le k, l \le r$, minus one in each degree between r and $2r$ since $(K+L)^m$ times the $K^k L^l$ relation gives $K^r L^m$ in terms of other monomials.

This yields a total of $2r^2 + r$ terms not known to be linearly dependent. If there are more relations, then there will be fewer linearly independent terms.

The dimension formula for family algebras tells us that we should be getting

$$dim_{I(\mathfrak{g})} C_{\mathfrak{g}}(\mathfrak{g}) = \sum_{\lambda \in Wt(\mathfrak{g})} m_{\mathfrak{g}}(\lambda)^2$$

For the adjoint representation, the weights with non-zero multiplicity are the roots, each with multiplicity 1, and 0, with multiplicity equal to the rank of the algebra. This gives us $r(r+1) + r^2 = 2r^2 + r$. Hence, since the relations given above limit us to at most $2r^2 + r$ linearly independent elements and any further relations would reduce that number, there cannot be any more relations.

Thus we can determine an $I(\mathfrak{g})$-linear basis for the family algebra in terms of K, L and S. Using the original basis M, R_2 and S, we rewrite the set as

$$M^m R_2^n \text{ for } m \le e_r + 1, n \le r - 1$$

$$R_2^m S R_2^n + R_2^n S R_2^m \text{ for } m \le n \le r-2$$

$$R_2^m S R_2^n - R_2^n S R_2^m \text{ for } m < n \le r-1$$

Note that we can define an element R_k for $k \le r$, where $R_k = K_r + L_k$, which in turn can be written as R_2^k plus other terms. Hence we can write write our basis as

$$M^m R_k \text{ for } m \le e_r + 1, k \le r-1$$

$$R_m S R_n + R_n S R_m \text{ for } m \le n \le r-2$$

$$R_m S R_n - R_n S R_m \text{ for } m < n \le r-1$$

In our example of A_3, we have the following basis

$$1, M, R_2, M^2, MR_2, R_2^2, S, M^3, M^2R_2, MR_2^2, R_2S + SR_2,$$

$$R_2S - SR_2, M^4, M^3R_2, M^2R_2^2, R_2SR_2, R_2^2S - SR_2^2$$

$$M^4R_2, M^3R_2^2, R_2^2SR_2 - R_2SR_2^2, M^4R_2^2$$

6.6 Generalized Exponents

For the irreducible component of $\mathfrak{g} \otimes \mathfrak{g}^\vee$ with highest weight λ, there is a projection operator P_λ that projects from $\mathfrak{g} \otimes \mathfrak{g}^\vee$ to the component of type V_λ. See [Cv08] for details. Using the Killing form, we identify $\mathfrak{g}^\vee$ with $\mathfrak{g}$ and consider $\mathfrak{g} \otimes \mathfrak{g}$. As a $\mathfrak{g}$-module this decomposes into $\wedge^2\mathfrak{g}$ and $S^2\mathfrak{g}$, the alternating and symmetric tensor square respectively, which then further decompose into irreducible representations.

For $r = 1$, $\wedge^2\mathfrak{g}$ is isomorphic to $\mathfrak{g}$ itself, and hence is the adjoint representation, with generalized exponent 1. $S^2\mathfrak{g}$ decomposes into a trivial representation and a representation of dimension 5; these representations have generalized exponents 0 and 2 respectively.

For $r = 2$, $\wedge^2\mathfrak{g}$ decomposes into a copy $\mathfrak{g}$, with generalized exponents 1 and 2, and two dual 10-dimensional representations with weights $3\omega_1$ and $3\omega_2$ respectively and each with generalized exponent 3. $S^2\mathfrak{g}$ decomposes into the trivial representation with generalized exponent 0, another copy of $\mathfrak{g}$, again with generalized exponents 1 and 2, and a 27-dimensional representation with generalized exponents $2,3$ and 4. See [Ro01] for details. Note that Rozhkovskaya uses a different basis, generated by harmonic elements. Her M_1 is proportional to M, her N_1 is proportional to R_2, and her N_2 is proportional to $3R_2^2 + 3M^2 + S + c_1$.

For $r \geq 3$, the decomposition of $\mathfrak{g} \otimes \mathfrak{g}$ is uniform. $\wedge^2\mathfrak{g}$ decomposes into a copy of $\mathfrak{g}$ and two dual representations with highest weights $2\omega_1 + \omega_{r-1}$ and $\omega_2 + 2\omega_r$ respectively, while $S^2\mathfrak{g}$ decomposes into the trivial representation, another copy of $\mathfrak{g}$, and two representations with highest weights $\omega_2 + \omega_{r-1}$ and $2\omega_1 + 2\omega_r$ respectively. In an orthonormal basis for $\mathfrak{g}$, the corresponding elements of the family algebra are actually symmetric or antisymmetric as matrices.

The $L^k R^l$ reduction relation gives us a relation $\sim$ on elements in $V_{\omega_2+\omega_{r-1}}$; applying the differential operator $D = \left(\frac{\partial}{x^\alpha} c_2\right) \frac{\partial}{\partial x_\alpha}$ gives a relation equivalent to the multiples of the $L^k R^l$ relation times $L + R$, modulo the L_{r+1} and R_{r+1} relations. Since D transforms as the trivial representation, D applied to both sides of $\sim$ again gives a relation between elements of the $V_{\omega_2+\omega_{r-1}}$ representation; hence we get that the generalized exponents of $V_{\omega_2+\omega_{r-1}}$ plus a copy of $\{r,\ldots,2r\}$ gives the generalized exponents of $V_{2\omega_1+2\omega_r}$. Along with the fact that $V_{\omega_1+\omega_r}$ has generalized exponents $1,\ldots,r$ gives us enough information to get the full set of generalized exponents for the representations in question, given in table 1. Note that the two copies of $V_{\omega_1+\omega_r}$ each give an independent set of harmonic basis elements, one symmetric, one antisymmetric. We get that $P_V(q)$ is equal to the Kostka polynomial for V, which are computable from Young Tableaux [DLT94]. Hence we can easily check the results given.

Table 6.1: Generalized Exponents in $C_{\omega_1+\omega_r}(A_r)$

V	$P_V(q)$
V_0	1
$V_{\omega_1+\omega_r}$	$q[r]_q$
$V_{\omega_2+\omega_{r-1}}$	$q^2\frac{[r+1]_q[r-2]_q}{[2]_q}$
$V_{2\omega_1+2\omega_r}$	$q^2\binom{r+1}{2}_q$
$V_{2\omega_1+\omega_{r-1}}$	$q^3\binom{r}{2}_q$
$V_{\omega_2+2\omega_r}$	$q^3\binom{r}{2}_q$

Table 6.2: Generalized Exponents in $C_{\omega_1+\omega_3}(A_3)$

V	$P_V(q)$
V_0	1
$V_{\omega_1+\omega_r}$	$q+q^2+q^3$
$V_{2\omega_2}$	q^2+q^4
$V_{2\omega_1+2\omega_3}$	$q^2+q^3+2q^4+q^5+q^6$
$V_{2\omega_1+\omega_2}$	$q^3+q^4+q^5$
$V_{\omega_2+2\omega_3}$	$q^3+q^4+q^5$

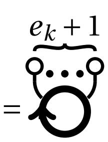
Chapter 7

The B_r, C_r case

The cases of B_r and C_r end up very similar, so we treat them both here. We start with C_r since it is somewhat simpler. As with the previous chapter, we use the case of $r = 3$ as an example.

7.1 Diagrams

As in the A_r case, we can write the primitive Casimir operators as traces

$c_k =$ 

Because the invariant changes sign every time it passes an adjoint edge, we get that the odd-degree traces vanish, matching the fact that C_r only has odd-degree exponents and hence even degree primitive Casimir elements. For C_r, the exponents are $e_i = 2i - 1$, so that $c_k = tr_V(M_d^{2i})$.

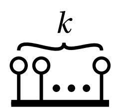
Similar to the A_r case, we have a Cayley-Hamilton identity on our matrices in the reference representation. Defining

$$d_k = \sum_{2n_i m_i = k} \frac{1}{m_i!}\left(-\frac{c_{n_i}}{2n_i}\right)^{m_i}$$

where $m_i n_i$ indicates a sum over distinct n_i, we get that

$$\sum_k d_{2r-k} Q^k = 0$$

where

$Q^k =$ 

We call this the matrix Cayley-Hamilton identity, to distinguish it from the Casimir Cayley-Hamilton identity

$$\sum_2 n_i m_i = 2r + 2\frac{1}{m_i!}\left(-\frac{c_{n_i}}{2n_i}\right)^{m_i} = 0$$

which we get by multiplying the matrix Cayley-Hamilton identity by M^2 and then taking traces in the $2r$-dimensional representation.

The adjoint projection for getting rid of internal adjoint edges is also different:

Note the directions of the symplectic forms; the first term on the right-hand side has both symplectic forms attached to the top edge, where they cancel.

Now we wish to show that the tensor invariants in $(T(C_r))^{Sp(r)}$ are generated by tensor products of traces over the reference representation.

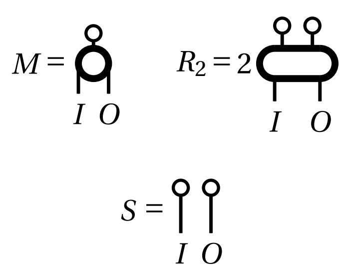
All finite dimensional representations of C_r can be written in terms of the reference representation, so we only have to worry about tensors with adjoint and reference edges. The vertices are Clebsches between the adjoint and $V \otimes V^\vee$, and the Levi-Civita tensor on V.

Note that the reference representation, being of dimension $2r$, has a Levi-Civita tensor with $2r$ vectors coming out of it. Moreover, taking r copies of the symplectic form and antisymmetrizing all of the edges yields a multiple of the Levi-Civita tensor. Hence the Levi-Civita tensor can be replaced by the symplectic form. Thus we only have loops of the reference edges with adjoint edges attached, i.e. traces over the reference representation.

7.2 Generators

The main result about the generators for C_r is that there are again three of them:

Theorem 7.2.1 (Generators)**.** *The family algebra for the adjoint representation of C_r is generated by*



Because of the symplectic form, we get that L and R are no longer independent elements of the family algebra; in particular, $M = L = -R$ and there is no other independent degree 1 family algebra element.

Proof. For ease of calculation, again we use an alternate object instead of R_2. We write $T_{k,l}$ to be a trace attached to the I vertex, k dotted vertices, the O vertex, and then l dotted vertices, in that order. Thus M is $T_{1,0}$. In this notation, $R_2 = 2T_{2,0}$.

Using the symplectic form to swap the direction of the reference tensor, we get that

$$T_{k,l} = (-1)^{k+l} T_{l,k}$$

and via the adjoint projector we get

$$T_{i,j} T_{k,l} = \frac{1}{2}(T_{i+k,j+l} + (-1)^{k+l} T_{i+l,j+k})$$

In particular,

$$M T_{k,l} = \frac{1}{2}(T_{k+1,l} + (-1)^{k+l} T_{l+1,k})$$

We see the sufficiency of the generators as given by noting that

$$M^2 = \frac{1}{2}(T_{2,0} - T_{1,1})$$

so $T_{2,0}$ and $T_{1,1}$ can be generated from M and R_2, and then that

$$T_{k+1,l} = M T_{k,l} - M T_{l-1,k+1} + T_{2,0} T_{k,l-1}$$

so we can generate any $T_{k,l}$ via inducting from our base cases.

The only remaining possible elements are those where the I and O vertices are connected to unconnected traces. These we can achieve by S. In particular, we can realize an element where the I vertex is attached to a trace of degree k and the O vertex to a trace of degree l by $T_{k-1,0}ST_{l-1,0}$. □

7.3 Relations

Several relations are familiar from the A_r case:

Theorem 7.3.1. *The following relations hold for* C_r*:*

$$SM = MS = 0$$

$$R_2 M = MR_2$$

$$\sum_{k=0}^{r} d_{2r-2k} T_{2k,0} = 0$$

Define

$$Q_{2k} = \sum_{l=0}^{2k} T_{l,2k-l} - \sum_{l=0}^{k-1} T_{2l,0} S T_{2k-2l-2,0}$$

Then

$$\sum_{k=0}^{r} d_{2r-2k} Q_{2k} = 0$$

In our example, the r-dependent relations become

$$-T_{6,0} + d_2 T_{4,0} + d_4 T_{2,0} + d_6 = 0$$

$$2T_{6,0} + 2T_{5,1} + 2T_{4,2} + T_{3,3} - T_{4,0}S - T_{2,0}ST_{2,0} - ST_{4,0}$$

$$= d_2(2T_{4,0} + 2T_{3,1} + T_{2,2} - T_{2,0}S - ST_{2,0}) + d_4(2T_{2,0} + T_{1,1} - S) + d_6$$

The first two relations can be seen by expanding out the relevant diagrams, or in the case of the second relation by expanding out the $T_{k,l}$ relations.

