A Geometric Approach to the Study of the Cartesian Stiffness Matrix

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Abstract

The stiffness of a rigid body subject to conservative forces and moments is described by a tensor, whose components are best described by a 6×6 Cartesian stiffness matrix. We derive an expression that is independent of the parameterization of the motion of the rigid body using methods of differential geometry. The components of the tensor with respect to a basis of twists are given by evaluating the tensor on a pair of basis twists. We show that this tensor depends on the choice of an affine connection on the Lie group, $SE(3)$. In addition, we show that the definition of the Cartesian stiffness matrix used in the literature [2, 6] implicitly assumes an asymmetric connection and this results in an asymmetric stiffness matrix in a general loaded configuration. We prove that by choosing a symmetric connection we always obtain a symmetric Cartesian stiffness matrix. Finally, we derive stiffness matrices for different connections and illustrate the calculations using numerical examples.

1 Introduction

This paper considers the static analysis of conservative systems in which the associated potential energy, Φ, is a function of position only. Specifically, we consider two rigid bodies, one fixed and the other free, connected by a coupling so that the system is conservative. The coupling may consist of linear springs [4], or multiple articulations with flexible joints, as is often the model for multilegged walking machines [9], multifingered grasps [10], and compliant mechanisms [7]. If the configuration of the moving rigid body is described by n generalized coordinates, $q_1, q_2, ..., q_n$, the system is in equilibrium when the partial derivative of Φ with respect to each generalized coordinate, q_i, vanishes. If the system is subjected to external forces and moments, the *ith* generalized force, Q_i, is given by:

$$Q_i = \frac{\partial \Phi}{\partial q_i} \tag{1}$$

and it no longer vanishes. The stiffness matrix of the system describes the changes in the generalized forces with changes in the generalized coordinates. There is no ambiguity in the definition of the stiffness matrix because the configuration space of the rigid body is identified with the vector space $\mathbb{R}^n$ and we know exactly how to differentiate in $\mathbb{R}^n$. The stiffness matrix, K, can be obtained by differentiating Equation (1):

$$K_{ij} = \frac{\partial^2 \Phi}{\partial q_i \partial q_j} \tag{2}$$

Because the stiffness matrix consists of the second partial derivatives of the potential energy with respect to the generalized coordinates, it is always symmetric.

In a Cartesian coordinate system, the velocities are described in terms of a six dimensional basis consisting of zero pitch twists (pure rotations) along the axes of the reference frame and infinite pitch twists (pure translations) parallel to the axes of the reference frame. The forces and moments are expressed as components with respect to a similar basis of wrenches. The changes in forces and moments with respect to small motions along the basis twists are given by a 6×6 Cartesian stiffness matrix.

The definition of a 6×6 stiffness matrix using an appropriate set of six generalized coordinates (for example, three Euler angles and three linear displacement variables) and Equations (1,2) suffers from two potential difficulties. First, the stiffness matrix depends on the parameterization (generalized coordinates q_i) in Equation (2). A different choice of generalized coordinates would give us a different stiffness matrix. Second, the resulting matrix representation is only local. It is well known that the set of rigid body motions cannot be globally represented by a single set of generalized coordinates.

The first objective of the paper is to establish a framework that allows us to define the Cartesian stiffness matrix precisely without requiring a particular choice of coordinates. We use a differential geometric setting to achieve this goal. The set of rigid body motions is a Lie group [8] and therefore a six-dimensional differentiable manifold. Any position and orientation of the rigid body is a point on this manifold. The generalized velocity vector or the twist of the rigid body is a tangent vector at the point, while the generalized force vector or the wrench can be thought of as a cotangent vector at that point. Because the stiffness matrix requires differentiation of the generalized force vector on a manifold, we need to introduce additional structure on the manifold. Roughly speaking, the operation of differentiation necessitates the comparison of tangent vectors or cotangent vectors at two nearby, but *different* points before taking a limit. Since tangent (cotangent) vectors at a point belong to a tangent (cotangent) space that is different from the tangent (cotangent) space at another point, we need to endow the manifold with an *affine connection* that allows us to "connect" two distinct tangent (cotangent) spaces. We will explore choices of affine connections that make sense for practical applications and we will show that the Cartesian stiffness matrices defined by Griffis and Duffy [4], Ciblak and Lipkin [2], and Howard, Žefran and Kumar [6] implicitly assume an affine connection, the so-called $(-)$-connection of Cartan [1].

The second objective of the paper is concerned with the symmetry of Cartesian stiffness matrices. It was observed by Griffis and Duffy [4] and Ciblak and Lipkin [2], that the Cartesian stiffness matrix associated with a linear elastic coupling between two rigid bodies is, in general, asymmetric if the resulting forces and moments do not sum to zero. Howard, Žefran and Kumar [6] extended these results to arbitrary conservative systems and showed a general method to define a symmetric Cartesian stiffness matrix. We will prove in this paper that the stiffness matrix derived from a symmetric affine connection will always be symmetric. We will also show that the only reason for the asymmetry of the stiffness matrix observed in prior work [2, 4, 6] is the asymmetry of the connection used to compute the stiffness matrix.

2 Kinematics and differential geometry

2.1 Lie groups and Lie algebras

Consider a rigid body moving in free space. Assume any inertial reference frame {F} fixed in space and a frame {M} fixed to the body at point O'. At each instance, the configuration (position and orientation) of the rigid body can be described by a homogeneous transformation matrix corresponding to the displacement from frame {F} to frame {M}. These transformations (or displacements) form a Lie group $SE(3)$, the special Euclidean group in three-dimensions [12]:

$$SE(3) = \left\{ A \mid A = \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix}, R \in \mathbb{R}^{3\times 3}, d \in \mathbb{R}^3, R^T R = I, \det(R) = 1 \right\}.$$

It is easy to show [8, 12] that $SE(3)$ is a group with the standard matrix multiplication operation and that it is a six-dimensional manifold, and therefore a Lie group.

On a Lie group, the tangent space at the group identity element ({F} coincident with {M}) has the structure of a Lie algebra. The Lie algebra of $SE(3)$ is denoted by $se(3)$ and is given by:

$$se(3) = \left\{ \begin{bmatrix} \Omega & v \\ 0 & 0 \end{bmatrix}, \Omega \in \mathbb{R}^{3\times 3}, v \in \mathbb{R}^3, \Omega^T = -\Omega \right\}. \tag{3}$$

A 3×3 skew-symmetric matrix Ω can be uniquely identified with a vector $\omega \in \mathbb{R}^3$ so that for an arbitrary vector $x \in \mathbb{R}^3$, $\Omega x = \omega \times x$, where $\times$ is the cross product in $\mathbb{R}^3$. Each element $T \in se(3)$ can be thus identified with a vector pair $\{\omega, v\}$.

