

ESSAYS IN MACROECONOMIC DYNAMICS AND THE FINANCIAL MARKET

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For My Parents

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ABSTRACT

ESSAYS IN MACROECONOMIC DYNAMICS AND THE FINANCIAL MARKET

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Jesús Fernández-Villaverde

This thesis explores the important link between macroeconomic dynamics and the financial sector. The first essay studies Epstein-Zin preferences, which are found to be able to account for both aggregate macroeconomic dynamics and asset prices. In the first essay, I compare different solution methods for computing dynamic stochastic general equilibrium (DSGE) models with Epstein-Zin preferences and stochastic volatility. I show that perturbation methods are an attractive approach for computing this class of problems. The second essay emphasizes the importance of frictions in the financial market on real economic activities. The model studies the international business cycle co-movements when financial frictions are present. The model can account for the positive and sizable cross-country correlations of output, investment and hours worked in the data.

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Chapter 1

Introduction

This thesis studies the connection between aggregate macroeconomic dynamics and the financial sector. Since the work by Mehra and Prescott (1985), it is well known that standard real business cycle models have difficulty in explaining asset prices. The introduction of recursive preferences, such as Epstein-Zin preferences adds extra flexibility and helps the model to explain asset pricing patterns. Given the importance of Epstein-Zin preferences, the first essay compares different solution methods for computing the equilibrium of DSGE models with Epstein and Zin preferences and stochastic volatility. I solve the stochastic neoclassical growth model with recursive preferences and stochastic volatility using four different approaches: second- and third-order perturbation, Chebyshev polynomials, and value function iteration. I document the performance of the methods in terms of computing time, implementation complexity, and accuracy. Our main finding is that perturbations are competitive in terms of accuracy with Chebyshev polynomials and value function iteration, while being several orders of magnitude faster to run. Therefore, I conclude that perturbation methods are an attractive approach for computing this class of problems.

The second essay looks into the importance of financial frictions on real activities. I build a two-country DSGE model to study the quantitative impact of financial frictions on business cycle co-movements when investors have foreign asset exposure. An investor in each country holds capital in both countries and faces a leverage constraint on her debt. I show quantitatively that financial frictions along with foreign asset exposure give rise to a multiplier effect that amplifies the transmission of shocks between countries. The key mechanism is that a negative shock in the home country reduces the wealth of investors in both countries which tightens their leverage constraints, leading to a fall in the investment, consumption, and hours worked in the

foreign country. Compared to the existing literature, which tends to produce either negative or positive but small cross-country correlations, this model produces positive and sizable correlations that are consistent with the data. The model can account for two thirds of the output correlation, most of the employment correlation and a positive investment correlation. In addition, the model also shows that, consistent with empirical findings, when investors have more foreign asset exposure in the other country, the output correlation between the two countries increases.

Chapter 2

Computing DSGE Models with Recursive Preferences and Stochastic Volatility

2.1 Introduction

This paper compares different solution methods for computing the equilibrium of dynamic stochastic general equilibrium (DSGE) models with recursive preferences and stochastic volatility (SV). Both features have become very popular in finance and in macroeconomics as modelling devices to account for business cycle fluctuations and asset pricing. Recursive preferences, as those first proposed by Kreps and Porteus (1978) and later generalized by Epstein and Zin (1989 and 1991) and Weil (1990), are attractive for two reasons. First, they allow us to separate risk aversion and intertemporal elasticity of substitution (EIS). Second, they offer the intuitive appeal of having preferences for early or later resolution of uncertainty (see the reviews by Backus *et al.*, 2004 and 2007, and Hansen *et al.*, 2007, for further details and references). SV generates heteroskedastic aggregate fluctuations, a basic property of many time series such as output (see the review by Fernández-Villaverde and Rubio-Ramírez, 2010), and adds extra flexibility in accounting for asset pricing patterns. In fact, in an influential paper, Bansal and Yaron (2004) have argued that the *combination* of recursive preferences and SV is the key for their proposed mechanism, long-run risk, to be successful at explaining asset pricing.

But despite the popularity and importance of these issues, nearly nothing is known about the numerical properties of the different solution methods that solve equilibrium models with recursive preferences and SV. For example, we do not know how well value function iteration (VFI) performs or how good local approximations are compared with global ones.¹ Similarly, if we want to estimate the model, we need to assess which solution method is sufficiently reliable yet quick enough to make the exercise feasible. This paper attempts to fill this gap in the literature, and therefore, it complements previous work by Aruoba *et al.* (2006), in which a similar exercise is performed with the neoclassical growth model with CRRA utility function and constant volatility.

We solve and simulate the model using four main approaches: perturbation (of second and third order), Chebyshev polynomials, and VFI. By doing so, we span most of the relevant methods in the literature. Our results provide a strong guess of how

¹Also, remember that the most common solution algorithm in the DSGE literature, (log-) linearization, cannot be applied, since it makes us miss the whole point of recursive preferences or SV: the resulting (log-) linear decision rules are certainty equivalent and do not depend on risk aversion or volatility.

some other methods not covered here, such as finite elements, would work (rather similar to Chebyshev polynomials but more computationally intensive). We report results for a benchmark calibration of the model and for alternative calibrations that change the variance of the productivity shock, the risk aversion, and the intertemporal elasticity of substitution. In that way, we study the performance of the methods both for cases close to the CRRA utility function with constant volatility and for highly non-linear cases far away from the CRRA benchmark. For each method, we compute decision rules, the value function, the ergodic distribution of the economy, business cycle statistics, the welfare costs of aggregate fluctuations, and asset prices. Also, we evaluate the accuracy of the solution by reporting Euler equation errors.

We highlight four main results from our exercise. First, all methods provide a high degree of accuracy. Thus, researchers who stay within our set of solution algorithms can be confident that their quantitative answers are sound.

Second, perturbations deliver a surprisingly high level of accuracy with considerable speed. Both second- and third-order perturbations perform remarkably well in terms of accuracy for the benchmark calibration, being competitive with VFI or Chebyshev polynomials. For this calibration, a second-order perturbation that runs in a fraction of a second does nearly as well in terms of the average Euler equation error as a VFI that takes ten hours to run. Even in the extreme calibration with high risk aversion and high volatility of productivity shocks, perturbation works at a more than acceptable level. Since, in practice, perturbation methods are the only computationally feasible method to solve the medium-scale DSGE models used for policy analysis that have dozens of state variables (as in Smets and Wouters, 2007), this finding has an outmost applicability. Moreover, since implementing second- and third-order perturbations is feasible with off-the-shelf software like **Dynare**, which requires minimum programming knowledge by the user, our findings may induce many researchers to explore recursive preferences and/or SV in further detail. Two final advantages of perturbation are that, often, the perturbed solution provides insights about the economics of the problem and that it might be an excellent initial guess for VFI or for Chebyshev polynomials.

Third, Chebyshev polynomials provide a terrific level of accuracy with reasonable computational burden. When accuracy is most required and the dimensionality of the state space is not too high, as in our model, they are the obvious choice.

Fourth, we were disappointed by the poor performance of VFI, which, com-

pared with Chebyshev, could not achieve a high accuracy even with a large grid. This suggests that we should relegate VFI to solving those problems where non-differentiabilities complicate the application of the previous methods.

The rest of the paper is organized as follows. In section 2, we present our test model. Section 3 describes the different solution methods used to approximate the decision rules of the model. Section 4 discusses the calibration of the model. Section 5 reports numerical results and section 6 concludes. An appendix provides some additional details.

2.2 The Stochastic Neoclassical Growth Model with Recursive Preferences and SV

We use the stochastic neoclassical growth model with recursive preferences and SV in the process for technology as our test case. We select this model for three reasons. First, it is the workhorse of modern macroeconomics. Even more complicated New Keynesian models with real and nominal rigidities, such as those in Woodford (2003) or Christiano *et al.* (2005), are built around the core of the neoclassical growth model. Thus, any lesson learned with it is likely to have a wide applicability. Second, the model is, except for the form of the utility function and the process for SV, the same test case as in Aruoba *et al.* (2006). This provides us with a set of results to compare to our findings. Three, the introduction of recursive preferences and SV make the model both more non-linear (and hence, a challenge for different solution algorithms) and potentially more relevant for practical use. For example, and as mentioned in the introduction, Bansal and Yaron (2004) have emphasized the importance of the combination of recursive preferences and time-varying volatility to account for asset prices.

The description of the model is straightforward, and we just go through the details required to fix notation. There is a representative household that has preferences over streams of consumption, c_t , and leisure, $1 - l_t$, represented by a recursive function of the form:

$$U_t = \max_{c_t, l_t} \left[(1 - \beta) (c_t^v (1 - l_t)^{1-v})^{\frac{1-\gamma}{\theta}} + \beta (\mathbb{E}_t U_{t+1}^{1-\gamma})^{\frac{1}{\theta}} \right]^{\frac{\theta}{1-\gamma}} \quad (2.1)$$

The parameters in these preferences include β , the discount factor, v , which controls

labor supply, γ , which controls risk aversion, and:

$$\theta = \frac{1 - \gamma}{1 - \frac{1}{\psi}}$$

where ψ is the EIS. The parameter θ is an index of the deviation with respect to the benchmark CRRA utility function (when $\theta = 1$, we are back in that CRRA case where the inverse of the EIS and risk aversion coincide).

The household's budget constraint is given by:

$$c_t + i_t + \frac{b_{t+1}}{R_t^f} = w_t l_t + r_t k_t + b_t$$

where i_t is investment, R_t^f is the risk-free gross interest rate, b_t is the holding of an uncontingent bond that pays 1 unit of consumption good at time $t+1$, w_t is the wage, l_t is labor, r_t is the rental rate of capital, and k_t is capital. Asset markets are complete and we could have also included in the budget constraint the whole set of Arrow securities. Since we have a representative household, this is not necessary because the net supply of any security is zero. Households accumulate capital according to the law of motion $k_{t+1} = (1 - \delta)k_t + i_t$ where δ is the depreciation rate.

The final good in the economy is produced by a competitive firm with a Cobb-Douglas technology $y_t = e^{z_t} k_t^\zeta l_t^{1-\zeta}$ where z_t is the productivity level that follows:²

$$z_t = \lambda z_{t-1} + e^{\sigma_t} \varepsilon_t, \varepsilon_t \sim \mathcal{N}(0, 1).$$

The innovation ε_t is scaled by a SV level σ_t , which evolves as:

$$\sigma_t = (1 - \rho)\bar{\sigma} + \rho\sigma_{t-1} + \eta\omega_t, \omega_t \sim \mathcal{N}(0, 1)$$

where $\bar{\sigma}$ is the unconditional mean level of σ_t , ρ is the persistence of the processes, and η is the standard deviation of the innovations to σ_t . Our specification is parsimonious

²Stationarity is the natural choice for our exercise. If we had a deterministic trend, we would only need to adjust β in our calibration below (and the results would be nearly identical). If we had a stochastic trend, we would need to rescale the variables by the productivity level and solve the transformed problem. However, in this case, it is well known that the economy fluctuates less than when $\lambda < 1$, and therefore, all solution methods would be closer, limiting our ability to appreciate differences in their performance.

and it introduces only two new parameters, ρ and η . At the same time, it captures some important features of the data (see a detailed discussion in Fernández-Villaverde and Rubio-Ramírez, 2010). Another important point is that, with SV, we have two innovations, an innovation to technology, ε_t , and an innovation to the standard deviation of technology, ω_t . Finally, the economy must satisfy the aggregate resource constraint $y_t = c_t + i_t$.

The definition of equilibrium is standard and we skip it in the interest of space. Also, both welfare theorems hold, a fact that we will exploit by jumping back and forth between the solution of the social planner's problem and the competitive equilibrium. However, this is only to simplify our derivations. It is straightforward to adapt the solution methods described below to solve problems that are not Pareto optimal.

Thus, an alternative way to write this economy is to look at the value function representation of the social planner's problem in terms of its three state variables, capital k_t , productivity z_t , and volatility, σ_t :

$$\begin{aligned} V(k_t, z_t, \sigma_t) = \max_{c_t, l_t} & \left[(1 - \beta) (c_t^v (1 - l_t)^{1-v})^{\frac{1-\gamma}{\theta}} + \beta (\mathbb{E}_t V^{1-\gamma}(k_{t+1}, z_{t+1}, \sigma_{t+1}))^{\frac{1}{\theta}} \right]^{\frac{\theta}{1-\gamma}} \\ \text{s.t. } & c_t + k_{t+1} = e^{z_t} k_t^\zeta l_t^{1-\zeta} + (1 - \delta) k_t \\ & z_t = \lambda z_{t-1} + e^{\sigma_t} \varepsilon_t, \varepsilon_t \sim \mathcal{N}(0, 1) \\ & \sigma_t = (1 - \rho) \bar{\sigma} + \rho \sigma_{t-1} + \eta \omega_t, \omega_t \sim \mathcal{N}(0, 1). \end{aligned}$$

Then, we can find the pricing kernel of the economy

$$m_{t+1} = \frac{\partial V_t / \partial c_{t+1}}{\partial V_t / \partial c_t}.$$

Now, note that:

$$\frac{\partial V_t}{\partial c_t} = (1 - \beta) V_t^{1-\frac{1-\gamma}{\theta}} v \frac{(c_t^v (1 - l_t)^{1-v})^{\frac{1-\gamma}{\theta}}}{c_t}$$

and:

$$\frac{\partial V_t}{\partial c_{t+1}} = \beta V_t^{1-\frac{1-\gamma}{\theta}} (\mathbb{E}_t V_{t+1}^{1-\gamma})^{\frac{1}{\theta}-1} \mathbb{E}_t \left(V_{t+1}^{-\gamma} (1 - \beta) V_{t+1}^{1-\frac{1-\gamma}{\theta}} v \frac{(c_{t+1}^v (1 - l_{t+1})^{1-v})^{\frac{1-\gamma}{\theta}}}{c_{t+1}} \right)$$

where in the last step we use the result regarding $\partial V_t / \partial c_t$ forwarded by one period.

Cancelling redundant terms, we get:

$$m_{t+1} = \frac{\partial V_t / \partial c_{t+1}}{\partial V_t / \partial c_t} = \beta \left(\frac{c_{t+1}^v (1 - l_{t+1})^{1-v}}{c_t^v (1 - l_t)^{1-v}} \right)^{\frac{1-\gamma}{\theta}} \frac{c_t}{c_{t+1}} \left(\frac{V_{t+1}^{1-\gamma}}{\mathbb{E}_t V_{t+1}^{1-\gamma}} \right)^{1-\frac{1}{\theta}}. \quad (2.2)$$

This equation shows how the pricing kernel is affected by the presence of recursive preferences. If $\theta = 1$, the last term,

$$\left(\frac{V_{t+1}^{1-\gamma}}{\mathbb{E}_t V_{t+1}^{1-\gamma}} \right)^{1-\frac{1}{\theta}} \quad (2.3)$$

is equal to 1 and we get back the pricing kernel of the standard CRRA case. If $\theta \neq 1$, the pricing kernel is twisted by (2.3).

We identify the net return on equity with the marginal net return on investment:

$$R_{t+1}^k = \zeta e^{z_{t+1}} k_{t+1}^{\zeta-1} l_{t+1}^{1-\zeta} - \delta$$

with expected return $\mathbb{E}_t [R_{t+1}^k]$.

2.3 Solution Methods

We are interested in comparing different solution methods to approximate the dynamics of the previous model. Since the literature on computational methods is large, it would be cumbersome to review every proposed method. Instead, we select those methods that we find most promising.

Our first method is perturbation (introduced by Judd and Guu, 1992 and 1997 and nicely explained in Schmitt-Grohé and Uribe, 2004). Perturbation algorithms build a Taylor series expansion of the agents' decision rules. Often, perturbation methods are very fast and, despite their local nature, highly accurate in a large range of values of the state variables (Aruoba *et al.*, 2006). This means that, in practice, perturbations are the only method that can handle models with dozens of state variables within any reasonable amount of time. Moreover, perturbation often provides insights into the structure of the solution and on the economics of the model. Finally, linearization and log-linearization, the most common solution methods for DSGE models, are particular cases of a perturbation of first order.

We implement a second- and a third-order perturbation of our model. A first-order perturbation is useless for our investigation: the resulting decision rules are certainty equivalent and, therefore, they depend on ψ but not on γ or σ_t . In other words, the first-order decision rules of the model with recursive preferences coincide with the decision rules of the model with CRRA preferences with the same ψ and $\bar{\sigma}$ for *any* value of γ or σ_t . We need to go, at least, to second-order decision rules to have terms that depend on γ or σ_t and, hence, allow recursive preferences or SV to play a role. Since the accuracy of second-order decision rules may not be high enough and, in addition, we want to explore time-varying risk premia, we also compute a third-order perturbation. As we will document below, a third-order perturbation provides enough accuracy without unnecessary complications. Thus, we do not need to go to higher orders.

The second method is a projection algorithm with Chebyshev polynomials (Judd, 1992). Projection algorithms build approximated decision rules that minimize a residual function that measures the distance between the left- and right-hand side of the equilibrium conditions of the model. Projection methods are attractive because they offer a global solution over the whole range of the state space. Their main drawback is that they suffer from an acute curse of dimensionality that makes it challenging to extend it to models with many state variables. Among the many different types of projection methods, Aruoba *et al.* (2006) show that Chebyshev polynomials are particularly efficient. Other projection methods, such as finite elements or parameterized expectations, tend to perform somewhat worse than Chebyshev polynomials, and therefore, in the interest of space, we do not consider them.

Finally, we compute the model using VFI (Epstein and Zin, 1989, show that a version of the contraction mapping theorem holds in the value function of the problem with recursive preferences). VFI is slow and it suffers as well from the curse of dimensionality, but it is safe, reliable, and well understood. Thus, it is a natural default algorithm for the solution of DSGE models.

2.3.1 Perturbation

We describe now each of the different methods in more detail. We start by explaining how to use a perturbation approach to solve DSGE models using the value function of the household. We are not the first to explore the perturbation of value function prob-

lems. Judd (1998) already presents the idea of perturbing the value function instead of the equilibrium conditions of a model. Unfortunately, he does not elaborate much on the topic. Schmitt-Grohé and Uribe (2005) employ a perturbation approach to find a second-order approximation to the value function that allows them to rank different fiscal and monetary policies in terms of welfare. However, we follow Binsbergen *et al.* (2009) in their emphasis on the generality of the approach.³

To illustrate the procedure, we limit our exposition to deriving the second-order approximation to the value function and the decision rules of the agents. Higher-order terms are derived analogously, but the algebra becomes too cumbersome to be developed explicitly (in our application, the symbolic algebra is undertaken by **Mathematica**, which automatically generates Fortran 95 code that we can evaluate numerically). Hopefully, our steps will be enough to allow the reader to understand the main thrust of the procedure and obtain higher-order approximations by herself.

First, we rewrite the exogenous processes in terms of a perturbation parameter χ ,

$$\begin{aligned} z_t &= \lambda z_{t-1} + \chi e^{\sigma_t} \varepsilon_t \\ \sigma_t &= (1 - \rho)\bar{\sigma} + \rho\sigma_{t-1} + \chi\eta\omega_t. \end{aligned}$$

When $\chi = 1$, which is just a normalization, we are dealing with the stochastic version of the model. When $\chi = 0$, we are dealing with the deterministic case with steady state k_{ss} , $z_{ss} = 0$, and $\sigma_{ss} = \bar{\sigma}$. Also, it is convenient for the algebra below to define a vector of states in differences with respect to the steady state:

$$s_t = (k_t - k_{ss}, z_{t-1}, \varepsilon_t, \sigma_{t-1} - \sigma_{ss}, \omega_t, \chi)$$

where s_{it} is the i - th component of this vector at time t for $i \in \{1, \dots, 6\}$. Then, we can write the social planner's value function, $V(s_t)$, and the decision rules for

³The perturbation method is related to Benigno and Woodford (2006) and Hansen and Sargent (1995). Benigno and Woodford present a linear-quadratic (LQ) approximation to solve optimal policy problems when the constraints of the problem are non-linear (see also Levine *et al.*, 2007). This allows them to find the correct local welfare ranking of different policies. Our perturbation can also deal with non-linear constraints and obtains the correct local approximation to welfare and policies, but with the advantage that it is easily generalizable to higher-order approximations. Hansen and Sargent (1995) modify the LQ problem to adjust for risk. In that way, they can handle some versions of recursive utilities. Hansen and Sargent's method, however, requires imposing a tight functional form for future utility and to surrender the assumption that risk-adjusted utility is separable across states of the world. Perturbation does not suffer from those limitations.

consumption, $c(s_t)$, investment, $i(s_t)$, capital, $k(s_t)$, and labor, $l(s_t)$, as a function of s_t .

Second, we note that, under differentiability assumptions, the second-order Taylor approximation of the value function around $s_t = \mathbf{0}$ (the vectorial zero) is:

$$V(s_t) \simeq V_{ss} + V_{i,ss}s_t^i + \frac{1}{2}V_{ij,ss}s_t^i s_t^j$$

where:

1. Each term $V_{...,ss}$ is a scalar equal to a derivative of the value function evaluated at $\mathbf{0}$: $V_{ss} \equiv V(\mathbf{0})$, $V_{i,ss} \equiv V_i(\mathbf{0})$ for $i \in \{1, \dots, 6\}$, and $V_{ij,ss} \equiv V_{ij}(\mathbf{0})$ for $i, j \in \{1, \dots, 6\}$,
2. We use the tensors $V_{i,ss}s_t^i = \sum_{i=1}^6 V_{i,ss}s_{i,t}$ and $V_{ij,ss}s_t^i s_t^j = \sum_{i=1}^6 \sum_{j=1}^6 V_{ij,ss}s_{i,t}s_{j,t}$, which eliminate the symbol $\sum_{i=1}^6$ when no confusion arises.

We can extend this notation to higher-order derivatives of the value function. This expansion could also be performed around a different point of the state space, such as the mode of the ergodic distribution of the state variables. In section 5, we discuss this point further.

