Quantum imaging and inverse scattering

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We consider the inverse scattering problem that arises in two-photon quantum imaging with interferometric measurements. We show that the two-point correlation function of the field contains information about the scattering medium at a spatial frequency of twice the Rayleigh bandwidth. The linearized inverse problem, however, yields reconstructions with a resolution of $\lambda/2$, where λ is the wavelength of light.

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The development of methods for optical imaging using nonclassical states of light is a topic of fundamental interest and considerable applied importance. Such so-called quantum imaging methods exploit quantum interference effects (correlations) to improve the performance of lithography [1,2], spectroscopy [3], and microscopy [4,5]. For instance, in imaging using two-photon entangled states, it is possible to break the Rayleigh diffraction-limit of $\lambda/2$, where λ is the wavelength of light [1,6–9]. This technique takes advantage of the fact that a two-photon state has twice the energy of the corresponding single-photon state, which leads to a twofold increase in resolution. Alternatively, it is possible to utilize entanglement due to post-detection selection to realize a comparable enhancement in resolution [10–17]. In either case, the superresolution that is achieved is due to visualization of quantum correlations of the electromagnetic field via interferometry. Thus, the resulting images contain information about the medium under investigation. However, they are not tomographic nor are they directly related to the optical properties of the medium.

In this Letter, we consider the inverse scattering problem that arises in two-photon quantum imaging with interferometric measurements. We show that quantum multipoint correlation functions contain information about the dielectric susceptibility of a scattering medium at spatial frequencies that exceed the Rayleigh bandwidth. The linearized inverse problem, however, yields reconstructions with a resolution of $\lambda/2$.

We begin by considering the experiment illustrated in Fig. 1, where two single-photon sources at the positions $\mathbf{y}_1$ and $\mathbf{y}_2$ illuminate a medium of interest. The resulting scattered photons are registered by point detectors at the positions $\mathbf{x}_1$ and $\mathbf{x}_2$ (such that only one photon is registered by each detector), and the outputs of the detectors are correlated. The sources are assumed to be noninteracting two-level atoms with ground and excited states $|0_i\rangle$ and $|1_i\rangle$, respectively, where $i = 1, 2$. We further assume that the atoms are initially in their excited states and that they radiate single photons by spontaneous emission. Thus, the positive-frequency part of the electric-field operator contains contributions from each of the sources and is of the form

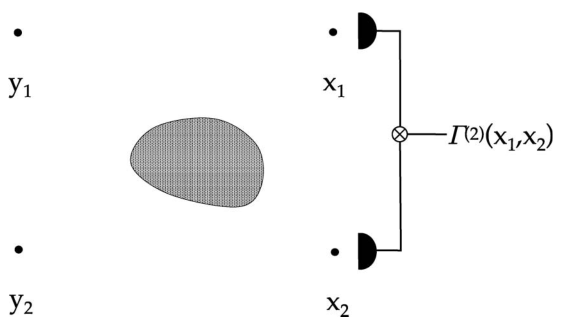

Fig. 1. Illustrating the experiment.

$$E^{(+)}(\mathbf{x}) = \frac{1}{\sqrt{2}}[G(\mathbf{x},\mathbf{y}_1)|0_1\rangle\langle 1_1| + G(\mathbf{x},\mathbf{y}_2)|0_2\rangle\langle 1_2|]. \quad (1)$$

Here $|0_i\rangle\langle 1_i|$ is the lowering operator for the ith atom and $G(\mathbf{x},\mathbf{y})$ is the Green's function, which corresponds to the field at the point $\mathbf{x}$ due to a unit amplitude point source at $\mathbf{y}$. The Green's function obeys the equation

$$\nabla^2 G(\mathbf{x},\mathbf{y}) + k_0^2(1 + 4\pi\eta(\mathbf{x}))G(\mathbf{x},\mathbf{y}) = -4\pi\delta(\mathbf{x}-\mathbf{y}), \quad (2)$$

where η is the generally complex dielectric susceptibility of the medium and, for simplicity, we ignore the vector properties of the optical field.