For the third relation, we have the matrix Cayley-Hamilton relation, which, as in the A_r case, gives us a relation on reference edges connected by adjoint edges to only dotted vertices. For C_r, the identity only involves even powers of the matrix, which translates to an even number of dotted vertices. $T_{2k,0}$ involves a reference edge connected to $2k$ dotted vertices, and thus we get the third relation.

The fourth relation comes from taking the decomposition of $tr(M_d^{2r+2})$ into primitive Casimir operators, interpreting it as diagrams, and replacing dotted vertices with I and O vertices, analogous to the $L^m R^n$ relation for A_r.

Note that the third relation has no mentions of S, and the fourth relation has both $ST_{2k,0}$ and $T_{2k,0}S$. Thus if P times the third relation yields R_2^l of the fourth relation, we get that $l = r$ and thus we get that at the very least the fourth relation yields r relations that are independent of the third relation.

We now use a counting argument. We can form objects of the forms $T_{i,j}$ and $T_{k,0}ST_{l,0}$. We note that if k or l is odd, then $T_{k,0}ST_{l,0}$ vanishes, due to the symplectic form. So we really only have $T_{i,j}$ and $T_{2k,0}ST_{2l,0}$. By the third relation, we can limit $i + j$ to be less than $2r$, and we can limit $i \leq j$ since $T_{i,j}$ and $T_{j,i}$ are not independent. We can similarly limit k and l to be less than r. So we have $T_{i,j}$ for $0 \leq i \leq j \leq 2r-1$ and $T_{2k,0}ST_{2l,0}$ for $0 \leq k, l \leq r-1$. This yields a total of $3r^2 + r$ elements, from which the fourth relation removes another r elements, to yield $3r^2$ linearly independent elements. By the dimension formula for family algebras, we should be getting $r^2 + 2r^2 = 3r^2$ elements. Since there are no more elements to remove, there are no more relations.

We will thus take as our basis $T_{i,j}$ for $0 \le i \le j \le 2r-1$, $T_{2k,0}ST_{2l,0} + T_{2l,0}ST_{2k,0}$ for $k, l \le r-2$ and $T_{2k,0}ST_{2l,0} - T_{2l,0}ST_{2k,0}$ for $k, l \le r-1$. To make it more in line with the results for other Lie algebras, we write this as

$$M^m R_2^n \text{ for } m \le e_r + 1, n \le r-1$$

$$R_2^m S R_2^n + R_2^n S R_2^m \text{ for } m \le n \le r-2$$

$$R_2^m S R_2^n - R_2^n S R_2^m \text{ for } m < n \le r-1$$

Note that we can define an element R_k for $k \le r$ as the trace that attaches to the I vertex, $e_k - 2$ dotted vertices, and then to the O vertex which in turn can be written as R_2^k plus other terms. Hence we can write write our basis as

$$M^m R_k \text{ for } m \le e_r + 1, k \le r-1$$

$$R_m S R_n + R_n S R_m \text{ for } m \le n \le r-2$$

$$R_m S R_n - R_n S R_m \text{ for } m < n \le r-1$$

So for C_3, the basis is

$$1, M, M^2, R_2, S, M^3, MR_2, M^4, M^2R_2, R_2^2, R_2S + SR_2, R_2S - SR_2,$$

$$M^5, M^3R_2, MR_2^2, M^6, M^4R_2, M^2R_2^2, R_2SR_2, R_2^2S - SR_2^2, M^5R_2,$$

$$M^3R_2^2, M^6R_2, M^4R_2^2, R_2^2SR_2 - R_2SR_2^2, M^5R_2^2, M^6R_2^2$$

Note that the only difference, at least in the labelling, between this and the basis for the family algebra for A_3 is the maximum power of M allowed.

7.4 B_r

Now we address the B_r case. We have a change in the adjoint projector:

We still get a sign change whenever we move the bilinear form past an adjoint edge and hence the primitive Casimir operators are all of even degree and follow the same Casimir Cayley-Hamilton identity. The change in the adjoint projector and the symmetry of the bilinear form make the $T_{k,l}$ objects for the family algebra for B_r follow the same rules as the ones for the C_r family algebra. So we get that the family algebra for C_r and the family algebra for B_r are almost isomorphic. There is one slight difference in the relation from the matrix Cayley-Hamilton identity.

Since the reference representation for B_r is $2r+1$-dimensional, we expect that the matrix Cayley-Hamilton relation has degree $2r+1$, and indeed it does, with no relation of lower degree working for all elements of B_r. Hence we get a relation

$$\sum_k d_{2r+1-(2k+1)} T_{2k+1,0} = 0$$

However, this relation can itself be reduced to a relation in lower degree. First we note that $d_{2r+1-(2k+1)} = d_{2r-2k}$. Secondly, we note that

$$T_{2k+1,0} = M \sum_{l=0}^{2k} T_{2k-l,l}$$

We further note that

$$\sum_{l=1}^{2k} T_{2k-l,l} = 2M \sum_{j=1}^{k} T_{2k-2j,2j-1}$$

Hence we look at

$$Q = \sum_{k=0}^{r} d_{2r-2k}\left(T_{2k,0} + 2M\sum_{j=1}^{k} T_{2k-2j,2j-1}\right)$$

Multiplying this by M yields the matrix Cayley-Hamilton relation, i.e. this expression times M vanishes. Now we look at this object restricted to the Cartan subalgebra.

Since M is invertible on the vector part, since the entry in the x_α, x^α position is $\alpha^\vee$, we get that since MQ vanishes on the vector part, Q must vanish on the vector part. We also note that M vanishes on the torus part, so the torus part of Q is

$$\sum_{k=0}^{r} d_{2r-2k} T_{2k,0}$$

Since the primitive Casimir operators for B_r restricted to the Cartan subalgebra are identical to the primitive Casimir operators for C_r restricted to the Cartan subalgebra, we get that since $\sum_{k=0}^{r} d_{2r-2k} T_{2k,0}$ vanishes on the torus for C_r, it also must vanish for B_r.

So Q restricted to the Cartan subalgebra must vanish on both the vector and torus parts, and hence vanishes everywhere. Since the restriction map is an injection, Q itself must vanish. Hence we have a relation in degree $2r$ rather than $2r+1$.

Again, this relation involves no terms of S, so again the Casimir Cayley-Hamilton identity yields r separate relations, and the counting argument above still holds. Hence we get that the relations for B_r are as follows:

Theorem 7.4.1. *The following relations hold for B_r:*

$$SM = MS = 0$$

$$R_2 M = M R_2$$

$$\sum_{k=0}^{r} d_{2r-2k}\left(T_{2k,0} + 2M\sum_{j=1}^{k} T_{2k-2j,2j-1}\right) = 0$$

Define

$$Q_{2k} = \sum_{l=0}^{2k} T_{l,2k-l} - \sum_{l=0}^{k-1} T_{2l,0} S T_{2k-2l-2,0}$$

Then

$$\sum_{k=0}^{r} d_{2r-2k} Q_{2k} = 0$$

Hence the family algebra for B_r has the same $I(\mathfrak{g})$-linear basis as the family algebra for C_r, with the two family algebras differing only in the algebraic relations.

In terms of the example of B_3, we have the first r-dependent relation being

$$-T_{6,0} - 2M(T_{4,1} + T_{2,3} + T_{0,5}) + d_2 T_{4,0} + 2d_2 M(T_{2,1} + T_{0,3}) + d_4 T_{2,0} + 2d_4 M T_{0,1} + d_6 = 0$$

and the other r-dependent relation being identical to the case for C_3. Similarly, the basis elements are identical to those for C_3.

7.5 Generalized Exponents

The decomposition of the tensor square of the adjoint representation into irreducible components is very similar for B_r and C_r. Both decompose into a symmetric part and an antisymmetric part, with the symmetric part decomposing into four irreducible components and the antisymmetric part decomposing into two. See [Cv08] for details and explicit projection operators.

Applying the projection operators to the basis elements computed above, we get that $P_2(T_{2k,0}ST_{2l,0} + T_{2l,0}ST_{2k,0})$ depends only on $k+l$, and that $P_2(T_{i,j})$ is a linear combination of the $P_2(T_{2k,0}ST_{2l,0} + T_{2l,0}ST_{2k,0})$ terms. So we get that the component corresponding to V_2 is spanned by $P_2(T_{2k,0}S + ST_{2k,0})$. Note that while there is a relation involving $T_{2r-2,0}S + ST_{2r-2,0}S$, that relation occurs in V_2 and thus we get that $P_2(T_{2r-2,0}S + ST_{2r-2,0})$ is linearly-independent from the set of $P_2(T_{2k,0}S + ST_{2k,0})$ for $k < r-1$.

Since V_2 corresponds to S^2V, we get that V_2 should have highest generalized exponent $2r$ and have r generalized exponents. Hence we get that the generalized exponents for V_2 are thus $q^2[r]_{q^2}$.

$$\begin{aligned} P_4(T_{2k+1,2l+1}) &= -P_4(T_{2k,0}ST_{2l,0} + T_{2l,0}ST_{2k,0}) \\ &= -\frac{1}{6}(T_{2k,0}ST_{2l,0} + T_{2l,0}ST_{2k,0}) - \frac{2}{3}T_{2k+1,2l+1} \end{aligned}$$

and $P_4(T_{2k,2l}) = 0$. So V_4 has a generalized exponent for each pair k,l such that $k,l \leq r-2$. Hence $P_{V_4}(q) = q^2\binom{r}{2}_{q^2}$.

V_1 is the trivial representation, and thus has a single generalized exponent of degree 0. So all the rest of the degrees of the symmetric basis elements give generalized exponents for V_3. Thus we have that the generalized exponents of V_3 are $q^2[r+1]_{q^2}[r-1]_{q^2}$. The antisymmetric elements yield degrees $q[r]_{q^2}^2 + q^4\binom{r}{2}_{q^2}$. The adjoint representation has exponents $q[r]_{q^2}$, so we are left with V_6 having exponents $q^3[3]_q\binom{r}{2}_{q^2}$.

Table 7.1: Generalized Exponents in $C_{\mathfrak{g}}(B_r/C_r)$

V	$P_V(q)$
V_1	1
V_2	$q^2[r]_{q^2}$
V_3	$q^2[r+1]_{q^2}[r-1]_{q^2}$
V_4	$q^2\binom{r}{2}_{q^2}$
V_5	$q[r]_{q^2}$
V_6	$q^3[3]_q\binom{r}{2}_{q^2}$

Table 7.2: Generalized Exponents in $C_{\mathfrak{g}}(B_3/C_3)$

V	$P_V(q)$
V_1	1
V_2	$q^2+q^4+q^6$
V_3	$q^2+2q^4+2q^6+2q^8+q^{10}$
V_4	$q^2+q^4+q^6$
V_5	$q+q^3+q^5$
V_6	$q^3+q^4+2q^5+q^6+2q^7+q^8+q^9$

Chapter 8

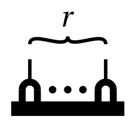
The D_r case

Here we use both $r = 3$ and $r = 4$ as examples, as they have somewhat different behavior, and also because $r = 3$ has already been discussed, via the $A_r \cong D_r$ isomorphism.

8.1 Diagrams

The first fundamental theorem for $SO(2r)$ acting on the $2r$-dimensional representation tells us that the only invariants are a symmetric bilinear form and a Levi-Civita tensor. Thus the projection from $V \otimes V^\vee$ to the adjoint representation is the same as that for B_r. Unlike in the B_r case, the Levi-Civita tensor has an even number of edges coming out of it, and so we can have an element of $(T(D_r))^{SO(2r)}$ with a single Levi-Civita tensor in it, and unlike in the C_r case, the Levi-Civita tensor cannot be reduced to the bilinear form. As a result, the set of primitive Casimir operators for D_r cannot be expressed solely as traces over the reference representation. Instead, we have traces of

$2k$ adjoint edges for $1 \leq k \leq r-1$, and a degree r Casimir, called the Pfaffian, built from the Levi-Civita tensor.

