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Stress Tensor Perturbations in Conformal Field Theory

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We reconsider the problem of deforming a conformal field theory to a neighboring theory which is again critical. An invariant formulation of this problem is important for understanding the underlying symmetry of string theory. We give a simple derivation of A. Sen's recent formula for the change in the stress tensor and show that, when correctly interpreted, it is coordinate-invariant. We give the corresponding superconformal perturbation for superfield backgrounds and explain why it has no direct analog for spin-field backgrounds.

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1. Introduction

Suppose that one were handed the space of solutions to Yang-Mills on a compact manifold. This is a large and complicated space: large because it contains all gauge copies, complicated because it contains all the instanton moduli spaces. One is now told that this space sits in a still larger, but much simpler, space as the solutions to a simple differential equation. How does one discover that the larger space is the space of Yang-Mills connections, particularly if the solution space is presented in some awkward coordinates having little relation to the convenient A_μ^a?

This hypothetical situation may seem far-fetched, but it is of course our present dilemma in string theory. We have a clear characterization of the *solution* space as the space of (super) conformal field theories with $c = 0$ and a specific ghost sector. We can for example characterize such theories by weights and operator products, or sometimes by spacetime background fields. These coordinates may well be unrelated to the good coordinates on the full configuration space.

In the Yang-Mills case a good move would be to examine the solution space for its *symmetries*. In fact we can find the full symmetry group of configuration space just by examining *solution* space. Indeed every nontrivial gauge transformation acts nontrivially on some solution. We can then seek coordinates on which the symmetries act in a simple way, then discover the full space of connections. Can we do as much for string theory?

Certainly our first steps should be to get an abstract characterization of infinitesimal gauge symmetry, one not tied to weak background fields about flat spacetime. But we can take a hint from string perturbation theory. There it is known that perturbation by *spurious* states decouple, and so can be interpreted as symmetry transformations. Accordingly we abstract the idea that at least some BRST-exact deformations of conformal field theories should be regarded as small gauge transformations. Finding the Lie algebra of these transformations for closed strings is a challenging task; good coordinates must be found in which the structure 'constants' are constant.

Moreover it is not clear that all BRST-exact deformations will be symmetries. Indeed by studying sigma models Evans and Ovrut have found an important condition for a deformation to be a gauge deformation [1]. We will return to this point, but let us note here that a key element of their analysis was the explicit construction of the change in the stress tensor as we deform the theory. It is this change which we will reconsider here.

The stress tensor is an operator-valued (2,0)-form which we build for any $c = 0$ CFT. It is a convenient probe for distinguishing different theories. Conversely we can think of $T(z)$ as partially *defining* the theory, and attempt to *construct* a new theory by modifying $T(z)$. This is the approach taken in [1]. However we regard T, it is a useful device not only for studying symmetry in string theory [2][1], but also for investigating the background-dependence of string field theory [3] and the structure of auxiliary fields and their symmetries in string-induced supergravity theories [4][5][6][7].

Also of course the deformation of CFT is a problem of independent mathematical interest. There is a generalization of Kodaira-Spencer theory in which the BRST operator plays the role of the Čech differential. In this language the gauge deformations are given by trivial Čech classes.

Recently A. Sen has given a formula for the perturbation of the stress tensor as we change the theory [3]:

$$L_n + \delta L_n = L_n - \frac{\Delta}{2\pi i} \oint_{|z|=\epsilon} \mathrm{d}\bar{z}\; z^{n+1} \Phi_{z\bar{z}}(z,\bar{z}) \quad . \tag{1.1}$$

Here $\Phi_{z\bar{z}}$ is a conformal field of weight $(1,1)$ and Δ is a small number. Various features of this formula are at first sight puzzling. The stress tensor should characterize the theory, but δL_n seems to depend on a cutoff ϵ, and indeed on a coordinate z. The coordinate-dependence enters both through the field Φ and through the choice of contour, which matters since Φ is not holomorphic. This contour arises also in the formula of [1]

$$T(z) + \delta T(z) = T(z) + \Delta\Phi(z,\bar{z}) \;, \qquad \text{on the cylinder} \tag{1.2}$$

which is valid only on one equal-time contour $\tau = \text{const}$ and so again appears coordinate-dependent. Finally in [7] we find (this time on $|z| = \text{const}$)

$$T_{zz}(z) + \delta T_{zz}(z) = T_{zz}(z) + \frac{\bar{z}}{z}\; \Phi_{z\bar{z}}(z,\bar{z}) \quad \text{on the plane} \tag{1.3}$$

which not only seems to be coordinate-dependent, but also gives an apparent singularity in the o.p.e. $T(z)\bar{T}(\bar{w})$ [7]!