Fernández-Villaverde *et al.* (2010) show that many of these terms $V_{...,ss}$ are zero (for instance, those implied by certainty equivalence in the first-order component). More directly related to this paper, Binsbergen *et al.* (2009) demonstrate that γ does not affect the values of any of the coefficients except $V_{66,ss}$ and also that $V_{66,ss} \neq 0$. This result is intuitive, since the value function of a risk-averse agent is in general affected by uncertainty and we want to have an approximation with terms that capture this effect and allow for the appropriate welfare ranking of decision rules. Indeed, $V_{66,ss}$ has a straightforward interpretation. At the deterministic steady state with $\chi = 1$ (that is, even if we are in the stochastic economy, we just happen to be exactly at the steady state values of all the other states), we have:

$$V(0, 0, 0, 0, 0, 1) \simeq V_{ss} + \frac{1}{2}V_{66,ss}$$

Hence $\frac{1}{2}V_{66,ss}$ is a measure of the welfare cost of the business cycle, that is, of how much utility changes when the variance of the productivity shocks is at steady-state value σ_{ss} instead of zero (note that this quantity is not necessarily negative). This

term is an accurate evaluation of the third order of the welfare cost of business cycle fluctuations because all of the third-order terms in the approximation of the value function either have coefficient values of zero or drop when evaluated at the deterministic steady state.

This cost of the business cycle can easily be transformed into consumption equivalent units. We can compute the percentage decrease in consumption τ that will make the household indifferent between consuming $(1 - \tau) c_{ss}$ units per period with certainty or c_t units with uncertainty. To do so, note that the steady-state value function is just $V_{ss} = c_{ss}^v (1 - l_{ss})^{1-v}$, which implies that:

$$c_{ss}^v (1 - l_{ss})^{1-v} + \frac{1}{2} V_{66,ss} = ((1 - \tau) c_{ss})^v (1 - l_{ss})^{1-v}$$

or:

$$V_{ss} + \frac{1}{2} V_{66,ss} = (1 - \tau)^v V_{ss}$$

Then:

$$\tau = 1 - \left[1 + \frac{1}{2} \frac{V_{66,ss}}{V_{ss}} \right]^{\frac{1}{v}}.$$

We are perturbing the value function in levels of the variables. However, there is nothing special about levels and we could have done the same in logs (a common practice when linearizing DSGE models) or in any other function of the states. These changes of variables may improve the performance of perturbation (Fernández-Villaverde and Rubio-Ramírez, 2006). By doing the perturbation in levels, we are picking the most conservative case for perturbation. Since one of the conclusions that we will reach from our numerical results is that perturbation works surprisingly well in terms of accuracy, that conclusion will only be reinforced by an appropriate change of variables.⁴

The decision rules can be expanded in the same way. For example, the second-order approximation of the decision rule for consumption is, under differentiability

⁴This comment begets the question, nevertheless, of why we did not perform a perturbation in logs, since many economists will find it more natural than using levels. Our experience with the CRRA utility case (Aruoba *et al.*, 2006) and some computations with recursive preferences not included in the paper suggest that a perturbation in logs does slightly worse than a perturbation in levels.

assumptions:

$$c(s_t) \simeq c_{ss} + c_{i,ss}s_t^i + \frac{1}{2}c_{ij,ss}s_t^i s_t^j$$

where we have followed the same derivative and tensor notation as before.

As with the approximation of the value function, Binsbergen *et al.* (2009) show that γ does not affect the values of any of the coefficients except $c_{66,ss}$. This term is a constant that captures precautionary behavior caused by risk. This observation tells us two facts. First, a linear approximation to the decision rule does not depend on γ (it is certainty equivalent), and therefore, if we are interested in recursive preferences, we need to go at least to a second-order approximation. Second, given some fixed parameter values, the difference between the second-order approximation to the decision rules of a model with CRRA preferences and a model with recursive preferences is a constant. This constant generates a second, indirect effect, because it changes the ergodic distribution of the state variables and, hence, the points where we evaluate the decision rules along the equilibrium path. These arguments demonstrate how perturbation methods can provide analytic insights beyond computational advantages and help in understanding the numerical results in Tallarini (2000). In the third-order approximation, all of the terms on functions of χ^2 depend on γ .

Following the same steps, we can derive the decision rules for labor, investment, and capital. In addition we have functions that give us the evolution of other variables of interest, such as the pricing kernel or the risk-free gross interest rate R_t^f . All of these functions have the same structure and properties regarding γ as the decision rule for consumption. In the case of functions pricing assets, the second-order approximation generates a constant risk premium, while the third-order approximation creates a time-varying risk premium.

Once we have reached this point, there are two paths we can follow to solve for the coefficients of the perturbation. The first procedure is to write down the equilibrium conditions of the model plus the definition of the value function. Then, we take successive derivatives in this augmented set of equilibrium conditions and solve for the unknown coefficients. This approach, which we call equilibrium conditions perturbation (ECP), gets us, after n iterations, the n -th-order approximation to the value function and to the decision rules.

A second procedure is to take derivatives of the value function with respect to states and controls and use those derivatives to find the unknown coefficient. This

approach, which we call value function perturbation (VFP), delivers after $(n + 1)$ steps, the $(n + 1)$ -th order approximation to the value function and the n -th order approximation to the decision rules.⁵ Loosely speaking, ECP undertakes the first step of VFP by hand by forcing the researcher to derive the equilibrium conditions.

The ECP approach is simpler but it relies on our ability to find equilibrium conditions that do not depend on derivatives of the value function. Otherwise, we need to modify the equilibrium conditions to include the definitions of the derivatives of the value function. Even if this is possible to do (and not particularly difficult), it amounts to solving a problem that is equivalent to VFP. This observation is important because it is easy to postulate models that have equilibrium conditions where we cannot get rid of all the derivatives of the value function (for example, in problems of optimal policy design). ECP is also faster from a computational perspective. However, VFP is only trivially more involved because finding the $(n + 1)$ -th-order approximation to the value function on top of the n -th order approximation requires nearly no additional effort.

The algorithm presented here is based on the system of equilibrium equations derived using the ECP. In the appendix, we derive a system of equations using the VFP. We take the first-order conditions of the social planner. First, with respect to consumption today:

$$\frac{\partial V_t}{\partial c_t} - \lambda_t = 0$$

where λ_t is the Lagrangian multiplier associated with the resource constraint. Second, with respect to capital:

$$-\lambda_t + \mathbb{E}_t \lambda_{t+1} \left(\zeta e^{z_{t+1}} k_{t+1}^{\zeta-1} l_{t+1}^{1-\zeta} + 1 - \delta \right) = 0.$$

Third, with respect to labor:

$$\frac{1-v}{v} \frac{c_t}{(1-l_t)} = (1-\zeta) e^{z_t} k_t^{\zeta} l_t^{-\zeta}.$$

Then, we have $\mathbb{E}_t m_{t+1} \left(\zeta e^{z_t} k_{t+1}^{\zeta-1} l_{t+1}^{1-\zeta} + 1 - \delta \right) = 1$ where m_{t+1} was derived in equation

⁵The classical strategy of finding a quadratic approximation of the utility function to derive a linear decision rule is a second-order example of VFP (Anderson *et al.*, 1996). A standard linearization of the equilibrium conditions of a DSGE model when we add the value function to those equilibrium conditions is a simple case of ECP. This is done, for instance, in Schmitt-Grohé and Uribe (2005).

(2.2). Note that, as explained above, the derivatives of the value function in (2.2) are eliminated.

Once we substitute for the pricing kernel, the augmented equilibrium conditions are:

$$\begin{aligned}
V_t - \left[(1 - \beta) (c_t^v (1 - l_t)^{1-v})^{\frac{1-\gamma}{\theta}} + \beta (\mathbb{E}_t V^{1-\gamma}(k_{t+1}, z_{t+1}))^{\frac{1}{\theta}} \right]^{\frac{\theta}{1-\gamma}} &= 0 \\
\mathbb{E}_t \left[\beta \left(\frac{c_{t+1}}{c_t} \right)^{\frac{1-\gamma}{\theta}-1} \left(\frac{V_{t+1}^{1-\gamma}}{\mathbb{E}_t V_{t+1}^{1-\gamma}} \right)^{1-\frac{1}{\theta}} \left(\zeta e^{z_{t+1}} k_{t+1}^{\zeta-1} l_{t+1}^{1-\zeta} + 1 - \delta \right) \right] - 1 &= 0 \\
\frac{1-v}{v} \frac{c_t}{(1-l_t)} &= (1-\zeta) e^{z_t} k_t^{\zeta} l_t^{1-\zeta} = 0 \\
\mathbb{E}_t \beta \left(\frac{c_{t+1}}{c_t} \right)^{\frac{1-\gamma}{\theta}-1} \left(\frac{V_{t+1}^{1-\gamma}}{\mathbb{E}_t V_{t+1}^{1-\gamma}} \right)^{1-\frac{1}{\theta}} R_t^f - 1 &= 0 \\
c_t + i_t - e^{z_t} k_t^{\zeta} l_t^{1-\zeta} &= 0 \\
k_{t+1} - i_t - (1-\delta) k_t &= 0
\end{aligned}$$

plus the law of motion for productivity and volatility. Note that all the endogenous variables are functions of the states and that we drop the max operator in front of the value function because we are already evaluating it at the optimum. Thus, a more compact notation for the previous equilibrium conditions as a function of the states is:

$$F(\mathbf{0}) = \mathbf{0}$$

where $F : \mathbb{R}^6 \rightarrow \mathbb{R}^8$.

In steady state, $m_{ss} = \beta$ and the set of equilibrium conditions simplifies to:

$$\begin{aligned}
V_{ss} &= c_{ss}^v (1 - l_{ss})^{1-v} \\
(\zeta k_{ss}^{\zeta-1} l_{ss}^{1-\zeta} + 1 - \delta) &= 1/\beta \\
\frac{1-v}{v} \frac{c_{ss}}{(1-l_{ss})} &= (1-\zeta) k_{ss}^{\zeta} l_{ss}^{1-\zeta} \\
R_{ss}^f &= 1/\beta \\
c_{ss} + i_{ss} &= k_{ss}^{\zeta} l_{ss}^{1-\zeta} \\
i_{ss} &= \delta k_{ss}
\end{aligned}$$

a system of 6 equations on 6 unknowns, V_{ss} , c_{ss} , k_{ss} , i_{ss} , l_{ss} , and R_{ss}^f that can be easily solved (see the appendix for the derivations). This steady state is identical to the steady state of the real business cycle model with a standard CRRA utility function and no SV.

To find the first-order approximation to the value function and the decision rules, we take first derivatives of the function F with respect to the states s_t and evaluate them at $\mathbf{0}$:

$$F_i(\mathbf{0}) = \mathbf{0} \text{ for } i \in \{1, \dots, 6\}.$$

This step gives us 48 different first derivatives (8 equilibrium conditions times the 6 variables of F). Since each dimension of F is equal to zero for all possible values of s_t , their derivatives must also be equal to zero. Therefore, once we substitute the steady-state values and forget about the exogenous processes (which we do not need to solve for), we have a quadratic system of 36 equations on 36 unknowns: $V_{i,ss}$, $c_{i,ss}$, $i_{i,ss}$, $k_{i,ss}$, $l_{i,ss}$, and $R_{i,ss}^f$ for $i \in \{1, \dots, 6\}$. One of the solutions is an unstable root of the system that violates the transversality condition of the problem and we eliminate it. Thus, we keep the solution that implies stability.

To find the second-order approximation, we take derivatives on the first derivatives of the function F , again with respect to the states and the perturbation parameter:

$$F_{ij}(\mathbf{0}) = \mathbf{0} \text{ for } i, j \in \{1, \dots, 6\}.$$

This step gives us a new system of equations. Then, we plug in the terms that we already know from the steady state and from the first-order approximation and we get that the only unknowns left are the second-order terms of the value function and other functions of interest. Quite conveniently, this system of equations is linear and it can be solved quickly. Repeating these steps (taking higher-order derivatives, plugging in the terms already known, and solving for the remaining unknowns), we can get any arbitrary order approximation. For simplicity, and since we checked that we were already obtaining a high accuracy, we decided to stop at a third-order approximation (we are particularly interested in applying the perturbation for estimation purposes and we want to document how a third-order approximation is accurate enough for many problems without spending too much time deriving higher-order terms).

2.3.2 Projection

Projection methods take basis functions to build an approximated value function and decision rules that minimize a residual function defined by the augmented equilibrium conditions of the model. There are two popular methods for choosing basis functions: finite elements and the spectral method. We will present only the spectral method for several reasons: first, in the neoclassical growth model the decision rules and value function are smooth and spectral methods provide an excellent approximation. Second, spectral methods allow us to use a large number of basis functions, with the consequent high accuracy. Third, spectral methods are easier to implement. Their main drawback is that since they approximate the solution with a spectral basis, if the decision rules display a rapidly changing local behavior or kinks, it may be difficult for this scheme to capture those local properties.

Our target is to solve the decision rule for labor and the value function $\{l_t, V_t\}$ from the two conditions:

$$\mathcal{H}(l_t, V_t) = \begin{bmatrix} u_{c,t} - \beta \left(\mathbb{E}_t V_{t+1}^{1-\gamma} \right)^{\frac{1}{\theta}-1} \mathbb{E}_t \left[V_{t+1}^{\frac{(1-\gamma)(\theta-1)}{\theta}} u_{c,t+1} \left(\zeta e^{z_{t+1}} k_{t+1}^{\zeta-1} l_{t+1}^{1-\zeta} + 1 - \delta \right) \right] \\ V_t - \left[(1-\beta)(c_t^v(1-l_t^v))^{\frac{1-\gamma}{\theta}} + \beta \mathbb{E}_t (V_{t+1}^{1-\gamma})^{\frac{1}{\theta}} \right]^{\frac{\theta}{1-\gamma}} \end{bmatrix} = \mathbf{0}$$

where, to save on notation, we define $V_t = V(k_t, z_t, \sigma_t)$ and:

$$u_{c,t} = \frac{1-\gamma}{\theta} v \frac{(c_t^v(1-l_t^v))^{\frac{1-\gamma}{\theta}}}{c_t}$$

Then, from the static condition

$$c_t = \frac{v}{1-v} (1-\zeta) e^{z_t} k_t^{\zeta} l_t^{-\zeta} (1-l_t)$$

and the resource constraint, we can find c_t and k_{t+1} .

Spectral methods solve this problem by specifying the decision rule for labor and the value function $\{l_t, V_t\}$ as linear combinations of weighted basis functions:

$$\begin{aligned} l(k_t, z_j, \sigma_m; \rho) &= \sum_i \rho_{ijm}^l \psi_i(k_t) \\ V(k_t, z_j, \sigma_m; \rho) &= \sum_i \rho_{ijm}^V \psi_i(k_t) \end{aligned}$$

where $\{\psi_i(k)\}_{i=1,\dots,n_k}$ are the n_k basis functions that we will use for our approximation along the capital dimension and $\rho = \{\rho_{ijm}^l, \rho_{ijm}^V\}_{i=1,\dots,n_k; j=1,\dots,J; m=1,\dots,M}$ are unknown coefficients to be determined. In this expression, we have discretized the stochastic processes σ_t for volatility and z_t for productivity using Tauchen's (1986) method as follows. First, we have a grid of M points $G_\sigma = \{e^{\sigma^1}, e^{\sigma^2}, \dots, e^{\sigma^M}\}$ for σ_t and a transition matrix Π^M with generic element $\pi_{i,j}^M = \text{Prob}(e^{\sigma_{t+1}} = e^{\sigma_j} | e^{\sigma_t} = e^{\sigma_i})$. The grid covers 3 standard deviations of the process in each direction. Then, for each M point, we find a grid with J points $G_z^m = \{z_1^m, z_2^m, \dots, z_J^m\}$ for z_t and transition matrixes $\Pi^{J,m}$ with generic element $\pi_{i,j}^{J,m} = \text{Prob}(z_{t+1}^m = z_j^m | z_t^m = z_i^m)$. Again, and conditional on e^{σ^m} , the grid covers 3 standard deviations in each direction. Values for the decision rule outside the grids G_σ and G_z^m can be approximated by interpolation. We make the grids for z_t depend on the level of volatility m to adapt the accuracy of Tauchen's procedure to each conditional variance (although this forces us to interpolate when we switch variances). Since we set $J = 25$ and $M = 5$, the approximation is quite accurate along the productivity axis (we explored other choices of J and M to be sure that our choice was sensible).

A common choice for the basis functions are Chebyshev polynomials because of their flexibility and convenience. Since their domain is $[-1,1]$, we need to bound capital to the set $[\underline{k}, \bar{k}]$, where $\underline{k}$ ($\bar{k}$) is chosen sufficiently low (high) to bind only with extremely low probability, and define a linear map from those bounds into $[-1,1]$. Then, we set $\psi_i(k_t) = \tilde{\psi}_i(\phi_k(k_t))$ where $\tilde{\psi}_i(\cdot)$ are Chebyshev polynomials and $\phi_k(k_t)$ is the linear mapping from $[\underline{k}, \bar{k}]$ to $[-1,1]$.

By plugging $l(k_t, z_j, \sigma_m; \rho)$ and $V(k_t, z_j, \sigma_m; \rho)$ into $\mathcal{H}(l_t, V_t)$, we find the residual function:

$$\mathcal{R}(k_t, z_j, \sigma_m; \rho) = \mathcal{H}(l(k_t, z_j, \sigma_m; \rho), V(k_t, z_j, \sigma_m; \rho))$$

We determine the coefficients ρ to get the residual function as close to $\mathbf{0}$ as possible. However, to do so, we need to choose a weight of the residual function over the space (k_t, z_j, σ_m) . A collocation point criterion delivers the best trade-off between speed and accuracy (Fornberg, 1998) by making the residual function exactly equal to zero in $\{k_i\}_{i=1}^{n_k}$ roots of the n_k -th order Chebyshev polynomial and in the Tauchen points (also, by the Chebyshev interpolation theorem, if an approximating function is exact at the roots of the n_k -th order Chebyshev polynomial, then, as $n_k \rightarrow \infty$, the approximation error becomes arbitrarily small). Therefore, we just need to solve

the following system of $n_k \times J \times M \times 2$ equations:

$$\mathcal{R}(k_i, z_j, \sigma_m; \rho) = \mathbf{0} \text{ for any } i, j, m \text{ collocation points}$$

on $n_k \times J \times M \times 2$ unknowns ρ . We solve this system with a Newton method and an iteration based on the increment of the number of basis functions. First, we solve a system with only three collocation points for capital and 125 ($125 = 25 \times 5$) points for technology. Then, we use that solution as a guess for a system with more collocation points for capital (with the new coefficients being guessed to be equal to zero) and iterate on the procedure. We stop the iteration when we have 11 polynomials in the capital dimension (therefore, in the last step we solve for $2,750 = 11 \times 25 \times 5 \times 2$ coefficients). The iteration is needed because otherwise the residual function is too cumbersome to allow for direct solution of the 2,750 final coefficients.

2.3.3 Value Function Iteration

Our final solution method is VFI. Since the dynamic algorithm is well known, our presentation is most brief. Consider the following Bellman operator:

$$\begin{aligned} TV(k_t, z_t, \sigma_t) &= \max_{c_t, l_t} \left[(1 - \beta) (c_t^v (1 - l_t)^{1-v})^{\frac{1-\gamma}{\theta}} + \beta (\mathbb{E}_t V^{1-\gamma}(k_{t+1}, z_{t+1}, \sigma_{t+1}))^{\frac{1}{\theta}} \right]^{\frac{\theta}{1-\gamma}} \\ \text{s.t. } c_t + k_{t+1} &= e^{z_t} k_t^\zeta l_t^{1-\zeta} + (1 - \delta) k_t \\ z_t &= \lambda z_{t-1} + e^{\sigma_t} \varepsilon_t, \varepsilon_t \sim \mathcal{N}(0, 1) \\ \sigma_t &= (1 - \rho) \bar{\sigma} + \rho \sigma_{t-1} + \eta \omega_t, \omega_t \sim \mathcal{N}(0, 1). \end{aligned}$$

To solve for this Bellman operator, we define a grid on capital, $G_k = \{k_1, k_2, \dots, k_M\}$, a grid on volatility and on the productivity level. The grid on capital is just a uniform distribution of points over the capital dimension. As we did for projection, we set a grid $G_\sigma = \{e^{\sigma_1}, e^{\sigma_2}, \dots, e^{\sigma_M}\}$ for σ_t and a transition matrix Π^M for volatility and M grids $G_z^m = \{z_1^m, z_2^m, \dots, z_J^m\}$ for z_t and transition matrixes $\Pi^{J,m}$ using Tauchen's (1986) procedure. The algorithm to iterate on the value function for this grid is:

1. Set $n = 0$ and $V^0(k_t, z_t, \sigma_t) = c_{ss}^v (1 - l_{ss})^{1-v}$ for all $k_t \in G_k$ and all $z_t \in G_z$.

2. For every $\{k_t, z_t, \sigma_t\}$, use the Newton method to find c_t^*, l_t^*, k_{t+1}^* that solve:

$$\begin{aligned} c_t &= \frac{v}{1-v} (1-\zeta) e^{z_t} k_t^\zeta l_t^{1-\zeta} (1-l_t) \\ (1-\beta) v \frac{(c_t^v (1-l_t)^{1-v})^{\frac{1-\gamma}{\theta}}}{c_t} &= \beta \left(\mathbb{E}_t (V_{t+1}^n)^{1-\gamma} \right)^{\frac{1}{\theta}-1} \mathbb{E}_t \left[(V_{t+1}^n)^{-\gamma} V_{1,t+1}^n \right] \\ c_t + k_{t+1} &= e^{z_t} k_t^\zeta l_t^{1-\zeta} + (1-\delta) k_t \end{aligned}$$

3. Construct V^{n+1} from the Bellman equation:

$$V^{n+1} = ((1-\beta)(c_t^{*v} (1-l_t^*)^{1-v})^{\frac{1-\gamma}{\theta}} + \beta (\mathbb{E}_t (V(k_{t+1}^*, z_{t+1}, \sigma_{t+1})^{1-\gamma}))^{\frac{1}{\theta}})^{\frac{\theta}{1-\gamma}}$$

4. If $\frac{|V^{n+1}-V^n|}{|V^n|} \geq 1.0e^{-7}$, then $n = n + 1$ and go to 2. Otherwise, stop.

To accelerate convergence and give VFI a fair chance, we implement a multigrid scheme as described by Chow and Tsitsiklis (1991). We start by iterating on a small grid. Then, after convergence, we add more points to the grid and recompute the Bellman operator using the previously found value function as an initial guess (with linear interpolation to fill the unknown values in the new grid points). Since the previous value function is an excellent grid, we quickly converge in the new grid. Repeating these steps several times, we move from an initial 23,000-point grid into a final one with 375,000 points (3,000 points for capital, 25 for productivity, and 5 for volatility).