$Pf=$ 

Note that since the $2r$ reference edges going into the Levi-Civita tensor are fully antisymmetrized, the adjoint edges attached to the Pfaffian are automatically fully symmetrized as a tensor, unlike the traces which have to be symmetrized separately.

We now have a decomposition of $tr(M_d^{2r})$ into lower order Casimir operators. We get

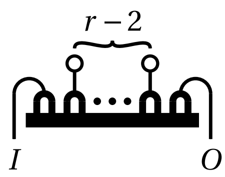
$$\sum_{n_i m_i = 2r} \prod_i \frac{1}{m_i!} \left(-\frac{tr(M_d)^{n_i}}{n_i} \right)^{m_i} = 4(-1)^r \left(\frac{Pf}{r!} \right)^2$$

The rest of the elements of $(T(D_r))^{SO(2r)}$ are either tensor products of traces, or tensor products of traces with tensors that like the Pfaffian are built from a single Levi-Civita tensor, only with multiple adjoint edges attached to each reference edge instead of just one. Note that since the Levi-Civita tensor is fully antisymmetric, the ends of the reference edges are antisymmetrized, so there must be an odd number of adjoint edges attached to each one.

8.2 Generators

The generators of B_r can be used to generate all of the elements of the family algebra that do not involve the Pfaffian, but always yield diagrams that only have traces. To get the diagrams that involve the Pfaffian, we add a fourth generator, P:

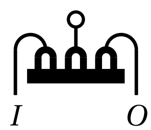
$P =$



Since the Pfaffian is independent of the trace Casimir operators, we cannot hope to build P out of the elements written as M, R_2 and S. So we have to have it as a fourth generator.

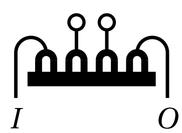
In the case of $r = 3$, the Pfaffian looks like

$P =$



In the $r = 4$ case, it looks like

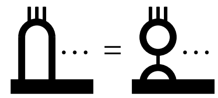
$P =$



8.3 Sufficiency of the Generators

The same reasoning as used for B_r shows that any element of the family algebra built only from traces in the reference representation can be generated by M, R_2 and S. Hence we consider elements that contain a Levi-Civita tensor.

Suppose our element A has a single Levi-Civita tensor, and assume that for all elements B such that $\deg(B) < \deg(A)$, B can be generated by M, R_2, S and P. All the reference edges going out of the Levi-Civita tensor must go back in, and must have an odd number of adjoint edges attached. The reference edges are all fully antisymmetrized, so for a given loop we can insert an adjoint edge projector.



Suppose we have such a loop L with $2k+1$ adjoint edges attached to it. If the I vertex is attached to the loop, then we can insert an adjoint projector and write the family algebra element as $A = T_{m,2k-m}B$ where B has one copy of the Levi-Civita tensor. By construction, $\deg(B) = \deg(A) - 2k$ and furthermore has only one adjoint edge connected to the loop corresponding to the loop L on our original element. Similarly, if the O vertex is attached to the loop, then we can construct a B such that $A = BT_{m,2k-m}$ for some m. Since $\deg(Q) < \deg(A)$ in both cases, Q must be generated by M, R_2, S and P, and thus A must also be generated by them.

Now consider the case where both the I and O vertices are attached to loops that only have one adjoint edge attached to them. Using the symmetrization process outlined in chapter 3, we can construct $\overline{A}$ and $\widetilde{A}$. $\overline{A}$ can be written in terms of primitive Casimir operators, i.e. in terms of M, R_2, S and P. The terms in $\widetilde{A}$ either have the I vertex attached to a loop with only one adjoint edge attached or attached to a loop with multiple adjoint edges attached. In the latter case, we can rewrite the term as $T_{m,2k-m}B$ for some B of lower degree, while in the former case we get another copy of A, since the reference edges connected to the Pfaffian are fully antisymmetrized, so the loops are distinguishable only by how many adjoint edges are attached and to what the other ends of those adjoint edges attach. So we have a nonzero multiple of A being equal to something expressible in terms of M, R_2, S and P, plus terms of the form $T_{m,2k-m}B$ where B has degree less than A. Since $\deg(B) < \deg(A)$ for all of the B, they must be generated by M, R_2, S and P. Thus A must also be generated by them.

Hence M, R_2, S and P generate the entire family algebra.

8.4 Relations

Several of the relations for M, R_2 and S are identical to those from B_r:

$$MR_2 = R_2M$$

$$MS = SM = 0$$

We have a relation coming from the Cayley-Hamilton identity for Casimir operators:

$$2PSP + 2PfP + \sum_{k=0}^{r-1} d_{2r-2k-2}Q_{2k} = 0$$

Expanding out Q_{2k} gives us that we can read this relation as expressing $R_2^{r-2}S + SR_2^{r-2}$ in terms of other elements.

The relation also gives us that $R_2^{r-1}S$ decomposes, since the resulting diagram has a trace of degree $2r$ attached to the I vertex but not the O vertex, and thus is it symmetrizes to $tr(M_d^{2r})$, which decomposes, and all of the diagrams in the symmetrization are the equal. Similarly, SR_2^{r-1} also decomposes.

We also have additional relations involving P:

$$PS + PT_{1,1} = 2Pf$$

Defining $P_{2k} = Q_{2k} - \sum_{l=0}^{k-1} T_{2l+1,2k-2l-1}$ for $k \geq 1$ and $P_0 = 1$, we get

$$P^2 + \sum_{k=0} (-1)^{r-1} d_{2r-4-2k} P_{2k} = 0$$

So our possibly linearly independent elements are

$$M^m R_2^n \text{ for } m \leq 2r-2, n \leq r-2$$

$$M^m P \text{ for } m \leq 2r-2$$

$$R_2^m S R_2^n + R_2^n S R_2^m \text{ for } m \leq n \leq r-3$$

$$P S R_2^m + R_2^m S P \text{ for } m \leq r-3$$

$$R_2^m S R_2^n - R_2^n S R_2^m \text{ for } m < n \leq r-2$$

$$P S R_2^m - R_2^m S P \text{ for } m \leq r-2$$

$$PSP$$

Similarly to A_r, B_r and C_r, we can define R_k to be an element related to the primitive Casimir operators, although here we have to be careful about indexing. For $k \neq \lceil \frac{r}{2} \rceil$, we define R_k to be the trace that attaches to the I vertex, then to the $e_k - 1$ dotted vertices, and then to the O vertex, while for $R_{\lceil \frac{r}{2} \rceil}$ we use P. Then we can rewrite the basis as

$$M^m R_k \text{ for } m \leq e_r + 1, k \leq r-1$$

$$R_m S R_n + R_n S R_m \text{ for } m \leq n \leq r-2$$

$$R_m S R_n - R_n S R_m \text{ for } m < n \leq r-1$$

This gives a total of $3r^2 - 2r$ elements, which matches the number of elements given by the dimension formula. Note that we could expand PSP by expanding the two Levi-Civita tensors into antisymmetrized reference edges, but including PSP makes the full set easier to describe, as well as better matching the list of linearly independent elements of $A_3 = D_3$. In particular, for D_3 we have

$$1, M, P, M^2, MP, R_2, S, M^3, M^2P, MR_2, PS + SP,$$

$$PS - SP, M^4, M^3P, M^2R_2, PSP, R_2S - SR_2,$$

$$M^5, M^4P, M^3R_2, PSR_2 - R_2SP, M^4R_2$$

The P here corresponds to R_2 for A_3, with R_2 here taking the place of R_2^2 for A_3. For $r = 4$, we have

$$1, M, M^2, R_2, P, S, M^3, MR_2, MP,$$

$$M^4, M^2R_2, M^2P, R_2^2, R_2S + SR_2, PS + SP,$$

$$R_2S - SR_2, PS - SP, M^5, M^3R_2, M^3P, MR_2^2,$$

$$M^6, M^4R_2, M^4P, M^2R_2^2, R_2SR_2, PSR_2 + R_2SP, PSP,$$

$$R_2^2S - SR_2^2, PSR_2 - R_2SP, M^5R_2, M^5P, M^3R_2^2,$$

$$M^6R_2, M^6P, M^4R_2^2, R_2^2SR_2 - R_2SR_2^2, PSR_2^2 - R_2^2SP,$$

$$M^5R_2^2, M^6R_2^2$$

8.5 Generalized Exponents

With a few low-dimensional exceptions, for D_r the $\mathfrak{g} \otimes \mathfrak{g}$ representation decomposes identically to the B_r case: the symmetric subspace of $\mathfrak{g} \otimes \mathfrak{g}$ decomposes into four irreducible representations, one being the trivial representation, and the antisymmetric subspace decomposes into two representations, one being the adjoint representation. The projection operators are identical, except instances of $2r+1$ are replaced by $2r$. So we label the six representations P_1 through P_6, matching the notation from B_r. For D_3, the antisymmetric subspace decomposes into three irreducible representations due to the Pfaffian. This matches the decomposition of A_3. For $r > 3$, the Pfaffian does not appear in any projection operators, so there six components listed are all irreducible. The projections of the elements involving only M, R_2 and S are the same as in the B_r case, so we can just examine the elements involving P.

$P_2(PM^{2k})$ and $P_2(PSR_2^k)$ both decompose into lower order terms. $P_2(PSP)$ can be written in terms of P_2 applied to elements generated by M, R_2 and S. Hence V_2 has generalized exponents $P_{V_2}(q) = q^2[r-1]_{q^2}$, matching the B_r case but with $[r-1]_{q^2}$ instead of r due to the decomposition of the degree $2r$ trace.

$P_4(PM^{2k})$ reduces to PSR_2^k for k positive, as does $P_4(PSP)$. $P_4(PSR_2^k)$ expands as PSR_2^k plus lower order terms. Hence we get that V_4 has generalized exponents

$$P_{V_4}(q) = q^2\binom{r-1}{2} + q^{r-2}[r-1]_{q^2}$$

V_3 then has the rest of the positive degree symmetric elements, giving it generalized exponents

$$P_{V_3}(q) = q^2[r]_{q^2}[r-2]_{q^2} + q^r[r-1]_{q^2} + q^{2r-2}$$

The adjoint representation has exponents $P_{V_5}(q) = q[r-1]_{q^2} + q^{r-1}$ and therefore V_6 has the remaining antisymmetric degrees, giving it generalized exponents

$$P_{V_6}(q) = q^3[3]_q \binom{r-1}{2}_{q^2} + q^r[2r-1]_q$$

Table 8.1: Generalized Exponents in $C_{\mathfrak{g}}(D_r)$

V	$P_V(q)$
V_1	1
V_2	$q^2[r-1]_{q^2}$
V_3	$q^2[r]_{q^2}[r-2]_{q^2} + q^r[r-1]_{q^2} + q^{2r-2}$
V_4	$q^2\binom{r-1}{2} + q^{r-2}[r-1]_{q^2}$
V_5	$q[r-1]_{q^2} + q^{r-1}$
V_6	$q^3[3]_q\binom{r-1}{2}_{q^2} + q^r[2r-3]_q$

Table 8.2: Generalized Exponents in $C_{\mathfrak{g}}(D_3)$

V	$P_V(q)$
V_1	1
V_2	q^2+q^4
V_3	$q^2+q^3+2q^4+q^5+q^6$
V_4	$q+q^2+q^3$
V_5	$q+q^2+q^3$
V_6	$2q^3+2q^4+2q^5$

In the $r=3$ case, P_4 is a copy of the adjoint representation, and P_6 decomposes into two antisymmetric representations that are dual to each other, giving the results for A_3.

Table 8.3: Generalized Exponents in $C_{\mathfrak{g}}(D_4)$

V	$P_V(q)$
V_1	1
V_2	$q^2+q^4+q^6$
V_3	$q^2+3q^4+4q^6+3q^8+q^{10}$
V_4	$2q^2+2q^4+2q^6$
V_5	$q+2q^3+q^5$
V_6	$q^3+2q^4+3q^5+2q^6+3q^7+2q^8+q^9$

Chapter 9

Restrictions to Subalgebras

9.1 The Weyl Group Action

The torus part is made of $r \times r$ matrices, and as a Weyl-group module is $\mathfrak{h} \otimes \mathfrak{h}$. So it decomposes as $1 \oplus S^2\mathfrak{h} \oplus \lambda^2\mathfrak{h}$, where $S^2\mathfrak{h}$ is the traceless symmetric square of $\mathfrak{h}$ and $\lambda^2\mathfrak{h}$ is the second alternating power. For the non-simply laced algebras B_r, C_r, G_2 and F_4 we further divide the vector part into a long part, where the coordinates are long roots, and a short part, where the coordinates are short roots.