We recall that the correlation functions of the field are given by expectations of normally ordered products of field operators [18]:

$$\Gamma^{(1)}(\mathbf{x}) = \langle\psi|E^{(-)}(\mathbf{x})E^{(+)}(\mathbf{x})|\psi\rangle, \quad (3)$$

$$\Gamma^{(2)}(\mathbf{x}_1,\mathbf{x}_2) = \langle\psi|E^{(-)}(\mathbf{x}_1)E^{(-)}(\mathbf{x}_2)E^{(+)}(\mathbf{x}_2)E^{(+)}(\mathbf{x}_1)|\psi\rangle, \quad (4)$$

where $|\psi\rangle = |1_1, 1_2\rangle$ and $E^{(-)}$ denotes the negative-frequency part of the electric-field operator that is given by $E^{(-)} = [E^{(+)}]^\dagger$. We note that $\Gamma^{(2)}(\mathbf{x}_1,\mathbf{x}_2)$ is proportional to the probability of detecting one photon at $\mathbf{x}_1$ and a second photon at $\mathbf{x}_2$, which can be measured by correlating the outputs of the detectors. Making use of Eq. (1), we obtain

$$\Gamma^{(1)}(\mathbf{x}) = \frac{1}{2}[|G(\mathbf{x},\mathbf{y}_1)|^2 + |G(\mathbf{x},\mathbf{y}_2)|^2], \quad (5)$$

where we have utilized the assumption that the atoms are noninteracting, which corresponds to putting $\langle 0_1|0_2\rangle = 0$. We also find that

$$\Gamma^{(2)}(\mathbf{x}_1,\mathbf{x}_2)=\frac{1}{4}|G(\mathbf{x}_1,\mathbf{y}_1)G(\mathbf{x}_2,\mathbf{y}_2)+G(\mathbf{x}_1,\mathbf{y}_2)G(\mathbf{x}_2,\mathbf{y}_1)|^2. \tag{6}$$

We note that, when $\mathbf{y}_1=\mathbf{y}_2=\mathbf{y}$, $\Gamma^{(1)}(\mathbf{x})=|G(\mathbf{x},\mathbf{y})|^2$ is proportional to the intensity measured by a point detector at $\mathbf{x}$ due to a point source at $\mathbf{y}$ and is thus a classical quantity. Finally, we introduce the connected correlation function, which is defined by

$$\Gamma_c^{(2)}(\mathbf{x}_1,\mathbf{x}_2)=\Gamma^{(2)}(\mathbf{x}_1,\mathbf{x}_2)-\Gamma^{(1)}(\mathbf{x}_1)\Gamma^{(1)}(\mathbf{x}_2). \tag{7}$$

By using Eqs. (5) and (6), we obtain

$$\Gamma_c^{(2)}(\mathbf{x}_1,\mathbf{x}_2)=\frac{1}{4}G(\mathbf{x}_1,\mathbf{y}_1)G^*(\mathbf{x}_1,\mathbf{y}_2)G^*(\mathbf{x}_2,\mathbf{y}_1)G(\mathbf{x}_2,\mathbf{y}_2)+c.c.. \tag{8}$$

We now compute the correlation functions for the case of a small spherical scatterer of radius $a\ll\lambda$, which we treat as a point scatterer. The susceptibility is then given by $\eta(\mathbf{x})=\alpha_0\delta(\mathbf{x}-\mathbf{x}_0)$, where α_0 is the polarizability of the sphere and $\mathbf{x}_0$ is its center. The Green's function G obeys the integral equation

$$G(\mathbf{x},\mathbf{y})=G_0(\mathbf{x},\mathbf{y})+k_0^2\int G_0(\mathbf{x},\mathbf{z})\eta(\mathbf{z})G(\mathbf{z},\mathbf{y})\mathrm{d}^3z, \tag{9}$$

where G_0 is the free-space Green's function, which is given by

$$G_0(\mathbf{x},\mathbf{y})=\frac{e^{ik_0|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|}. \tag{10}$$