2.4 Calibration

We now select a benchmark calibration for our numerical computations. We follow the literature as closely as possible and select parameter values to match, in the steady state, some basic observations of the U.S. economy (as we will see below, for the benchmark calibration, the means of the ergodic distribution and the steady-state values are nearly identical). We set the discount factor $\beta = 0.991$ to generate an annual interest rate of around 3.6 percent. We set the parameter that governs labor supply, $\theta = 0.357$, to get the representative household to work one-third of its time. The elasticity of output to capital, $\zeta = 0.3$, matches the labor share of national

income. A value of the depreciation rate $\delta = 0.0196$ matches the ratio of investment-output. Finally, $\lambda = 0.95$ and $\log \bar{\sigma} = 0.007$ are standard values for the stochastic properties of the Solow residual. For the SV process, we pick $\rho = 0.9$ and $\eta = 0.06$, to match the persistence and standard deviation of the heteroskedastic component of the Solow residual during the last 5 decades.

Since we want to explore the dynamics of the model for a range of values that encompasses all the estimates from the literature, we select four values for the parameter that controls risk aversion, γ , 2, 5, 10, and 40, and two values for EIS ψ , 0.5, and 1.5, which bracket most of the values used in the literature (although many authors prefer smaller values for ψ , we found that the simulation results for smaller ψ 's do not change much from the case when $\psi = 0.5$). We then compute the model for all eight combinations of values of γ and ψ , that is $\{2, 0.5\}$, $\{5, 0.5\}$, $\{10, 0.5\}$, and so on. When $\psi = 0.5$ and $\gamma = 2$, we are back in the standard CRRA case. However, in the interest of space, we will report only a limited subset of results that we find are the most interesting ones.

We pick as the benchmark case the calibration $\{\gamma, \psi, \log \bar{\sigma}, \eta\} = \{5, 0.5, 0.007, 0.06\}$. These values reflect an EIS centered around the median of the estimates in the literature, a reasonably high level of risk aversion, and the observed volatility of productivity shocks. To check robustness, we increase, in the extreme case, the risk aversion, the average standard deviation of the productivity shock, and the standard deviation of the innovations to volatility to $\{\gamma, \psi, \log \bar{\sigma}, \eta\} = \{40, 0.5, 0.021, 0.1\}$. This case combines levels of risk aversion that are in the upper bound of all estimates in the literature with huge productivity shocks. Therefore, it pushes all solution methods to their limits, in particular, making life hard for perturbation since the interaction of the large precautionary behavior induced by γ and large shocks will move the economy far away from the deterministic steady state. We leave the discussion of the effects of $\psi = 1.5$ for the robustness analysis at the end of the next section.

2.5 Numerical Results

In this section we report our numerical findings. First, we present and discuss the computed decision rules. Second, we show the results of simulating the model. Third, we report the Euler equation errors. Fourth, we discuss the effects of changing the EIS and the perturbation point. Finally, we discuss implementation and computing

time.

2.5.1 Decision Rules

One of our first results is the decision rules and the value function of the agent. Figure 2.1 plots the decision rules for consumption, labor supply, capital, and the value function in the benchmark case when $z_t = 0$ and $\sigma_t = \bar{\sigma}$ computed over a capital interval centered on the steady-state level of capital of 9.54 with a width of $\pm 40\%$, [5.72,13.36]. We selected an interval for capital big enough to encompass all the simulations in our sample. Similar figures could be plotted for other values of z_t and σ_t . We omit them because of space considerations.

Since all methods provide nearly indistinguishable answers, we observe only one line in all figures. It is possible to appreciate very tiny differences in labor supply between second-order perturbation and the other methods only when capital is far from its steady-state level. Monotonicity of the decision rules is preserved by all methods. We must be cautious, however, mapping differences in choices into differences in utility. The Euler error function below provides a better view of the welfare consequences of different approximations.

We see bigger differences in the decision rules and value functions as we increase the risk aversion and the variance of innovations. Figure 2.2 plots the decision rules and value functions for the extreme calibration. In this figure, we change the interval where we compute our decision rules to [3,32] (roughly 1/3 and 3 times the steady-state capital) because, owing to the high variance of the calibration, the equilibrium paths fluctuate through much wider ranges of capital.

We highlight several results. First, all the methods deliver similar results in our original interval for the benchmark calibration. Second, as we go far away from the steady state, VFI and the Chebyshev polynomial still overlap with each other (and, as shown by our Euler error computations below, we can roughly take them as the “exact” solution), but second- and third-order approximations start to deviate. Third, the decision rule for consumption and the value function approximated by the third-order perturbation changes from concavity into convexity for values of capital bigger than 15. This phenomenon (also documented in Aruoba *et al.* 2006) is due to the poor performance of local approximation when we move too far away from the expansion point and the polynomials begin to behave wildly. In any case, this issue

is irrelevant because, as we will show below, the problematic region is visited with nearly zero probability.

2.5.2 Simulations

Applied economists often characterize the behavior of the model through statistics from simulated paths of the economy. We simulate the model, starting from the deterministic steady state, for 10,000 periods, using the decision rules for each of the eight combinations of risk aversion and EIS discussed above. To make the comparison meaningful, the shocks are common across all paths. We discard the first 1,000 periods as a burn-in to eliminate the transition from the deterministic steady state of the model to the middle regions of the ergodic distribution of capital. This is usually achieved in many fewer periods than the ones in our burn-in, but we want to be conservative in our results. The remaining observations constitute a sample from the ergodic distribution of the economy.

For the benchmark calibration, the simulations from all of the solution methods generate almost identical equilibrium paths (and therefore we do not report them). We focus instead on the densities of the endogenous variables as shown in figure 3. Given the low risk aversion and SV of the productivity shocks, all densities are roughly centered around the deterministic steady-state value of the variable. For example, the mean of the distribution of capital is only 0.2 percent higher than the deterministic value. Also, capital is nearly always between 8.5 and 10.5. This range will be important below to judge the accuracy of our approximations.

Table 2.2 reports business cycle statistics and, because DSGE models with recursive preferences and SV are often used for asset pricing, the average and variance of the (quarterly) risk-free rate and return on capital. Again, we see that nearly all values are the same, a simple consequence of the similarity of the decision rules.

The welfare cost of the business cycle is reported in table 2.3 in consumption equivalent terms. The computed costs are actually negative. Besides the Jensen's effect on average productivity, this is also due to the fact that when we have leisure in the utility function, the indirect utility function may be convex in input prices (agents change their behavior over time by a large amount to take advantage of changing productivity). Cho and Cooley (2000) present a similar example. Welfare costs are comparable across methods. Remember that the welfare cost of the business

cycle for the second- and third-order perturbations is the same because the third-order terms all drop or are zero when evaluated at the steady state.

When we move to the extreme calibration, we see more differences. Figure 2.4 plots the histograms of the simulated series for each solution method. Looking at quantities, the histograms of consumption, output, and labor are the same across all of the methods. The ergodic distribution of capital puts nearly all the mass between values of 6 and 15. This considerable move to the right in comparison with figure 3 is due to the effect of precautionary behavior in the presence of high risk aversion, large productivity shocks, and high SV. Capital also visits low values of capital more than in the benchmark calibration because of large, persistent productivity shocks. In any case, the translation is more pronounced to the right than to the left.

Table 2.4 reports business cycle statistics. Differences across methods are minor in terms of means (note that the mean of the risk-free rate is lower than in the benchmark calibration because of the extra accumulation of capital induced by precautionary behavior). In terms of variances, the second-order perturbation produces less volatility than all other methods. This suggests that a second-order perturbation may not be good enough if we face high variance of the shocks and/or high risk aversion. A third-order perturbation, in comparison, eliminates most of the differences and delivers nearly the same implications as Chebyshev polynomials or VFI.

Finally, table 2.5 presents the welfare cost of the business cycle. Now, in comparison with the benchmark calibration, the welfare cost of the business cycle is positive and significant, slightly above 1.1 percent. This is not a surprise, since we have both a large risk aversion and productivity shocks with an average standard deviation three times as big as the observed one. All methods deliver numbers that are close.

2.5.3 Euler Equation Errors

While the plots of the decision rules and the computation of densities and business cycle statistics that we presented in the previous subsection are highly informative, it is also important to evaluate the accuracy of each of the procedures. Euler equation errors, introduced by Judd (1992), have become a common tool for determining the quality of the solution method. The idea is to observe that, in our model, the

intertemporal condition:

$$u_{c,t} = \beta(\mathbb{E}_t V_{t+1}^{1-\gamma})^{\frac{1}{\theta}-1} \mathbb{E}_t \left(V_{t+1}^{\frac{(\gamma-1)(1-\theta)}{\theta}} u_{c,t+1} R(k_t, z_t, \sigma_t; z_{t+1}, \sigma_{t+1}) \right) \quad (2.4)$$

where $R(k_t, z_t, \sigma_t; z_{t+1}, \sigma_{t+1}) = 1 + \zeta e^{z_{t+1}} k_{t+1}^{\zeta-1} l_{t+1}^{1-\zeta} - \delta$ is the gross return of capital given states k_t, z_t, σ_t , and realizations z_{t+1} and σ_{t+1} should hold exactly for any given k_t , and z_t . However, since the solution methods we use are only approximations, there will be an error in (2.4) when we plug in the computed decision rules. This Euler equation error function $EE^i(k_t, z_t, \sigma_t)$ is defined, in consumption terms:

$$EE^i(k_t, z_t, \sigma_t) = 1 - \frac{\left[\frac{\beta(\mathbb{E}_t (V_{t+1}^i)^{1-\gamma})^{\frac{1}{\theta}-1} \mathbb{E}_t \left((V_{t+1}^i)^{\frac{(\gamma-1)(1-\theta)}{\theta}} u_{c,t+1}^i R(k_t, z_t, \sigma_t; z_{t+1}, \sigma_{t+1}) \right)}{\frac{1-\gamma}{\theta} v (1-l_t^i)^{(1-v)} \frac{1-\gamma}{\theta}} \right]^{\frac{1}{v \frac{1-\gamma}{\theta} - 1}}}{c_t^i}$$

This function determines the (unit free) error in the Euler equation as a fraction of the consumption given the current states and solution method i . Following Judd and Guu (1997), we can interpret this error as the optimization error incurred by the use of the approximated decision rule and we report the absolute errors in base 10 logarithms to ease interpretation. Thus, a value of -3 means a \$1 mistake for each \$1000 spent, a value of -4 a \$1 mistake for each \$10,000 spent, and so on.

Figure 2.5 displays a transversal cut of the errors for the benchmark calibration when $z = 0$ and $\sigma_t = \bar{\sigma}$. Other transversal cuts at different technology and volatility levels reveal similar patterns. The first lesson from figure 5 is that all methods deliver high accuracy. We know from figure 3 that capital is nearly always between 8.5 and 10.5. In that range, the (log10) Euler equation errors are at most -5, and most of the time they are even smaller. For instance, the second- and third-order perturbations have an Euler equation error of around -7 in the neighborhood of the deterministic steady state, VFI of around -6.5, and Chebyshev an impressive -11/-13. The second lesson from figure 5 is that, as expected, global methods (Chebyshev and VFI) perform very well in the whole range of capital values, while perturbations deteriorate as we move away from the steady state. For second-order perturbation, the Euler error in the steady state is almost four orders of magnitude smaller than on the boundaries. Third-order perturbation is around half an order of magnitude more accurate than

second-order perturbation over the whole range of values (except in a small region close to the deterministic steady state).

There are two complementary ways to summarize the information from Euler equation error functions. First, in the second column of table 2.6 , we report the maximum error in our interval (capital between 60 percent and 140 percent of the steady state and the grids for productivity and volatility). The maximum Euler error is useful because it bounds the mistake owing to the approximation. Both perturbations have a maximum Euler error of around -2.7, VFI of -3.1, and Chebyshev, an impressive -9.8. We read this column as indicating that all methods perform adequately. The second procedure for summarizing Euler equation errors is to integrate the function with respect to the ergodic distribution of capital and productivity to find the average error.⁶ We can think of this exercise as a generalization of the Den Haan–Marcet test (Den Haan and Marcet, 1994). We report our results in the third column of table 2.6 . Both perturbations have roughly the same performance (around -5.3), VFI a slightly better -6.4, while Chebyshev polynomials do fantastically well at -10.4 (the average loss of welfare is \$1 for each \$500 billion). But even an approximation with an average error of \$1 for each \$200,000, such as the one implied by third-order perturbation, must suffice for most relevant applications.

We repeat our exercise for the extreme calibration. Figure 2.6 displays the results for the extreme case. As we did when we computed the decision rules of the agents, we have changed the capital interval to [3,32]. Now, perturbations worsen more as we get further away from the deterministic steady state. However, in the relevant range of values of capital of [6,17], where, as reported in figure 4, nearly all the mass of the ergodic distribution is, we still have Euler equation errors smaller than -3 and, hence, probably small enough for most applications of interest. The performance of VFI deteriorates around one order of magnitude with respect to our benchmark calibration. Chebyshev polynomials suffer more in relative terms (they started at a quite outstanding accuracy level), but they still deliver the smallest errors in nearly all the relevant range of capital.

Table 2.7 reports maximum Euler equation errors and their integrals. The max-

⁶There is the technical consideration of which ergodic distribution to use for this task, since this is an object that can only be found by simulation. We use the ergodic simulation generated by VFI, which slightly favors this method over the other ones. However, we checked that the results are robust to using the ergodic distributions coming from the other methods.

imum Euler equation error is large for perturbation methods while it is rather small using Chebyshev polynomials. However, given the very large range of capital used in the computation, this maximum Euler error provides a too negative view of accuracy. We find the integral of the Euler equation error to be much more instructive. With a second-order perturbation, we have -4.02 and with a third-order perturbation we have -4.12. To evaluate this number, remember that we have extremely high risk aversion and large productivity shocks. Even in this challenging environment, perturbations deliver a high degree of accuracy. VFI does not display a big loss of precision compared to the benchmark case. On the other hand, Chebyshev polynomials deteriorate somewhat, but the accuracy it delivers it is still of \$1 out of each \$1 million spent.

2.5.4 Robustness: Changing the EIS and Changing the Perturbation Point

In the results we reported above, we kept the EIS equal to 0.5, a conventional value in the literature, while we modified the risk aversion and the volatility of productivity shocks. However, since some researchers prefer higher values of the EIS (see, for instance, Bansal and Yaron, 2004, a paper that we have used to motivate our investigation), we also computed our model with $\psi = 1.5$. Basically our results were unchanged. To save on space, we concentrate only on the Euler equation errors (decision rules and simulation paths are available upon request). In table ??, we report the maxima of the Euler equation errors and their integrals with respect to the ergodic distribution. The relative size and values of the entries in this table are quite similar to the entries in table 2.6 (except, partially, VFI that performs a bit better).

Table ?? repeats the same exercise for the extreme calibration. Again, the entries in the table are very close to the ones in table 2.7 (and now, VFI does not perform better than when $\psi = 0.5$).

As a final robustness test, we computed the perturbations not around the deterministic steady state (as we did in the main text), but around a point close to the mode of the ergodic distribution of capital. This strategy, if perhaps difficult to implement because of the need to compute the mode of the ergodic distribution,⁷ could

⁷For example, the algorithm of finding a perturbation around the steady state, simulate from it, find a second perturbation around the model of the implied ergodic simulation, and so on until convergence, may not settle in any fixed point. In our exercise, we avoid this problem because we

deliver better accuracy because we approximate the value function and decision rules in a region where the model spends more time. As we suspected, we found only trivial improvements in terms of accuracy. Moreover, expanding at a point different from the deterministic steady state has the disadvantage that the theorems that ensure the convergence of the Taylor approximation might fail (see theorem 6 in Jin and Judd, 2002).

2.5.5 Implementation and Computing Time

We briefly discuss implementation and computing time. For the benchmark calibration, second-order perturbation and third-order perturbation algorithms take only 0.02 second and 0.05 second, respectively, in a 3.3GHz Intel PC with Windows 7 (the reference computer for all times below), and it is simple to implement: 664 lines of code in Fortran 95 for second order and 1133 lines of code for third order, plus in both cases, the analytical derivatives of the equilibrium conditions that Fortran 95 borrows from a code written in **Mathematica** 6.⁸ The code that generates the analytic derivatives has between 150 to 210 lines, although **Mathematica** is much less verbose. While the number of lines doubles in the third order, the complexity in terms of coding does not increase much: the extra lines are mainly from declaring external functions and reading and assigning values to the perturbation coefficients. An interesting observation is that we only need to take the analytic derivatives once, since they are expressed in terms of parameters and not in terms of parameter values. This allows Fortran to evaluate the analytic derivatives extremely fast for new combinations of parameter values. This advantage of perturbation is particularly relevant when we need to solve the model repeatedly for many different parameter values, for example, when we are estimating the model. For completeness, the second-order perturbation was also run in **Dynare** (although we had to use version 4.0, which computes analytic derivatives, instead of previous versions, which use numerical derivatives that are not accurate enough for perturbation). This run was a double-check of the code

have the ergodic distribution implied by VFI. This is an unfair advantage for perturbations at the mode of the ergodic distribution but it makes our point below about the lack of improvement in accuracy even stronger.

⁸We use lines of code as a proxy for the complexity of implementation. We do not count comment lines.

and a test of the feasibility of using off-the-shelf software to solve DSGE models with recursive preferences.

The projection algorithm takes around 300 seconds, but it requires a good initial guess for the solution of the system of equations. Finding the initial guess for some combination of parameter values proved to be challenging. The code is 652 lines of Fortran 95. Finally, the VFI code is 707 lines of Fortran 95, but it takes about ten hours to run.

2.6 Conclusions

In this paper, we have compared different solution methods for DSGE models with recursive preferences and SV. We evaluated the different algorithms based on accuracy, speed, and programming burden. We learned that all of the most promising methods (perturbation, projection, and VFI) do a fair job in terms of accuracy. We were surprised by how well simple second-order and third-order perturbations perform even for fairly non-linear problems. We were impressed by how accurate Chebyshev polynomials can be. However, their computational cost was higher and we are concerned about the curse of dimensionality. In any case, it seems clear to us that, when accuracy is the key consideration, Chebyshev polynomials are the way to go. Finally, we were disappointed by VFI since even with 125,000 points in the grid, it only did marginally better than perturbation and it performed much worse than Chebyshev polynomials in our benchmark calibration. This suggests that unless there are compelling reasons such as non-differentiabilities or non-convexities in the model, we better avoid VFI.

A theme we have not developed in this paper is the possibility of interplay among different solution methods. For instance, we can compute extremely easily a second-order approximation to the value function and use it as an initial guess for VFI. This second-order approximation is such a good guess that VFI will converge in few iterations. We verified this idea in non-reported experiments, where VFI took one-tenth of the time to converge once we used the second-order approximation to the value function as the initial guess. This approach may even work when the true value function is not differentiable at some points or has jumps, since the only goal of perturbation is to provide a good starting point, not a theoretically sound approximation. This algorithm may be particularly useful in problems with many

state variables. More research on this type of hybrid method is a natural extension of our work.

We close the paper by pointing out that recursive preferences are only one example of a large class of non-standard preferences that have received much attention by theorists and applied researchers over the last several years (see Backus, Routledge, and Zin, 2004). Having fast and reliable solution methods for this class of new preferences will help researchers to sort out which of these preferences deserve further attention and to derive empirical implications. Thus, this paper is a first step in the task of learning how to compute DSGE models with non-standard preferences.

2.7 Appendices

2.7.1 Steady State of the Model

To solve the system:

$$\begin{aligned}
 V_{ss} &= c_{ss}^v (1 - l_{ss})^{1-v} \\
 (\zeta k_{ss}^{\zeta-1} l_{ss}^{1-\zeta} + 1 - \delta) &= 1/\beta \\
 \frac{1-\nu}{\nu} \frac{c_{ss}}{(1-l_{ss})} &= (1-\zeta) k_{ss}^{\zeta} l_{ss}^{-\zeta} \\
 m_{ss} R_{ss}^f &= 1/\beta \\
 c_{ss} + i_{ss} &= k_{ss}^{\zeta} l_{ss}^{1-\zeta} \\
 i_{ss} &= \delta k_{ss}
 \end{aligned}$$

note first that:

$$\frac{k_{ss}}{l_{ss}} = \left(\frac{1}{\zeta} \left(\frac{1}{\beta} - 1 + \delta \right) \right)^{\frac{1}{\zeta-1}} = \Omega$$

Now, from the leisure-consumption condition:

$$\frac{c_{ss}}{1-l_{ss}} = \frac{v}{1-v} (1-\zeta) \Omega^{\zeta} = \Phi \Rightarrow c_{ss} = \Phi (1-l_{ss})$$

Then:

$$c_{ss} + \delta k_{ss} = k_{ss}^{\zeta} l_{ss}^{1-\zeta} = \Omega^{\zeta} l_{ss} \Rightarrow c_{ss} = (\Omega^{\zeta} - \delta \Omega) l_{ss}$$

and:

$$\begin{aligned}
 \Phi (1-l_{ss}) &= (\Omega^{\zeta} - \delta \Omega) l_{ss} \Rightarrow \\
 l_{ss} &= \frac{\Phi}{\Omega^{\zeta} - \delta \Omega + \Phi} \\
 k_{ss} &= \frac{\Phi \Omega}{\Omega^{\zeta} - \delta \Omega + \Phi}
 \end{aligned}$$

from which we can find V_{ss} and i_{ss} .

2.7.2 Value Function Perturbation (VFP)

We mentioned in the main text that instead of perturbing the equilibrium conditions of the model, we could directly perturb the value function in what we called value function perturbation (VFP). To undertake the VFP, we write the value function as:

$$V(k_t, z_t, \sigma_t; \chi) = \max_{c_t, l_t} \left[(1 - \beta) (c_t^v (1 - l_t)^{1-v})^{\frac{1-\gamma}{\theta}} + \beta \mathbb{E}_t V^{1-\gamma}(k_{t+1}, z_{t+1}, \sigma_{t+1}; \chi)^{\frac{1}{\theta}} \right]^{\frac{\theta}{1-\gamma}}$$

To find a second-order approximation to the value function, we take derivatives of the value function with respect to controls (c_t, l_t) , states (k_t, z_t, σ_t) , and the perturbation parameter χ . We collect these 6 equations, together with the resource constraint, the value function itself, and the exogenous processes in a system:

$$\tilde{F}(k_t, z_t, \chi) = \mathbf{0}$$

where the hat over F emphasizes that now we are dealing with a slightly different set of equations than the F in the main text.