Since $se(3)$ is a vector space, any element can be expressed as a 6×1 vector of components corresponding to a chosen basis. The standard basis that will be used throughout the paper is:

$$\begin{aligned}
L_1 &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad
L_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad
L_3 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
L_4 &= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad
L_5 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad
L_6 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\end{aligned} \tag{4}$$

This basis has the property that the components of an element $T \in se(3)$ are given precisely by the vector pair $\{\omega, v\}$ mentioned above.

The product operation on a Lie algebra is called a Lie bracket. The Lie bracket of two elements $T_1, T_2 \in se(3)$ is defined by the matrix commutator:

$$[T_1, T_2] = T_1T_2 - T_2T_1 \tag{5}$$

The coefficients C_{ij}^k corresponding to the Lie brackets of the basis vectors:

$$[L_i, L_j] = \sum_k C_{ij}^k L_k, \tag{6}$$

are called *structure constants* of the Lie algebra. The nonzero structure constants for the basis (4) are:

$$\begin{aligned} C_{12}^3 = C_{31}^2 = C_{23}^1 = C_{15}^6 = C_{26}^4 = C_{34}^5 = C_{42}^6 = C_{53}^4 = C_{61}^5 = 1 \\ C_{21}^3 = C_{13}^2 = C_{32}^1 = C_{51}^6 = C_{62}^4 = C_{43}^5 = C_{24}^6 = C_{35}^4 = C_{16}^5 = -1. \end{aligned} \tag{7}$$

The motion of a rigid body can be described by a curve $A(t) : \mathbb{R} \to SE(3)$. The tangent vector to this curve, $\frac{dA}{dt}$, is the velocity of the rigid body. The tangent vector $\frac{dA}{dt}$ can be mapped to an element $T(t)$ of the Lie algebra $se(3)$ by:

$$T(t) = A(t)^{-1}\dot{A}(t) = \begin{bmatrix} R^T\dot{R} & -R^T\dot{d} \\ 0 & 0 \end{bmatrix}. \tag{8}$$

If $T(t)$ is expressed in the basis (4), the vector of components $\{\omega(t), v(t)\}$ has the following physical significance. $\omega(t)$ is the angular velocity of the rigid body while $v(t)$ is the linear velocity of the origin of the frame {M} (the point O' on the rigid body), both expressed in the body-fixed reference frame {M}. Thus, $T(t)$ is the *instantaneous twist* [11] in the body fixed frame. The Lie algebra $se(3)$ is isomorphic to the set of all twists [12].

It can be shown that the twist $T(t)$ derived in Equation (8) does not depend on the choice of the inertial frame {F}. If the inertial frame is displaced by $Q \in SE(3)$, the trajectory $A(t)$ is transformed into $QA(t)$. In the terminology of group theory, the transformation is a *left translation* of each element $A(t)$ by Q. A straightforward computation shows that the twist in the new inertial frame remains unchanged:

$$T(t) = [QA(t)]^{-1}\left[Q\dot{A}(t)\right] = A(t)^{-1}\dot{A}(t).$$

2.2 Left invariant vector fields

A differentiable vector field on a manifold is a smooth assignment of a tangent vector to each element of the manifold. At each point, a vector field defines a unique *integral curve* to which it is tangent [3]. Formally, a vector field X is a derivation operator which, given any real-valued differentiable function h, returns its derivative along the integral curves of X. If $(\xi^1, \xi^2, \ldots, \xi^6)$ is a set of coordinates for the manifold $SE(3)$, and the integral curve $A(t)$ can be expressed in these coordinates as $A(t) = (\xi_A^1(t), \xi_A^2(t), \ldots, \xi_A^6(t))$, the vector field X can be expressed at a point $A_0 = A(t_0)$ in the so called coordinate form (note that a repeated index implies a summation over that index):

$$X|_{A_0} = \left.\frac{d\xi_A^i(t)}{dt}\frac{\partial(.)}{\partial \xi^i}\right|_{t_0} = X^i|_{A_0}\frac{\partial(.)}{\partial \xi^i} \tag{9}$$

$E_i = \frac{\partial(.)}{\partial \xi^i}$ are vector fields that correspond to derivatives along the coordinate curves and form at every point a basis for the tangent space at that point. This basis is called the *coordinate basis.* $X^i\big|_{A_0} = \left.\frac{d\xi_A^i(t)}{dt}\right|_{t_0}$ are the components of X with respect to this basis. The vector field X operating on the function h thus yields:

$$X(h)|_{A_0} = \left.\frac{dh(A(t))}{dt}\right|_{t_0} = \left.\frac{\partial h}{\partial \xi^i}\frac{d\xi_A^i(t)}{dt}\right|_{t_0} = X^i|_{A_0}\frac{\partial h}{\partial \xi^i} \tag{10}$$

Alternatively, since every element of $SE(3)$ is a homogeneous matrix, we can use the usual matrix calculus to obtain the matrix representation of a tangent vector at a point $A_0 = A(t_0)$:

$$X|_{A_0} = \left.\frac{dA}{dt}\right|_{t_0} \tag{11}$$

Thus, the components $X^i\big|_{A_0}$ can be computed by writing out the coordinate basis vectors E_i in the matrix form:

$$X|_{A_0} = \left.\frac{dA}{dt}\right|_{t_0} = \left.\frac{\partial A}{\partial \xi^i}\right|_{A_0}\left.\frac{d\xi_A^i(t)}{dt}\right|_{t_0} = X^i|_{A_0}\left.\frac{\partial A}{\partial \xi^i}\right|_{A_0} \tag{12}$$

In this paper, we will be particularly interested in *left invariant vector fields* on $SE(3)$. From any twist, $T \in se(3)$, we can generate a differentiable vector field, $\hat{T}$, by assigning to all $A \in SE(3)$ a vector, $\hat{T}\big|_A$, in the tangent space at that point, given in the matrix notation of Equation (11) by:

$$\hat{T}\Big|_A = AT, \tag{13}$$

We say that $\hat{T}$ is the left invariant vector field generated by the Lie algebra element T.