Each part of an $\mathfrak{h}$-restricted family algebra element is W-invariant, where W acts on $S(\mathfrak{h})$ in the natural way. Since the coordinates of each part form a representation of W, we can find the W-irreducible components of the subspace of $\mathfrak{g}$ generated by those coordinates. There is a module $\mathfrak{M}(\mathfrak{h})$ such that $S(\mathfrak{h}) \cong (S(\mathfrak{h}))^W \otimes \mathfrak{M}(\mathfrak{h})$ and $\mathfrak{M}(\mathfrak{h})$ affords the regular representation; for an irreducible W-representation U, the fake degrees

[Lu77] are the exponents with multiplicity of the polynomial

$$P_U(q) = \sum_k \dim(Hom_W(U, \mathfrak{M}_K(\mathfrak{h}))q^k$$

and are analogous to generalized exponents for a Lie algebra. They describe the degrees of a basis for $(B_V(\mathfrak{h}))^W$, although they do not explicitly give a way to find a basis. Usually generic degrees are the objects calculated, but fake degrees are computable from generic degrees [Lu77].

Elements of $(B_V(\mathfrak{h}))^W$ can be turned into elements of $C_\mathfrak{g}(\mathfrak{g})$ by either adding elements on different parts together and trying to adjust for the lack of G-invariance or by multiplying an element by a G-invariant object whose Cartan restriction vanishes on all but one part and then extending from $\mathfrak{h}$ to $\mathfrak{g}$.

Here we show how $P_V(q)$ for the representations V calculated above can be expressed in terms of the fake degrees of V^T. We use P to denote both generalized exponents of $\mathfrak{g}$ representations and fake degrees of W representations. The decomposition of V^T into representations of W can be extracted by setting $q = 1$, with the labels of P denoting the labels of the representation. For A_r, the subscripts indicate partitions. For BC_r and D_r, the subscripts indicate pairs of partitions (allowing empty partitions or entries of 0) α and β such that $|\alpha| + |\beta| = r$; for BC_r $l(\alpha) = l(\beta) + 1$ and for $D_r, l(\alpha) = l(\beta)$. The $-$ in $\phi_{r,-}$ indicates the empty partition.

The decomposition for the exceptional Lie algebras will be discussed in the next chapter.

Table 9.1: Generalized exponents as fake degrees for A_r

V	$P_V(q)$
V_0	$P_{(r+1)}(q)$
$V_{\omega_1+\omega_r}$	$P_{r1}(q)$
$V_{\omega_2+\omega_{r-1}}$	$P_{(r-1)2}(q)$
$V_{2\omega_1+2\omega_r}$	$q^2P_{(r+1)}(q)+q^2P_{r1}+q^2P_{(r-1)2}(q)$
$V_{2\omega_1+\omega_{r-1}}$	$P_{(r-1)11}(q)$
$V_{\omega_2+2\omega_{r-1}}$	$P_{(r-1)11}(q)$

Table 9.2: Generalized exponents as fake degrees for BC_r

V	$P_V(q)$
V_1	$P_{r,-}(q)$
V_2	$q^2P_{r,-}(q)+P_{(r-1)1,0}(q)$
V_3	$q^2P_{r,-}(q)+(2q^2+q^{2r})P_{(r-1)1,0}(q)+q^2P_{(r-2)2,0}(q)+P_{(r-2)11,00}(q)$
V_4	$P_{(r-2)0,2}(q)$
V_5	$P_{(r-1)0,1}(q)$
V_6	$P_{(r-2)00,11}(q)+q^{2r-2}P_{(r-1)0,1}(q)+P_{(r-2)1,1}(q)$

Table 9.3: Generalized exponents as fake degrees for D_r

V	$P_V(q)$
V_1	$P_{r,0}(q)$
V_2	$P_{(r-1)1,00}(q)$
V_3	$q^2 P_{r,0}(q) + q^2 P_{(r-1)1,00}(q) + (q^2 + q^4) P_{(r-2)2,00}(q)$
V_4	$P_{(r-2),2}(q) + q^{r-1} P_{(r-1),1}(q)$
V_5	$P_{(r-1),1}(q)$
V_6	$P_{(r-2)0,11}(q) + P_{(r-2)1,10}(q)$

9.2 Restriction to Maximal Subalgebras

Directly computing with the exceptional Lie algebras can be difficult due to their size and the complexity of their descriptions. Thus it is often easier to consider classical maximal subalgebras of the exceptional Lie algebras, and then try to extend results on the classical cases to the exceptional cases. Note that the maximal subalgebras are not always simple, but can at least be chosen to be direct sums of classical simple Lie algebras.

For $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$, pick a maximal subalgebra $\mathfrak{k} \subset \mathfrak{g}$ with the same Cartan subalgebra such that the closure of the roots of $\mathfrak{k}$ under W_G is the set of roots of $\mathfrak{g}$. We can take a basis of $\mathfrak{h}$ and root vectors of $\mathfrak{k}$ and $\mathfrak{g}$ respectively as basis elements for $\mathfrak{k}$ and $\mathfrak{g}$; denote by V the span of the root vectors of $\mathfrak{g}$ that are not root vectors of $\mathfrak{k}$. Let $K \subset G$ be the group corresponding to $\mathfrak{k}$.

We get a homomorphism $res : S(\mathfrak{g}) \to S(\mathfrak{k})$ by sending V to 0. We also get a map $Mat(\mathfrak{g}) \to Mat(\mathfrak{k})$ by sending $A \in Mat(\mathfrak{g})$ to the submatrix where the coordinates are both in $\mathfrak{k}$. Combining these two maps gives a map

$$Res : (Mat(\mathfrak{g}) \otimes S(\mathfrak{g}))^G \to (Mat(\mathfrak{k}) \otimes S(\mathfrak{k}))^K$$

In a matrix in $(Mat(\mathfrak{g}) \otimes S(\mathfrak{g}))^G$, for a non-zero entry with one coordinate in $\mathfrak{k}$ and one in V, the entry must be in the kernel of res. Hence we get that Res is an algebra homomorphism. For $A \in (Mat(\mathfrak{g}) \otimes S(\mathfrak{g}))^G$, denote $A_{\mathfrak{k}} = Res(A)$.

Lemma 9.2.1. *Res is an injection.*

Proof. Suppose that $A_{\mathfrak{k}} = B_{\mathfrak{k}}$. Then $A_{\mathfrak{k}} - B_{\mathfrak{k}} = 0_{\mathfrak{k}}$. So $A - B$ has entries in $S(V)$ on the $\mathfrak{k}$ submatrix. When restricting to $\mathfrak{h}$, this restricts to 0 on the $\mathfrak{k}$ submatrix. Since W_G allows us to move roots of $\mathfrak{g}$ to roots of $\mathfrak{k}$, we get that $A - B$ restricted to $\mathfrak{h}$ must also be 0 on the V submatrix. Hence, since the restriction of an element of $(Mat(\mathfrak{g}) \otimes S(\mathfrak{g}))^G$ to $\mathfrak{h}$ has non-zero entries only on the Cartan submatrix and the diagonal, we get that $A - B$ restricted to $\mathfrak{h}$ is 0 everywhere. But since

$$(Mat(\mathfrak{g}) \otimes S(\mathfrak{g}))^G \otimes F(\mathfrak{g}) \cong (Mat(\mathfrak{g}) \otimes S(\mathfrak{h}))^{W_G} \otimes F_G(\mathfrak{h})$$

and since the base field doesn't change if an element is 0 or not, we get that restriction to $\mathfrak{h}$ is an injection. Hence $A - B = 0$.

Therefore Res is an injection. □

Using this injection, we can prove the following useful lemma:

Lemma 9.2.2 (The Vector Restriction Lemma)**.** *For $A \in C_{\mathfrak{g}}(\mathfrak{g})$, if $Res(A)$ vanishes on the torus, then A is a multiple of M.*

Proof. The exact descriptions of the family algebras for the classical cases has been handled in the previous chapters. The observation to make is that the fake degrees of the W-representations that $\mathfrak{h} \otimes \mathfrak{h}$ decomposes into match the degrees given by the elements of the family algebra that do not vanish on the torus, and hence anything that does vanish on the torus is a multiple of M.

We handle the exceptional cases by reducing to the classical case. For $\mathfrak{g} \neq G_2, E_6$, we suppose that we have an element A with vanishing torus part. Then the restriction of

A to maximal subalgebra $\mathfrak{k}$ also has vanishing torus part. For F_4, we use $\mathfrak{k} = B_4$, for E_7 we use A_7 and for E_8 we use D_8. We look at the part of the vector component of A that correspond to roots of $\mathfrak{k}$. By the first part of the restriction lemma, this part is a multiple of M restricted to this part and hence can be written as $M|_{\mathfrak{k}}P|_{\mathfrak{k}}$. $P|_{\mathfrak{k}}$ is $W(\mathfrak{k})$-invariant, and since both $A|_{\mathfrak{k}}$ and $M|_{\mathfrak{k}}$ are invariant under the subgroup of $W(\mathfrak{g})$ that fixes $\mathfrak{k}$, $P|_{\mathfrak{k}}$ is also invariant under that subgroup of $W(\mathfrak{g})$ and hence can be extended to the entire root system of $\mathfrak{g}$; we denote the extension by $\widetilde{P}$. Since the extension of $M|_{\mathfrak{k}}$ by $W(\mathfrak{g})$ is just M, we get that $A = M\widetilde{P}$.

The cases of G_2 and E_6 have to be handled separately, since E_6 has no maximal subalgebras that are simple, while G_2 has A_2 but the image of the roots of A_2 under $W(G_2)$ misses some roots of G_2. So we consider the case of family algebras for semisimple Lie algebras.

For a semisimple algebra $\mathfrak{g} \oplus \mathfrak{k}$, the adjoint group is GxK where G and K are the adjoint groups for $\mathfrak{g}$ and $\mathfrak{k}$ respectively. For a representation of $\mathfrak{g} \oplus \mathfrak{k}$ that decomposes as $U \oplus V$ where $\mathfrak{g}$ acts trivially on V and $\mathfrak{k}$ acts trivially on U, we can write the corresponding family algebra by distributing $End(U \oplus V)$ and noting that G leaves fixed V, $V^\vee$ and $S(\mathfrak{k})$, and similarly for K:

$$\begin{aligned}(End(U \oplus V) \otimes S(\mathfrak{g} \oplus \mathfrak{k}))^{G\times K} =& (End(U) \otimes S(\mathfrak{g}))^G \otimes I(\mathfrak{k}) \oplus (U \otimes S(\mathfrak{g}))^G \otimes (V^v \otimes S(\mathfrak{k}))^K \\ &\oplus (V \otimes S(\mathfrak{k}))^K \otimes (U^\vee \otimes S(\mathfrak{g}))^G \oplus (End(V) \otimes S(\mathfrak{k}))^K \otimes I(\mathfrak{g})\end{aligned}$$

So we get that the family algebra of $U \oplus V$ breaks into four blocks, depending on if the coordinates are in U or V. So it remains a free module over $I(\mathfrak{g} \oplus \mathfrak{k}) = I(\mathfrak{g}) \otimes I(\mathfrak{k})$.

For G_2, we use the maximal subalgebra $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{k}_2$, where $\mathfrak{k}_1 \cong \mathfrak{k}_2 \cong A_1$, with one A_1 containing a pair of long roots and the other a pair of short roots. The family algebra decomposes into four pieces, as above. For E_6, we use the maximal subalgebra $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{k}_2 \oplus \mathfrak{k}_3$, where $\mathfrak{k}_1 \cong \mathfrak{k}_2 \cong \mathfrak{k}_3 \cong A_2$, and adjoint group $K_1 \times K_2 \times K_3$. We have that the family algebra of the adjoint representation of $\mathfrak{k}$ decomposes into nine pieces, three of which are copies of the family algebra for A_2 tensored with two extra copies of $S(A_2)$. Note that for $X \in \mathfrak{k}_i$ and $Y \in \mathfrak{k}_j$ for $i \neq j$, the orbit of X under K_i spans $\mathfrak{k}_i$ and thus the orbit of (X, Y) under K_i spans $(\mathfrak{k}_i, Y)$.