In fact all of these formulas are correct when suitably interpreted. We will rederive (1.1) from scratch using a geometrical approach which absolutely *guarantees* that the $c=0$ Virasoro algebra will be satisfied, without any calculations. The reader may want to pass directly to this derivation, section three. We then argue that (1.1) is in fact coordinate-independent, again without calculation. Along the way we will review the approach to CFT presented by G. Segal, and in particular the notion of conformal field. This will make it clear why no formula like (1.1) can be expected to work when Φ is a spin field, or space-time fermion vertex operator, as found empirically in [5][7]. We will however generalize to the case where Φ is a spacetime boson; this is very easy with our geometrical construction.

2. Conformal Fields

2.1. Operator formalism

We must briefly review some key ideas codified in [8], all of which essentially appear in [9][10] and elsewhere.

A conformal field theory with $c=0$ is essentially a machine taking Riemann surfaces with holes to vectors in a state space $\mathcal{H}$, its dual $\mathcal{H}^*$, or their various tensor products. Let us make this a bit more precise. If Σ is a Riemann surface without holes, suppose ζ is a local complex coordinate, a function $\zeta : \Sigma \to \mathbf{C}$ well defined in some neighborhood of $\zeta = 0$. We momentarily suppose that $\zeta^{-1} : \mathbf{C} \to \Sigma$ is well defined throughout the unit disk $D = \{|z| < 1\}$ where z is the standard coordinate on $\mathbf{C}$. Then we can delete the disk $D_\zeta = \zeta^{-1}(D)$ from Σ to get $\Sigma \backslash D_\zeta$. If ζ^{-1} is not well-defined on all of D we can always rescale ζ to a new ζ' which is.

A CFT then assigns to Σ a number, the partition function; to (Σ, ζ) a vector $|\Sigma, \zeta\rangle \in \mathcal{H}$; to $(\Sigma, \zeta_1, \zeta_2)$ a bivector $|\Sigma, \zeta_1, \zeta_2\rangle \in \mathcal{H} \otimes \mathcal{H}$, and so on. A perhaps obvious point which will later prove crucial is that the vector $|\Sigma, \zeta_1, \zeta_2 \dots \zeta_k\rangle \in \mathcal{H}^{\otimes k}$ depends only on the *isomorphism class* of $(\Sigma, \zeta_1, \dots, \zeta_k)$ as a Riemann surface with chosen local coordinates. Thus if $(\widetilde{\Sigma}, z_1, \dots, z_k)$ is another Riemann surface with coordinates and if we find an analytic isomorphism

$$\Phi\colon \Sigma \backslash \left(D_{\zeta_1} \cup D_{\zeta_2} \cdots\right) \xrightarrow{\sim} \widetilde{\Sigma} \backslash \left(D_{z_1} \cup D_{z_2} \cdots\right) , \quad \text{with} \quad \zeta_i = z_i \circ \Phi \tag{2.1}$$

on some neighborhoods of the punctures, then the two vectors must agree. This requirement concisely summarizes both conformal invariance and modular invariance. We may phrase it even more concisely as the requirement of "no additional data." All of our constructions must operate on $(\Sigma, \zeta_1, \dots)$ with no choices of additional data (metric, marking, *etc.*) on Σ.

Consider the sphere $\mathbf{P}^1$ regarded as the z-plane plus a point. Then z, z^{-1} are coordinates well-defined and centered at 0, ∞, so $|\mathbf{P}^1, az, (az)^{-1}\rangle \in \mathcal{H} \otimes \mathcal{H}$ for $|a| > 1$. Considering the isomorphism $z \to z^{-1}$ we see that this bivector is *symmetric*. We require of any CFT that it be a *nondegenerate* form on $\mathcal{H}^* \otimes \mathcal{H}^*$ and so (taking $a \to 1$) defines a bilinear metric on $\mathcal{H}$.[1] Accordingly we can attach to each hole on Σ an orientation: if it

[1] We will have no need for the *hermitian* metric induced by the Minkowski structure [8]. Evidently, all our constructions are formal since $\mathcal{H}$ is infinite-dimensional. In practice $\mathcal{H}$ is graded and finite-dimensional in each grade, or else the tensor product of this with something simple.

matches the induced orientation we use the vector $|\Sigma, \zeta\rangle$ above; otherwise we use the transpose vector $\langle\Sigma, \zeta| \in \mathcal{H}^*$ constructed using the metric. Similarly with two holes $|\Sigma, \zeta_1, \zeta_2\rangle$ can be converted into an *operator* on $\mathcal{H}$. In particular $|\mathbf{P}^1, z, z^{-1}\rangle$ is by construction $\mathbf{1}$.