By construction, the set of left invariant vector fields is isomorphic to the Lie algebra $se(3)$. $\hat{L}_1, \hat{L}_2, \ldots, \hat{L}_6$, the left invariant vector fields generated by the basis twists, are a basis for the space of all left invariant vector fields. Moreover, *any* vector field X can be expressed as a linear combination of the left invariant vector fields:

$$X = \sum_{i=1}^{6} T^i \hat{L}_i \tag{14}$$

where the coefficients T^i are real-valued functions. If X is the velocity vector $\frac{dA}{dt}$ on $SE(3)$, we can identify the vector at each point with a twist in $se(3)$:

$$\frac{d}{dt}(A(t)) = A(t)\,T(t) = A(t)\left(\sum_{i=1}^{6} T^i(t) L_i\right) = \sum_{i=1}^{6} T^i(t)(A(t)\,L_i) = \sum_{i=1}^{6} T^i(t)\,\hat{L}_i\Big|_{A(t)},$$

where $\hat{L}_i\Big|_{A(t)}$ is the vector value of the vector field $\hat{L}_i$ at the point $A(t)$. Thus, if the velocity of the rigid body, $\dot{A}$, is expressed in the basis $\hat{L}_1, \ldots, \hat{L}_6$, its components are equal to the components of the instantaneous twist:

$$\omega(t) = [T^1(t), T^2(t), T^3(t)]^T, \qquad v(t) = [T^4(t), T^5(t), T^6(t)]^T.$$

Clearly, if $X = \frac{dA}{dt}$ is a left invariant vector field, T^i are constants, which implies the twist is constant. The integral curve of the vector field corresponds to a screw motion with that twist.

In the paper, it will be necessary to deal with two sets of basis vector fields. We will call $\hat{L}_1, \ldots, \hat{L}_6$ *the basis of left invariant vector fields (BLIVF).* The components of the velocity vector with respect to this basis at a point yields the instantaneous twist. The other basis will be the coordinate basis E_i. In Section 2.4, we will show a simple example that illustrates the change of basis computations.

2.3 Twists and wrenches

If a force f and a moment τ (about the origin, O') act on a rigid body, we refer to the vector pair, $W = \{\tau, f\}$, as a *wrench.* If a wrench W acts on a rigid body that undergoes a twist, T, over a time interval Δt, the work done by the wrench is given by

$$\Delta E = (f^T v + \tau^T \omega)\Delta t,$$

which is a scalar. Wrenches therefore belong to the dual of the vector space of twists, $se^*(3)$.

Recall that $\frac{dA}{dt}$, the velocity vector at the point $A(t)$, belongs to the tangent space of $SE(3)$ at that point, and the twist, $T(t)$, is the representation of the velocity vector in the Lie algebra, $se(3)$. Similarly, we can think of a generalized force vector acting at time t as a vector in the cotangent space at the point $A(t)$, and a wrench as the corresponding representation of the force in $se^*(3)$, the dual space of the Lie algebra. Further, the velocity of a moving rigid body defines a vector field and the generalized force acting on the rigid body is a *one-form* [14] on the manifold $SE(3)$.

Given a basis for the vector fields, there exists a natural basis for one-forms, called *the dual basis.* If $\{\hat{L}_i\}$ is a basis for the vector fields, the dual basis for the one-forms, $\{\hat{\lambda}^i\}$, satisfies:

$$<\hat{\lambda}^i\,;\,\hat{L}_j>= \delta^i_j, \tag{15}$$

where $<\hat{\lambda}^i\,;\,\hat{L}_j>$ represents the action of a one-form $\hat{\lambda}^i$ on a vector field $\hat{L}_j$ and δ^i_j is the *Kronecker Delta.* If a one-form $F = W_i\,\hat{\lambda}^i$ is expressed in the dual basis, it is easy to see that its action on a vector field $V = V^j\hat{L}_j$ is given by $<F\,;\,V>= W_i\,V^i$. Thus, the components of F with respect to the basis $\hat{\lambda}^i$ are given by:

$$W_i =<F\,;\,\hat{L}_i> \tag{16}$$

Just as the components of a velocity vector field with respect to the basis $\{\hat{L}_i\}$ yield the components of the instantaneous twist, the components of the force one-form in the basis $\{\hat{\lambda}^i\}$ are exactly the components of the corresponding wrench in $se^*(3)$.

Similarly, given the coordinate basis for the vector fields (see Equation (9)), $E_i = \frac{\partial(.)}{\partial\xi^i}$, there is a corresponding dual basis $d\xi^i$ given by:

$$<d\xi^i\,;\,\frac{\partial(.)}{\partial\xi^j}>= \delta^i_j.$$

The force one-form F and its components F_i with respect to this basis are given by:

$$F = F_i \, d\xi^i \tag{17}$$

where

$$F_i = < F \, ; \, E_i > \tag{18}$$

2.4 Example

This example illustrates the relationship between the basis of left invariant vector fields (BLIVF) and the coordinate basis for a given coordinate system, and that between the corresponding dual bases for one-forms. Consider a convenient coordinate chart $\xi = (\xi^1, \xi^2, \xi^3, \xi^4, \xi^5, \xi^6)$ for $SE(3)$ such that the homogeneous transformation matrix is given by

$$A(\xi) = Rot(x, \xi^1) \times Rot(y, \xi^2) \times Rot(z, \xi^3) \times Trans(x, \xi^4) \times Trans(y, \xi^5) \times Trans(z, \xi^6),$$

according to the notation of [13]. A straightforward computation reveals the rotation R and the translation d are given by:

$$R = \begin{bmatrix} C_{\xi^2} \, C_{\xi^3} \, C_{\xi^1} - S_{\xi^3} \, S_{\xi^1} & -C_{\xi^2} \, C_{\xi^1} \, S_{\xi^3} - C_{\xi^3} \, S_{\xi^1} & C_{\xi^1} \, S_{\xi^2} \\ C_{\xi^1} \, S_{\xi^3} + C_{\xi^2} \, C_{\xi^3} \, S_{\xi^1} & C_{\xi^3} \, C_{\xi^1} - C_{\xi^2} \, S_{\xi^3} \, S_{\xi^1} & S_{\xi^2} \, S_{\xi^1} \\ -C_{\xi^3} \, S_{\xi^2} & S_{\xi^2} \, S_{\xi^3} & C_{\xi^2} \end{bmatrix},$$

$$d = \begin{bmatrix} z \, C_{\xi^1} \, S_{\xi^2} + y \, \left(C_{\xi^2} \, C_{\xi^1} \, S_{\xi^3} - C_{\xi^3} \, S_{\xi^1}\right) + x \, \left(C_{\xi^2} \, C_{\xi^3} \, C_{\xi^1} - S_{\xi^3} \, S_{\xi^1}\right) \\ z \, S_{\xi^2} \, S_{\xi^1} + x \, \left(C_{\xi^1} \, S_{\xi^3} + C_{\xi^2} \, C_{\xi^3} \, S_{\xi^1}\right) + y \, \left(C_{\xi^3} \, C_{\xi^1} - C_{\xi^2} \, S_{\xi^3} \, S_{\xi^1}\right) \\ z \, C_{\xi^2} - x \, C_{\xi^3} \, S_{\xi^2} + y \, S_{\xi^2} \, S_{\xi^3} \end{bmatrix},$$

where $C_{(.)}$ and $S_{(.)}$ denote cos(.) and sin(.) respectively.