So suppose that we have an element of $C_{\mathfrak{k}}(\mathfrak{k})$ which vanishes on the torus. This element vanishes on (h_1, h_2) where h_1 and h_2 are $\mathfrak{k}_i$ and $\mathfrak{k}_j$ respectively. If $i \neq j$, then by the above all entries in $(\mathfrak{k}_i \otimes \mathfrak{k}_j^\vee)$ vanish. If $i = j$, then we're in a copy of the family algebra of mk_i tensored with $S(\mathfrak{k}_m)^{K_m} \otimes S(\mathfrak{k}_n)^{K_n}$ for i, m, n distinct, and thus since the vector restriction lemma holds for $\mathfrak{k}_i$, we get that the $\mathfrak{k}_i$ vector part of the family algebra element thus of the form $M_i P_i$. Because $W(\mathfrak{g})$ intertwines the actions of the $\mathfrak{k}_i$, we get that the element that the action of $W(\mathfrak{g})$ sends the P_i to each other, and hence $M_i P_i$ extends to a $W(\mathfrak{g})$-invariant element on the vector part of the family algebra of $\mathfrak{g}$, as was desired. $\square$

Chapter 10

The Exceptional Lie algebras

10.1 Invariants

The exceptional Lie algebras are not uniform in many senses, so we give a table listing the data for the reference representation and the exponents:

Table 10.1: Exponents for the Exceptional Lie algebras

$\mathfrak{g}$	$\dim V$	Exponents
G_2	7	$1, 5$
F_4	26	$1, 5, 7, 11$
E_6	27	$1, 4, 5, 7, 8, 11$
E_7	56	$1, 5, 7, 9, 11, 13, 17$
E_8	248	$1, 7, 11, 13, 17, 19, 23, 29$

Unlike for the classical Lie algebras, the reference representations of the exceptional Lie algebras carry invariants of degree higher than 2. For G_2, the 7 dimensional representation has a degree 2 symmetric invariant and a degree 3 antisymmetric invariant. For F_4, the 26 dimensional representation has a degree 2 symmetric invariant and a degree 3 symmetric invariant. For E_6, the 27 dimensional representation has a degree 3 symmetric invariant. For E_7, the 56 dimensional representation has a degree 2 antisymmetric invariant and a degree 4 symmetric invariant. For E_8 the 248 dimensional representation has a degree 2 symmetric invariant and a degree 3 antisymmetric invariant. The existence of these higher degree invariants gives us elements of $(T(\mathfrak{g}))^G$ that are not obviously traces over the reference representation.

10.2 The Decomposition of $\mathfrak{g} \otimes \mathfrak{g}$

The decomposition of $\mathfrak{g} \otimes \mathfrak{g}$ into irreducible representations is uniform for the five exceptional Lie algebras. There are three symmetric representations and two antisymmetric representations. Of the three symmetric representations, one is the trivial representation, and one has highest weight twice that of the adjoint representation, so we label it $S^2 Adj$. The remaining symmetric representation we label Sym. Of the antisymmetric representations, one is the adjoint representation; the other we label $\wedge^2 Adj$.

Table 10.2: Decomposition of V^T

V	G_2	F_4	E_r
$Triv$	$\phi_{1,0}$	$\phi_{1,0}$	$\phi_{1,0}$
Sym	$\phi_{2,2} \oplus \phi_{1,0}$	$\phi_{9,2} \oplus \phi_{1,0} \oplus \phi_{2,4}$	$\phi_{\binom{r}{2}-1,2}$
$S^2 Adj$	$\phi_{1,0} \oplus 2\phi_{2,2}$	$\phi_{1,0} \oplus 2\phi_{9,2} \oplus \phi_{2,4}$	$\phi_{1,0} \oplus \phi_{\binom{r}{2}-1,2} \oplus \phi_{N-\binom{r}{2},4}$
Adj	$\phi_{2,1}$	$\phi_{4,1}$	$\phi_{r,1}$
$\wedge^2 Adj$	$2\phi_{1,3} \oplus \phi_{2,1} \oplus \phi_{1,6}$	$2\phi_{8,3} \oplus \phi_{4,1} \oplus \phi_{6,6}$	$\phi_{N-r,3} \oplus \phi_{\binom{r-1}{2},e_2+1}$

where in the E_r column, N is the number of positive roots. Here the first subscript is the dimension of the corresponding representation, the second index is the lowest fake degree of the representation. The subscripts do not uniquely determine the representation, but they do uniquely determine the set of associated fake degrees.

10.3 General Structure

To find a linearly-independent basis, we restrict to the Cartan subalgebra, allowing us to decompose elements of the family algebra into torus and vector parts. The structure of the family algebras is first determined by determining the generators on the torus, in particular by showing that $(End(\mathfrak{h}) \otimes S(\mathfrak{h}))^W$ is generated by the torus parts of R_i and R_iSR_j. This in turn is done by showing that R_i for $i \leq r$ and $R_iSR_j + R_jSR_i$ for $i, j \leq r-1$ are linearly independent, and that $R_iSR_j - R_jSR_i$ for $i, j \leq r$ are linearly independent. There are r^2 elements of the given form, and using the isomorphism to $Mat_{\mathfrak{h}}(S(\mathfrak{h}))$ gives us that there can only be r^2 linearly independent elements, so the elements R_i, $R_iSR_j + R_jSR_i$ and $R_iSR_j - R_jSR_i$ would form a basis.

The linear independence of $R_iSR_j - R_jSR_i$ is proven in [So64], theorem 2, which describes the basis of the $\wedge^2\mathfrak{h}$ isotypic component of $S(\mathfrak{h})/I(W)$ as the order 2 minors of the Jacobian matrix for $c_1,\ldots,c_r$. As noted in the section on restriction to subalgebras, this leaves the symmetric parts of $End(\mathfrak{h})$, which decompose as $1 \oplus S^2\mathfrak{h}$. The fake degrees of both representations are known, calculated in [Ca], and match the degrees of R_i and R_iSR_j. Hence we only have to check the linear independence of R_i and $R_iSR_j + R_jSR_i$.

Once the generation of the torus parts by the torus parts of R_i and S is established, suppose that A is an element of the family algebra and for all elements B with smaller degree, B can be generated by M, R_i and S. The generation of the torus parts by R_i and S shows that there is some element $P(R_i, S)$ such that $A - P(R_i, S)$ vanishes on the torus. The vector restriction lemma then shows that $A - P(R_i, S) = MQ$ for some family algebra element Q. $\deg(Q) < \deg(A)$, so Q can be generated by M, R_i and S, so A can also be generated. Hence we get a generating set, M, R_i and S.

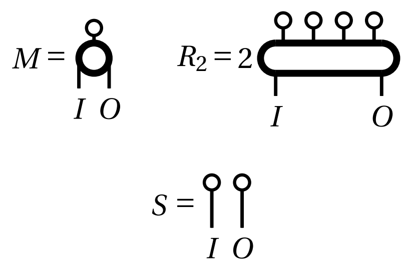
In terms of relations, for E_r the elements of the form M^kR_i for $k \leq e_r - 1$ match the fake degrees of the representations of W on the vector part, so since they are a minimal spanning set, they must be independent. Hence multiplying by M^2 gets a set of objects linearly independent from the aforementioned R_i and R_iSR_j. So our $I(\mathfrak{g})$-basis of our family algebra is R_i, R_iSR_j, M^kR_i for $1 \le k \le e_r + 1$.

For F_4 and G_2, the vector part splits in two, and so must be handled separately, but the resulting statement of the $I(\mathfrak{g})$-basis for the family algebra is the same.

10.4 Larger exceptional Lie algebras

The G_2 case acts differently from the other exceptional Lie algebras, so we list it first.

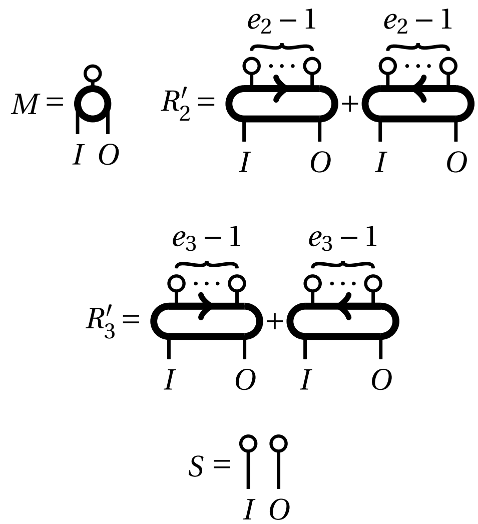
Theorem 10.4.1 (Generators for the family algebra of G_2)**.** *The family algebra $C_{\mathfrak{g}}(G_2)$ is generated by the following:*



The relations come from the Cayley-Hamilton relation on the 7-dimensional reference representation, which yield a degree 6 relation, as well as higher-degree relations coming from higher degree traces as there are no primitive Casimir elements in degree higher than 6. Unlike the classical cases, here the relations coming from higher-degree traces are sometimes independent of the Cayley-Hamilton relations.

Unlike G_2, the larger exceptional Lie algebras cannot be generated by the three generators listed above. Instead they appear more similar to the case of D_r, requiring a fourth generator due to the structure of the primitive Casimir invariants.

Theorem 10.4.2 (Generators for $C_{\mathfrak{g}}(\mathfrak{g})$ for $\mathfrak{g} = F_4, E_6, E_7, E_8$)**.** *For* $\mathfrak{g} = F_4, E_6, E_7$ *or* E_8, $C_{\mathfrak{g}}(\mathfrak{g})$ *is generated over* $I(\mathfrak{g})$ *by*



While for F_4, E_7 and E_8, the two terms in R'_2 are equal, just as in the G_2 case, and similarly for the two terms in R'_3, for E_6 the two terms are different, since the 27-dimensional reference representation of E_6 is not self-dual. Compare this to the A_r case, for which the reference representation is not self-dual, and contrast with the B_r/C_r and D_r cases, where it is.

For each of them, the elements R'_2 and R'_3 do not commute, but there are commuting elements R_2 and R_3 that can be written as R'_2 plus lower order terms and R'_3 plus lower order terms respectively.

As with the other cases, M is central and vanishes on the torus part, S vanishes on the vector part, and the other relations come from reductions of traces with degrees that aren't primitive Casimir operators. In particular, the terms of the form $R_2^m R_3^n$ are expressible in terms of other elements for $m(e_2-1)+n(e_3-1)$ not equal to $e_i - 1$

for some i, and $R_2^n R_3^m S + S R_2^m R_3^n$ is also expressible in terms of other elements when $m(e_2 - 1) + n(e_3 - 1) = e_r - 1$.

Chapter 11

G_2

The real compact group associated to G_2 is usually defined as the group of automorphisms of the octonions $\mathbb{O}$ as a division algebra over $\mathbb{R}$; we define G_2 to be the complexification of the Lie algebra of this group. Since the real part of $\mathbb{O}$ is fixed, we get an irreducible 7-dimensional representation of the automorphism group, which we complexify to get an irreducible representation of G_2. We use this as the reference representation.

The 7-dimensional representation has two primitive invariant forms, a symmetric one g_{ab} corresponding to the norm on the octonions, and an antisymmetric trilinear form f_{abc} corresponding to the imaginary component of the multiplication. Writing e_a for a basis of the imaginary part of $\mathbb{O}$, we can write our invariant forms as

$$e_a e_b = -g_{ab} + f_{abc} g^{cd} e_d$$

The torus part of G_2 decomposes as $\phi_{1,0} \oplus \phi_{2,2} \oplus \phi_{1,6}$, while the vector parts decompose as $\phi_{1,0} \oplus \phi_{2,2} \oplus \phi_{2,1} \oplus \phi_{1,3}$. $\phi_{2,1}$ has fake degrees $q + q^5$, $\phi_{2,2}$ has fake degrees $q^2 + q^6$ and $\phi_{1,k}$ has fake degree q^k. Note that there are actually several inequivalent representations denoted by $\phi_{1,6}$, $\phi_{2,2}$ and $\phi_{1,3}$, but they have the same fake degrees so we do not distinguish between them.