Given two Riemann surfaces with coordinates $(\Sigma_L, \zeta_1^L, \ldots)$, $(\Sigma_R, \zeta_1^R, \ldots)$ we can sew them in the usual way by removing $\{\zeta_i^L = 0\}$ and $\{\zeta_j^R = 0\}$ and identifying $P_L \sim P_R$ when

$$\zeta_i^L(P_L) = 1/\zeta_j^R(P_R) \quad . \tag{2.2}$$

This construction is natural, *i.e.* the isomorphism class of the joined surface depends only on the isomorphism classes of the original surfaces under (2.1). Then it makes sense to demand of a CFT that the vector associated to the joined surface should be the product of the vectors associated to the original surfaces:

$$|(\Sigma_L, \zeta_1^L, \ldots)\infty_{ij}(\Sigma_R, \zeta_1^R, \ldots)\rangle = \langle\Sigma_L, \zeta_1^L, \ldots|\Sigma_R, \zeta_i^R, \ldots\rangle_{ij} \tag{2.3}$$

where the notation means the dual pairing of the i-th copy of $\mathcal{H}^*$ with the j-th copy of $\mathcal{H}$; we choose opposite orientations for these holes. ∞_{ij} denotes the geometrical operation of sewing.

The sewing axiom just expresses locality of field theory. In path integral language it says that we must be able to cut spacetime, impose matching boundary conditions on each side of the cut, do two separate path integrals, and then sum over all boundary data to obtain the original path integral.

These three axioms—no additional data, nondegeneracy, and "sewing" (2.3) —are the main ingredients in conformal field theory.[2] We should note, however, that when we consider *families* of CFT's, for example in the deformation problem, a new subtlety will arise: the vector space $\mathcal{H}$ will itself depend on the theory. Suppose we consider only infinitesimal deformations by $\Delta \ll 1$ about a generic theory, so that all the $\mathcal{H}_\Delta$ can be identified. In general there will still be some freedom in *how* we identify them, *i.e.* in trivializing the bundle of state spaces over theory space. We can readily see this freedom in the above axioms. Given a CFT, let us construct a new one by letting $|\Sigma, \zeta\rangle^\sim \equiv U|\Sigma, \zeta\rangle$, where U is a constant invertible matrix. Then ${}^\sim\langle\Sigma, \zeta| = \langle\Sigma, \zeta \mid U^{-1}$ and so ${}^\sim\langle\Sigma_L, \zeta^L \mid \Sigma_R, \zeta^R\rangle^\sim = \langle\Sigma_L, \zeta^L \mid \Sigma_R, \zeta^R\rangle$, so the "new" theory again obeys the sewing axiom. In fact we have done nothing but change the framing for $\mathcal{H}$, a trivial passive transformation. All operators $|\Sigma, \zeta_1, \zeta_2\rangle$ simply suffer an inner automorphism by U.

[2] The other axioms of [8] will not be of interest here.

Passive coordinate transformations in general have no physical significance. In some cases however, they imply true (dynamical) symmetries. For example in special relativity those coordinate transformations preserving the metric are symmetries because the metric is the only geometrical object which must be chosen to define the action. Similarly Evans and Ovrut found that some, but not all, inner automorphisms correspond to low-energy symmetries of string theory. In background-independent language their operator U must be generated by the contour integral of a local current. It would be extremely interesting to work backwards and find the hidden geometrical object (analogous to the 4-metric hiding in Maxwell's equations) which is preserved when this criterion is met. Just as in relativity one can then explore what happens when this object is made dynamical.

2.2. Virasoro algebra

Again consider the sphere $\mathbf{P}^1$. The surface $(\mathbf{P}^1, \varphi\circ z, z^{-1})$ has the property that when joined to $(\Sigma, \zeta_1, \ldots)$ at the i-th hole it yields $(\Sigma, \zeta_1, \ldots, \varphi\circ\zeta_i, \ldots)$. We define the generator ℓ_n of coordinate changes by the map

$$\varphi(z) = z - \epsilon z^{n+1} \quad ,$$

for small ϵ. The operator $|\mathbf{P}^1, \varphi\circ z, z^{-1}\rangle$ is then close to $\mathbf{1}$. Expanding it as

$$\left|\mathbf{P}^1, z - \sum_n \epsilon_n z^{n+1}, z^{-1}\right\rangle \equiv \mathbf{1} + \sum_n (\epsilon_n L_{-n} + \bar{\epsilon}_n \bar{L}_{-n}) \tag{2.4}$$

defines operators L_n and $\bar{L}_n$. We have dropped order $|\epsilon|^2$ terms, but since the ϵ are complex we must expand in both ϵ and $\bar{\epsilon}$, where $\bar{\epsilon}$ is the complex conjugate. Note that $\bar{L}_n$ need not be conjugate to L_n. In (2.4) we take the second puncture to live in $\mathcal{H}^*$, the first in $\mathcal{H}$. Thus the sewing property (2.3) says that

$$|\Sigma, \zeta - \epsilon\zeta^{n+1}\rangle = (\mathbf{1} + \epsilon L_{-n} + \bar{\epsilon}\bar{L}_{-n})|\Sigma, \zeta\rangle \ , \qquad n \geq 0 \quad . \tag{2.5}$$

We can similarly define L_{-n} by (2.4) for $n < 0$; such transformations then change the shape of Σ or the location of the hole instead of just changing the shape of the hole as in (2.5).