The directional derivatives, E_i, form a coordinate basis for the velocity vector. At any point A, these vectors are given by:

$$E_i(\xi) = \frac{\partial A}{\partial \xi^i}.$$

The left invariant vector fields $\hat{L}_1, \hat{L}_2, \ldots, \hat{L}_6$ can be found using Equation (13). At the point A,

$$\hat{L}_i = A(\xi) L_i$$

Since the left invariant vector fields are also a basis for the tangent space at any point, the E_i can everywhere be expressed as a linear combination of the $\hat{L}_j$:

$$\begin{aligned} E_1 &= C_{\xi^2} \, C_{\xi^3} \, \hat{L}_1 - C_{\xi^2} \, S_{\xi^3} \, \hat{L}_2 + S_{\xi^2} \, \hat{L}_3 - \left(\xi^5 \, S_{\xi^2} + \xi^6 \, C_{\xi^2} \, C_{\xi^3}\right) \hat{L}_4 \\ &+ \left(\xi^4 \, S_{\xi^2} - \xi^6 \, C_{\xi^2} \, C_{\xi^3}\right) \hat{L}_5 + \left(\xi^5 \, C_{\xi^2} \, C_{\xi^3} + \xi^4 \, C_{\xi^2} \, S_{\xi^3}\right) \hat{L}_6 \\ E_2 &= S_{\xi^3} \, \hat{L}_1 + C_{\xi^3} \, \hat{L}_2 + \xi^6 \, C_{\xi^3} \, \hat{L}_4 - \xi^6 \, S_{\xi^3} \, \hat{L}_5 + \left(\xi^5 \, S_{\xi^3} - \xi^4 \, C_{\xi^3}\right) \hat{L}_6 \\ E_3 &= \hat{L}_3 - \xi^5 \, \hat{L}_4 + \xi^4 \, \hat{L}_5 \end{aligned}$$

$$
\begin{aligned}
E_4 &= \hat{L}_4 \\
E_5 &= \hat{L}_5 \\
E_6 &= \hat{L}_6
\end{aligned}
\tag{19}
$$

It is straightforward to solve for the $\hat{L}_j$ in terms of the E_i. The exception is at the singular points for the coordinate system given by:

$$\cos \xi^2 = 0.$$

It is well-known that any coordinate chart for $SE(3)$ will suffer from such singularities. At such singular points, a different coordinate chart must be used. Hence an expression for a stiffness matrix derived from a coordinate system cannot be valid everywhere on $SE(3)$ and therefore must be a local result. However, our definition of the Cartesian stiffness matrix will be based on the $BLIVF$ and will be valid everywhere on $SE(3)$.

We can let $F = \hat{\lambda}^i$ in Equation (17), and use Equation (18) to write the basis one-forms $\hat{\lambda}^i$ in terms of $d\xi^j$:

$$
\begin{aligned}
\hat{\lambda}^1 &= C_{\xi^2}\, C_{\xi^3}\, d\xi^1 + S_{\xi^3}\, d\xi^2 \\
\hat{\lambda}^2 &= -C_{\xi^2}\, C_{\xi^3} d\xi^1 \;+\; C_{\xi^3} d\xi^2 \\
\hat{\lambda}^3 &= S_{\xi^2}\, d\xi^1 + d\xi^3 \\
\hat{\lambda}^4 &= -\left(\xi^5\, S_{\xi^2} + \xi^6\, C_{\xi^2}\, C_{\xi^3}\right) d\xi^1 \;+\; \xi^6\, C_{\xi^3} d\xi^2 \;-\; \xi^5 d\xi^3 \;+\; d\xi^4 \\
\hat{\lambda}^5 &= \left(\xi^4\, S_{\xi^2} - \xi^6\, C_{\xi^2}\, C_{\xi^3}\right)\, d\xi^1 \;-\; \xi^6\, S_{\xi^3} d\xi^2 \;+\; \xi^4\, d\xi^3 \;+\; d\xi^4 \\
\hat{\lambda}^6 &= \left(\xi^5\, C_{\xi^2}\, C_{\xi^3} + \xi^4\, C_{\xi^2}\, S_{\xi^3}\right)\, d\xi^1 \;+ \left(\xi^5\, S_{\xi^3} - \xi^4\, C_{\xi^3}\right) \;+\; d\xi^6
\end{aligned}
\tag{20}
$$

Once again, it is easy to solve for the $d\xi^j$ in terms of the $\hat{\lambda}^i$ except at the singular points where $\cos \xi^2 = 0$.

2.5 Forces in a potential field

The force one-form, F, corresponding to a potential field Φ for a conservative system is given by:

$$F = -d\Phi. \tag{21}$$

This is basically the generalization of the gradient in $\mathbb{R}^3$. In a coordinate system ξ^i, the components of the one-form with respect to the basis one-forms $d\xi^i$ are simply $-\frac{\partial \Phi}{\partial \xi^i}$:

$$F = -\frac{\partial \Phi}{\partial \xi^i} d\xi^i \tag{22}$$

To obtain the wrench that corresponds to the force one-form F at a point $A \in SE(3)$, the one form must be expressed in the basis $\hat{\lambda}^i$ that is dual to the basis twists:

$$F = W_i\, \hat{\lambda}^i \tag{23}$$

We can see from Equation (15) that the components of the wrench, W_i are given by

$$W_i = < F\, ;\, \hat{L}_i > = -\hat{L}_i(\Phi) \tag{24}$$

Thus, in the coordinate chart used in the example of Section 2.4, these components can be obtained either by substituting for the $d\xi^i$ in terms of $\hat{\lambda}^i$ in Equation (22) or by directly computing the directional derivatives in Equation (24).

3 The Cartesian Stiffness Matrix

3.1 Introduction

Broadly speaking, a Cartesian stiffness matrix consists of components each of which describes how a component of a wrench acting on a rigid body changes as the body moves along a basis twist. This suggests that the wrench has to be differentiated along the left invariant vector fields generated by the basis twists. However, a wrench belongs to the co-tangent space $se^*(3)$, the dual of the space of twists, and therefore, is only defined at a point (the identity element). It cannot be differentiated along a vector field (which is defined over the entire manifold). We must therefore formalize this notion of "changes in the components of a wrench" in terms of differentiation of the associated force one-form F. Further, the operation of differentiating a force one-form will necessitate the comparison of cotangent vectors at two nearby, but different points, before taking a limit. A formal definition of a Cartesian stiffness matrix must entail a framework for differentiation on $SE(3)$, and a suitable recipe in terms of a derivative of a force one-form.