The elements $1, S, R_2$ and $R_2 S - S R_2$ are all linearly independent on the torus, and hence the torus parts of these elements generate the torus parts of all elements of the family algebra. So now we can determine the linearly independent elements on the vector part.

By the fake degrees, there is only 1 linearly independent order 4 element when restricted to the short roots, but since we can construct M^4 and R_2, there must be some nonzero element P of the family algebra of order 4 that vanishes on the short roots. Hence M^4 restricted to the short roots can be written in terms of R_2 and lower order terms. Similarly there must be some element nonzero Q of order 6 that vanishes on the long roots, and hence M^4 restricted to the long roots can be written in terms of R_2 and lower order terms. P is not a multiple of Q.

Using the fact that R_2^2 is not amongst the list of linearly independent torus elements above, we get that the following must be linearly independent on the short roots:

$$1, M^2, R_2$$

and the same set is also linearly independent on the long roots. Denote this set $\{A_i\}_i$

Since the set is linearly independent on the long roots, the set $\{PA_i\}_i$ is also linearly independent, and vanishes on the short roots. No linear combination of the A_i vanishes on the short roots, so the set $\{A_i, PA_i\}_i$ is linearly independent. Since M^2 is nonzero on both the short and long roots, $\{M^2A_i, M^2PA_i\}_i$ are linearly independent, and are themselves linearly independent from the set of linearly independent torus elements above, since they all have vanishing torus part. Hence we have all of the linearly independent symmetric pieces.

An $I(G_2)$ basis of the family algebra can be written as

$$M^m R_2^n \text{ for } 0 \le m \le 6, n = 0, 1$$

$$S$$

$$R_2 S - S R_2$$

Considering R_1 as a scalar, we get

$$M^m R_n \text{ for } 0 \le m \le e_2, n \le 2$$

$$R_m S R_n + R_n S R_m \text{ for } 1 \le m \le n \le 1$$

$$R_m S R_n - R_n S R_m \text{ for } 1 \le m < n \le 2$$

The generalized exponents for the representations in question have already been computed by Pieter Mostert [Mo12] and we can compare them to both the basis above and the fake degrees.

Table 11.1: Generalized Exponents in $C_{\mathfrak{g}}(G_2)$

V	$P_V(q)$
$Triv$	$1 = P_{1,0}(q)$
Sym	$q^2[3]_q^2 = P_{2,2}(q) + q^4 P_{1,0}(q)$
$S^2 Adj$	$q^2[5]_{q^2} = q^2 P_{1,0}(q) + q^2 + q^4 P_{2,2}(q)$
Adj	$q[2]_{q^4} = P_{2,1}(q)$
$\wedge^2 Adj$	$q[4]_{q^2} + q^6 = (1+q^4)P_{1,3}(q) + q^4 P_{2,1}(q) + P_{1,6}(q)$

Chapter 12

F_4

F_4 can be thought of as the Lie algebra of the isometry group of the octonionic projective plane, or as the Lie algebra of the automorphism group of traceless part of the Albert algebra. It has a symmetric bilinear form and a symmetric trilinear form, both from the multiplication in the Albert algebra: using $e_1, \ldots, e_{26}$ as the basis of the traceless part of the Algebra algebra, for $v = v^a e_a$ and $w = w^b e_b$ we get $v \circ = g_{ab} v^a w^b I_3 + d_{abc} g^{cd} v^a w^b e_d$, where I_3 is the $3x$3-identity matrix.

Alternatively, we view F_4 as $B_4 \oplus \Delta$, where Δ is the spinor representation of B_4 with Lie bracket given by $[X, s] = X.s$ for $X \in B_4$ and $s \in \Delta$, and $[s, t]$ is defined by $\langle X, [s, t] \rangle_{B_4} = -\langle X.t, s \rangle_{\Delta}$. In this setup, the 26-dimensional representation decomposes as $\mathbb{C} \oplus V_9 \oplus \Delta$ where V_9 is the standard 9-dimensional representation of B_4; here B_4 acts on each component via the usual representation, while for $s \in \Delta$, $s.a = as$ for $a \in \mathbb{C}$, $s.v = v(s)$ for $v \in V_9$ where the right-hand side is the usual action of V_9 on Δ, and then for $t \in \Delta$,

$s.t$ is defined by $\langle u, s.t\rangle_{V_9} = -\langle u.t, s\rangle_{\Delta}$.

To describe the trilinear form, it is easiest to view everything in terms of $so(8)$ representations, with Δ decomposing as Δ^+ and Δ^-. the two spinor representations of $so(8)$, and V_9 as $V_8 \oplus \mathbb{C}$. We now get that the 26-dimensional representation can be described as $S \oplus V_1 \oplus V_2 \oplus V_3$, where $V_1 = V_8$, $V_2 = \Delta^+$, $V_3 = \Delta^-$, and S is the subspace of $\mathbb{C}^3$ such that $s_1 + s_2 + s_3 = 0$ for $(s_1, s_2, s_3) \in S$.

The bilinear form on S is given by $(s, t) = \frac{1}{2}(s_1 t_1 + s_2 t_2 + s_3 t_3)$, and so the bilinear form on the 26-dimensional representation is the sum of the bilinear forms on each part.

The trilinear form becomes

$$(v_1, v_2, v_3) = f(v_1, v_2, v_3) \text{ for } v_1 \in V_1, v_2 \in V_2, v_3 \in V_3$$

$$(s, u, v) = s_i(u, v) \text{ for } u, v \in V_i$$

$$(r, s, t) = -(r_1 s_1 t_1 + r_2 s_2 t_2 + r_3 s_3 t_3) \text{ for } r, s, t \in S$$

where f is the triality map from $V_8 \otimes \Delta^+ \oplus \Delta^-$ to $\mathbb{C}$, with the scaling such that $|f(x, y, z)| \leq 1$ when $\|x\| = \|y\| = \|z\| = 1$ and the bound is attained for some triple.

with permutations of the arguments above giving the same values and all other triples of arguments giving 0.

The presence of the bilinear form gives us that all odd degree Casimir operators vanish, while the cubic invariant gives us that the quartic and degree 10 Casimir operators reduce. We are left with exponents of $1, 5, 7$, and 11, giving us primitive Casimir operators of degrees 2, 6, 8 and 12.

Because F_4 is not simply laced, there are two root lengths, long and short, so the vector part of the family algebra decomposes into a long root part and a short root part. Hence we have three parts. Following [Ca], we write the W-representations as $\phi_{a,b}$ where a is the dimension and b is the lowest fake degree. Writing the fake degrees as exponents of q-multiplicities $Q_{a,b}$, the torus decomposes as $\phi_{1,0} \oplus \phi_{9,2} \oplus \phi_{6,6}$. The vector parts each decompose as $\phi_{1,0} \oplus \phi_{9,2} \oplus \phi_{2,4} \oplus \phi_{4,1} \oplus \phi_{8,3}$. Note that the $\phi_{2,4}$ and $\phi_{8,3}$ labels actually each denote two inequivalent representations, but although the $\phi_{2,4}$ of the long roots is not equivalent to the $\phi_{2,4}$ of the short roots, the fake degrees are the same, and similarly for the $\phi_{8,3}$ representations, so the distinction is ignored here. Also note that except for the trivial and the adjoint representation, none of these $W(F_4)$ representations correspond nicely to representations of F_4 itself. They do provide an upper-limit for the number of linearly-independent elements we can have in each degree, though. See Table 12.1 for the fake degrees of these representations.

R_3 is of degree 6 and hence cannot be generated by M, S and R_2. Hence we need four generators. $R_2'R_3' - R_3'R_2'$ has degree 10, but $\phi_{6,6}$ has no fake degree of 10, so $R_2'R_3' - R_3'R_2'$ must be reducible to lower order terms. Hence we can define an R_2 and an R_3 of degrees 3 and 4 respectively that commute and that, along with M and S, generate R_2' and R_3'.

The torus parts of the set R_i for $i \leq 4$, $R_iSR_j + R_jSR_i$ for $i \leq j \leq 3$ and $R_iSR_j - R_jSR_i$ for $i < j \leq 4$ form a basis of the torus parts of the algebra, as expected in the general strategy. Now we determine the linearly independent elements on the vector part.

Table 12.1: Fake degrees in $End(F_4)^T$

$\phi_{a,b}$	$P_{a,b}(q)$
$\phi_{1,0}$	1
$\phi_{9,2}$	$q^2+q^4+2q^6+q^8+2q^{10}+q^{12}+q^{14}$
$\phi_{6,6}$	$q^6+q^8+q^{12}+q^{16}+q^{18}$
$\phi_{2,4}$	q^4+q^8
$\phi_{4,1}$	$q+q^5+q^7+q^{11}$
$\phi_{8,3}$	$q^3+q^5+q^7+2q^9+q^{11}+q^{13}+q^{15}$

By the fake degrees, there are no more than 2 linearly independent order 6 elements when restricted to the short roots, but since we can construct M^6, R_3 and M^2R_2, there must be some nonzero element P of the family algebra of order 6 that vanishes on the short roots. Hence M^6 restricted to the short roots can be written in terms of R_3, M^2R_2 and lower order terms. Similarly there must be some element nonzero Q of order 6 that vanishes on the long roots, and hence M^6 restricted to the long roots can be written in terms of R_3, M^2R_2 and lower order terms. P is not a multiple of Q.

Using the fact that R_2^2 and R_3^2 are not amongst the list of linearly independent torus elements above, we get that the following must be linearly independent on the short roots:

$$1, M^2, M^4, R_2, M^2R_2, R_3, M^4R_2, M^2R_3, R_2R_3, M^4R_3, M^2R_2R_3, M^4R_2R_3$$

and the same set is also linearly independent on the long roots. Denote this set $\{A_i\}_i$ Since the set is linearly independent on the long roots, the set $\{PA_i\}_i$ is also linearly independent, and vanishes on the short roots. No linear combination of the A_i vanishes on the short roots, so the set $\{A_i, PA_i\}_i$ is linearly independent. Since M^2 is nonzero on both the short and long roots, $\{M^2A_i, M^2PA_i\}_i$ are linearly independent, and are themselves linearly independent from the set of linearly independent torus elements above, since they all have vanishing torus part. Hence we have all of the linearly independent symmetric pieces.

The symmetric representations, $Triv, Sym$ and S^2Adj, have 0-weight multiplicities 1, 12 and 21 respectively. Unfortunately, the elements corresponding to parts of the $W(F_4)$ representation do not at all match those corresponding to the various representations of F_4 since the decomposition into torus and vector parts is not F_4-invariant. Hence the determination of which degrees give generalized exponents for which representation is nontrivial despite knowing the decompositions of the 0-weight spaces of the F_4-representations into $W(F_4)$-representations.

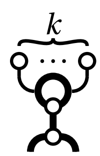
To determine the generalized exponents of the symmetric representations we examine the family algebra of the reference representation V of F_4. $V \otimes V$ decomposes as $Triv \oplus V \oplus Sym \oplus Adj \oplus Asym$, where $Asym$ is a 273-dimensional representation. The first three representations belong to S^2V, the latter two to $\wedge^2V$.

The family algebra $C_V(F_4)$, restricted to the maximal torus, splits into a 2×2 matrix corresponding to the 0-weight space of V and a set of 1×1 matrix algebras corresponding to the nonzero weights. Analogous to the case of $C_{\mathfrak{g}}(\mathfrak{g})$, we call the matrix the torus part of the family algebra, and the 1×1 matrix algebras the vector part.

The torus part decomposes into $W(F_4)$ representations as $\phi_{1,0} \oplus \phi_{2,4} \oplus \phi_{1,12}$. The vector part decomposes as $\phi_{1,0} \oplus \phi_{2,4} \oplus \phi_{9,2} \oplus \phi_{4,1} \oplus \phi_{8,3}$. The $\phi_{1,12}$, $\phi_{4,1}$ and $\phi_{8,3}$ are antisymmetric, the rest are symmetric.

We write the trilinear form on V as a white dot with three reference edges attached. Note that the trilinear form is precisely the projector from $V \otimes V$ to V.

V is a small representation, in the sense of Broer, so its generalized exponents are 4 and 8. Thus for the following diagram we get that



vanishes for k odd and for $k = 0$ or 2, and for all other k can be written in terms of the diagrams for $k = 4$ and $k = 8$. We denote the corresponding elements of the family algebra as P and Q. Note that the restrictions of P and Q to the torus are linearly independent.