We see from (2.4) that the L_n are universal operators quite independent of the surface Σ to which we may apply them in (2.5). Let us consider two such transformations in succession. If $\psi(z) = z - \delta z^{m+1}$, then $\psi \circ \varphi(z) = z - \epsilon z^{n+1} - \delta z^{m+1} + \epsilon\delta(m+1)z^{n+m+1}$. We will drop order ϵ^2 or δ^2 terms. We thus get

$$|\mathbf{P}^1, z-\epsilon z^{n+1}-\delta z^{m+1}+\epsilon\delta(m+1)z^{n+m+1}, z^{-1}\rangle = (\mathbf{1}+\delta L_{-m}+\bar{\delta}\bar{L}_{-m})(\mathbf{1}+\epsilon L_{-n}+\bar{\epsilon}\bar{L}_{-n}) \quad .$$

We can now write the lhs as $\mathbf{1} - (m+1)(\epsilon\delta L_{-n-m} + \bar{\epsilon}\bar{\delta}\bar{L}_{-n-m})$ plus terms symmetric under $\epsilon \leftrightarrow \delta$, $n \leftrightarrow m$. Making the exchange and subtracting we thus find that

$$[L_n, L_m] = (n-m)L_{n+m}, \quad [\bar{L}_n, \bar{L}_m] = (n-m)\bar{L}_{n+m} \ , \quad [L_n, \bar{L}_m] = 0 \quad , \tag{2.6}$$

the algebra $\mathrm{Vect} \oplus \overline{\mathrm{Vect}}$ of meromorphic vector fields on $\mathbf{C}$ and its conjugate.

We have repeated this well-known derivation for two reasons. First, we wished to emphasize that the algebra (2.6) arises simply as a direct geometrical consequence of the Lie bracket algebra of vector fields. We derived it purely from the axioms of section 2.1. When those axioms are satisfied and L_n is defined via (2.4) there is *no need* to do any computation to verify (2.6), nor (we will see) the Ward identity. Secondly, there is no need to pretend that $z, \bar{z}$ are somehow independent in order to get two commuting copies of Vect.[3]

We note in passing that the surface $(\mathbf{P}^1, z - \epsilon z^{n+1}, z^{-1})$ is isomorphic to $(\mathbf{P}^1, z, z^{-1} - \epsilon(z^{-1})^{-n+1})$. Equating the two states obtained from (2.5), we get $\mathbf{1} + \epsilon L_{-n} + \bar{\epsilon}\bar{L}_{-n} = 1 + \epsilon L_n^T + \bar{\epsilon}\bar{L}_n^T$ where the operator adjoint comes from taking the dual on the second copy of $\mathcal{H}$. Thus $L_n^T = L_{-n}$ again follows from axioms already stated, and we have

$$\langle \Sigma, \zeta - \epsilon\zeta^{n+1}| = \langle \Sigma, \zeta|(\mathbf{1} + \epsilon L_n + \bar{\epsilon}\bar{L}_n) \quad . \tag{2.7}$$

Now that we have an operator $L_0 + \bar{L}_0$ which rescales the coordinate ζ, we see that we need not literally cut out the unit disk D_ζ from Σ to define $|\Sigma, \zeta\rangle$. Instead we may rescale ζ to get some conveniently small disk $D_{\zeta/q}$ and let $|\Sigma, \zeta\rangle = q^{-(L_0+\bar{L}_0)}|\Sigma, \zeta/q\rangle$.

Using the L_n, $\bar{L}_n$ we can classify states as usual [9]. We now wish to recall why a primary state $\psi \in \mathcal{H}$ of weight $(h, \bar{h})$ gives rise to a rank $(h, \bar{h})$ tensor field $\langle\psi(P)\rangle_\Sigma$ on Σ. Choose any point on Σ and a coordinate ζ centered there. Then the number $\langle \Sigma, \zeta \mid \psi\rangle$

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[3] The reader may ask how a central term can arise. In fact when $c \neq 0$ the axioms of 2.1 fail; the vector $|\Sigma, \zeta\rangle$ is only projectively defined [8].