It can be seen that the solution to Eq. (9) is of the form

$$G(\mathbf{x},\mathbf{y})=G_0(\mathbf{x},\mathbf{y})+\alpha k_0^2G_0(\mathbf{x},\mathbf{x}_0)G_0(\mathbf{x}_0,\mathbf{y}), \tag{11}$$

a result which is obtained by resummation of the perturbation series derived from Eq. (9) [19]. The renormalized polarizability α is given by the expression

$$\alpha=\frac{\alpha_0}{1-\alpha_0k_0^2/(\pi a)+i\alpha_0k_0^3}, \tag{12}$$

which includes radiative corrections to the Lorentz–Lorenz form of the polarizability.

First, we calculate $\Gamma^{(1)}$ for the case of illumination by a single-photon source at $\mathbf{y}$. If the source and the detector are in the far field of the scatterer, then, by using the asymptotic form of the Green's function

$$G_0(\mathbf{x},\mathbf{y})\sim\frac{e^{ik_0r}}{r}e^{-ik_0\hat{\mathbf{x}}\cdot\mathbf{y}},\qquad |\mathbf{x}|\gg|\mathbf{y}|, \tag{13}$$

where $r=|\mathbf{x}|$, we find that

$$\Gamma^{(1)}(\mathbf{x})\propto\cos[k_0(\hat{\mathbf{x}}+\hat{\mathbf{y}})\cdot\mathbf{x}_0-k_0R]+\cdots. \tag{14}$$

Here $R=|\mathbf{x}|+|\mathbf{y}|$, and the ellipsis denotes lower-frequency terms. We thus see that a direct imaging experiment, in which the spatial dependence of $\Gamma^{(1)}$ is mapped, can detect spatial frequencies of $2k_0$, which corresponds to a resolution of $\lambda/2$, consistent with the Rayleigh limit of classical optics. Next, we calculate the connected correlation function $\Gamma_c^{(2)}$. By making use of Eqs. (8) and (11), we obtain in the far-field limit

$$\Gamma_c^{(2)}(\mathbf{x}_1,\mathbf{x}_2)\propto\cos[k_0(\hat{\mathbf{x}}_1+\hat{\mathbf{x}}_2+\hat{\mathbf{y}}_1+\hat{\mathbf{y}}_2)\cdot\mathbf{x}_0]+\cdots. \tag{15}$$

Evidently, direct imaging of $\Gamma_c^{(2)}$ provides access to spatial frequencies of size $4k_0$ from measured data, which leads to an enhancement in resolution by a factor of 2 in comparison to the Rayleigh limit. That is, there is sufficient information to characterize the scatterer, even if only half of the spatial frequencies required for classical imaging are employed. Although the above calculation was carried out for the case of point scatterers, it can be seen that the result applies, more generally, to scattering by an extended object.

We now consider the inverse problem of recovering η from far-field measurements of $\Gamma_c^{(2)}$. To proceed, we introduce a complex phase ϕ defined so that

$$G(\mathbf{x},\mathbf{y})=G_0(\mathbf{x},\mathbf{y})e^{\phi(\mathbf{x},\mathbf{y})}. \tag{16}$$

Within the accuracy of the first Rytov approximation [20], ϕ is given by

$$\phi(\mathbf{x},\mathbf{y})=\frac{k_0^2}{G_0(\mathbf{x},\mathbf{y})}\int G_0(\mathbf{x},\mathbf{z})\eta(\mathbf{z})G_0(\mathbf{z},\mathbf{y})\mathrm{d}^3z. \tag{17}$$

In the far-field limit, ϕ becomes

$$\phi(\mathbf{x},\mathbf{y})=A(\mathbf{x},\mathbf{y})\tilde{\eta}(k_0(\hat{\mathbf{x}}+\hat{\mathbf{y}})), \tag{18}$$

where $\tilde{\eta}(\mathbf{k})=\int\exp(-i\mathbf{k}\cdot\mathbf{x})\eta(\mathbf{x})\mathrm{d}^3x$ is the Fourier transform of η and

$$A(\mathbf{x},\mathbf{y})=\frac{k_0^2}{G_0(\mathbf{x},\mathbf{y})}\frac{e^{ik_0(|\mathbf{x}|+|\mathbf{y}|)}}{|\mathbf{x}||\mathbf{y}|}. \tag{19}$$