We note that M_d, treated as an element of $C_V(F_4)$, vanishes on the torus. Hence, by the same reasoning for generating $C_{\mathfrak{g}}(\mathfrak{g})$ elements on the vector part from $W(\mathfrak{g})$-invariant elements on the vector part, multiplying a set of linearly independent $W(F_4)$-invariant elements yields linearly independent elements of $C_V(F_4)$ that vanish on the torus. Since we have that P and Q restricted to the torus are linearly independent, they are linearly independent from the elements of $C_V(F_4)$ that vanish on the torus.

Hence the generalized exponents of the symmetric parts of $C_V(F_4)$ are

$$(1+q^2)P_{1,0}+(1+q^2)P_{2,4}+q^2P_{9,2}$$

Since we have that q^0 is the generalized exponent for the trivial representation and $P_{2,4}$ gives the generalized exponents for V, we get that the generalized exponents for Sym are $q^2P_{1,0}+q^2P_{2,4}+q^2P_{9,2}$.

Returning to the case of $C_{\mathfrak{g}}(F_4)$, we can now determine the generalized exponents of S^2Adj by elimination. We get

$$P_{Triv}(q)=q^0$$

$$P_{Sym}(q)=q^2P_{1,0}+q^2P_{2,4}+q^2P_{9,2}$$

$$P_{S^2Adj}(q)=q^8P_{1,0}+q^8P_{2,4}+(1+q^8)P_{9,2}$$

We can rewrite the last two expressions as

$$P_{Sym}(q)=P_{9,2}+q^8P_{1,0}+q^8P_{2,4}$$

$$P_{S^2Adj}(q)=q^2P_{1,0}+q^2P_{2,4}+(q^2+q^8)P_{9,2}$$

to better match the results for the other exceptional Lie algebras.

The antisymmetric basis elements are the ones that come from $\wedge^2\mathfrak{h}$ or that vanish on the torus. As with the symmetric elements, we get that $\{MA_i, MPA_i\}_i$ are linearly independent, and this in turn gives us that the $M^{2k+1}, M^{2k+1}R_2, M^{2k+1}R_3$ and $M^{2k+1}R_4$ are linearly independent for $k\leq 5$.

Table 12.2: Generalized Exponents in $C_{\mathfrak{g}}(F_4)$

V	$P_V(q)$
$Triv$	1
Sym	$P_{9,2}(q)+q^8P_{1,0}(q)+q^8P_{2,4}(q)$
S^2Adj	$q^2P_{1,0}(q)+q^2P_{2,4}(q)+(q^2+q^8)P_{9,2}(q)$
Adj	$P_{4,1}(q)$
$\wedge^2Adj$	$(1+q^6)P_{8,3}(q)+q^6P_{4,1}(q)+P_{6,6}(q)$

Chapter 13

E_6

The reference representation of E_6 is 27-dimensional, bearing a symmetric trilinear form derived from the multiplication on the Albert algebra. Using $e_1, \ldots, e_{26}$ to denote the 26 traceless basis elements, we write e_0 for the identity element and extend d_{abc} by $d_{0ab} = g_{ab}$ for $a, b \neq 0$, and $d_{00a} = 0$. E_6 is then the group that preserves this extended d_{abc} without regard to preserving g_{ab}.

Alternatively, we can write E_6 as a module over $(A_2)^3$. We take $(A_2)_1, (A_2)_2$ and $(A_2)_3$ to be three copies of A_2, with standard representations U_1, U_2 and U_3. Then

$$E_6 = (A_2)_1 \oplus (A_2)_2 \oplus (A_2)_3 \oplus U_1 \otimes U_2 \otimes U_3 \oplus U_1^\vee \otimes U_2^\vee \otimes U_3^\vee$$

The bracket of an element of one of the $(A_2)_i$ with something in either $U_1 \otimes U_2 \otimes U_3$ or $U_1^\vee \otimes U_2^\vee \otimes U_3^\vee$ is given by the usual action of $(A_2)_i$ on U_i or $U_i^\vee$. We write

$$[u_1 \otimes u_2 \otimes u_3, v_1^\vee \otimes v_2^\vee \otimes v_3^\vee] = \sum_i (u_i \otimes v_i^\vee - tr(u_i \otimes v_i^\vee)) v_j^\vee(u_j) v_k^\vee(u_k)$$

where i, j and k cycle through $1,2,3$.

The 27-dimensional representation of E_6 then becomes $U_1 \otimes U_2^\vee \oplus U_2 \otimes U_3^\vee \oplus U_3 \otimes U_1^\vee$. The action of the three copies of A_2 is the usual action. Define a pair of maps $\phi : U_i^\vee \otimes U_i^\vee \to U_i$ and $\phi^\vee : U_i \otimes U_i \to U_i^\vee$ via the determinant on U_i. Then the action of $U_1 \otimes U_2 \otimes U_3$ on $U_i \otimes U_j^\vee$ is given by

$$u_1 \otimes u_2 \otimes u_3 . v_i \otimes v_j^\vee = v_j^\vee(u_j) u_k \otimes \phi^\vee(u_i, v_i)$$

And similarly for $U_1^\vee \otimes U_2^\vee \otimes U_3^\vee$,

$$u_1^\vee \otimes u_2^\vee \otimes u_3^\vee . v_i \otimes v_j^\vee = u_i^\vee(v_i)\phi(u_j^\vee, v_j^\vee) \otimes u_k^\vee$$

The trilinear form now looks like

$$(u_1 \otimes u_2^\vee, v_2 \otimes v_3^\vee, w_3 \otimes w_1^\vee) = u_2^\vee(v_2) v_3^\vee(w_3) w_1^\vee(u_1)$$

with all permutations of the three arguments giving the same value, and all other triples of arguments giving 0.

The lack of a bilinear form allows for odd-degree nonvanishing Casimir operators, and indeed the degrees of the primitive Casimir operators for E_6 are $2,5,6,8,9,12$, with corresponding exponents $1,4,5,7,8,11$.

Again following [Ca], we get that the torus part decomposes as $\phi_{1,0} \oplus \phi_{20,2} \oplus \phi_{15,5}$. Unlike F_4, E_6 is simply laced, so we only have one vector part, which decomposes as $\phi_{1,0} \oplus \phi_{20,2} \oplus \phi_{15,4} \oplus \phi_{6,1} \oplus \phi_{30,3}$. See Table 13.1 for the fake degrees of these representations.

Table 13.1: Fake degrees in $End(E_6)^T$

$\phi_{a,b}$	$P_{a,b}$
$\phi_{1,0}$	1
$\phi_{20,2}$	$q^2+q^3+q^4+q^5+2q^6+q^7+2q^8$ $+2q^9+2q^{10}+q^{11}+2q^{12}+q^{13}+q^{14}+q^{15}+q^{16}$
$\phi_{15,5}$	$q^5+q^6+q^8+2q^9+q^{11}+3q^{12}+q^{13}+2q^{15}+q^{16}+q^{18}+q^{19}$
$\phi_{15,4}$	$q^4+q^6+q^7+q^8+2q^{10}+q^{11}+q^{12}+q^{13}+2q^{14}+q^{16}+q^{17}+q^{18}+q^{20}$
$\phi_{6,1}$	$q+q^4+q^5+q^7+q^8+q^{11}$
$\phi_{30,3}$	$q^3+q^5+q^6+2q^7+q^8+3q^9+2q^{10}+3q^{11}+2q^{12}$ $+3q^{13}+2q^{14}+3q^{15}+q^{16}+2q^{17}+q^{18}+q^{19}+q^{21}$

R_3 is of degree 4 and hence cannot be generated by M, S and R_2. Hence we need four generators. $R_2'R_3' - R_3'R_2'$ has degree 7, but $\phi_{15,5}$ doesn't have a fake degree of 7, and hence $R_2'R_3' - R_3'R_2'$ must be reducible to lower order terms. Hence we can define an R_2 and an R_3 of degrees 3 and 4 respectively that commute and that, along with M and S, generate R_2' and R_3'.

The torus parts of the set R_i for $i \le 6$, $R_iSR_j + R_jSR_i$ for $i \le j \le 5$ and $R_iSR_j - R_jSR_i$ for $i < j \le 6$ form a basis of the torus parts of the algebra, as expected in the general strategy. Thus we get that the entire family algebra is generated by M, S and the R_i, with the following elements being linearly independent:

$$M^iR_j \text{ for } i \le 12, j \le 6$$

$$R_iSR_j + R_jSR_i \text{ for } i \leq j \leq 5$$

$$R_iSR_j - R_jSR_i \text{ for } i < j \leq 6$$

As in previous cases, multiplying an element by S on both sides yields S times an element of $I(E_6)$, so we can define d_{22}, d_{23} and d_{223} in $I(E_6)$ by

$$d_{22}S = SR_2^2S$$

$$d_{23}S = SR_2R_3S$$

$$d_{223}S = SR_2^2R_3S$$

d_{22} has degree 8, d_{23} has degree 9 and d_{223} has degree 12. Moreover, the set

$$\{c_1, c_2, c_3, d_{22}, d_{23}, d_{223}\}$$

is algebraically independent, so d_{22} must be a $\mathbb{C}^*$-multiple of c_4 plus products of lower degree primitive Casimir operators, d_{23} must be a $\mathbb{C}^*$-multiple of c_5 plus products of lower degree primitive Casimir operators, and d_{223} must be a $\mathbb{C}^*$-multiple of c_6 plus products of lower degree primitive Casimir operators. Hence, since we can write R_4, R_5 and R_6 in terms of c_4, c_5 and c_6, they can be generated by M, S, R_2 and R_3. Hence the entire family algebra can be generated over $I(E_6)$ by M, S, R_2 and R_3, as stated in the theorem.

Table 13.2: Generalized Exponents in $C_{\mathfrak{g}}(E_6)$

V	$P_V(q)$
$Triv$	1
Sym	$P_{20,2}$
S^2Adj	$q^2P_{1,0} + q^2P_{15,4} + q^2P_{20,2}$
Adj	$P_{6,1}$
$\wedge^2Adj$	$P_{30,3} + P_{15,5}$

Chapter 14

E_7

The reference representation of E_7 is 56-dimensional, bearing an antisymmetric bilinear form and a symmetric quartic form. We can view it as the Grassman component of a supervector space whose even part is Euclidean $\mathbb{R}^4$, with the metric on $\mathbb{R}^4$ becoming the bilinear form and the determinant on $\mathbb{R}^4$ becoming the quartic form.

Alternatively, we can write E_7 as a module over A_7. We take V to be the standard 8-dimensional representation of A_7, and we take an identification of $\wedge^8 V$ with $\mathbb{C}$, along with the induced identification of $\wedge^8 V^\vee$ with $\mathbb{C}$. This gives us identifications of $\wedge^k V$ with $\wedge^{8-k} V^\vee$.

E_7 decomposes as $E_7 = A_7 \oplus \wedge^4 V$, with V being the standard 8-dimensional representation of A_7. For $X \in A_7$ and $v \in \wedge^4 V$, we define $[X, v] = X.v$, and for $u, v \in \wedge^4 V$, we define $[u, v]$ by $\langle X, [u, v]\rangle_{A_7} = -(X.u) \wedge v$

We can view the 56-dimensional representation as $\wedge^2 V \oplus \wedge^2 V^\vee$. The action of A_7

is the usual action on $\wedge^2 V$ and $\wedge^2 V^\vee$, while the action of $u \in \wedge^4 V$ on $t \in \wedge^2 V$ is $u.t = u \wedge t \in \wedge^6 V \cong \wedge^2 V^\vee$ and the action of u on $\wedge^2 V^\vee$ is $u.t = u \wedge t \in \wedge^6 V^\vee \cong \wedge^2 V$.