has very little dependence on ζ. Transforming ζ by ℓ_n, $n>0$ has no effect at all because $L_n\psi=0$, $n>0$, while $\zeta'=(1+\epsilon)\zeta$ gives a factor of

$$1-\epsilon h-\bar{\epsilon}\bar{h}=\left(\frac{\partial\zeta'}{\partial\zeta}\right)^{-h}\left(\frac{\partial\bar{\zeta}'}{\partial\bar{\zeta}}\right)^{-\bar{h}}$$

since $L_0\psi=h\psi$, $\bar{L}_0\psi=\bar{h}\psi$. Hence the form

$$\langle\psi(P)\rangle_\Sigma\equiv\langle\Sigma,\zeta\mid\psi\rangle(\mathrm{d}\zeta|_P)^h(\mathrm{d}\bar{\zeta}|_P)^{\bar{h}}\ ;\quad \zeta(P)=0 \tag{2.8}$$

is independent of the choice of ζ. Varying P we get a tensor field on Σ.

When ψ is *not* primary then we must indicate its ζ-dependence explicitly. One convenient notation [11] is

$$\langle{:}\psi(P){:}_\zeta\rangle_\Sigma\equiv\langle\Sigma,\zeta\mid\psi\rangle\ ;\qquad \zeta(P)=0\quad .$$

This makes sense, since a change of ζ changes the mode expansion of ψ and is precisely a change of normal ordering.

2.3. Operator fields

In quantum field theory it is often important to think of fields not just in terms of their correlations, but as *operator*-valued differential forms, mapping a suitable "in" state space to an "out" space. Normally this poses few problems. We choose a Lorentz frame and equal-time hyperplanes; we restrict functional integrals to the region between $t=\pm T$ with appropriate vacuum boundary conditions on each, eventually taking $T\to\infty$. Changing to a different Lorentz frame changes the hyperplanes, but this is easily compensated. A unitary operator $U(\Lambda)$ changes the state associated to one hyperplane to the other. Since the theory is Lorentz invariant, nothing changes if we subject everything to the transformation Λ, and so for example scalar field operators obey

$$U(\Lambda)^\dagger\Phi(x)U(\Lambda)=\Phi(\Lambda^{-1}x) \tag{2.9}$$

in the full interacting theory.

In quantum gravity we have much more symmetry. Spacetime can be very complicated; the initial and final surfaces may be arbitrary hypersurfaces. But CFT lies somewhere between these two extremes. Spacetime is again complicated, but we can take all of our hypersurfaces to be of the form $|\zeta|=1$ where ζ is some analytic coordinate. We already know the analog of $U(\Lambda)$ for changes of ζ, so we again get a Ward identity like (2.9).

Specifically given a CFT and a primary ψ, we define the field operator Ψ on the plane by

$$\Psi(P;z) \equiv \langle \mathbf{P}^1, z, z^{-1}, u \mid \psi\rangle (\mathrm{d}u|_P)^h (\mathrm{d}\bar{u}|_P)^{\bar{h}} \quad . \tag{2.10}$$

Here P is a point on the z-plane and u is any coordinate centered at P; z is the usual coordinate centered at the origin.[4] As before the choice of u drops out. Note however that $\Psi(P;z)$ has a *functional* dependence on z. Since we agree to think of larger radius (closer to ∞) as 'later' time, we make (2.10) into an operator by taking the dual on the copy of $\mathcal{H}$ associated to z, the coordinate centered on 0. Thus Ψ eats 'in' state coming from 0.

One simple choice of u in (2.10) is $u = z - z(P)$; then we abbreviate

$$\Psi_{zz\ldots}(P) \equiv \langle \mathbf{P}^1, z, z^{-1}, z - z(P) \mid \psi\rangle_P \quad . \tag{2.11}$$

As usual we write h unbarred and $\bar{h}$ barred indices. Note that (2.11) does not treat the in- and out-points symmetrically. The formulas (2.10), (2.11) make it clear that the conformal field Ψ depends not only on $\psi \in \mathcal{H}$ but also on the theory in question: if we identify the state spaces of two theories then the same ψ yields two very different fields. For example changing the hamiltonian $L_0 + \bar{L}_0$ changes the dependence of Ψ on the radius $|z|$.

Again we get a Ward identity analogous to (2.9). It says that since only the conformal structure enters into CFT, transforming *everything* in (2.11) by an isomorphism changes nothing; this is of course just the "no extra choices" axiom of section 2.1 again. Clearly it is not enough to transform Ψ as a tensor: we must also transform the in- and out-slices. Using (2.4), (2.7), we at once get

$$\Psi_{z'z'\ldots}(P') = \Psi_{zz\ldots}(P) \qquad \text{where} \quad z(P) = z'(P') \tag{2.12}$$

or

$$0 = z(P)^{n+1}\, \frac{\partial}{\partial z(P)}\, \Psi_{z\ldots}(P) + h(n+1) z(P)^n \Psi_{z\ldots}(P) - [L_n, \Psi_{z\ldots}(P)] \quad , \tag{2.13}$$

which is the usual Ward identity [9]. The commutator comes because $z^{-1} \mapsto z^{-1} + \epsilon(z^{-1})^{-n+1}$, but we take a dual on the inner hole's state space. Again we have given