Using the above results and Eq. (8), we find that

$$\Gamma_c^{(2)}(\mathbf{x}_1,\mathbf{x}_2)=\Gamma_0(\mathbf{x}_1,\mathbf{x}_2)e^{\Phi(\mathbf{x}_1,\mathbf{x}_2)}+c.c., \tag{20}$$

where

$$\Gamma_0(\mathbf{x}_1,\mathbf{x}_2)=\frac{1}{4}G_0(\mathbf{x}_1,\mathbf{y}_1)G_0^*(\mathbf{x}_1,\mathbf{y}_2)G_0^*(\mathbf{x}_2,\mathbf{y}_1)G_0(\mathbf{x}_2,\mathbf{y}_2), \tag{21}$$

$$\begin{aligned}\Phi(\mathbf{x}_1,\mathbf{x}_2)=&A(\mathbf{x}_1,\mathbf{y}_1)\tilde{\eta}(k_0(\hat{\mathbf{x}}_1+\hat{\mathbf{y}}_1))\\&+A^*(\mathbf{x}_1,\mathbf{y}_2)\tilde{\eta}^*(k_0(\hat{\mathbf{x}}_1+\hat{\mathbf{y}}_2))\\&+A^*(\mathbf{x}_2,\mathbf{y}_1)\tilde{\eta}^*(k_0(\hat{\mathbf{x}}_2+\hat{\mathbf{y}}_1))\\&+A(\mathbf{x}_2,\mathbf{y}_2)\tilde{\eta}(k_0(\hat{\mathbf{x}}_2+\hat{\mathbf{y}}_2)).\end{aligned} \tag{22}$$

Evidently, by varying the directions $\hat{\mathbf{x}}_1$, $\hat{\mathbf{x}}_2$, $\hat{\mathbf{y}}_1$, $\hat{\mathbf{y}}_2$ for different values of the radii $|\mathbf{x}_1|$ and $|\mathbf{x}_2|$, we can determine $\tilde{\eta}(\mathbf{k})$ for $|\mathbf{k}|\le 2k_0$. That is, it is possible to recover a band-limited approximation to η with bandwidth $2k_0$

from measurements of $\Gamma_c^{(2)}$. This bandwidth corresponds to a spatial resolution of $\lambda/2$, which is the same as would be obtained by solving the inverse problem using as data measurements of $\Gamma^{(1)}$ or, equivalently, intensity measurements of the scattered field [21,22].

It is important to emphasize that the above result depends upon the use of the first Rytov approximation, which leads to a linearization of the inverse problem. We conjecture that resolution beyond the Rayleigh limit is unlikely to be obtained by solving the nonlinear inverse problem.

We close with a few remarks. (i) The requirement that precisely one photon is registered by each detector is an essential aspect of our method. This postdetection selection mechanism forces the entanglement of the initially uncorrelated photons and is responsible for the quantum mechanical nature of the measurement. (ii) We also note that calculations along the same lines as presented herein, indicate that access to frequencies of size $2Nk_0$ can be obtained from experiments carried out with N single-photon sources and N detectors. (iii) Although in our model the electromagnetic field is quantized, the interaction of the field with the scattering medium is treated classically. It would be of interest to extend our results to the case in which the medium consists of a collection of two-level atoms. In this context, the inverse problem would consist of recovering the position-dependent number density of the atoms.

In conclusion, we have studied the inverse scattering problem that arises in two-photon imaging with interferometric measurements. We have found that the quantum two-point correlation function of the field contains information about the scattering medium at a spatial frequency of twice the Rayleigh bandwidth. The corresponding linearized inverse problem, however, yields reconstructions with a resolution of $\lambda/2$.

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