Let $a, b, c, d \in \wedge^2 V$ and $a^\vee, b^\vee c^\vee, d^\vee \in \wedge^2 V^\vee$. Then we have that the bilinear form looks like

$$(a, b) = (a^\vee, b^\vee) = 0,\ (a, b^\vee) = -(b^\vee, a) = b^\vee(a)$$

and the quadratic form looks like

$$(a, b, c, d) = a \wedge b \wedge c \wedge d \in \wedge^8 V \cong \mathbb{C}$$

$$(a, b, c, d^\vee) = 0$$

$$(a, b, c^\vee, d^\vee) = \frac{1}{2} c^\vee(a) d^\vee(b) + \frac{1}{2} d^\vee(a) c^\vee(b) - (c^\vee \wedge d^\vee)(a \wedge b)$$

$$(a, b^\vee, c^\vee, d^\vee) = 0$$

$$(a^\vee, b^\vee, c^\vee, d^\vee) = a^\vee \wedge b^\vee \wedge c^\vee \wedge d^\vee \in \wedge^8 V^\vee \cong \mathbb{C}$$

with permutations of the arguments treated via the full symmetry of the quadratic form.

The existence of the bilinear form forces all odd-degree Casimir elements to vanish, and we are left with the degrees of the primitive Casimir elements being $2, 6, 8, 10, 12, 14$ and 18, with corresponding exponents $1, 5, 7, 9, 11, 13$, and 17.

The torus part of the family algebra decomposes as $\phi_{1,0} \oplus \phi_{27,2} \oplus \phi_{21,6}$, and the vector part as $\phi_{1,0} \oplus \phi_{27,2} \oplus \phi_{35,4} \oplus \phi_{7,1} \oplus \phi_{56,3}$. See Table 14.1 for the fake degrees of these representations.

R_3 is of degree 6 and hence cannot be generated by M, S and R_2. Hence we need four

Table 14.1: Fake degrees in $End(E_7)^T$

$\phi_{a,b}$	$P_{a,b}$
$\phi_{1,0}$	1
$\phi_{27,2}$	$q^2+q^4+2q^6+2q^8+3q^{10}+3q^{12}+3q^{14}$ $+3q^{16}+3q^{18}+2q^{20}+2q^{22}+q^{24}+q^{26}$
$\phi_{21,6}$	$q^6+q^8+q^{10}+2q^{12}+2q^{14}+2q^{16}$ $+3q^{18}+2q^{20}+2q^{22}+2q^{24}+q^{26}+q^{28}+q^{30}$
$\phi_{35,4}$	$q^4+q^6+2q^8+2q^{10}+3q^{12}+3q^{14}+4q^{16}+3q^{18}$ $+4q^{20}+3q^{22}+3q^{24}+2q^{26}+2q^{28}+q^{30}+q^{32}$
$\phi_{7,1}$	$q+q^5+q^7+q^9+q^{11}+q^{13}+q^{17}$
$\phi_{56,3}$	$q^3+q^5+2q^7+3q^9+4q^{11}+5q^{13}+6q^{15}+6q^{17}$ $+6q^{19}+6q^{21}+5q^{23}+4q^{25}+3q^{27}+2q^{29}+q^{31}+q^{33}$

generators. $R'_2R'_3 - R'_3R'_2$ has degree 10; while $\phi_{21,6}$ has a fake degree of 10, it only has one of them, so $R'_2R'_3 - R'_3R'_2$ must be reducible to $R_4S - SR_4$ and lower order terms. Hence (once we prove that the torus part of R_4 can be generated by R'_2, R'_3 and S) we can define an R_2 and an R_3 of degrees 4 and 6 respectively that commute and that, along with M and S, generate R'_2 and R'_3.

The torus parts of the set R_i for $i \le 7$, $R_iSR_j + R_jSR_i$ for $i \le j \le 6$ and $R_iSR_j - R_jSR_i$ for $i < j \le 7$ form a basis of the torus parts of the algebra, as expected in the general strategy. Thus we get that the entire family algebra is generated by M, S and the R_i, with the following elements being linearly independent:

$$M^i R_j \text{ for } i \leq 18, j \leq 7$$

$$R_i S R_j + R_j S R_i \text{ for } i \leq j \leq 6$$

$$R_i S R_j - R_j S R_i \text{ for } i < j \leq 7$$

As in previous cases, multiplying an element by S on both sides yields S times an element of $I(E_7)$, so we can define d_{22}, d_{23}, d_{33} and d_{233} in $I(E_7)$ by

$$d_{22} S = S R_2^2 S$$

$$d_{23} S = S R_2 R_3 S$$

$$d_{33} S = S R_3^2 S$$

$$d_{233} S = S R_2 R_3^2 S$$

d_{22} has degree 10, d_{23} has degree 12, d_{33} has degree 14 and d_{223} has degree 18. Moreover, the set

$$\{c_1, c_2, c_3, d_{22}, d_{23}, d_{33}, d_{233}\}$$

is algebraically independent, so d_{22} must be a $\mathbb{C}^*$-multiple of c_4 plus products of lower degree primitive Casimir operators, d_{23} must be a $\mathbb{C}^*$-multiple of c_5 plus products of lower degree primitive Casimir operators, d_{33} must be a $\mathbb{C}^*$-multiple of c_6 plus products of lower degree primitive Casimir operators, and d_{233} must be a $\mathbb{C}^*$-multiple of c_7 plus products of lower degree primitive Casimir operators. Hence, since we can write R_4, R_5, R_6 and R_7 in terms of c_4, c_5, c_6 and c_7, they can be generated by M, S, R_2 and

R_3. Hence the entire family algebra can be generated over $I(E_7)$ by M, S, R_2 and R_3, as stated in the theorem.

Note that R_2^3 also has degree 12, but the $\phi_{27,2}$ representation only allows 3 degree 12 elements. Since $R_3^2, R_2R_3S + SR_2R_3$ and $R_2SR_3 + R_3SR_2$ are all linearly-independent on the torus, R_2^3 must be expressible as a $\mathbb{C}$-linear combination of $R_3^2, R_2R_3S + SR_2R_3$ and $R_2SR_3 + R_3SR_2$ plus lower order terms.

Table 14.2: Generalized Exponents in $C_{\mathfrak{g}}(E_7)$

V	$P_V(q)$
$Triv$	1
Sym	$P_{27,2}$
S^2Adj	$q^2P_{1,0} + q^2P_{35,4} + q^2P_{27,2}$
Adj	$P_{7,1}$
$\wedge^2 Adj$	$P_{56,3} + P_{21,6}$

Chapter 15

E_8

The reference representation of E_8 is the adjoint representation, bearing a symmetric bilinear form (the Killing form) and an antisymmetric trilinear form (the structure constants). Because the reference representation is the adjoint representation, there is no nice description of the invariant forms other than as themselves.

Alternatively, we can write E_8 as a module over D_8. E_8 decomposes as $E_8 = D_8 \oplus S$, where S is one of the two spinor representations of D_8; it doesn't matter which one. The Lie bracket is extended by $[X, s] = X.s$ for $X \in D_8$ and $s \in S$, and for $s, t \in S$, $[s, t]$ is defined by $\langle X, [s, t]\rangle_{D_4} = -\langle X.t, s\rangle_S$. The Lie bracket also provides the action of E_8 on itself.

The existence of the bilinear form forces all odd-degree Casimir elements to vanish, and we are left with the degrees of the primitive Casimir elements being $2, 8, 12, 14$, $18, 20, 24$, and 30, with corresponding exponents $1, 7, 11, 13, 17$, $19, 23$, and 29.

The torus part of the family algebra decomposes as $\phi_{1,0} \oplus \phi_{35,2} \oplus \phi_{28,8}$. The vector part decomposes as $\phi_{1,0} \oplus \phi_{35,2} \oplus \phi_{84,4} \oplus \phi_{8,1} \oplus \phi_{112,3}$. See Table 15.1 for the fake degrees of these representations.

R_3 is of degree 10 and hence cannot be generated by M, S and R_2. Hence we need

Table 15.1: Fake degrees in $End(E_8)^T$

$\phi_{a,b}$	$P_{a,b}$
$\phi_{1,0}$	1
$\phi_{35,2}$	$q^2+q^6+q^8+q^{10}+2q^{12}+2q^{14}+q^{16}+3q^{18}+2q^{20}+2q^{22}+3q^{24}$ $+2q^{26}+2q^{28}+3q^{30}+q^{32}+2q^{34}+2q^{36}+q^{38}+q^{40}+q^{42}+q^{46}$
$\phi_{28,8}$	$q^8+q^{12}+q^{14}+2q^{18}+2q^{20}+3q^{24}+q^{26}+q^{28}+4q^{30}$ $+q^{32}+q^{34}+3q^{36}+2q^{40}+2q^{42}+q^{46}+q^{48}+q^{52}$
$\phi_{84,4}$	$q^4+q^6+q^8+2q^{10}+2q^{12}+2q^{14}+4q^{16}+3q^{18}+4q^{20}+5q^{22}+4q^{24}$ $+5q^{26}+6q^{28}+4q^{30}+6q^{32}+5q^{34}+4q^{36}+5q^{38}+4q^{40}+3q^{42}$ $+4q^{44}+2q^{46}+2q^{48}+2q^{50}+q^{52}+q^{54}+q^{56}$
$\phi_{8,1}$	$q+q^7+q^{11}+q^{13}+q^{17}+q^{19}+q^{23}+q^{29}$
$\phi_{112,3}$	$q^3+q^5+q^7+2q^9+2q^{11}+3q^{13}+4q^{15}+4q^{17}+5q^{19}+6q^{21}+6q^{23}$ $+7q^{25}+7q^{27}+7q^{29}+7q^{31}+7q^{33}+7q^{35}+6q^{37}+6q^{39}+5q^{41}$ $+4q^{43}+4q^{45}+3q^{47}+2q^{49}+2q^{51}+q^{53}+q^{55}+q^{57}$

four generators. $R'_2R'_3 - R'_3R'_2$ has degree 16, but $\phi_{28,8}$ has no fake degree of 16, so $R'_2R'_3 - R'_3R'_2$ must be reducible to lower order terms. Hence we can define an R_2 and an R_3 of degrees 6 and 10 respectively that commute and that, along with M and S, generate R_2' and R_3'.

The torus parts of the set R_i for $i \leq 8$, $R_i S R_j + R_j S R_i$ for $i \leq j \leq 7$ and $R_i S R_j - R_j S R_i$ for $i < j \leq 8$ form a basis of the torus parts of the algebra, as expected in the general strategy. Thus we get that the entire family algebra is generated by M, S and the R_i, with the following elements being linearly independent:

$$M^i R_j \text{ for } i \leq 18, j \leq 7$$

$$R_i S R_j + R_j S R_i \text{ for } i \leq j \leq 6$$

$$R_i S R_j - R_j S R_i \text{ for } i < j \leq 7$$

As in previous cases, multiplying an element by S on both sides yields S times an element of $I(E_8)$, so we can define d_{22}, d_{23}, d_{222}, d_{223} and d_{2223} in $I(E_8)$ by

$$d_{22} S = S R_2^2 S$$

$$d_{23} S = S R_2 R_3 S$$

$$d_{222} S = S R_2^3 S$$

$$d_{223} S = S R_2^2 R_3 S$$

$$d_{2223} S = S R_2^3 R_3 S$$

d_{22} has degree 14, d_{23} has degree 18, d_{222} has degree 20, d_{223} has degree 24, and d_{2223} has degree 30. Moreover, the set $c_1, c_2, c_3, d_{22}, d_{23}, d_{222}$, d_{223} and d_{2223} is algebraically independent, so d_{22} must be a $\mathbb{C}^*$-multiple of c_4 plus products of lower degree primitive Casimir operators, d_{23} must be a $\mathbb{C}^*$-multiple of c_5 plus products of lower degree primitive Casimir operators, d_{222} must be a $\mathbb{C}^*$-multiple of c_6 plus products of lower degree primitive Casimir operators, d_{223} must be a $\mathbb{C}^*$-multiple of c_7 plus products of lower degree primitive Casimir operators, and d_{2223} must be a $\mathbb{C}^*$-multiple of c_8 plus products of lower degree primitive Casimir operators. Hence, since we can write R_4, R_5, R_6, R_7 and R_8 in terms of c_4, c_5, c_6, c_7 and c_8, they can be generated by M, S, R_2 and R_3. Hence the entire family algebra can be generated over $I(E_8)$ by M, S, R_2 and R_3, as stated in the theorem.

Table 15.2: Generalized Exponents in $C_{\mathfrak{g}}(E_8)$

V	$P_V(q)$
$Triv$	1
Sym	$P_{35,2}$
S^2Adj	$q^2P_{1,0} + q^2P_{84,4} + q^2P_{35,2}$
Adj	$P_{8,1}$
$\wedge^2Adj$	$P_{112,3} + P_{28,8}$

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