[4] As mentioned earlier it does not matter that the in- and out-surfaces both coincide at $|z| = 1$; we can rescale both using $q^{(L_0 + \bar{L}_0)}$.

this derivation to emphasize that (2.13) is is a purely geometrical fact, an automatic consequence of the axioms of section 2.1. The conditions (2.13) are very restrictive; most operator-valued forms on C are not conformal fields at all. Since the conditions depend on L_n we again see that the notion of conformal field depends on the CFT.

Note that the idea of conformal field elaborated here needs no modification for superfields. We consider the family of spheres $(\mathbf{P}^{1|1}_{NS}, (z,\theta), (z^{-1}, \pm iz^{-1}\theta), (z - z(\hat{P}) - \theta\theta(\hat{P}), \theta - \theta(\hat{P}))$ where $z(\hat{P})$, $\theta(\hat{P})$ are two constants and $\mathbf{P}^{1|1}_{NS}$ is the usual super sphere. Inserting a primary Neveu-Schwarz state at $\hat{P}$ gives us a superconformal tensor-valued field [12], and identities like (2.13) follow. For spin fields the situation is quite different. It now makes no sense to let a point $\hat{P}$ move around on a *fixed* super Riemann surface $\mathbf{P}^{1|1}_{R}$. This is because the superconformal structure is required to degenerate at the moving point $\hat{P}$ [13][14], and also at one of the fixed points $0, \infty$. We can certainly invent a family of SRS with coordinates which behaves in this way, but it will give a much more complicated formula than (2.13).

2.4. Stress tensor

We can now construct an example of a conformal field. Letting

$$T_{zz}(P) = \sum_{n=-\infty}^{\infty} z(P)^{-n-2} L_n \tag{2.14}$$

we find that (2.13) is satisfied with $(h, \bar{h}) = (2, 0)$, using (2.6). Eqn. (2.14) would of course seem strange if we expected the lhs to transform simply as a tensor, since the operators L_n on the rhs are the same in every coordinate system. T deserves to be called the stress tensor because the L_n, which generate conformal transformations, are moments of T

$$L_n = T[-\ell_n] \equiv \frac{1}{2\pi i} \oint z^{n+1} T_{zz}(z)\, \mathrm{d}z \tag{2.15}$$

just as the usual Lorentz generators are moments of the usual stress tensor.

3. Deformation

3.1. Conformal case

Let us try to deform our CFT while preserving the axioms of section 2.1. The easiest way to ensure the sewing property (2.3) is to modify the partition function by the insertion of a local field, then integrate over the insertion point. Sewing will be satisfied if we further specify that on a surface with holes we integrate the new field over $\Sigma \backslash (D_{\zeta_1} \cup D_{\zeta_2} \ldots)$ only. We stress that due to (2.2) we want to exclude *unit* disks, not ϵ-disks, from this integral.[5] Then cutting the partition function into $\langle \Sigma_L, \xi \mid \Sigma_R, \zeta \rangle$, the change is $\int_\Sigma \langle \Phi(P) \rangle$, which equals the inner product of

$$|\Sigma_R, \zeta\rangle_\Delta \equiv |\Sigma_R, \zeta\rangle_0 + \frac{\Delta}{2\pi i} \int_{P \in \Sigma_R \backslash D_\zeta} {}_P\langle \phi \mid \Sigma_R, \zeta, u \rangle_0 \, \mathrm{d}u|_P \mathrm{d}\bar{u}|_P \tag{3.1}$$

times the corresponding left state up to order Δ. u is a coordinate centered at P; Δ is a real constant.

We need to make (3.1) a bit more precise. To maintain conformal and modular invariance the rhs must depend only on the isomorphism class of (Σ_R, ζ^R). The only things which can be invariantly integrated on Σ are densities, or equivalently tensor fields of rank (1,1). Thus we see that ϕ must be taken to be a primary state of weight (1,1): a vertex operator. Finally, we generalize (3.1) in the obvious way to allow for several holes on the lhs.

One consequence of (3.1) is immediate. The state $|\mathbf{P}^1, z, z^{-1}\rangle_\Delta = |\mathbf{P}^1, z, z^{-1}\rangle_0$, since there is no area between the two unit disks. Since this state defines the metric, the latter does not change. More invariantly, (3.1) implies that some identification of state spaces $\mathcal{H}_0 \cong \mathcal{H}_\Delta$ has been made; we see that this identification corresponds to a *unitary* connection.