This is done by endowing the manifold with an *affine connection*. Given a curve $A(t)$ and a vector field X on the manifold, the affine connection specifies how the vector $X(t_1)$ in the tangent space at point $A(t_1)$ can be mapped to an element $X^{t_1}(t_2)$ of the tangent space at some other point $A(t_2)$. The vector $X^{t_1}(t_2)$ is called the parallel transport of X along $A(t)$. If such an affine connection is specified, we can define a *covariant derivative* of a vector field $X(t)$ along a curve $A(t)$ by:

$$\left.\frac{DX}{dt}\right|_{t_0} = \lim_{\epsilon \to 0} \frac{X^{t_0}(t_0+\epsilon) - X(t_0)}{\epsilon}, \tag{25}$$

Similarly, the covariant derivative, $\nabla_Y X$, of the vector field X with respect to another vector field Y, is another vector field whose value at any point $A(t_0)$ is obtained by letting $A(t)$ be the integral curve of Y passing though $A(t_0)$:

$$\nabla_Y X|_{A(t_0)} = \left.\frac{DX}{dt}\right|_{t_0} \tag{26}$$

If X and Y are chosen to be basis left invariant vector fields, the resulting covariant derivatives are of special interest. The coefficients Γ^k_{ji} of the covariant derivative of a basis vector field with respect to another basis vector field,

$$\nabla_{\hat{L}_i} \hat{L}_j = \Gamma^k_{ji} \hat{L}_k, \tag{27}$$

are called *Christoffel symbols*. The affine connection completely defines the Christoffel symbols. Conversely, the coefficients Γ^k_{ji} define the affine connection.

The covariant derivative $\nabla_U F$ of a one-form F can be defined through the covariant derivative of a vector field. In order to do this, we use a "generalization" of the Leibniz' rule. Let V be another arbitrary vector field. We then have:

$$\nabla_U < F\,;\, V > = < \nabla_U F\,;\, V > + < F\,;\, \nabla_U V > . \tag{28}$$

Since $< F\,;\, V >$ is a real-valued function, $\nabla_U < F\,;\, V > = U(< F\,;\, V >)$. The covariant derivative of F with respect to U is given in terms of its action on an arbitrary vector field V:

$$< \nabla_U F\,;\, V > = U(< F\,;\, V >) - < F\,;\, \nabla_U V > . \tag{29}$$

3.2 The Cartesian stiffness matrix and stiffness tensor

We will now define the Cartesian stiffness matrix for a rigid body subject to conservative forces that can be derived from a smooth potential function Φ. The components of this matrix reflect the rate of change of the components of the force one-form due to the change in the potential energy as the rigid body moves along any of the basis twists. The covariant derivative of the force one form, $d\Phi$, with respect to a left invariant velocity field, $\hat{L}_i$ yields a one-form, $\nabla_{\hat{L}_i} d\Phi$. The components of this one form with respect to the basis $\hat{\lambda}_i$ (that is dual to the $\hat{L}_i$) yields a wrench. This wrench is the change in wrench due to a motion along the basis twist L_i. This leads to the following definition for the coefficients of the stiffness matrix:

$$K_{ij} = < \nabla_{\hat{L}_j} d\Phi\,;\, \hat{L}_i > . \tag{30}$$

The minus sign is omitted to conform to the usual definition of the stiffness matrix in the literature. We can easily verify that K_{ij} are the components of a $(0,2)$ tensor.

The *stiffness tensor* $K = \nabla d\Phi$ can be defined in terms of its operation on two arbitrary vector fields X and Y:

$$K(X,Y) = < \nabla_Y d\Phi\,;\, X > . \tag{31}$$

Alternatively, we can expand the right hand side using Eq. (29):

$$\begin{aligned} K(X,Y) &= < \nabla_Y d\Phi\,;\, X > \\ &= Y(< d\Phi\,;\, X >) - < d\Phi\,;\, \nabla_Y X > \\ &= (YX - \nabla_Y X)(\Phi). \end{aligned} \tag{32}$$

The Cartesian stiffness matrix in (30) is thus given by:

$$K_{ij} = K(\hat{L}_i, \hat{L}_j) = (\hat{L}_j \hat{L}_i - \nabla_{\hat{L}_j} \hat{L}_i)(\Phi). \tag{33}$$

Thus, the following result is proved:

Proposition 3.1 *The component K_{ij} of the Cartesian stiffness matrix is obtained by evaluating the stiffness tensor $K = \nabla d\Phi$ on the pair of basis twists $\hat{L}_i$ and $\hat{L}_j$.*

Proposition 3.2 *The Cartesian stiffness matrix depends only on the basis twists and does not depend on the choice of the coordinates for $SE(3)$.*

Proof: This is immediately apparent from Equation (33). The entries of the stiffness matrix depend on the $(0,2)$ tensor K on $SE(3)$ and on the *BLIVF*, $\{\hat{L}_i\}$. By definition, the tensor K is independent of the choice of the coordinates. Further, the *BLIVF* are defined by the left translation of the basis twists in the Lie algebra $se(3)$. The left translation operation is again coordinate invariant. This completes the proof.

□

The independence of the Cartesian stiffness matrix from the parameterization of $SE(3)$ is also demonstrated by the fact that we never used a coordinate chart for $SE(3)$ in the derivations in this section. However, suppose the potential function is known in some coordinate system as a function of the coordinates ξ^i, and we are required to determine the Cartesian stiffness matrix. Because the *BLIVF* can be easily expressed in terms of the coordinate basis as shown in Section 2.4, the computation in Equation (33) is fairly straightforward. The end result will not depend on the choice of the coordinate system ξ^i.

In previous work [2, 5], the stiffness matrix is defined according to:

$$K_{ij} = \hat{L}_j \hat{L}_i(\Phi). \tag{34}$$

If we compare Eq. (34) with Eq. (33), it is immediately apparent that the term with the connection is missing in (34). Hence, we conclude that the definition of the stiffness matrix used in [2, 6] requires:

$$\nabla_{\hat{L}_i} \hat{L}_j = 0 \quad \forall i, j. \tag{35}$$

This equation completely specifies the connection ∇. If the vector fields are expressed with the basis twists $\{\hat{L}_i\}$, the Christoffel symbols for this connection all vanish:

$$\Gamma^k_{ji} = 0 \quad \forall i, j, k. \tag{36}$$

This connection is the $(-)$-connection of Cartan [1].

4 Symmetric connections and stiffness matrices

4.1 Introduction

We have seen that the stiffness matrix depends on the choice of an affine connection. In this section, we will discuss possible ways of defining an appropriate affine connection. First, we will discuss connections on $SE(3)$ that make sense from a geometric viewpoint in the next subsection. We will later discuss connections that are meaningful from a physical viewpoint in robotics. Finally, we will discuss the conditions under which the stiffness matrix is symmetric.

We will first require the definition of a symmetric connection that is available in any standard differential geometry text (see, for example, [3]). If for all vector fields X and Y, a connection ∇ satisfies:

$$\nabla_X Y - \nabla_Y X = [X, Y], \tag{37}$$

the connection is said to be *symmetric.* A possible physical interpretation for this concept is offered by the result in Proposition 4.1.