After solving for the steady state of this system, we take derivatives of the function $\tilde{F}$ with respect to k_t, z_t, σ_t , and χ :

$$\tilde{F}_i(k_{ss}, 0, \sigma_{ss}; 0) = \mathbf{0} \text{ for } i = \{1, 2, 3, 4\}$$

and we solve for the unknown coefficients. This solution will give us a second-order approximation of the value function but only a first-order approximation of the decision rules. By repeating these steps n times, we can obtain the $n + 1$ -order approximation of the value function and the n -order approximation of the decision rules. It is straightforward to check that the coefficients obtained by ECP and VFP are the same. Thus, the choice for one approach or the other should be dictated by expediency.

2.7.3 Tables and Figures

Table 2.1: Calibrated Parameters

Parameter	β	v	ζ	δ	λ	$\log \bar{\sigma}$	ρ	η
Value	0.991	0.357	0.3	0.0196	0.95	0.007	0.9	0.06

Table 2.2: Business Cycle Statistics - Benchmark Calibration

	c	y	i	$R^f(\%)$	$R^k(\%)$
	Mean				
Second-Order Perturbation	0.7253	0.9128	0.1873	0.9070	0.9078
Third-Order Perturbation	0.7257	0.9133	0.1875	0.9062	0.9069
Chebyshev Polynomial	0.7256	0.9130	0.1875	0.9063	0.9066
Value Function Iteration	0.7256	0.9130	0.1875	0.9063	0.9066
	Variance (%)				
Second-Order Perturbation	0.0331	0.1084	0.0293	0.0001	0.0001
Third-Order Perturbation	0.0330	0.1079	0.0288	0.0001	0.0001
Chebyshev Polynomial	0.0347	0.1117	0.0313	0.0001	0.0001
Value Function Iteration	0.0347	0.1117	0.0313	0.0001	0.0001

Table 2.3: Welfare Costs of Business Cycle - Benchmark Calibration

2nd-Order Pert.	3rd-Order Pert.	Chebyshev	Value Function
-2.0864e(-5)	-2.0864e(-5)	-3.2849e(-5)	-3.2849e(-5)

Table 2.4: Business Cycle Statistics - Extreme Calibration

	c	y	i	$R^f(\%)$	$R^k(\%)$
	Mean				
Second-Order Perturbation	0.7338	0.9297	0.1950	0.8432	0.8562
Third-Order Perturbation	0.7344	0.9311	0.1955	0.8416	0.8529
Chebyshev Polynomial	0.7359	0.9329	0.1970	0.8331	0.8402
Value Function Iteration	0.7359	0.9329	0.1970	0.8352	0.8403
	Variance (%)				
Second-Order Perturbation	0.2956	1.0575	0.2718	0.0004	0.0004
Third-Order Perturbation	0.3634	1.2178	0.3113	0.0004	0.0005
Chebyshev Polynomial	0.3413	1.1523	0.3425	0.0005	0.0006
Value Function Iteration	0.3414	1.1528	0.3427	0.0005	0.0006

Table 2.5: Welfare Costs of Business Cycle - Extreme Calibration

2nd-Order Pert.	3rd-Order Pert.	Chebyshev	Value Function
1.1278e-2	1.1278e-2	1.2855e-2	1.2838e-2

Table 2.6: Euler errors - Benchmark Calibration

	Max Euler Error	Integral of the Euler Errors
Second-Order Perturbation	-2.6294	-5.2350
Third-Order Perturbation	-2.7437	-5.3164
Chebyshev Polynomial	-9.7919	-10.4034
Value Function Iteration	-3.0848	-6.4039

Table 2.7: Euler errors - Extreme Calibration

	Max Euler Error	Integral of the Euler Errors
Second-Order Perturbation	-1.5188	-4.0195
Third-Order Perturbation	-1.6698	-4.1189
Chebyshev Polynomial	-4.8979	-5.9339
Value Function Iteration	-2.5186	-6.2870

Figure 2.1: Decision Rules and Value Function, benchmark calibration

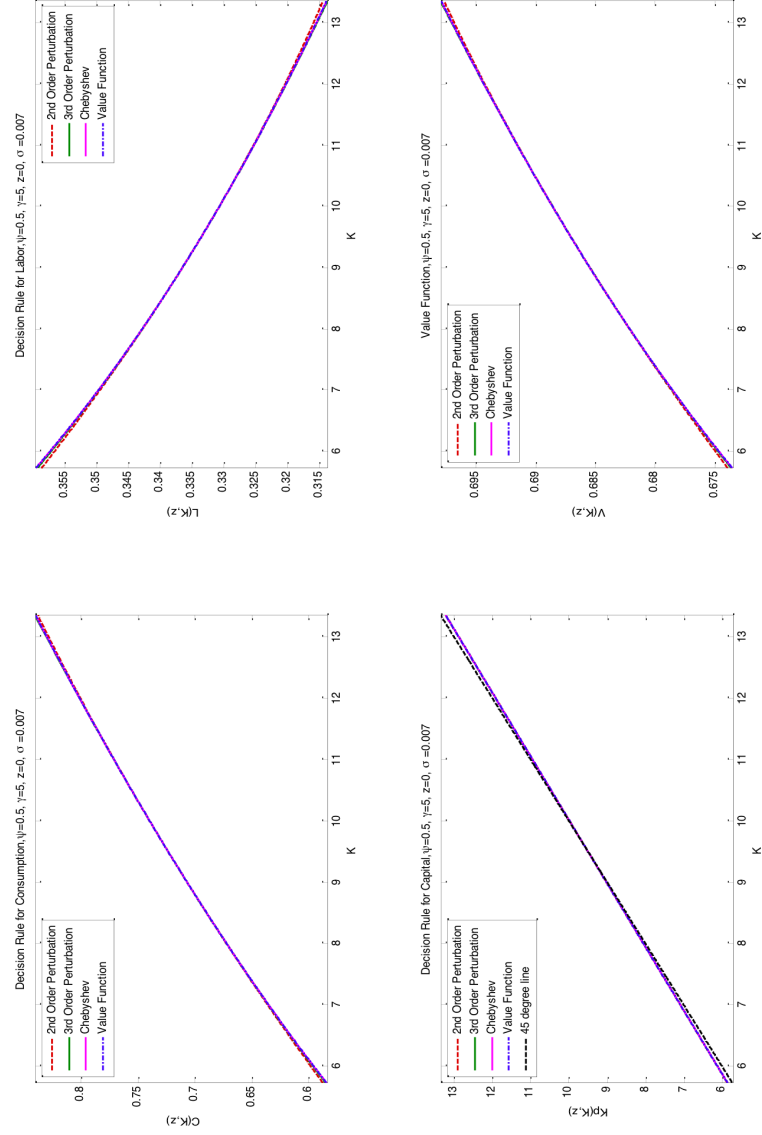


Figure 2.2: Decision Rules and Value Function, extreme calibration

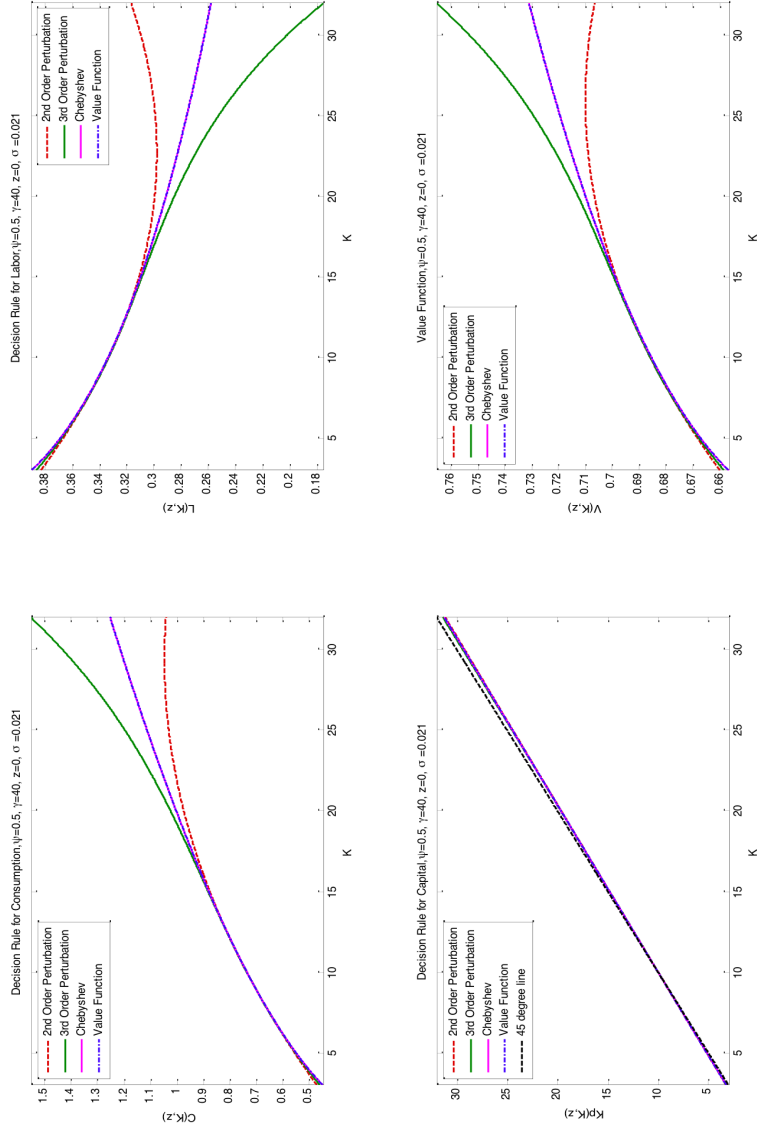


Figure 2.3: Densities, benchmark calibration

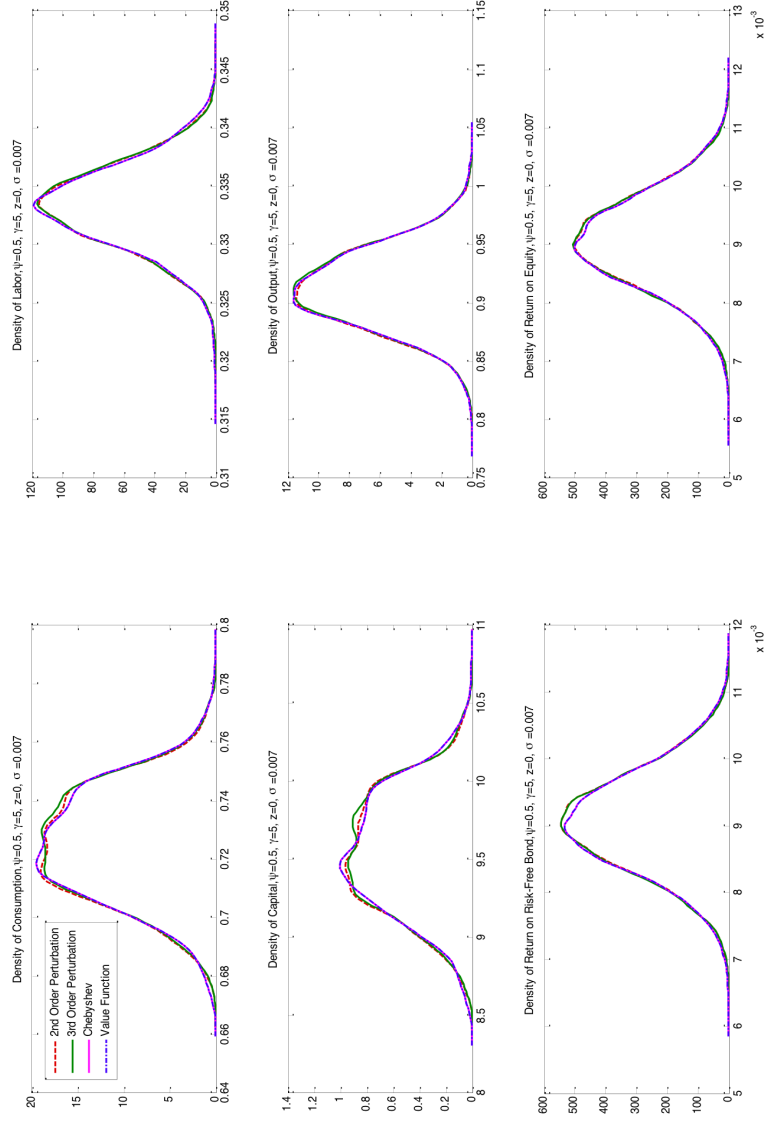


Figure 2.4: Densities, extreme calibration

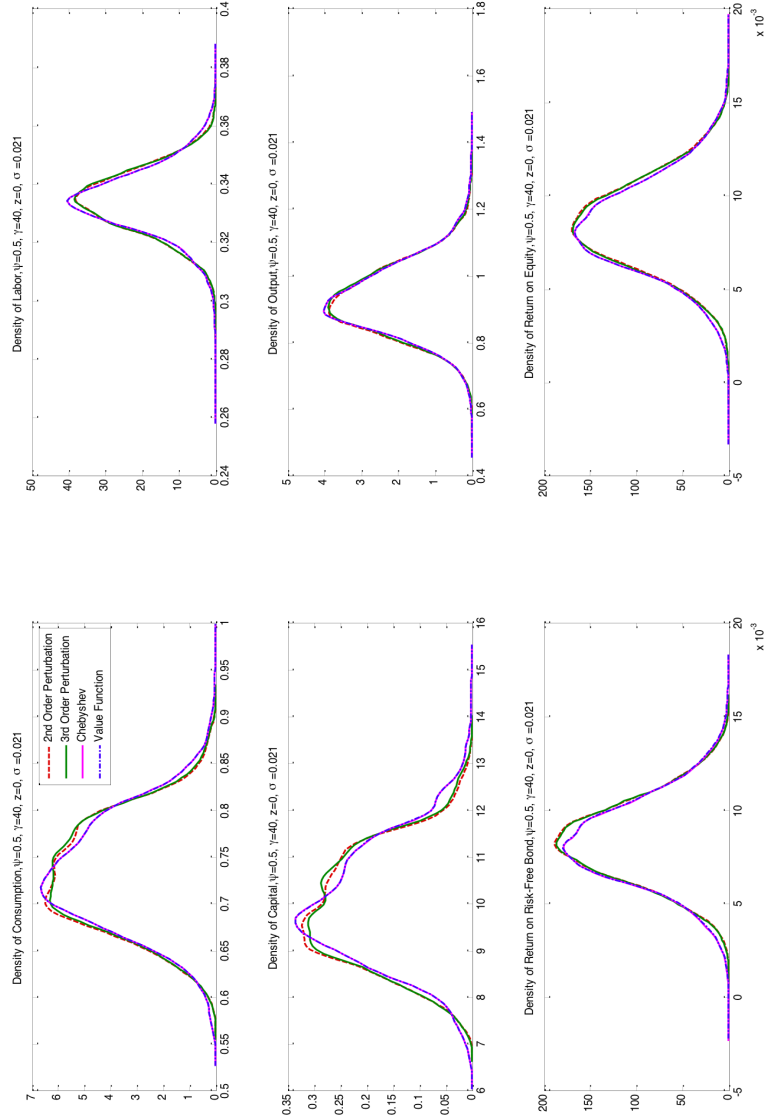


Figure 2.5: Euler Equation Error, benchmark calibration

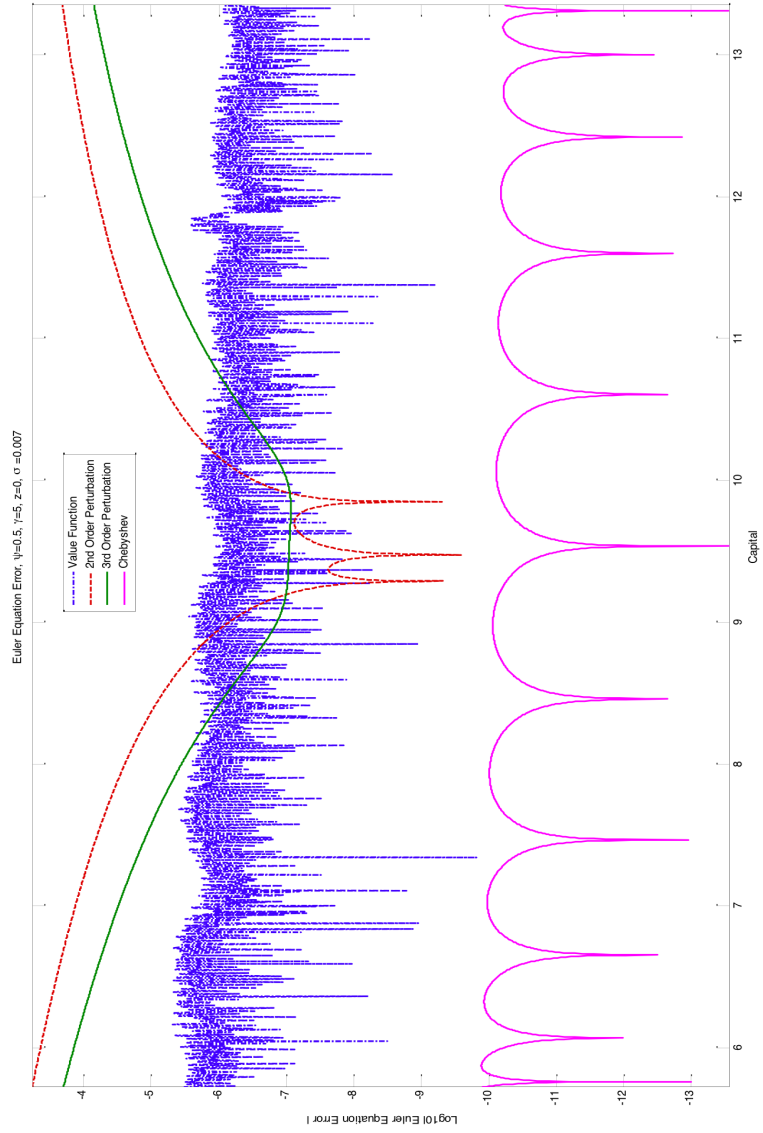
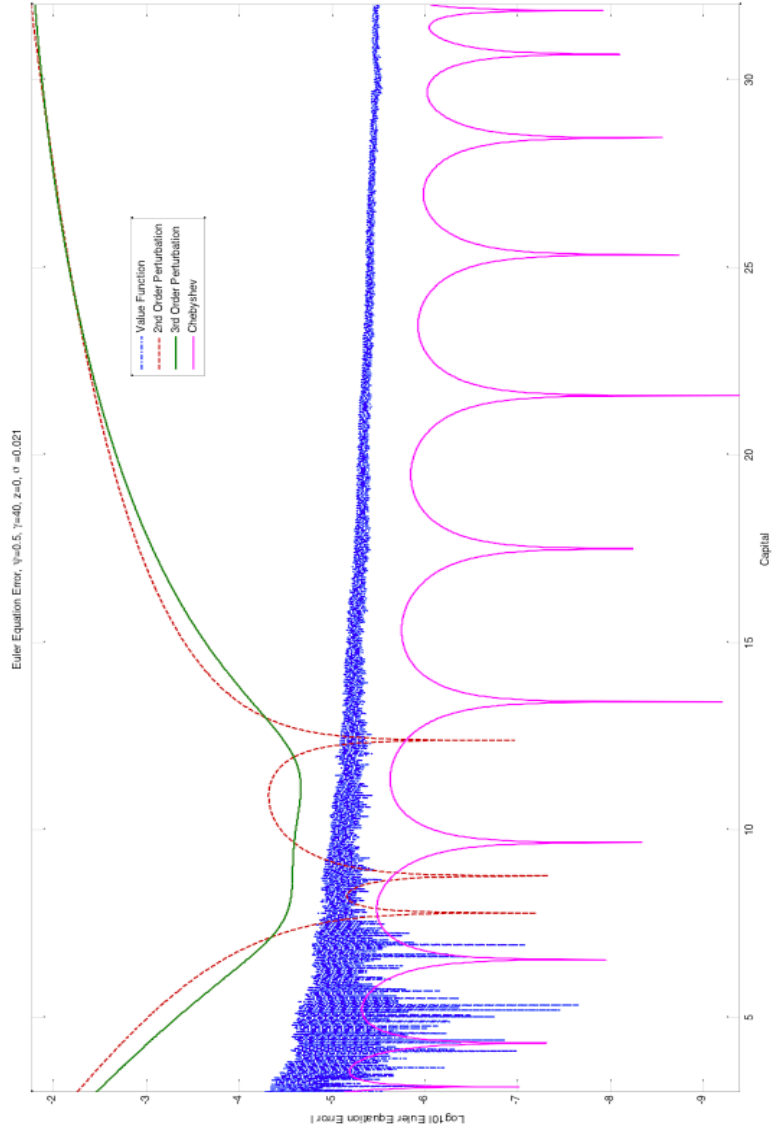


Figure 2.6: Euler Equation Errors, extreme calibration



Chapter 3

International Business Cycles and Financial Frictions

3.1 Introduction

The question this paper addresses is the quantitative impact of financial frictions on the business cycle co-movements between countries when investors have foreign asset exposure. The breakout and spread of the 2007 financial crisis highlights the importance of financial frictions for international business cycle co-movements: European investors who were exposed to US mortgage-backed securities experienced a fall in their net worth when the US market collapsed. The decline in net worth tightened their leverage constraint and led to a contraction in investment activities in Europe. To analyze this mechanism, this paper embeds this type of financial friction within an international real business cycle model and concludes that the presence of financial frictions helps the model do a better job of accounting for the correlations of output, investment and employment in the data. In addition, the model also shows that as foreign asset exposure increases, business cycles become more synchronized.

I build a two-country model where credit contracts are imperfectly enforceable and business cycles are driven by technology shocks. Each country has two types of agents: investor and saver. The investor holds both domestic and foreign capital. She receives risky returns by renting her capital to the market production firm. She also borrows from the domestic saver to finance her capital holdings. Because the investor cannot promise to repay her loans, she faces a leverage constraint that limits her loans to be smaller than a portion of the market value of her total capital holdings. The saver makes use of the domestic capital in home production and lends her savings to the investor. Both agents work at the market production firm. Since I am interested in evaluating business cycle implications quantitatively, I model explicitly endogenous labor supply and capital accumulation. These ingredients are important for two reasons. First, variation in hours contributes to most of the business cycle fluctuations. Second, financial frictions can generate a large amplification effect when capital is fixed. Introducing capital accumulation disciplines the exercise empirically.