We thus see that by taking the state space to be the same $\mathcal{H}$ as before and the state to be (3.1) we satisfy all three of our axioms and hence have a new CFT. As we have emphasized, the new theory has a new stress tensor which is guaranteed without any calculations to obey the Virasoro algebra. We can now write it down by combining (2.7) with (3.1).

Eqn. (2.7) says for $\zeta' = \zeta - \epsilon\zeta^{n+1}$

$$\begin{aligned} {}_0\langle \Sigma, \zeta'| &= {}_0\langle \Sigma, \zeta|(\mathbf{1} + \epsilon L_n + \bar{\epsilon}\bar{L}_n) + \mathcal{O}(\epsilon^2) \\ {}_\Delta\langle \Sigma, \zeta'| &= {}_\Delta\langle \Sigma, \zeta|(\mathbf{1} + \epsilon(L_n + \Delta X_n) + \bar{\epsilon}(\bar{L}_n + \Delta \bar{X}_n)) + \mathcal{O}(\epsilon^2) + \mathcal{O}(\Delta^2) \quad , \end{aligned} \tag{3.2}$$

[5] Since the actual size of the holes can be taken less than 1, we have no possibility of a singularity here.

where we write the modified generator as $L_n^{(\Delta)} \equiv L_n + \Delta X_n$ (recall Δ is real). Expanding both sides of (3.2) we get

$$
\begin{aligned}
{}_0\langle \Sigma, \zeta \mid (\epsilon X_n + \bar\epsilon \bar X_n) = & \frac{1}{2\pi i} \int_{Q \in \Sigma \setminus D_{\zeta'}} {}_0\langle \Sigma, \zeta, u | \phi \rangle_Q \, \mathrm{d}u|_Q \wedge \mathrm{d}\bar u|_Q \\
& - \frac{1}{2\pi i} \int_{Q \in \Sigma \setminus D_{\zeta}} {}_0\langle \Sigma, \zeta, u \mid \phi \rangle_Q \, \mathrm{d}u|_Q \wedge \mathrm{d}\bar u|_Q
\end{aligned}
\tag{3.3}
$$

where u is centered at Q. Thus X_n involves an integral over the signed area of $D_\zeta \setminus D_{\zeta'}$ (see Fig. 1). We dropped some terms of order ϵ^2 from the lhs of (3.3). Thus the response of the perturbed theory to a change of the region is the integral of the local perturbation over the new territory.

Fig. 1: The dotted regions get an extra minus sign in eqn. (3.3).

Since $D_\zeta \setminus D_{\zeta'}$ is very thin, both sides of (3.3) are of order ϵ. In fact to lowest order the integral is just the line integral along $|\zeta| = 1$ times the width of the region in Fig. 1. The width is just $-\frac{1}{2}(\epsilon z^n + \bar\epsilon \bar z^n)$ in the z-plane.

For example let Σ be the sphere with an additional hole. Since $(\mathbf{P}^1, z, z^{-1})$ just gives the unit operator, eqn. (3.3) gives

$$
\epsilon X_n + \bar\epsilon \bar X_n = \frac{1}{2\pi i} \int_0^{2\pi} \mathrm{d}\theta \cdot (-2i)(-\tfrac{1}{2})(\epsilon z^n + \bar\epsilon \bar z^n) \langle \mathbf{P}^1, z, z^{-1}, z - z(Q) | \phi \rangle_Q \, , \quad z(Q) = \mathrm{e}^{i\theta} \quad .
$$

We took $u = z - z(Q)$ and did the radial integral. Hence

$$
X_n = -\frac{1}{2\pi i} \oint_{|z(Q)|=1} \mathrm{d}\bar z \; z^{n+1} \Phi_{z\bar z}(Q) \quad ;
\tag{3.4}
$$

which is Sen's formula. Φ is the operator field of (2.11), in the *unperturbed* theory. Note that unlike (2.15), formula (3.4) requires a specific contour in the z-plane.

One can easily check that $L_n + \Delta X_n$, $\bar{L}_n + \Delta\bar{X}_n$ obey all of (2.6) as we argued they had to: using the methods of [15] we find $[L_n, X_m] = (1-m)X_{n+m}$ and $[\bar{L}_n, X_m] = 0$, from which (2.6) follows. One can also construct

$$T_{zz}^{(\Delta)}(z) = \sum_{-\infty}^{\infty} z^{-n-2}(L_n + \Delta X_n) \quad . \tag{3.5}$$

Unlike formulas (1.2), (1.3) this is valid throughout the plane and it is manifestly holomorphic in z, by fiat. Also (3.5) reduces to (1.3) for $|z| = 1$, or to (1.2) on the cylinder. And the Ward identity (2.13) is satisfied for $L_n^{(\Delta)}$ and $T_{zz}^{(\Delta)}(z)$, as we argued had to happen; in particular the operator product of $T^{(\Delta)}$ with $\bar{T}^{(\Delta)}$ is nonsingular in contrast to [7]. The root cause of the various apparent paradoxes with (3.4), (3.5) have to do with the fact that Φ in (3.4) is a conformal field for the original theory while $T^{(\Delta)}$ is a conformal field for the perturbed theory. As we stressed before these are distinct notions. Note that by itself the second term of (3.5) is not a conformal field in *either* sense, nor should it be.