4.2 Affine connections on $SE(3)$

Žefran, Kumar and Croke [15] consider possible affine connections on $SE(3)$. In particular, they define the *kinematic connection* that allows the computation of the acceleration of a rigid body from the covariant derivative of the velocity vector field along the curve describing the motion. This connection is symmetric and is given in terms of the *BLIVF* by the Christoffel symbols:

$$\begin{aligned} \Gamma^3_{21} = \Gamma^2_{13} = \Gamma^1_{32} &= \frac{1}{2}, & \Gamma^3_{12} = \Gamma^2_{31} = \Gamma^1_{23} &= -\frac{1}{2}, \\ \Gamma^6_{51} = \Gamma^4_{62} = \Gamma^5_{43} &= 1, & \Gamma^6_{42} = \Gamma^4_{53} = \Gamma^5_{61} &= -1, \end{aligned} \tag{38}$$

all other symbols being zero. This connection can be easily seen to be symmetric:

$$\Gamma^k_{ji} - \Gamma^k_{ij} = C^k_{ij}. \tag{39}$$

where C^k_{ij} are the structure constants in Equation (8). Because the Christoffel symbols with respect to a basis of left invariant vector fields are constants, the connection is also left invariant. In other words, it does not depend on the choice of the inertial frame.

An important symmetric connection on any Lie group is the $0-$connection of Cartan [1]:

$$\nabla_{\hat{L}_i} \hat{L}_j = \frac{1}{2}[\hat{L}_i, \hat{L}_j]. \tag{40}$$

The fact that it is symmetric follows immediately from its definition. This connection is specially important in the context of $SE(3)$ because the geodesics defined by this connection are screw motions [15]. This connection can be established to be bi-invariant [1, 15]. In other words, it is independent of the definition of the body-fixed and inertial frames. We will call it the *bi-invariant connection.*

4.3 Connection induced by the flat connection on R^6

When we deal with generalized coordinates (ξ^i) of a system, the coordinates define a Euclidean space and there is a natural connection on R^n. This is the so called *flat connection* that makes the directional derivatives $\frac{\partial}{\partial \xi^i}(.)$ (covariantly) constant along any coordinate curve. In other words, using the notation $E_i = \frac{\partial(.)}{\partial \xi^i}$, the connection is defined by:

$$\nabla_{E_i} E_j = 0, \qquad i, j = 1, \ldots, 6. \tag{41}$$

In order to see the physical implications, consider the stiffness matrix of a six degree of freedom manipulator in which the actuated joints are controlled so that the manipulator behaves like a passive elastic structure. Let $\xi = \left(\xi^1, \ldots, \xi^6\right)^T$ be the vector of joint coordinates. The potential energy of the system is given by $\Phi(\xi)$. Consider a nonsingular configuration, $\xi_0 = \left(\xi_0^1, \ldots, \xi_0^6\right)^T$, which we can take to be zero without any loss of generality. Choose an end-effector frame {M} which is fixed to the last link of the manipulator and set the inertial reference frame {F} to be the position of the end-effector frame at $\xi_0 = \left(\xi_0^1, \ldots, \xi_0^6\right)^T = (0, 0, \ldots, 0)^T$. Thus the point ξ_0 in the configuration space corresponds to the identity element of $SE(3)$, and in some neighborhood U, $SE(3)$ is (locally) parameterized by the joint coordinates ξ.

We calculate the stiffness matrix using the affine connection defined in Equation (41). This connection is clearly symmetric because the condition (37) is trivially satisfied — the Lie brackets of coordinate basis vectors vanish and from Equation (41),

$$\nabla_{E_i} E_j - \nabla_{E_j} E_i = 0. \tag{42}$$

The entries of the stiffness matrix $\mathcal{K}$ are therefore given by Equation (32):

$$\begin{aligned} \mathcal{K}_{ij} &= (E_j E_i - \nabla_{E_j} E_i)\Phi \\ &= E_j E_i(\Phi) \\ &= \frac{\partial^2 \Phi}{\partial \xi^j \xi^i} \end{aligned} \tag{43}$$

We get the familiar expression for the joint stiffness matrix $\mathcal{K}$ (distinct from the Cartesian stiffness matrix, K). Because we are working in the space of generalized coordinates (R^6), the stiffness matrix is nothing but the familiar Hessian given by the matrix of second partial derivatives.

To find the Cartesian stiffness matrix in $SE(3)$ that corresponds to the connection defined in (41) we need the components of the Jacobian, J. If they are denoted by γ_i^j, the components of the twists are T^j, and the joint rates are $\dot{\xi}^i$, then:

$$T^j = \gamma_i^j \, \dot{\xi}^i.$$

Since the velocity of the end effector is given by $T^j \hat{L}_j = \dot{\xi}^i E_i$, it is clear from the above equation that:

$$E_i = \gamma_i^j \hat{L}_j. \tag{44}$$

Let α_i^j denote the element of the inverse Jacobian, $[\alpha_i^j] = [\gamma_i^j]^{-1}$. We can therefore write:

$$\hat{L}_i = \alpha_i^j E_j. \tag{45}$$

We can now compute the Christoffel symbols of the connection (41) in terms of the *BLIVF*:

$$\begin{aligned} \nabla_{\hat{L}_j} \hat{L}_i &= \nabla_{(\alpha_j^l E_l)} (\alpha_i^m E_m) \\ &= (\alpha_j^l E_l)(\alpha_i^m) E_m + \alpha_i^m \alpha_j^l \nabla_{E_l} E_m \\ &= \alpha_j^l \frac{\partial \alpha_i^m}{\partial \xi^l} \gamma_m^k \hat{L}_k \end{aligned} \tag{46}$$

where the last equality is obtained by using (44) and definition of the connection (41). The Christoffel symbols with respect to the *BLIVF* are therefore:

$$\Gamma_{ij}^k = \alpha_j^l \frac{\partial \alpha_i^m}{\partial \xi^l} \gamma_m^k. \tag{47}$$

Thus the same connection which appears "flat" in the coordinate basis E_i, is given by Equation (47) in the *BLIVF*.

It is important to note that in the case of a manipulator, there is a physical basis for choosing the generalized coordinates and this defines the connection (41). But in general, there is no *a priori* basis for defining this connection, which depends on the choice of coordinates. Even in the case of a manipulator, where there is a natural choice of coordinates, the connection coefficients in (47) depend on the Jacobian matrix which in turn depends on the position of the end effector in the workspace. Thus, the connection is not invariant with respect to changes in the body or inertial reference frames.