The financial frictions and foreign asset exposure in this model together generate a multiplier effect that amplifies the transmission of shocks across countries. Output correlation across countries is driven up through this financial channel. When a negative technology shock hits the domestic market, the demand for capital in the home country falls, which forces down the price of domestic capital. The price decline leads to a tightening of investors' leverage constraint in both countries. Borrowing is

reduced globally and therefore demand for capital in the foreign country also declines. Prices of foreign assets fall, triggering another round of decline in investment and output. A multiplier effect arises since the decline in investment lowers asset prices and investors' net worth, further pushing down investment. With the presence of the financial frictions and foreign asset exposure, the shock spills over from one country to another and thus drives up the business cycle correlations.

To judge the empirical relevance of my framework, I conduct a quantitative exercise aimed at exploring whether the existence of financial frictions can improve the model's ability to account for cross-country correlations of output, employment and investment. I calibrate the model to match the data from the US and the rest of the industrial world. The model is then solved using an iterative second-order perturbation method developed by Heathcote and Perri (2009). This is because when agents have multiple assets, in the steady-state where risk is absent, the returns on the assets are the same. Therefore the portfolio shares are not determinate and we need to use information from higher-order perturbation to pin down steady-state portfolios.

The main findings of the paper are the following. First, the simulation result shows that the presence of financial frictions together with foreign asset exposure improves the business cycle co-movements along several dimensions: the calibrated model produces positive and sizable correlations of output, investment and employment. The model produces an output correlation of 0.4, which accounts for two-thirds of the output correlation in the data. The model also indicates an employment correlation of 0.41, which is close to 0.43 in the data. Moreover, the model predicts a positive investment correlation of 0.64, which is closer to the data than the model without financial frictions. Compared to the previous literature which tends to predict either negative or positive but relatively small business cycle correlations, this model makes good progress by taking financial frictions into account.

Second, substantial differences exist in impulse response functions between versions of the model with and without financial frictions. Let me take the IRFs for hours as an example; other IRFs will be discussed later in the main text. When the leverage constraint is present, after a decline in productivity in country 1, hours fall in both countries. Hours fall in country 1 because of lower wages. Hours fall in country 2 because of the leverage constraint. Since the fall in productivity leads to a decline in the asset price in country 1, which tightens the leverage constraint of country 2's investor, capital used in country 2's production is reduced. Hence hours in country 2

also fall. However, in the case where financial frictions are absent, when productivity in country 1 falls, country 1's hours decline but country 2's hours increase because country 2 is relatively more productive.

Third, this model also predicts that when the investor increases her foreign asset exposure to the other country, the output correlation between the two countries increases. This result is consistent with the evidence documented in Imbs (2006) that output correlations rise with financial integration.

This paper is related to several strands of the literature. The first strand addresses the co-movements of international business cycles. Backus, Kehoe, and Kydland (1992) showed that in a complete market model, output, investment and labor are negatively correlated because of efficient allocation of resources across countries. Baxter and Crucini (1995), Kollmann (1996), and Heathcote and Perri (2002) introduced incomplete markets. However, they find that incomplete markets do not help much in matching the business cycle correlations in the data, because there is little need for insurance markets.

The second strand is a recent and growing literature analyzing financial frictions in an open economy context, including Gertler, Gilchrist, and Natalucci (2007), Faia (2007) and Devereux and Yetman (2010). Gertler, Gilchrist, and Natalucci (2007) builds a small open economy model with credit frictions to explore the connection between the exchange rate regime and financial distress in the case of the 1997 Korea crisis. Faia (2007) studies financial frictions in a two-country DSGE model showing that business cycle synchronization increases when economies have similar financial structures, while it decreases with the degree of financial openness. However, these two papers and the previous literature did not study the impact of financial frictions when the constrained agents have foreign capital exposure.

The paper by Devereux and Yetman (2010) is the closest to my work in that it studies financial frictions and capital portfolio choice in a two-country model. In contrast to my paper, their model lacks capital accumulation and endogenous labor choice, which are the key ingredients for business cycle fluctuations.

The third strand is the international portfolio choice literature, pioneered by Tille and Van Wincoop (2007) and Devereux and Sutherland (2008) with a recent contribution by Heathcote and Perri (2009). This literature uses higher order perturbation to solve optimal portfolio allocations in DSGE models.

The paper is organized as follows. In Section 2, I describe the model economy,

highlight the key mechanism and show how to solve this model. In Section 3, I discuss the calibration of the model. In Section 4, I present the main results. I compare the results from a model with financial frictions and a model without financial frictions. I also provide some intuition for the results. In Section 5, I provide several robustness checks. Section 6 concludes.

3.2 Model

In this section I outline a two-country, one-good international business cycle model. The world economy consists of two countries, home (country 1) and foreign (country 2), which are the same size. Each country has three sectors: a household sector, a market production sector and a capital producer sector. The household sector is populated with two types of infinitely lived agents: investor and saver. The investor and saver are distinct from each other in order to motivate lending and borrowing. Adding the market production sector allows agents to derive returns from capital and labor. Moreover, I have the capital producer to facilitate the introduction of variation in capital price.

I assume that capital is mobile across the countries but labor is immobile across the countries. The following subsections detail the economic choice faced by agents in the two economies, the structure of production and the relevant market clearing conditions.

3.2.1 Household

There are two types of households in the model: an investor and a saver. The investors can buy the capital installed both domestically and abroad. They rent the capital to the market production firm and receive a risky return. At the same time, they can also borrow from domestic savers to finance their capital holdings. Investors account for a fraction n of all households. The rest of the households participate only in the domestic bond market and I refer to them as savers. Similar to the assumption made in Bernanke, Gertler and Gilchrist (1999), I assume that investors have the ability to transform capital into a factor that can be used in the market good production. However, since savers do not have this ability, they will purchase capital to be used only in home production. Savers are assumed to be more patient than investors such

that in equilibrium, savers always want to lend to investors. Finally, the credit friction comes in the form of a leverage constraint: the debt that investors borrow cannot exceed a certain fraction of their total asset value.

Investor

Investors in each country i choose consumption c_{it}^I , provide labor services l_{it}^I , and make a portfolio choice among domestic capital, foreign capital and domestic debt. Their utility is given by the following expression:

$$E_t \sum_{t=0}^{\infty} [\beta(C_{it}^I, L_{it}^I)]^t \frac{1}{1-\gamma} \left(c_{it}^I - \psi^I \frac{(l_{it}^I)^{1+\theta}}{1+\theta} \right)^{1-\gamma} \quad i = 1, 2 \quad (3.1)$$

The Greenwood-Hercowitz-Huffman (GHH) preference is widely used in the open economy literature. Early work includes Correia, Neves and Rebelo (1995). GHH preference is chosen because there is no wealth effect on labor supply. As a result, only a substitution effect operates on hours and suggests that the path of hours will closely follow that of output.¹ To ensure stationary equilibrium, I follow Mendoza (1991) to assume an endogenous discount factor.

$$\beta(C_{it}^I, L_{it}^I) = \left(1 + C_{it}^I - \psi^I \frac{(L_{it}^I)^{1+\theta}}{1+\theta} \right)^{-\omega^I}$$

The discount factor is external in the sense that a household takes $\beta(C_{it}^I, L_{it}^I)$ as exogenous. (C_{it}^I and L_{it}^I are the aggregate level of consumption and hours of investors.) As shown in Schmitt-Grohe and Uribe (2003), internalizing the discount factor makes negligible quantitative differences.

The period budget constraint of a representative investor is given by

$$\begin{aligned} c_{it}^I + q_{it}^k k_{ii,t+1}^I + q_{jt}^k k_{ij,t+1}^I &= w_{it} l_{it}^I + q_{it}^b B_{it+1}^I - B_{it}^I \\ &+ ((1-\delta)q_{it}^k + R_{it}^k) k_{ii,t}^I + ((1-\delta)q_{jt}^k + R_{jt}^k - \tau_t) k_{ij,t}^I \end{aligned} \quad (3.2)$$

Here q_{it}^k denotes the price of capital in country i , q_{it}^b denotes the price of a bond in

¹GHH preferences are commonly used in the open economy literature, dating back to Mendoza (1991) and Devereux, Gregory and Smith (1992). Recent examples include Mendoza and Smith (2002) and Raffo (2009).

country i ($q_{it}^b = \frac{1}{1+R_{it}^b}$ where R_{it}^b is the risk-free rate), k_{ijt+1}^I denotes the capital in country j held by an investor from country i . In each period, the investor receives a return R_{it}^k (R_{jt}^k) by renting the capital to the market production firm in country i (j). She also receives labor income by supplying labor to the market production firm. She then sells the capital after depreciation back to the capital producer at price q_{it}^k (q_{jt}^k). By assumption, the investor is less patient than the saver; therefore, in equilibrium she will always borrow from the saver at the risk-free rate to finance the purchase of capital for the next period.

In order to introduce home bias in the investor's capital holdings, following Tille and Wincoop (2007), I assume that there is some extra uncertainty about the capital return from the foreign market. This represents the fact that it is more difficult for a domestic investor to invest in the foreign market, and therefore, she faces a riskier return. Specifically, the return from the foreign country that the investor receives is subject to²

$$\tau_t \sim N(0, \chi\sigma^2)$$

I assume that the investor may default on her debt; thus she always has to put down collateral against her debt. That is, the investor faces a collateral constraint (or leverage constraint) that restricts her debt to be smaller than a fraction κ of the value of the asset offered as collateral.

$$B_{it+1}^I \leq \kappa(q_{it}^k k_{ii,t+1}^I + q_{jt}^k k_{ij,t+1}^I) \quad \text{where } 0 \leq \kappa \leq 1 \quad (3.3)$$

Here B_{it+1}^I denotes the amount of debt that she can borrow from the domestic saver and κ controls the leverage ratio. This form of leverage constraint is in the style of Kiyotaki and Moore (1997) and Mendoza and Smith (2002). Since the debt level is linked directly to the investor's total asset value, any fluctuation in either country's capital price will have an immediate impact on the borrowing capacity of the investors in both countries. Therefore, both the leverage constraint and the foreign capital exposure are the key ingredients that help to amplify the transmission of technology shocks across countries.

²It is quite standard to introduce exogenous financial frictions, such as Tille and Van Wincoop (2007) and Devereux and Sutherland (2009).

The FOCs for the investor are

$$q_{it}^k U_{c_{i,t}}^I = \beta_t^I E_t U_{c_{i,t+1}}^I ((1 - \delta) q_{it+1}^k + R_{it+1}^k) + \kappa \mu_{it} q_{it}^k \quad (3.4)$$

$$q_{jt}^k U_{c_{i,t}}^I = \beta_t^I E_t U_{c_{i,t+1}}^I ((1 - \delta) q_{jt+1}^k + R_{jt+1}^k - \tau_{t+1}) + \kappa \mu_{it} q_{jt}^k \quad (3.5)$$

$$q_{it}^b U_{c_{i,t}}^I = \beta_t^I E_t U_{c_{i,t+1}}^I + \mu_{it} \quad (3.6)$$

$$w_{it} = \psi^I (l_{it}^I)^\theta \quad (3.7)$$

where μ_{it} is the Lagrange multiplier on country i 's leverage constraint. We can see that when μ_{it} is positive, the investor wants to borrow more from the saver but is constrained by the leverage constraint.

Saver

Consider a saver with GHH preferences described by

$$E_0 \sum_{t=0}^{\infty} [\beta(C_{it}^S, L_{it}^S)]^t u(c_{it}^{SM}, c_{it}^{SH}, l_{it}^{SM} l_{it}^{SH}) \quad (3.8)$$

where

$$u(c_{it}^{SM}, c_{it}^{SH}, l_{it}^{SM} l_{it}^{SH}) = \frac{1}{1 - \gamma} \left(c_{it}^S - \psi^S \frac{(l_{it}^S)^{1+\theta}}{1 + \theta} \right)^{1-\gamma} \quad (3.9)$$

and

$$c_{it}^S = (\lambda (c_{it}^{SM})^e + (1 - \lambda) (c_{it}^{SH})^e)^{1/e}$$

$$l_{it}^S = l_{it}^{SM} + l_{it}^{SH}$$

The period utility is defined over four arguments: c_{it}^{SM} is the consumption of a market good in country i , c_{it}^{SH} is the consumption of a home good, l_{it}^{SM} is labor time spent in market production and l_{it}^{SH} is labor time spent in home production. The elasticity of substitution between c_{it}^{SM} and c_{it}^{SH} is given by $\frac{1}{1-e}$. The discount factor is defined similarly to that of an investor:

$$\beta(C_{it}^S, L_{it}^S) = \left(1 + C_{it}^S - \psi^I \frac{(L_{it}^S)^{1+\theta}}{1 + \theta} \right)^{-\omega^S}$$

where ω^S represents the elasticity of the discount factor to the composite $1 + C_{it}^S - \psi^I \frac{(L_{it}^S)^{1+\theta}}{1+\theta}$.

At each date, the saver is subject to a market budget constraint that allocates total income between two uses: the purchase of the market consumption good and the purchase of household capital. Capital is sold back to the capital producer after being used in the home production. I assume that capital depreciates at rate δ . For simplicity, I assume it to be the same as the depreciation rate in the market production sector. The saver receives an interest payment on the bond she purchased. She also gets labor income by supplying labor to the market production firm. If w_{it} is the wage rate, and q_{it}^b is the price for bond, then the budget constraint can be written as

$$c_{it}^{SM} + q_{it}^k k_{ii,t+1}^S = w_{it} l_{it}^{SM} + (1 - \delta) q_{it}^k k_{ii,t}^S + q_{it}^b B_{it+1}^S - B_{it}^S \quad (3.10)$$

The saver is also subject to the home production constraint at each date

$$c_{it}^{SH} = G(k_{ii,t}^S, l_{it}^{SH}) \quad (3.11)$$

I assume that home production has a Cobb-Douglas production technology of the form

$$G(k_{ii,t}^S, l_{it}^{SH}) = (k_{ii,t}^S)^{\alpha_2} (l_{it}^{SH})^{1-\alpha_2} \quad (3.12)$$

Solving the saver's problem leads to the following FOCs:

$$q_{it}^k U_{c_m,t}^S = \beta_t^S E_t (U_{c_m,t+1}^S (1 - \delta) q_{it+1}^k + U_{c_h,t+1}^S G_K(k_{ii,t+1}^S, l_{it+1}^{SH})) \quad (3.13)$$

$$q_{it}^b U_{c_m,t}^S = \beta_t^S E_t U_{c_m,t+1}^S \quad (3.14)$$

$$w_{it} = \frac{\psi^S (l_{it}^{SM} + l_{it}^{SH})^\theta}{(c_{it}^S)^{1-e} \lambda (c_{it}^{SM})^{e-1}} \quad (3.15)$$

$$G_L(k_{ii,t}^S, l_{it}^{SH}) = \frac{\psi^S (l_{it}^{SM} + l_{it}^{SH})^\theta}{(c_{it}^S)^{1-e} (1 - \lambda) (c_{it}^{SH})^{e-1}} \quad (3.16)$$

3.2.2 Capital Producer

In each country, there is one representative capital producer who operates in a perfectly competitive market. At the end of period t , the capital producer purchases

final goods i_{it} and the undepreciated physical capital $(1 - \delta)k_{i,t}$ that has been used in period t 's production cycle. The capital producer uses these inputs to produce new installed capital $k_{i,t+1}$ using the following constant return to scale production technology

$$k_{i,t+1} = (1 - \delta)k_{i,t} + \phi\left(\frac{i_{i,t}}{k_{i,t}}\right) k_{i,t}$$

I assume that the construction of new capital goods is subject to adjustment costs, whereas the repair of old capital goods is not. The following specification for adjustment cost is adopted

$$\phi\left(\frac{i_{i,t}}{k_{i,t}}\right) = \frac{g_1}{1 - \pi} \left(\frac{i_{i,t}}{k_{i,t}}\right)^{1-\pi} + g_2$$

where $\phi(\cdot)$ is a positive, concave function. I denote the price of the new capital to be $q_{i,t}^k$, then the parameter π controls the elasticity of $q_{i,t}^k$ with respect to the investment to capital ratio. This specification allows the shadow price of installed capital to diverge from the price of an additional unit of capital, i.e., it permits variation in the price $q_{i,t}^k$. Similar to Kiyotaki and Moore (1997), the idea is to have asset price variability contribute to volatility in the investor's balance sheet.

Since the marginal rate of transformation from previously installed capital to new capital is unity, the price of old capital is also $q_{i,t}^k$. The firm's profits at time t is

$$\Pi_{i,t} = q_{i,t}^k k_{i,t+1} - q_{i,t}^k (1 - \delta)k_{i,t} - i_{i,t}$$

The capital producer therefore solves

$$\begin{aligned} \max_{k_t, i_t} \quad & \Pi_{i,t} = q_{i,t}^k k_{i,t+1} - q_{i,t}^k (1 - \delta)k_{i,t} - i_{i,t} \\ \text{s.t.} \quad & k_{i,t+1} = (1 - \delta)k_{i,t} + \phi\left(\frac{i_{i,t}}{k_{i,t}}\right) k_{i,t} \end{aligned}$$

Solving the maximization problem above leads to the following expression for capital price

$$q_{i,t}^k = \frac{1}{\phi'\left(\frac{i_{i,t}}{k_{i,t}}\right)} \quad (3.17)$$

Moreover, the new installed capital produced in each country (k_{it}) is bought by three types of agent: the domestic investor (k_{iit}^I), the foreign investor (k_{jit}^I) and the domestic

saver (k_{it}^S).

$$\begin{aligned} k_{1t} &= nk_{11t}^I + nk_{21t}^I + (1-n)k_{11t}^S \\ k_{2t} &= nk_{12t}^I + nk_{22t}^I + (1-n)k_{22t}^S \end{aligned}$$

3.2.3 Production

The structure of the market production firm is straightforward. The firm lives for only one period and has a Cobb-Douglas production function in capital and labor. The production of the market good is subject to a stochastic technology shock z_{it}^m .

$$F(z_{it}^m, k_{i,t}^I, l_{it}) = e^{z_{it}^m} (k_{i,t}^I)^{\alpha_1} (l_{it})^{1-\alpha_1} \quad (3.18)$$

The firm rents capital from domestic and foreign investors

$$k_{i,t}^I = n(k_{ii,t}^I + k_{ji,t}^I) \quad (3.19)$$

and it also rents labor from the domestic investor and the domestic saver

$$l_{it} = nl_{it}^I + (1-n)l_{it}^{SM} \quad (3.20)$$

The optimality conditions for the firm are

$$w_{it} = F_L(z_{it}^m, k_{i,t}^I, l_{it}) \quad (3.21)$$

$$R_{it}^k = F_K(z_{it}^m, k_{i,t}^I, l_{it}) \quad (3.22)$$

I assume that the law of motion for the technology shock to market production is given by a stationary VAR of the form

$$\begin{bmatrix} z_{1t}^m \\ z_{2t}^m \end{bmatrix} = \begin{bmatrix} \rho_1^m & \rho_2^m \\ \rho_2^m & \rho_1^m \end{bmatrix} \begin{bmatrix} z_{1t-1}^m \\ z_{2t-1}^m \end{bmatrix} + \begin{bmatrix} \epsilon_{1t}^m \\ \epsilon_{2t}^m \end{bmatrix} \quad (3.23)$$

where ρ_1^m represents the persistence of the technology shock and ρ_2^m represents the spillover effect of the technology shock. The innovation follows

$$\begin{bmatrix} \epsilon_{1t}^m \\ \epsilon_{2t}^m \end{bmatrix} \sim N(0, \Sigma) \text{ with correlation matrix } \begin{bmatrix} \sigma_1^m & \phi^m \\ \phi^m & \sigma_2^m \end{bmatrix} \quad (3.24)$$

where ϕ^m is the correlation between the two technology shocks.

3.2.4 Market Clearing

There are two sets of market clearing conditions: the bond market clearing and the good market clearing. Since the bond market is assumed to be domestic, the total bond within a country is zero, which gives the following conditions,

$$nB_{1t+1}^I + (1-n)B_{1t+1}^S = 0 \quad (3.25)$$

$$nB_{2t+1}^I + (1-n)B_{2t+1}^S = 0 \quad (3.26)$$

Now I develop the aggregate resource constraint for this economy.

$$nc_{1t}^I + (1-n)c_{1t}^{SM} + nc_{2t}^I + (1-n)c_{2t}^{SM} + \bar{\tau}_t + i_{1t} + i_{2t} = F(z_{1t}^m, k_{1t}^I, l_{1t}) + F(z_{2t}^m, k_{2t}^I, l_{2t}) \quad (3.27)$$

The market good clearing gives that total market output is used in three aspects: total market consumption, total investment and the sum of the portfolio cost $\bar{\tau}_t$.

3.2.5 Model Mechanism

This section reviews the main mechanism of the model and highlights important parameters. I show that both financial frictions and foreign asset exposure are important in leading to the increase in business cycle co-movements. When a negative technology shock hits the domestic market, the demand for capital in the home country falls, which forces down investment and the price of domestic capital. The degree to which the price of capital falls depends on the parameter π , which controls the elasticity of price with respect to the investment to capital ratio. As π becomes larger, the capital price is more variable in response to a change in investment.

$$q_{i,t}^k = \frac{1}{\phi' \left(\frac{i_{i,t}}{k_{i,t}} \right)} = \frac{1}{g_1} \left(\frac{i_{i,t}}{k_{i,t}} \right)^\pi$$

From the investor's leverage constraint below, we see that the fall in the asset price leads to a tightening of investors' leverage constraint in both countries. Borrowing is thus reduced globally. Since the leverage ratio is $\frac{1}{1-\kappa}$ for a given κ , as κ becomes

bigger, the leverage ratio is higher. Hence the decline in global borrowing is steeper.

$$B_{it+1}^I \leq \kappa(q_{it}^k k_{ii,t+1}^I + q_{jt}^k k_{ij,t+1}^I) \quad \text{where } 0 \leq \kappa \leq 1$$

As borrowing falls globally, demand for capital in the foreign country also declines, which pushes down the price of the foreign asset, leading to another round of credit tightening. A multiplier effect arises, since the decline in investment lowers asset prices and investors' net worth, further pushing down investment.

From the equation above, we can see that by considering foreign exposure, the foreign asset price has an immediate effect on the balance sheet of domestic investors. Along with the presence of the financial frictions, the technology shock spills over from one country to another and thus drives up the business cycle correlations.

3.2.6 Solution Method

This model is solved using an iterative second-order perturbation method adopted from Heathcote and Perri (2009). The standard method to analyze DSGE model is to take a linear approximation around a deterministic steady-state. However, this method cannot be used to solve the current model. Because when we have more than one asset, in the steady-state the returns are the same across assets. Hence the portfolio shares are indeterminate: any share of domestic and foreign capital holdings will be consistent with the steady-state. The way to find a steady-state portfolio share is to use information from the higher order approximation. The detailed algorithm is documented in the Appendix.