4.4 Symmetry of the stiffness matrix

Proposition 4.1 *If the affine connection used in the definition of the stiffness tensor K is symmetric, then the stiffness tensor is symmetric.*

Proof: For a symmetric connection (Equation (37)), we obtain:

$$\begin{aligned} K(X,Y) &= (XY - \nabla_X Y)(\Phi) \\ &= (XY - [X,Y] - \nabla_Y X)(\Phi) = (YX - \nabla_Y X)(\Phi) = K(Y,X). \end{aligned} \tag{48}$$

□

If we look at the connection (35) used in the definition of the stiffness matrix in [2, 5], it is immediately apparent that it is not symmetric:

$$0 = \nabla_{\hat{L}_i} \hat{L}_j - \nabla_{\hat{L}_j} \hat{L}_i \neq [\hat{L}_i, \hat{L}_j]. \tag{49}$$

(See Equation (8) for the values of the Lie brackets $[\hat{L}_i, \hat{L}_j]$.) An immediate consequence of this fact is stated in the following proposition [2, 5].

Proposition 4.2 *The Cartesian stiffness matrix, as defined in [2, 5], is in general asymmetric.*

Proof: This statement is the consequence of the asymmetry of the connection (see Proposition 4.1). From Equation (34) we compute:

$$K_{ij} - K_{ji} = [\hat{L}_j, \hat{L}_i](\Phi). \tag{50}$$

Since the Lie brackets of the basis twists do not vanish, $K_{ij} \neq K_{ji}$. However, at points where the potential field Φ is stationary (at a stationary point, $X(\Phi) = 0$ for any vector field X) the stiffness matrix becomes symmetric.

□

5 Stiffness matrices with different connections

5.1 Symbolic calculations

We first evaluate the components of the stiffness tensor given by Equation (33),

$$K_{ij} = (\hat{L}_j \hat{L}_i - \nabla_{\hat{L}_j} \hat{L}_i)(\Phi),$$

with different connections. In order to facilitate the analysis and comparison of the stiffness matrices obtained by different connections, and to relate the Cartesian stiffness matrix to the standard Hessian matrix, we define a coordinate system $\xi = \left(\xi^1, \ldots, \xi^6\right)$. This coordinate chart is chosen so that the origin $(0,0,0,0,0,0)^T$ corresponds to the identity element in $SE(3)$, $A = I$, and the coordinate basis vector fields $E_i = \frac{\partial(.)}{\partial \xi^i}$ are such that:

$$E_i|_{\xi=0} = \hat{L}_i\Big|_{\xi=0} = L_i. \tag{51}$$

The coordinate chart in the example in Section 2.4 is one such chart and we will use it for our analysis. We will evaluate the components of the stiffness matrix at the identity element of $SE(3)$. In other words, the configuration of the rigid body of interest (the end effector) will be $A = I$.

We will need to write the BLIVF in terms of the E_i from Equation (19):

$$\hat{L}_i = \alpha_i^k E_k.$$

From this and Equation (33), we get:

$$\begin{aligned} K_{ij} &= \hat{L}_j \left(\alpha_i^k E_k\left(\Phi\right)\right) - \Gamma_{ij}^k \hat{L}_k(\Phi) \\ &= \alpha_j^l \alpha_i^k E_l\left(E_k\left(\Phi\right)\right) + \alpha_j^l E_l\left(\alpha_i^k\right) E_k\left(\Phi\right) - \Gamma_{ij}^k W_k \end{aligned}$$

where we have used the fact that the the components of the wrench are given by $W_k = \hat{L}_k(\Phi)$. At the identity, since $E_i|_{A=I} = \hat{L}_i\Big|_{A=I}$,

$$\alpha_i^j\Big|_{\xi=0} = \delta_i^j,$$

and the above expression reduces to:

$$\begin{aligned} K_{ij} &= \frac{\partial^2 \Phi}{\partial \xi^i \, \partial \xi^j} + E_j(\alpha_i^k) W_k - \Gamma_{ij}^k W_k \\ &= [K_I]_{ij} + [K_{II}]_{ij} + [K_{III}]_{ij} \end{aligned} \tag{52}$$

$[K_I]$, the first term on the right hand side of Equation (52) is the standard Hessian matrix. This is known to be symmetric (and positive definite) regardless of the loading. It is clearly independent of the connection. The components of the second term calculated at the identity can be written in matrix form:

$$[K_{II}] = \begin{bmatrix} 0 & 0 & 0 & 0 & w_6 & -w_5 \\ -w_3 & 0 & 0 & -w_6 & 0 & w_4 \\ w_2 & -w_1 & 0 & w_5 & -w_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{53}$$

This term is also independent of the connection, but is asymmetric.

The third term on the right hand side of Equation (52), $[K_{III}]$ depends on the connection. We use the superscripts **A**, **B**, **K**, and **F**, to refer to the asymmetric (-) connection, the bi-invariant connection, the kinematic connection, and the flat connection respectively. This term is given by one of the following three expressions:

$$[K^{\mathbf{A}}_{III}] = 0 \tag{54}$$

$$[K^{\mathbf{K}}_{III}] = \begin{bmatrix} 0 & -\frac{1}{2}w_3 & \frac{1}{2}w_2 & 0 & 0 & 0 \\ \frac{1}{2}w_3 & 0 & -\frac{1}{2}w_1 & 0 & 0 & 0 \\ -\frac{1}{2}w_2 & \frac{1}{2}w_1 & 0 & 0 & 0 & 0 \\ 0 & -w_6 & w_5 & 0 & 0 & 0 \\ w_6 & 0 & -w_4 & 0 & 0 & 0 \\ -w_5 & w_4 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{55}$$

$$[K^{\mathbf{B}}_{III}] = \frac{1}{2}\begin{bmatrix} 0 & -w_3 & w_2 & 0 & -w_6 & w_5 \\ w_3 & 0 & -w_1 & w_6 & 0 & -w_4 \\ -w_2 & w_1 & 0 & -w_5 & w_4 & 0 \\ 0 & -w_6 & w_5 & 0 & 0 & 0 \\ w_6 & 0 & -w_4 & 0 & 0 & 0 \\ -w_5 & w_4 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{56}$$

It is clear from these matrices that while $[K_{II}]$ introduces asymmetry into the Cartesian stiffness matrix, a symmetric connection introduces a nonzero $[K_{III}]$ matrix that symmetricizes the stiffness matrix.

5.2 Numerical example

We borrow the example of a Stewart-Gough Platform from [4] in which the floating platform ($\triangle rst$) is connected to the base platform ($\triangle OPQ$) through linear line springs. The line springs model the compliance in the actuators and the servo system. The joint coordinates ξ_i are chosen to be the lengths of the corresponding limbs (Os, Ps, Pt, Qt, Qr, and Or). The potential energy for the system is given by:

$$\Phi = \sum_{i=1}^{6} \frac{1}{2} k_i (\xi_i - l_i)^2, \tag{57}$$

where k_i are the spring constants and l_i are the lengths of the springs in the unloaded configuration.