3.3 Calibration

I now proceed to choose parameter values, setting some numbers on the basis of a priori information and setting others according to the steady-state conditions. A period in the model corresponds to one quarter. The sample period in the data is from 1972:1 to 2008:4. Table 3.1 gives a summary of the calibration.

3.3.1 Preference and Production Parameters

The intertemporal elasticity of substitution (IES) is set to 0.5, which is standard in the literature. The parameter ω^S , which controls the saver's discount factor, is set to 0.039 to match an annual interest rate of 4%. Following Bernanke, Gertler and Gilchrist (1999), I use the investor's discount factor to match an interest premium on borrowed funds of 2%, approximately the historical average spread between the prime lending rate and the six-month Treasury bill rate. This gives ω^I the value of 0.112. The implied steady-state discount factor for the saver is 0.99 and the implied steady-state discount factor for the investor is 0.97. For the elasticity of labor supply, in line with Greenwood, Hercowitz and Huffman (1988), I calibrate it to be 1.7, which corresponds to $\theta = 0.6$.

The depreciation rate δ is set to 0.025, corresponding to an annual depreciation rate of 10%. I now use α_1 (capital share of market production), α_2 (capital share of home production), ψ^I (investor's labor supply level), ψ^S (saver's labor supply level) and λ (share of market consumption good) to match the following five observations: the market capital-to-output ratio, the home capital-to-output ratio, the market hours for the investor, the market hours for the saver and the home hours for the saver. According to Greenwood, Rogerson and Wright (1995), the home capital to output ratio is 5, where home capital is defined as consumer durables plus residential structures. Since the total capital to output ratio is around 12, as given by Cooley and Prescott (1995), the market capital to output ratio is set to 7. I choose the hours worked for market production to be 0.33 and the hours spend on home production to be 0.25. This calibration gives a capital share of market production (α_1) of 0.29 and a capital share of home production (α_2) of 0.40.

The only preference parameter that is left unspecified is e , the elasticity of substitution between the market and home consumption good. A higher value of e means that the saver is more willing to substitute consumption of one sector's output with consumption of the other sector's output. The empirical evidence on e is controversial. Eichenbaum and Hansen (1990) suggest that the two goods are very close to perfect substitutes. Benhabib, Rogerson and Wright (1991) use PSID data to estimate this elasticity, which results in a value of 0.8 for e . McGrattan, Rogerson and Wright (1993) use a maximum likelihood approach to estimate a model with home production and they come up with a value of 0.429. For the benchmark model, I use

an intermediate value among existing estimates, $e = 0.9$. In what follows I will also consider several alternative values of e as a robustness test.

The elasticity of the capital price with respect to the investment to capital ratio, π , is taken to be 0.25, following Bernanke, Gertler and Gilchrist (1999). This is one of the key parameters in the model since capital price is crucial for determining the level of loans for investor and hence the global investment level. However, there is no firm consensus in the literature about what this parameter value should be. A reasonable assumption about the adjustment cost suggest that the value should lie within a range of 0. to 0.5.³ The parameter χ controls the variance of the extra risk from investing in the foreign country. When this risk is absent, only 14% of the investor's capital holdings are domestic, exhibiting a substantial bias against home capital. This observation is consistent with theory since when an agent's labor income is correlated with her home capital return, to diversify this risk the agent will take a larger position in the foreign country. I set χ to be 0.14 such that 75% of the investor's asset are domestic.

When the leverage constraint is binding, the leverage ratio is $\frac{1}{1-\kappa}$ for a given κ . In this model, I calibrate the leverage ratio to be 3, according to Dedola and Lombardo (2010). This number is higher than the leverage ratio used in Bernanke, Gertler and Gilchrist (1999), since I consider not only non-financial firms but also financial firms.

In this model, savers do not have access to the equity market; therefore, I calibrate the share of savers to match the fraction of the population who do not participate in the stock market. According to the Survey of Consumer Finances (2007), about half of US households have become stock owners. Therefore, I set the share of savers to be 0.5.

3.3.2 Technology Parameters

For the benchmark calibration, I follow the estimates from Heathcote and Perri (2004). They estimate the parameters of the bivariate shock process using estimates of Solow residuals. They subtract a common deterministic growth trend from Solow residuals and then estimate by least squares. In this case, the productivity shocks

³Jermann (1998) uses a value of 0.23. Christensen and Dib (2008) give an estimate of 0.59 by Bayesian estimation. Meier and Muller (2006) report an estimate of 0.65, close to Christensen and Dib (2008).

still display high persistence and positively correlated innovations, but they no longer find evidence of spillovers. This gives the following estimates

$$\begin{bmatrix} z_{1t}^m \\ z_{2t}^m \end{bmatrix} = \begin{bmatrix} 0.91 & 0. \\ 0. & 0.91 \end{bmatrix} \begin{bmatrix} z_{1t-1}^m \\ z_{2t-1}^m \end{bmatrix} + \begin{bmatrix} \epsilon_{1t}^m \\ \epsilon_{2t}^m \end{bmatrix}$$

where

$$\begin{bmatrix} \epsilon_{1t}^m \\ \epsilon_{2t}^m \end{bmatrix} \sim N(0, \Sigma) \text{ with correlation matrix } \begin{bmatrix} 0.006 & \\ 0.25 & 0.006 \end{bmatrix}$$

In the sensitivity analysis, I also use the productivity estimates from Backus, Kehoe and Kydland (1992), where there is some evidence of spillover. The estimates are

$$\begin{bmatrix} z_{1t}^m \\ z_{2t}^m \end{bmatrix} = \begin{bmatrix} 0.906 & 0.088 \\ 0.088 & 0.906 \end{bmatrix} \begin{bmatrix} z_{1t-1}^m \\ z_{2t-1}^m \end{bmatrix} + \begin{bmatrix} \epsilon_{1t}^m \\ \epsilon_{2t}^m \end{bmatrix}$$

and I maintain the same covariance matrix as in Heathcote and Perri (2004).

3.4 Results

In this section, I analyze the quantitative implications of my model. First, I report the moments generated by the model and compare them with the data. Second, I look at the impulse response functions (IRFs) of the technology shock to analyze the model mechanism.

3.4.1 Moments

The results of the simulation under the benchmark calibration are summarized in Table 3.2. The first column of Table 3.2 shows the statistics calculated from the data. Panels (A), (B) and (C) are calculated from US time series for the period of 1972:1 to 2008:4. The statistics from panel (D) represent the correlation of US series with series from the rest of the industrial world (which is an aggregate of Europe, Japan and Canada). The details of the aggregation of the rest of the world data are shown in the Appendix. Except for net exports, all series are logged and filtered by the Hodrick-Prescott filter with a smoothing parameter of 1600. In the table, output and consumption refer to market output and consumption while investment refers to

total investment.⁴

The third column of Table 3.2, "Model 2: Constrained with 25% Foreign Exposure," is our benchmark model with calibrations documented in Section 3. The second column of Table 3.2, "Model 1: Unconstrained," is the same as Model 2 except that the investor does not face a leverage constraint. The last column, "Model 3: Constrained with 86% Foreign Exposure," is the model where instead of imposing a 75% home bias, I let the investor fully diversify her portfolio such that, as shown in the calibration, she holds 86% of the capital in the foreign market.

We first compare the data with the results from the constrained economy (Model 2). As we see from the cross-country correlations in panel (D) of Table 3.2, the constrained economy produces two-thirds of the output correlation in the data. It also produces a positive correlation of investment and employment, among which the employment correlation is well matched to the data. When we compare the constrained economy (Model 2) with the unconstrained economy (Model 1), we can see an overall improvement in the cross-country correlations. The unconstrained economy predicts a consumption correlation and an output correlation that are too low, relative to the data. The constrained economy does better, predicting a higher level of output correlation. Although it overshoots the consumption correlation, it is still closer to the data than the unconstrained economy. In terms of investment and employment, both models predict positive correlations, while the constrained economy is closer to the moments in the data.

Overall, the model with constraint performs better in terms of the cross-country correlations. The presence of the leverage constraint increases the correlation of consumption, labor and output, while it decreases the correlation of investment. As will be shown in the IRF analysis in Section 4.2, those improvements are introduced exactly by the financial frictions.

In terms of the within-country moments, in general the model with constraint gives moments that are closer to the moments in the data. The constrained economy replicates the level of output volatility in the data; however, output in the unconstrained economy (2.54%) is more volatile than in the constrained economy (1.92%). The high volatility of the unconstrained economy is introduced by the frequent substitution between market and home consumption. In terms of relative volatility, both

⁴The definition of investment is consistent with that in the data.

models overpredict the volatility of consumption while underpredict the volatility of investment in the data. For the within country correlations, both models give positive correlations of net export with output while we see negative correlations in the data.

We then compare the difference induced by financial exposure. The investor in Model 2 holds 25% of capital in the foreign market, while the investor in Model 3 holds 86% of capital in the foreign market. The impact of this foreign asset exposure on the business cycle co-movements is immediate. If we look at the cross-country correlations, output correlation increases from 0.4 to 0.53. Consumption and labor also rise because of the increased synchronization of output. Investment correlation, on the other hand, falls. As foreign capital exposure increases, foreign asset prices will have a more profound impact on the debt level of the investor, which in turn influences domestic investment and output. Hence, the output correlation is driven up by increased foreign asset exposure.

3.4.2 Impulse Responses

In this section, I explain why the behavior of the three models differs. I analyze the response of the two-country economy to a one standard-deviation negative shock in country 1. As in all the subsequent figures, the time units on the graphs are to be interpreted as quarters.

Figures 3.1-3.4 show the impact of a one-standard-deviation decline in country 1's technology shock. The upper panel shows country 1's response and the lower panel shows country 2's response. In each plot, the solid line corresponds to the impulse response in the constrained economy (Model 2) and the dashed line corresponds to the impulse response in the unconstrained economy (Model 1). In the figure, output and consumption refer to market output and consumption while investment refers to total investment.

Leverage Constraint

We first analyze the response of the constrained economy (Model 2). When a negative shock hits, the demand for capital in country 1 immediately falls, leading to a 0.9% decline in investment in country 1. Following the weak demand for capital, the price of capital in country 1 falls 0.22%. Since investors hold leveraged portfolio across countries, the decline in asset prices in country 1 leads to a reduction in total wealth

for investors in both countries. Therefore, the leverage constraints are tightened globally and the debts that investors are eligible to lend are reduced. We observe a 0.6% decline of debt in country 1 and a 0.2% decline of debt in country 2. After the global decline of debt, not only do investors in country 1 have a weak demand for capital, but so do investors in country 2. Hence, investment and capital price fall in country 2 as well. The decline in capital price thus triggers another round of declines in investment and output.

Since the decline in the demand for capital reduces the income of the investor, the investor's consumption falls. As the savers suffer from a decline in their wage income, savers' consumption is also reduced. Overall, total market consumption in country 1 falls around 1.4% and that of country 2 falls around 0.2%.

Upon the negative shock to productivity, wages fall in country 1; hence, the investor and saver's labor supplies are reduced immediately in country 1. We observe a 0.8% decline in total market labor for country 1. For the market labor in country 2, since there is no wealth effect on investor's labor for GHH preferences, the investor's market labor supply in country 2 does not move. However, since the saver faces substitution between home and market consumption, the wealth effect and substitution effect both affect her labor supply.⁵ When the shock hits, the wealth effect dominates the substitution effect and the saver in country 2 increases her market labor supply. However, this effect is minimum as it only leads to a one basis-point movement. Next period, the substitution effect becomes larger and the saver's market labor supply falls.

The output of market production in country 1 falls by 1.4% in the next period, and through the transmission mechanism introduced by the leverage constraint, the output of market production in country 2 falls by around 0.1%.

In country 2, we also see an increase in the capital used in home production. Because the domestic saver holds a portfolio of domestic capital and a bond, a decline in the demand for the bond makes the saver shift her assets to domestic capital; therefore, the capital used in home production increases.

⁵From Equations (15) and (16) we see that consumption shows up in the FOC for saver's labor choice, therefore affecting the saver's labor decision.

No Leverage Constraint

We then analyze the response of the unconstrained economy (Model 1). When the investor is not facing a leverage constraint, an unexpected one-standard-deviation decline in country 1's productivity leads to a fall in the return on market capital in country 1. We observe a fall in the debt level in country 1, because home production becomes more productive than market production, making the saver shift her holdings from a bond to home capital. From Figure 3.1 we also observe an increase in the debt level in country 2. This is because the investor from country 2 suffers from her investment loss in country 1 caused by the low return on market capital and she does not face any form of collateral constraint; therefore she increases her debt to compensate for her investment loss. The changes in the debt level in country 2 also lead to a decline in the purchase of home capital in country 2, because the increase in the debt level indicates an increase in the bond holdings of the saver. As the saver holds more bonds, she rebalances her portfolio by reducing her exposure to home capital.

Because of the decline in productivity in country 1's market sector, country 2 now looks more productive. Market capital flows from country 1 to country 2; thus, Figure 3.3 shows an increase in market capital in country 2. Market output in country 2 follows a pattern similar to that of market capital: market output in country 2 increases after the shock. Investment and capital prices in both countries fall. In country 1, the decline in investment is mainly driven by market capital: investment in market capital falls because of a lower return. In country 2, the decline in investment is mainly driven by home capital: the investment in home capital falls because of the saver's portfolio balancing.

Comparison

After examining the two scenarios separately, we now put them together for comparison. There are several points to note. First, upon a negative technology shock to country 1, market output in country 2 declines in the constrained economy, whereas it increases in the unconstrained economy. The response of the unconstrained economy is similar to the situation in a standard model with complete markets: capital flows into the more productive country, leading to negative responses of the production factors. The effect of the financial frictions becomes apparent when we look at the

response of the constrained economy. The presence of the leverage constraint limits investors' ability to invest in both countries. Since they are constrained from getting more loans, they do not have many resources to invest; therefore, although country 2's investment opportunity is better, market capital in country 2 still declines.

Second, the decline of consumption in country 2 in the constrained economy is nearly three times as much as in the unconstrained economy. For the unconstrained economy, country 2's consumption declines only 0.06%. Since the investor is not constrained, she can borrow from the saver to cushion her investment loss; therefore, her consumption is barely affected. However, for the constrained economy, the investor cannot borrow as much as she wants; hence, consumption is affected to a greater degree, leading to a 0.18% decline.

Third, investment in country 2 falls less in the constrained economy than in the unconstrained economy. The reason is the following: investment is defined as total investment in the country which means that it is the sum of investment in market production and investment in home production. In the constrained economy, investment in the market sector falls because of the tightened leverage constraint for the investor. However, investment in the home sector rises because the saver shifts her portfolio from bonds to home capital. These two forces work against each other and the fall in market investment outweighs the increase in home investment, leading to an overall decline of investment. In the unconstrained economy, market investment in country 2 rises because relative productivity in country 1 is now higher. At the same time, investment in the home sector declines because now the saver shifts her portfolio from home capital to bonds. These two forces result in a decline in the total investment level in country 2 and the magnitude is larger than in the constrained economy.

Fourth, the debt levels in the two countries move in the same direction in the constrained economy, whereas the debt levels move in different directions in the unconstrained economy. For the constrained economy, debt falls in both countries because the leverage constraints in both countries are tightened. For the unconstrained economy, country 2's investor increases her debt to offset the loss in investment in country 1.

To briefly sum up, the differences discussed above are exactly introduced by the financial frictions. The financial frictions drive up output, consumption and employment correlations and drive down the investment correlation.

Degree of Foreign Exposure

In this section, I look at different degrees of foreign asset exposure. I compare two cases: Model 2, in which investors are holding 25% of their capital in the foreign market and Model 3, in which investors are holding 86% of their capital in the foreign market. Figures 3.5-3.8 show the impact of a one-standard-deviation decline in country 1's technology level. The upper panel shows country 1's response to the shock and the lower panel shows country 2's response. In each plot, the solid line corresponds to the impulse response for the 25% foreign exposure economy and the dashed line corresponds to the 86% foreign exposure economy.

Given the same level of decline in capital prices in country 1 for both economies (Figure 3.5), it is straightforward to see that the more foreign capital the investor holds, the more she suffers from tightening of the leverage constraint. This idea is confirmed in Figure 3.5, which shows the response of the debt level in country 2. We notice that when the investor holds 86% of foreign capital, her debt level falls three times more than in the case where she only holds 25% of foreign capital. The debt level further influences other economic activities, and output and consumption decrease. Therefore, a larger balance-sheet exposure to risky foreign capital results in the business cycles that are more synchronized between the two countries.

3.5 Sensitivity Analysis

In this section, we report the results of the sensitivity analysis with respect to some key parameters in the model. Specifically, I explore some alternative values for the investment adjustment cost, leverage ratio, shock process and elasticity of substitution between home and market goods.

3.5.1 Adjustment Cost

The parameter π controls the elasticity of the capital price with respect to the investment to capital ratio. As discussed in the calibration, the estimate of this elasticity varies a lot. A recent paper by Christensen and Dib (2008) estimates π to be 0.59 using data on investment. Other papers such as Meier and Muller (2006) give an even higher value of 0.65. Therefore, as a robustness check, we set π to 0.5, implying a larger investment adjustment cost and a slower response of investment. We also

set π to 100; in this case the adjustment cost is so large that investment does not move at all. The model is then reduced to a version where capital is fixed in each country, which is similar to the setup of Devereux and Yetman (2010).⁶ I argue that a significant difference exists between the model with and without capital accumulation. When capital cannot move across countries, the business cycle synchronization becomes stronger.

Table 3.3 shows the simulation results when π is 0.25, 0.5 and 100, respectively. As π increases and as we move from left to right of the table, we see an increase in the cross-country correlations in all macro variables. The important role that π plays in the propagation mechanism is obvious. When the investment adjustment cost becomes higher, the capital price responds more to a technology shock. Since the capital price has an immediate impact on the investor's balance sheet, it influences the level of loans and the investor's future investment decisions. Therefore, when the investment adjustment cost increases, business cycles are more synchronized. A higher adjustment cost, on the other hand, also implies that investment is less responsive to shocks. Therefore, we see a decline in the investment volatility.

3.5.2 Leverage Ratio

Now I experiment with a higher leverage ratio for the investor's leverage constraint. As shown in the previous section, the leverage constraint serves as an important channel for the propagation of technology shocks. From the leverage constraint below, we see that as κ becomes bigger, the bigger the impact the investor's asset value has on the eligible loans.

$$B_{it+1}^I \leq \kappa(q_{it}^k k_{ii,t+1}^I + q_{jt}^k k_{ij,t+1}^I)$$

Since many financial firms have higher leverage ratios, we set κ to 0.8, corresponding to a leverage ratio of 5. As seen from Table 3.4, output volatility increases compared to economy with lower leverage. There is also an increase in the cross-country correlations of consumption, output and labor. Consumption correlation increases 0.06, while output correlation and labor correlation increase 0.07.

⁶However, the two models are still not the same since this one has endogenous labor. Capital in this case can be interpreted as land, which is not mobile across countries but nevertheless can be owned by different investors.

3.5.3 Different Shock Process

In the benchmark calibration, there is no spillover between the two technology shocks. Therefore I conduct a sensitivity analysis regarding the spillovers. The calibration for the technology shock is taken from Backus, Kehoe and Kydland (1992), where the persistence of the shock is 0.906 and the spillover is 0.088. The covariance matrix for the innovation remains the same.

From the last column of Table 3.4, we observe an increase in the consumption correlation and a decline in output, investment and investment correlations. When there is spillover between technology shocks, the consumption correlation increases from 0.53 to 0.63. This is because a negative shock to one country signals that the other country's output will also decline in the future. Consumers in that country take this into account and lower their current consumption. Therefore, the consumption correlation goes up when a technology shock spills over from one country to the other. Kehoe and Perri (2002) find a similar effect in their paper.

3.5.4 Elasticity of Substitution between Goods

Since the estimates of the elasticity of substitution between home and market goods span a wide range, we experiment with different values of e : 0.9, 0.5 and 0.1. As e gets smaller, it is more difficult to substitute between the two goods. Table 3.5 shows the simulation results for different values of e . As e gets smaller, the correlations of output, labor and consumption become smaller, while the investment correlation becomes larger.

The decline in the correlations of consumption, output and labor becomes apparent when we look at the saver's FOC for labor choice. Because the saver substitutes between home and market goods, consumption starts to have an impact on the labor choice. In other words, the wealth effect starts to show up in the saver's labor choice. Therefore, as e gets smaller, the saver starts to supply more labor because she feels poorer. This then reduces the correlation of labor. Thus the correlation of output is also reduced. The increase in the investment correlation occurs because when e is close to zero, it gets very difficult to substitute between market and home consumption; therefore, the role of home production is reduced to a minimum. Thus we see an increase in the investment correlation.

3.6 Conclusions

This paper argues that financial frictions are important for international business cycles because they magnify the propagation of technology shocks across countries through the balance sheet of leveraged investors.

I have shown that incorporating financial frictions and exposure to foreign assets, which seems to be an important aspect of the recent financial crisis, helps us do a better job of accounting for business cycle correlations across countries. The calibrated model can explain two-thirds of the output correlations in the data. The employment correlation matches the correlation in the data well and the correlation of investment gets closer to the data compared to the model without financial frictions. Moreover, the model also shows that, consistent with the data, when investors have more foreign asset exposure to the other country, the output correlation between the two countries increases.

My study reaffirms the growing attention in the open economy literature to integrating financial market frictions in otherwise standard two-country models. I document the importance of including financial frictions and foreign asset exposure in the analysis. Since this model is able to replicate some key facts of international business cycles, I believe that this framework is a promising one for conducting further research, particularly on welfare analysis and the design of monetary and fiscal policies.

3.7 Appendices

3.7.1 Equilibrium

A competitive equilibrium is defined as a sequence of allocations $\{c_{it}^I, c_{it}^{SM}, c_{it}^{SH}, k_{ij,t+1}^I, k_{ii,t+1}^S, l_{it}^I, l_{it}^{SM}, l_{it}^{SH}, B_{it+1}^I, B_{it+1}^S\}$ and prices $\{q_{it}^k, q_{it}^b, w_{it}, R_{it}^k, \mu_{it}\}$ ($i = 1, 2$) such that both the representative household and the firm maximize and the market clears. The set of equilibrium conditions that characterize the time paths for the allocation and prices are given by the first order conditions for the households and the firm that follow, together with the market clearing conditions and the stochastic process for the technology. To save space, only equilibrium conditions for country 1 are shown below

$$c_{1t}^I + q_{1t}^k k_{11,t+1}^I + q_{2t}^k k_{12,t+1}^I = w_{1t} l_{1t}^I + q_{1t}^b B_{1t+1}^I - B_{1t}^I \quad (3.28)$$

$$+ ((1 - \delta)q_{1t}^k + R_{1t}^k)k_{11,t}^I + ((1 - \delta)q_{2t}^k + R_{2t}^k - \tau_t)k_{12,t}^I \quad (3.29)$$

$$B_{1t+1}^I \leq \kappa(q_{1t}^k k_{11,t+1}^I + q_{2t}^k k_{12,t+1}^I) \quad (3.30)$$

$$q_{1t}^k U_{c_{1,t}}^I = \beta(C_{1t}^I, L_{1t}^I) E_t U_{c_{1,t+1}}^I ((1 - \delta)q_{1t+1}^k + R_{1t+1}^k) + \kappa \mu_{1t} q_{1t}^k \quad (3.31)$$

$$q_{2t}^k U_{c_{1,t}}^I = \beta(C_{1t}^I, L_{1t}^I) E_t U_{c_{1,t+1}}^I ((1 - \delta)q_{2t+1}^k + R_{2t+1}^k - \tau_{t+1}) + \kappa \mu_{1t} q_{2t}^k \quad (3.32)$$

$$q_{1t}^b U_{c_{1,t}}^I = \beta(C_{1t}^I, L_{1t}^I) E_t U_{c_{1,t+1}}^I + \mu_{1t} \quad (3.33)$$

$$c_{1t}^{SM} + q_{1t}^k k_{11,t+1}^S = w_{1t} l_{1t}^{SM} + (1 - \delta)q_{1t}^k k_{11,t}^S + q_{1t}^b B_{1t+1}^S - B_{1t}^S \quad (3.34)$$

$$c_{1t}^{SH} = G(k_{11,t}^S, l_{1t}^{SH}) \quad (3.35)$$

$$q_{1t}^k U_{c_m,t}^S = \beta(C_{1t}^S, L_{1t}^S) E_t (U_{c_m,t+1}^S (1 - \delta)q_{1t+1}^k + U_{c_h,t+1}^S G_K(k_{11,t+1}^S, l_{1t+1}^{SH})) \quad (3.36)$$

$$q_{1t}^b U_{c_m,t}^S = \beta(C_{1t}^S, L_{1t}^S) E_t U_{c_m,t+1}^S \quad (3.37)$$

$$w_{1t} = \psi^I(l_{1t}^I)^\theta \quad (3.38)$$

$$w_{1t} = \frac{\psi^S(l_{1t}^{SM} + l_{1t}^{SH})^\theta}{(c_{1t}^S)^{1-e} \lambda (c_{1t}^{SM})^{e-1}} \quad (3.39)$$

$$G_L(k_{11,t}^S, l_{1t}^{SH}) = \frac{\psi^S(l_{1t}^{SM} + l_{1t}^{SH})^\theta}{(c_{1t}^S)^{1-e} (1 - \lambda) (c_{1t}^{SH})^{e-1}} \quad (3.40)$$

$$w_{1t} = F_L(k_{1,t}^I, l_{1t}) \quad (3.41)$$

$$R_{1t}^k = F_K(k_{1,t}^I, l_{1t}) \quad (3.42)$$

$$q_t = \frac{1}{\phi_1(i_t, k_t)k_t} \quad (3.43)$$

where μ_{it} is the Lagrange multiplier associated with the leverage constraint in country i and the total capital used in country 1 is

$$k_{1t} = nk_{11t}^I + nk_{21t}^I + (1-n)k_{11t}^S$$

The capital used by market production in country 1 is

$$k_{1t}^I = nk_{11t}^I + nk_{21t}^I$$

The law of motion for total capital in country 1 is

$$k_{1t+1} = (1-\delta)k_{1t} + \phi\left(\frac{i_{1t}}{k_{1t}}\right)k_{1t}$$

The world market clearing condition is

$$nc_{1t}^I + (1-n)c_{1t}^{SM} + nc_{2t}^I + (1-n)c_{2t}^{SM} + \bar{\tau}_t + i_{1t} + i_{2t} = F(k_{1t}^I, l_{1t}) + F(k_{2t}^I, l_{2t})$$

By Walras Law, the world market clearing condition is redundant.

3.7.2 Computation

This appendix describes an algorithm for computing the equilibrium portfolios in open economy DSGE models. To a large extent, existing open economy models ignore portfolio composition, analyzing financial linkage between countries in terms of net foreign assets, with no distinction made between assets and liabilities. There is a growing literature that tries to develop methods to solve portfolio problems in these models. This work has been pioneered by Devereux and Sutherland (2009) and Tille and Wincoop (2007) with a recent contribution by Heathcote and Perri (2009). The idea of these three methods is essentially the same: If we have more than one asset, then the asset returns in the steady-state are the same. Therefore, the portfolios are indeterminate in the steady-state. In order to use the perturbation method to solve the model, we need steady-state portfolio shares to perturb around. In general, we use information from second-order perturbations to determine the steady-state portfolios.

To be specific, in my model the steady-state returns to capital in market production are the same across countries. Therefore, although the total amount of capital used in market production is known, the distribution is indeterminate: home and foreign investors can hold an arbitrary portion of the total market capital. I use the algorithm developed by Heathcote and Perri (2009) in solving this model.

Step 1 : Calculate the non-stochastic symmetric steady-state equilibrium. We denote the steady-state as $[\lambda_{11}, \lambda_{22}, X, Y]$ where λ_{11} is the market capital in country 1 held by country 1 investors, λ_{22} is the market capital in country 2 held by country 2 investors. X is the steady-state of non-portfolio state variables and Y is the steady-state of non-portfolio control variables. The first order conditions pin down the value of X and Y , while any value of $\lambda_0 = \lambda_{11} = \lambda_{22}$ is consistent with the equilibrium.

Step 2: Compute the decision rules $\lambda_{11,t+1} = g_1(\lambda_{11,t}, \lambda_{22,t}, X_t)$, $\lambda_{22,t+1} = g_2(\lambda_{11,t}, \lambda_{22,t}, X_t)$, $X_{t+1} = g_3(\lambda_{11,t}, \lambda_{22,t}, X_t, \epsilon_{t+1})$, $Y_t = g_4(\lambda_{11,t}, \lambda_{22,t}, X_t)$ up to second order around the steady-state. The decision rules are computed using methods from Schmitt-Grohe and Uribe (2004). In order to apply their methods, I add a small quadratic adjustment cost for changing the portfolio from its steady-state. However, we do not know whether the steady-state portfolio λ_0 we guessed is the same as the average equilibrium portfolio in the true stochastic economy.

Step 3: Simulate the model for a large number of periods using the computed de-

cision rules from Step 2. Compare the average portfolio shares with the steady-state portfolio. If they are different, then we update the steady-state portfolio with the average portfolio and return to Step 2. If the difference between them is within a certain tolerance level, then that means the initial steady-state λ_0 is a good approximation of the long run portfolio holdings and we take it as the solution to our model.

This algorithm is tested in Heathcote and Perri (2009) by comparing it to the model solution where the analytical form of the portfolio is known. The comparison shows that this algorithm gives a good approximation to the model and enjoys a rapid convergence.

3.7.3 Data

The data series come from the OECD Quarterly National Accounts (QNA). For the US, GDP, consumption and investment correspond to Gross Domestic Product, Private plus Government Final Consumption Expenditure and Gross Fixed Capital Formation (all at constant prices). The employment data, coming from OECD Main Economic Indicators, use the (deseasonalized) civilian employment index series. The imports and exports series at constant prices are from OECD Quarterly National Accounts.

For the data of rest of the world, we construct an aggregate of Canada, Japan and 19 European countries. The 19 European countries include Austria, Belgium, Denmark, Finland, France, Germany, Greece, Iceland, Ireland, Italy, Luxembourg, Norway, Netherlands, Portugal, Spain, Sweden, Switzerland, Turkey and the United Kingdom. For GDP, consumption and investment, I aggregate all the countries to create a single fictional non-US country by first rebasing each series in 2005 national currency constant prices and then expressing everything in 2005 US dollars using PPP exchange rates.

Employment for the rest of the world is aggregated using constant weights that are proportional to the number of employed persons in each area in 2005. An employment series for the 19 European countries is not available before 2001; therefore I use employment for Austria, Finland, France, Germany, Italy, Norway, Portugal, Spain, Sweden, Switzerland and the United Kingdom between 1984:1 and 2000:4. For the period 1972:1 to 1983:4, I use aggregated employment data from the same set of countries between 1984:1 and 2000:4 except Portugal. For the period 1962:1 to 1971:4, I use aggregated data from Finland, Germany, Italy, Sweden and the United Kingdom. These were the only European countries for which I could find consistent and comparable employment series.

3.7.4 Tables and Figures

Table 3.1: Benchmark Calibration

Parameter	Description	Value
γ	inverse of IES	2
ω^I	controls investor's discount factor	0.112
ω^S	controls saver's discount factor	0.039
θ	controls elasticity of labor supply	0.6
α_1	capital share of market production	0.29
α_2	capital share of home production	0.40
ψ^I	controls level of investor's labor	3.08
ψ^S	controls level of saver's labor	1.32
λ	share of market good consumption	0.57
e	controls ES between home and market good	0.9
δ	depreciation	0.025
π	investment adjustment cost	0.25
χ	controls variance of risk of holding foreign capital	0.14
n	measure of investors	0.5
κ	controls leverage ratio	2/3

Note: The first column shows the parameters that need to be calibrated. The second column describes the parameters and the last column shows the calibrated values for the parameters.

Table 3.2: Model Moments - Benchmark Model

	Data	Model 1 Unconstrained	Model 2 Constrained 25% Foreign Exposure	Model 3 Constrained 86% Foreign Exposure
(A) Standard Deviation in %				
Output	2.06	2.54	1.92	1.78
Net Export	0.39	0.29	0.17	0.16
(B) Standard Deviation relative to Output				
Consumption	0.63	1.07	1.05	0.99
Investment	2.82	0.59	0.77	0.77
Labor	0.67	0.73	0.72	0.71
(C) Cross Correlation with Output				
Consumption	0.82	0.99	0.99	0.98
Labor	0.86	1	1	1
Investment	0.95	0.84	0.76	0.96
Net Export	-0.45	0.60	0.53	0.46
(D) Cross-Country Correlations				
Consumption	0.44	0.29	0.53	0.75
Output	0.61	0.24	0.40	0.53
Investment	0.46	0.82	0.64	0.30
Labor	0.43	0.23	0.41	0.55

Note: The first column shows the statistics calculated from the data. Panels (A), (B) and (C) are calculated from US time series for the period 1972:1 to 2008:4. The statistics from panel (D) represent the correlation of US series with series from the rest of the industrial world. The third column, "Model 2," is the benchmark model. The second column, "Model 1," is the same as Model 2 except that the investor does not face the leverage constraint. The last column, "Model 3," is the same as Model 2 except that the investors have more exposure to foreign capital.

Table 3.3: Sensitivity Analysis - Adjustment Cost

	Data	Benchmark Model	Sensitivity Test	
		$\pi = 0.25$	$\pi = 0.5$	$\pi = 100$
(A) Standard Deviation in %				
Output	2.06	1.92	2.04	2.41
Net Export	0.39	0.17	0.25	0.34
(B) Standard Deviation relative to Ouput				
Consumption	0.63	1.05	1.10	1.21
Investment	2.82	0.77	0.49	0.00
Labor	0.67	0.72	0.72	0.73
(C) Cross Correlation with Output				
Consumption	0.82	0.99	0.99	0.99
Labor	0.86	1	1	1
Investment	0.95	0.76	0.77	0.80
Net Export	-0.45	0.53	0.50	0.42
(D) Cross-Country Correlations				
Consumption	0.44	0.53	0.61	0.78
Output	0.61	0.40	0.46	0.62
Investment	0.46	0.64	0.79	0.93
Labor	0.43	0.41	0.48	0.63

Note: The first column shows the statistics calculated from the data. The second column is the benchmark model. The last two columns are for different values of the investment adjustment cost.

Table 3.4: Sensitivity Analysis - Leverage and Shocks

	Data	Benchmark Model	Sensitivity Test	
			High Leverage	BKK
(A) Standard Deviation in %				
Output	2.06	1.92	2.31	1.67
Net Export	0.39	0.17	0.23	0.30
(B) Standard Deviation relative to Ouput				
Consumption	0.63	1.05	1.13	1.13
Investment	2.82	0.77	0.57	0.45
Labor	0.67	0.72	0.73	0.70
(C) Cross Correlation with Output				
Consumption	0.82	0.99	0.99	0.98
Labor	0.86	1	1	1
Investment	0.95	0.76	0.77	0.72
Net Export	-0.45	0.53	0.50	0.57
(D) Cross-Country Correlations				
Consumption	0.44	0.53	0.59	0.63
Output	0.61	0.40	0.46	0.34
Investment	0.46	0.64	0.72	0.58
Labor	0.43	0.41	0.48	0.32

Note: The first column shows the statistics calculated from the data. The second column is the benchmark model. The third column is the model with a leverage ratio of 5. The last column is the model with the technology process from Backus, Kehoe and Kydland (1992).

Table 3.5: Sensitivity Analysis - IE of Goods

		Benchmark Model	Sensitivity Test	
	Data	$e = 0.9$	$e = 0.5$	$e = 0.1$
(A) Standard Deviation in %				
Output	2.06	1.92	1.55	1.45
Net Export	0.39	0.17	0.21	0.22
(B) Standard Deviation relative to Ouput				
Consumption	0.63	1.05	0.88	0.83
Investment	2.82	0.77	0.98	1.05
Labor	0.67	0.72	0.57	0.52
(C) Cross Correlation with Output				
Consumption	0.82	0.99	0.99	0.99
Labor	0.86	1	1	0.99
Investment	0.95	0.76	0.84	0.85
Net Export	-0.45	0.53	0.56	0.57
(D) Cross-Country Correlations				
Consumption	0.44	0.53	0.49	0.48
Output	0.61	0.40	0.34	0.32
Investment	0.46	0.64	0.72	0.73
Labor	0.43	0.41	0.28	0.23

Note: The first column shows the statistics calculated from the data. The second column is the benchmark model. The last two columns are for different values of elasticity of substitution between home and market goods.