The coordinates of the vertices of the triangles in a fixed coordinate system are given by:

$$O = [0.0 \;\; 0.0 \;\; 0.0]^T, \qquad P = [7.0 \;\; 0.0 \;\; 0.0]^T,$$
$$Q = [3.5 \;\; 6.0 \;\; 0.0]^T, \qquad r = [10 \;\; 4 \;\; 12]^T,$$
$$s = [14.0 \;\; 8.0 \;\; 16.0]^T, \quad t = [14.5 \;\; 1.1 \;\; 16.5]^T.$$

The joint stiffness for each joint is taken to be $k_1 = 10N/cm$, $k_2 = 20N/cm$, $k_3 = 30N/cm$, $k_4 = 40N/cm$, $k_5 = 50N/cm$, and $k_6 = 60N/cm$, so that the joint stiffness matrix, $\mathcal{K}$ is diagonal. The free lengths of the springs are $l_1 = 11cm$, $l_2 = 12cm$, $l_3 = 13cm$, $_4 = 14cm$, $l_5 = 15cm$, and $l_6 = 16cm$.

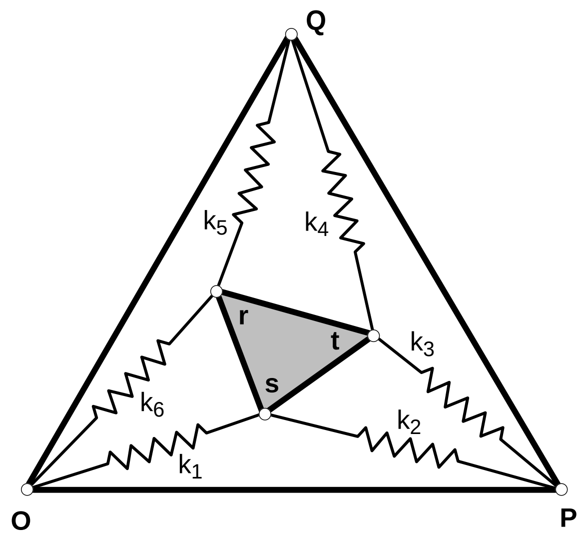

Figure 1: Top view of a parallel platform.

For the current configuration, the inverse of the Jacobian matrix is :

$$J^{-1} = \begin{bmatrix} 0.00 & 0.00 & 0.00 & 0.62 & 0.35 & 0.70 \\ 0.00 & -5.83 & 2.92 & 0.37 & 0.42 & 0.83 \\ 0.00 & -6.36 & 0.41 & 0.41 & 0.06 & 0.91 \\ 4.89 & -2.82 & -4.12 & 0.54 & -0.24 & 0.81 \\ 5.27 & -3.04 & -3.38 & 0.47 & -0.15 & 0.87 \\ 0.00 & 0.00 & 0.00 & 0.62 & 0.25 & 0.74 \end{bmatrix}.$$

The Cartesian stiffness matrix can be computed using different connections:

$$[K^{\mathbf{A}}] = \begin{bmatrix} 2141.4 & -1725.1 & -3895.1 & 206.9 & -581.5 & 466.5 \\ -2094.7 & 4107.4 & -336.5 & 303.5 & 5.3 & -836.8 \\ -1518.7 & 608.6 & 3263.9 & -239.5 & 516.7 & -212.2 \\ 206.9 & -202.4 & -180.2 & 80.0 & 5.2 & 75.6 \\ -75.5 & 5.3 & 212.0 & 5.2 & 39.3 & 5.2 \\ 407.2 & -532.2 & -212.2 & 75.6 & 5.2 & 150.6 \end{bmatrix},$$

$$[K^{\mathbf{K}}] = \begin{bmatrix} 2141.4 & -1909.9 & -2706.9 & 206.9 & -581.5 & 466.5 \\ -1909.9 & 4107.4 & 136.0 & 303.5 & 5.3 & -836.8 \\ -2706.9 & 136.0 & 3263.9 & -239.5 & 516.7 & -212.2 \\ 206.9 & 303.5 & -239.5 & 80.0 & 5.2 & 75.6 \\ -581.5 & 5.3 & 516.7 & 5.2 & 39.3 & 5.2 \\ 466.5 & -836.8 & -212.2 & 75.6 & 5.2 & 150.6 \end{bmatrix},$$

$$[K^{\mathbf{B}}] = \begin{bmatrix} 2141.4 & -1909.9 & -2706.9 & 206.9 & -328.5 & 436.9 \\ -1909.9 & 4107.4 & 136.0 & 50.6 & 5.3 & -684.5 \\ -2706.9 & 136.0 & 3263.9 & -209.9 & 364.3 & -212.2 \\ 206.9 & 50.6 & -209.9 & 80.0 & 5.2 & 75.6 \\ -328.5 & 5.3 & 364.3 & 5.2 & 39.3 & 5.2 \\ 436.9 & -684.5 & -212.2 & 75.6 & 5.2 & 150.6 \end{bmatrix},$$

and

$$[K^{\mathbf{F}}] = \begin{bmatrix} 2345.9 & -1354.5 & -1695.2 & 229.3 & -87.1 & 387.0 \\ -1354.5 & 2674.9 & 559.7 & -253.8 & -9.6 & -493.8 \\ -1695.2 & 559.7 & 1423.5 & -141.6 & 90.5 & -219.8 \\ 229.3 & -253.8 & -141.6 & 57.3 & 6.4 & 87.2 \\ -87.1 & -9.6 & 90.5 & 6.4 & 12.0 & 7.7 \\ 387.0 & -493.8 & -219.8 & 87.2 & 7.7 & 140.7 \end{bmatrix}.$$

Notice the four matrices are completely different except for the bottom right 3×3 matrix (consisting of translatory components) that is identical for $K^{\mathbf{A}}$, $K^{\mathbf{B}}$, and $K^{\mathbf{K}}$. The computations show that different connections lead to completely different stiffness matrices for the *same physical system.*

6 Conclusion

We presented a coordinate-free formulation of the Cartesian stiffness matrix for conservative mechanical systems in which the potential field Φ is a function of position only. We showed that it is necessary to define an affine connection on $SE(3)$ in order to compute the stiffness matrix. We showed that the definition of the Cartesian stiffness matrix used by previous researchers, $[K^{\mathbf{A}}]$, implicitly assumes a connection that is asymmetric. There are at least three other connections (and therefore definitions of the Cartesian stiffness matrix) that might make sense and are attractive because they are symmetric. In particular, $[K^{\mathbf{B}}]$, derived from a bi-invariant connection, and $[K^{\mathbf{K}}]$, derived from a left-invariant connection, are independent of the choice of coordinates. The third connection depends on the parameterization of $SE(3)$ and is not coordinate-free. But the resulting stiffness matrix $[K^{\mathbf{F}}]$ is useful in robotic applications in which the joint coordinates provide a natural parameterization for $SE(3)$.

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