Figure 3.1: IRF Comparison: Unconstrained vs Constrained Economy

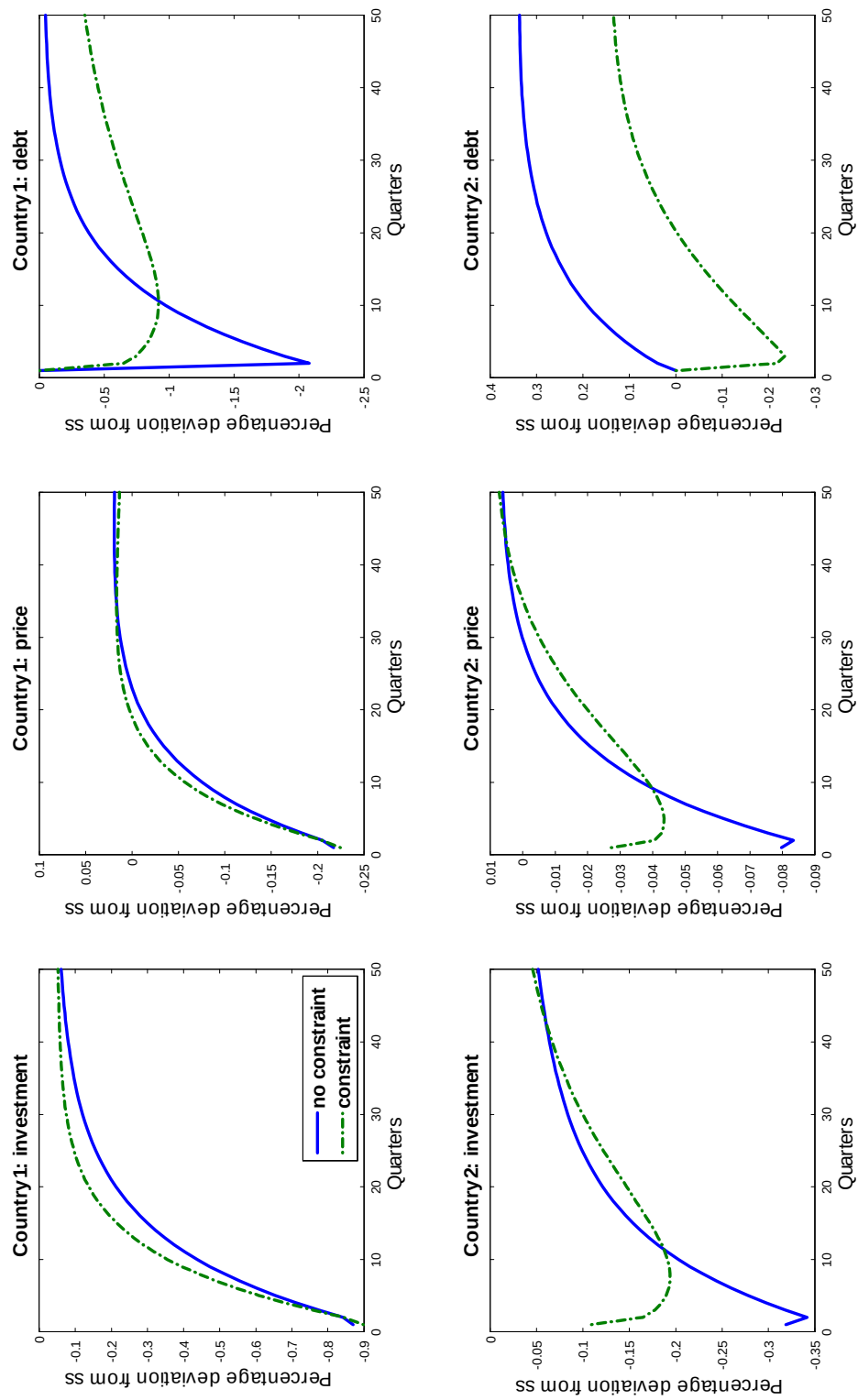


Figure 3.2: IRF Comparison: Unconstrained vs Constrained Economy

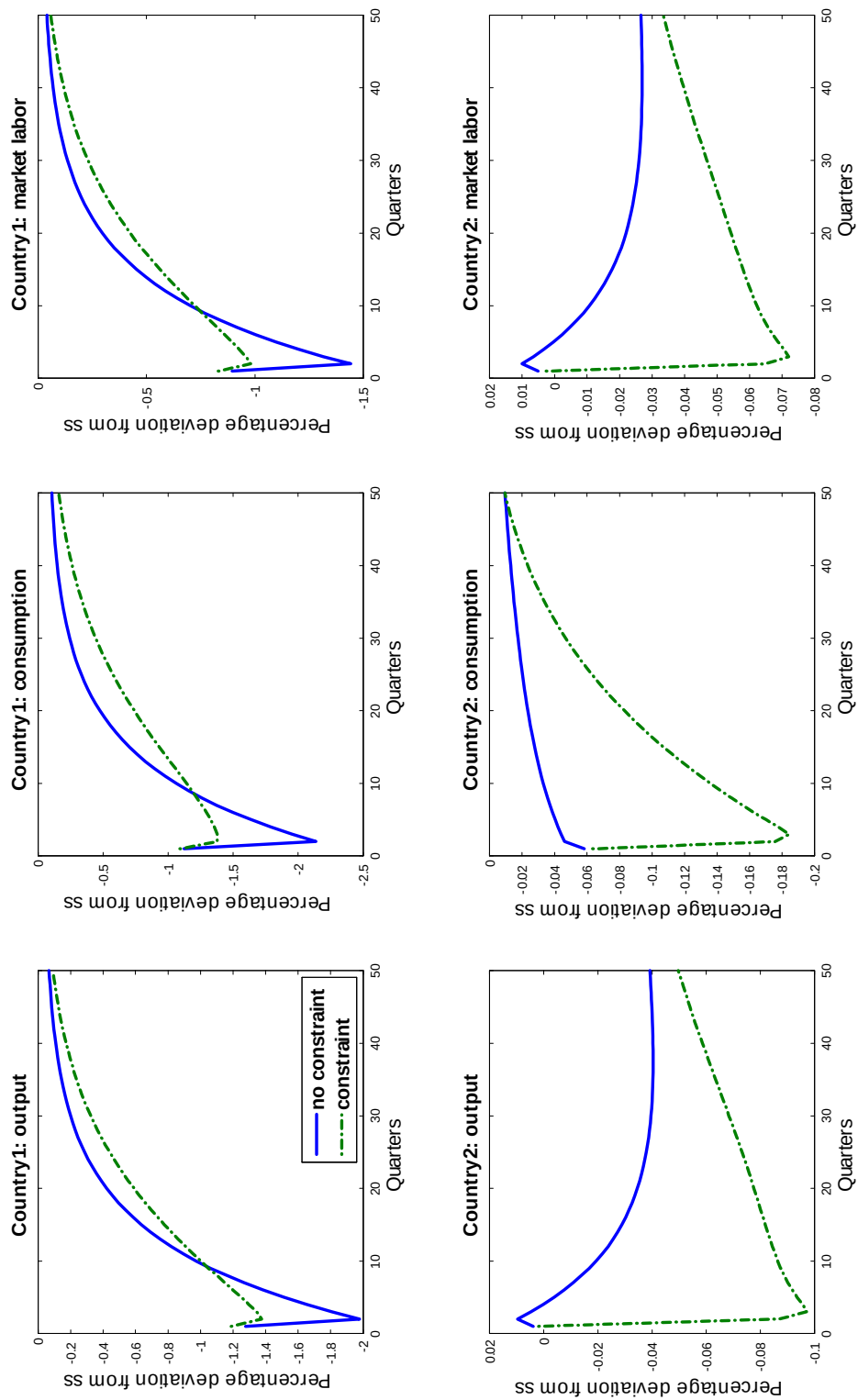


Figure 3.3: IRF Comparison: Unconstrained vs Constrained Economy

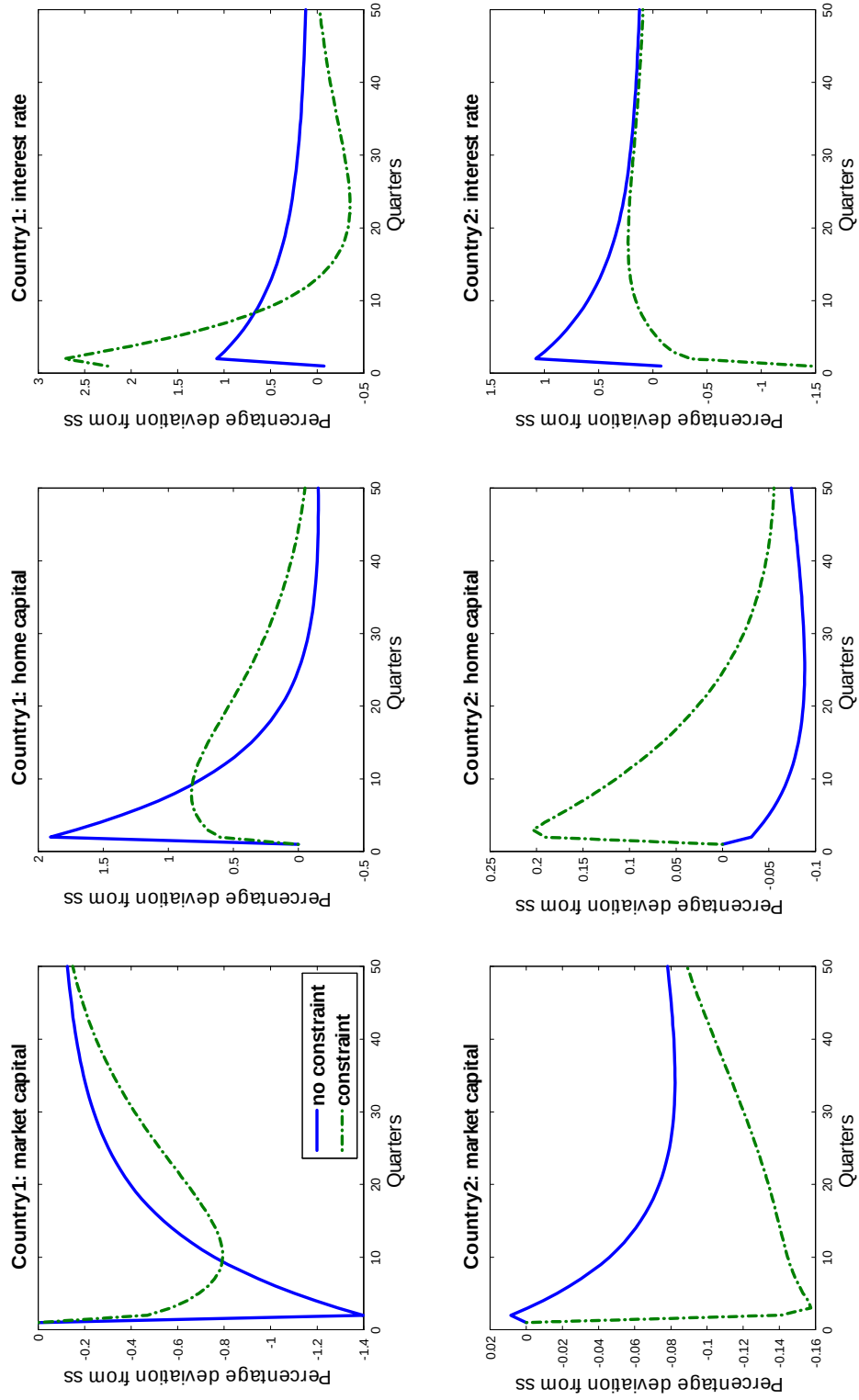


Figure 3.4: IRF Comparison: Unconstrained vs Constrained Economy

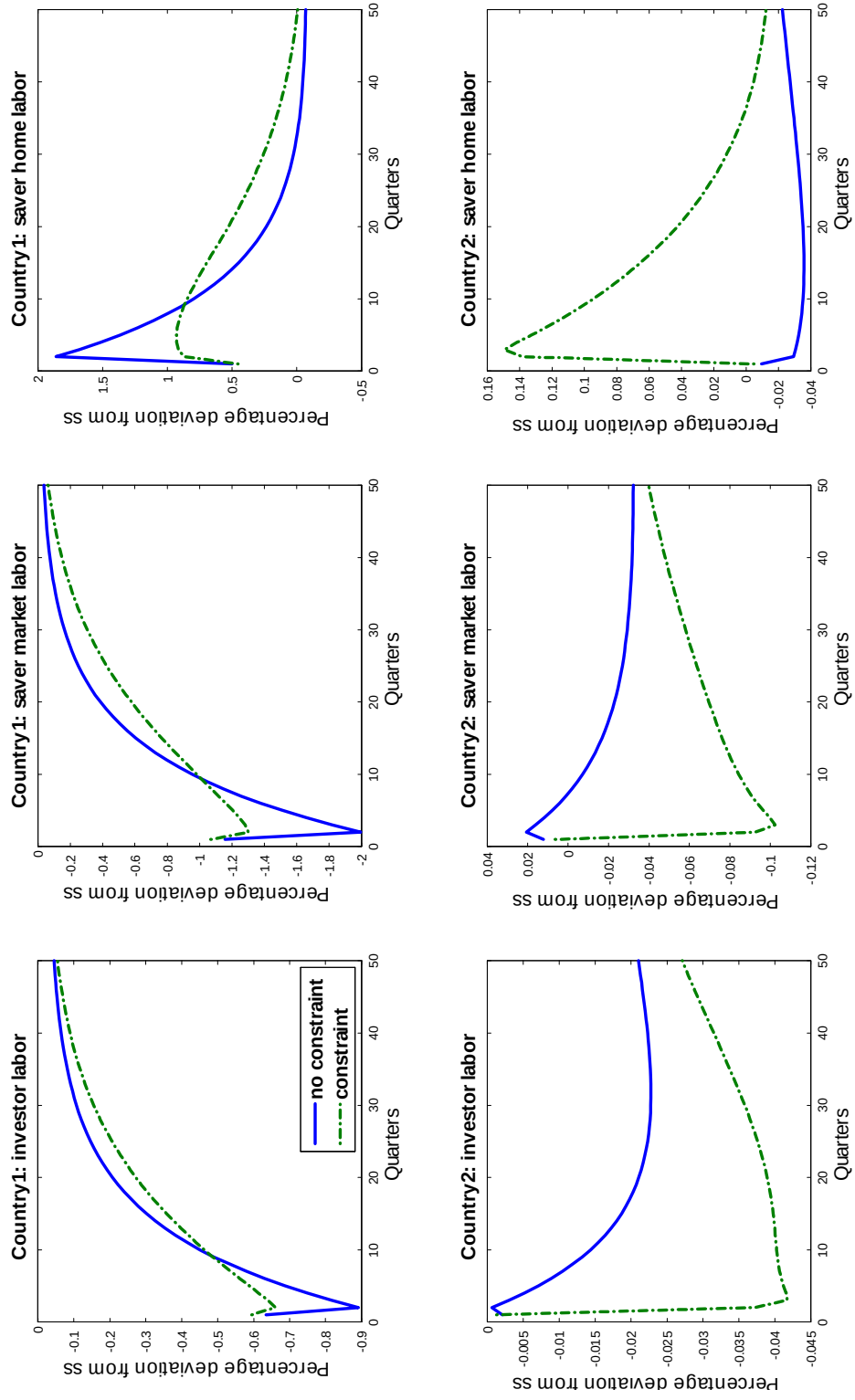


Figure 3.5: IRF for different degrees of foreign exposure

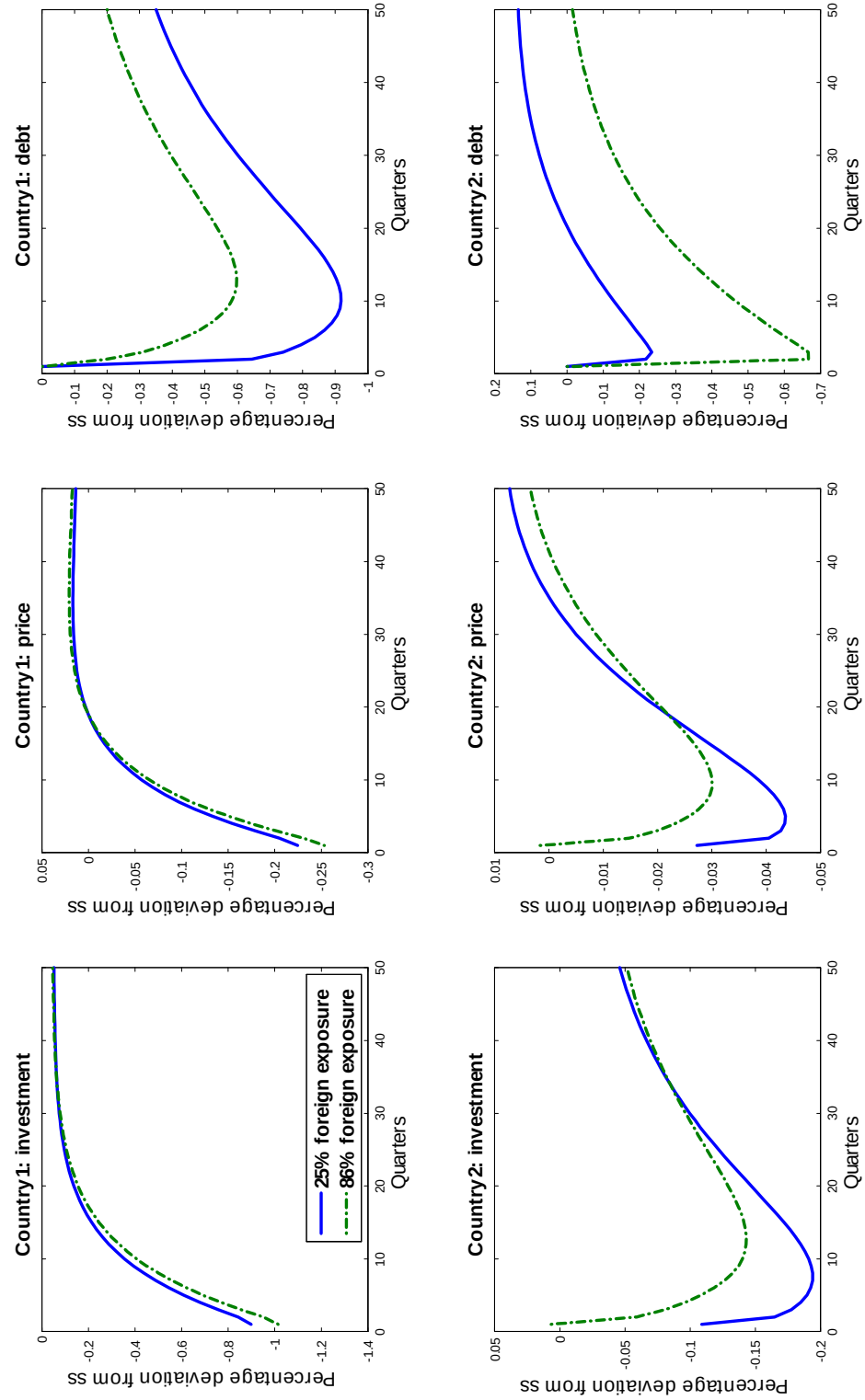


Figure 3.6: IRF for different degrees of foreign exposure

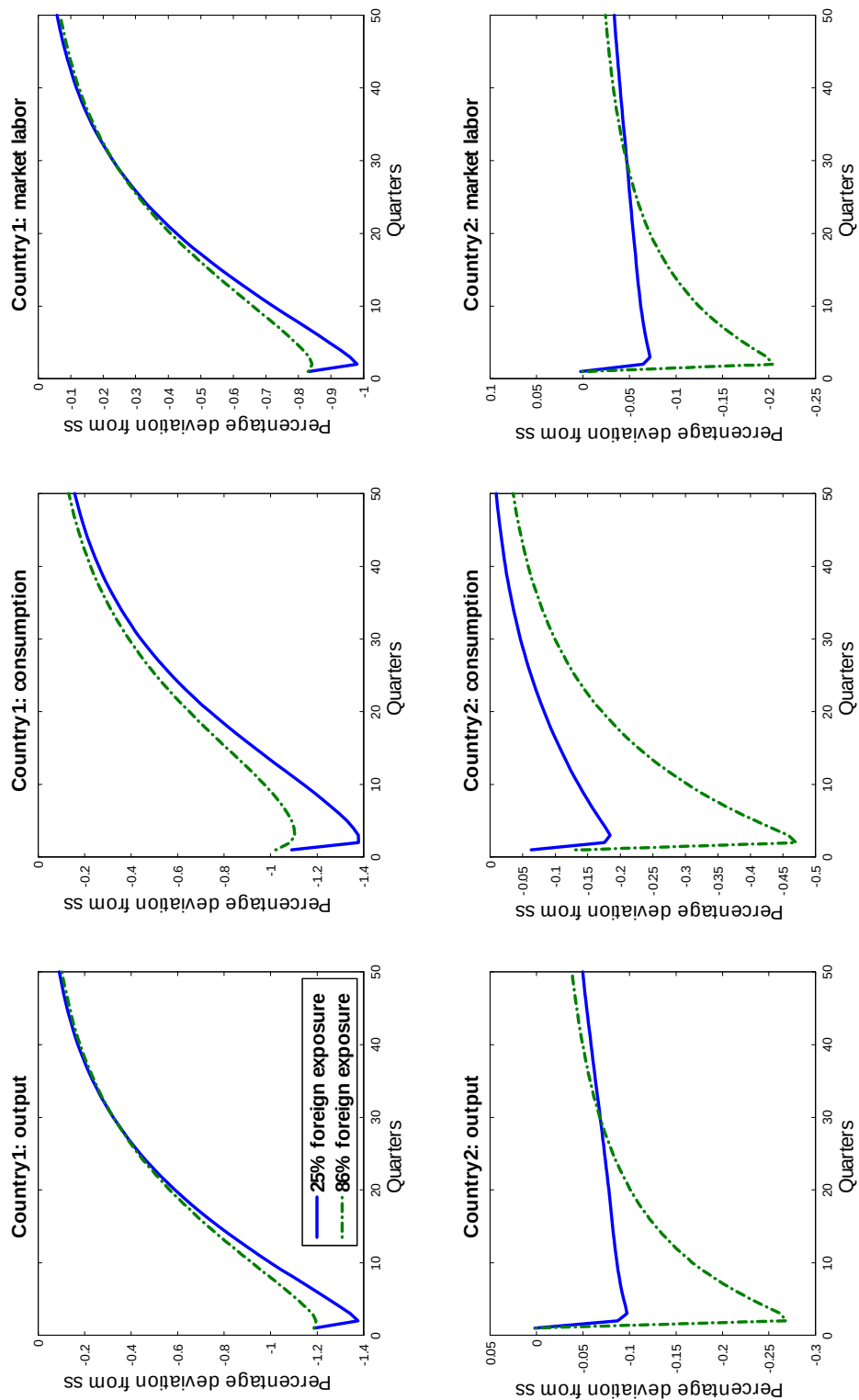


Figure 3.7: IRF for different degrees of foreign exposure

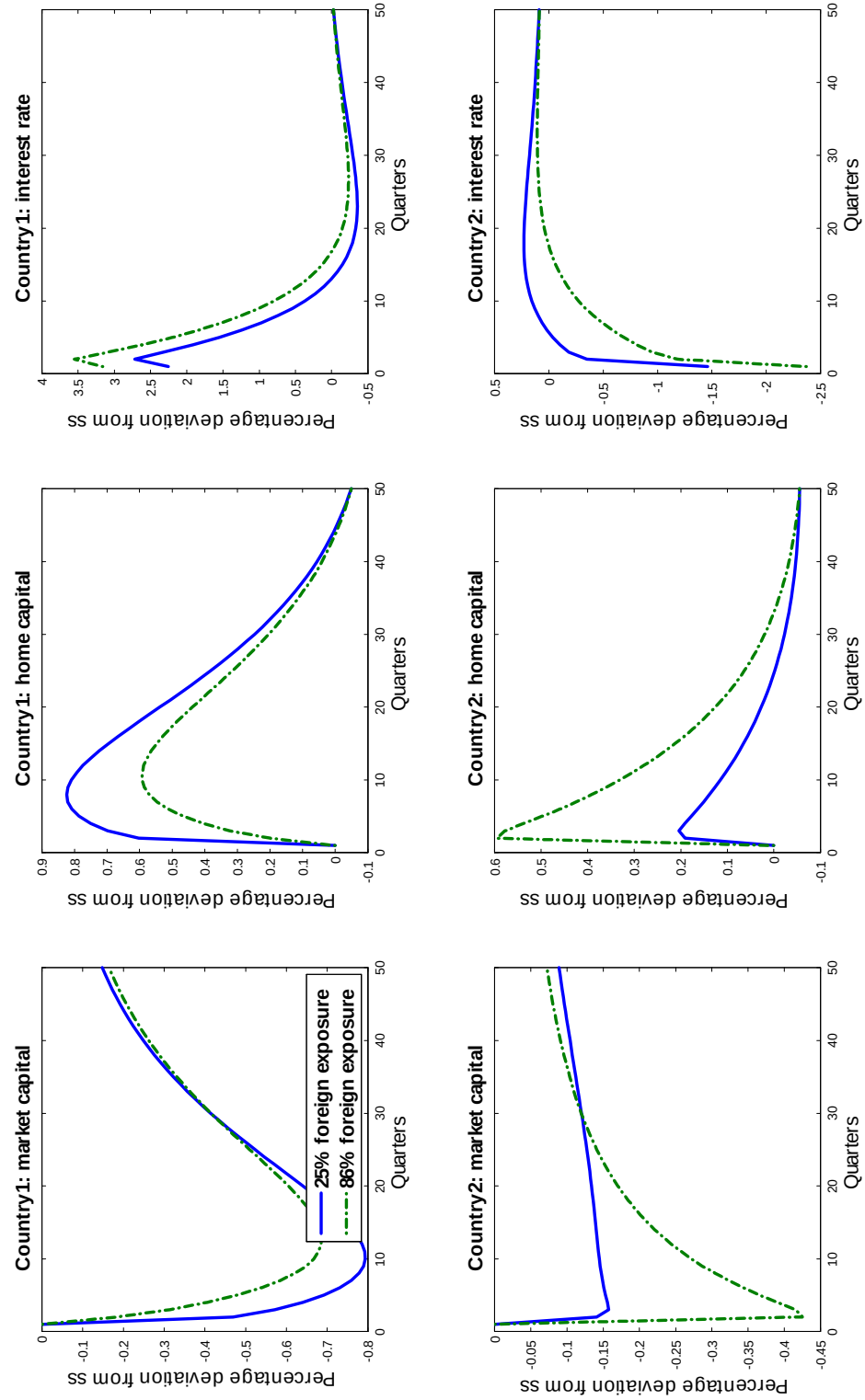
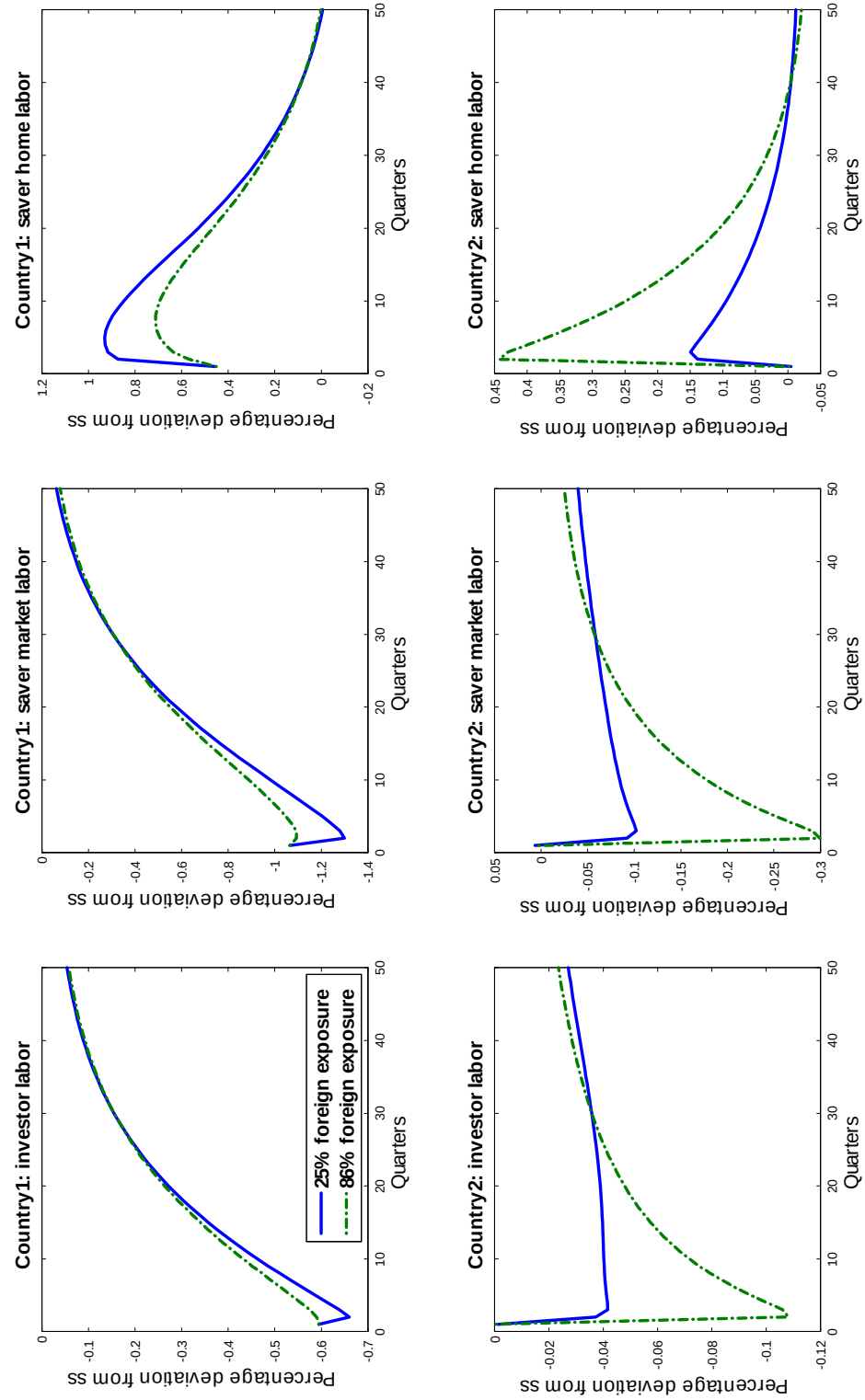


Figure 3.8: IRF for different degrees of foreign exposure



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