The real question about (3.4) is as we indicated whether it makes coordinate-invariant sense. Since (3.5) satisfies the criterion of no dependence on additional choices, this too should follow automatically. Let us verify it.

We want to compare (3.4) to

$$X'_n = -\frac{1}{2\pi i} \oint_{|z'(P)|=1} \mathrm{d}\bar{z}' (z')^{n+1} \Phi_{z'\bar{z}'}(P) \quad .$$

To compare this to X_n we need to account for all the coordinate dependence of Φ, including the constant-time slices. But under the map $z \to z'(z)$ a point Q with $|z(Q)| = 1$ goes to a point P with $z'(P) = z(Q)$; by (2.12) we have $\Phi_{z\bar{z}}(Q) = \Phi_{z'\bar{z}'}(P)$, and hence

$$X_n = X'_n = -\frac{1}{2\pi i} \int_0^{2\pi} \mathrm{d}\theta \mathrm{e}^{i\theta} \Phi_{z\bar{z}}(Q) \;, \quad z(Q) = \mathrm{e}^{i\theta} \quad .$$

One can easily verify this argument explicitly using the Ward identity. The point is that the disturbing contour dependence of (3.4) is eliminated by the derivative $\partial\Phi$ term in the transformation law of Φ.

3.2. Superconformal case

A superconformal field theory obeys axioms similar to those of section 2.1.[6] We associate states in $\mathcal{H}$ to $(\hat{\Sigma}, \mathbf{z}_1, \ldots)$ where $\mathbf{z}_i = (z_i, \theta_i)$ is a local superconformal coordinate. Corresponding to the small superconformal transformations $z \mapsto z + V^z(z,\theta)$, $\theta \mapsto \theta + V^\theta(z,\theta)$ we get generators L_n, G_k as before. We will confine our attention to "super" (or NS) punctures for reasons discussed earlier; then the index k is an integer plus one half. Specifically we define

$$\langle \hat{\Sigma}, (z - \epsilon z^{n+1}, \theta - \tfrac{1}{2}\theta\epsilon(n+1)z^n)| \equiv \langle \hat{\Sigma}, (z,\theta)|(\mathbf{1} + \epsilon L_n + \bar{\epsilon}\bar{L}_n) \tag{2.7$'$}$$

$$\left\langle \hat{\Sigma}, \left(z + \tfrac{1}{2}\alpha\theta z^{k+1/2}, \theta - \tfrac{1}{2}\alpha z^{k+1/2}\right)\right| \equiv \langle \hat{\Sigma}, (z,\theta)|(\mathbf{1} + \tfrac{1}{2}\alpha G_k + \tfrac{1}{2}\bar{\alpha}\bar{G}_k) \quad . \tag{3.6}$$

Exactly as before we find from the axioms two commuting copies of the Neveu-Schwarz algebra. Similarly define operator superfields by

$$\Psi_{\mathbf{z}\ldots}(\hat{P}) \equiv \langle \mathbf{P}^{1|1}_{NS}, (z,\theta), (z^{-1}, iz^{-1}\theta), (z - z(\hat{P}) - \theta\theta(\hat{P}), \theta - \theta(\hat{P}))|\psi\rangle \quad . \tag{2.11$'$}$$

Here $\mathbf{P}^{1|1}_{NS}$ is the usual super sphere obtained from the (z,θ)-plane, $z(\hat{P})$, $\theta(\hat{P})$ are constants regarded as the coordinates of a point $\hat{P}$, and ψ is any vector annihilated by L_n, $G_{n-1/2}$, $n > 0$. One shows that under changes of superconformal coordinates preserving $\hat{P}$ the quantity $\langle \Psi_{\mathbf{z}\ldots}(\hat{P})\rangle_{\hat{\Sigma}} \mathbf{dz}^{2h} \mathbf{d\bar{z}}^{2\bar{h}}$ is invariant while $\Psi_{\mathbf{z}\ldots}(\hat{P})$ obeys a rule like (2.13).

To deform the theory we write

$$|\hat{\Sigma}, \mathbf{z}\rangle_\Delta = |\hat{\Sigma}, \mathbf{z}\rangle_0 + \frac{\Delta}{2\pi i} \int_{\hat{Q} \in \Sigma \setminus D_{\mathbf{z}}} {}_{\hat{Q}}\langle \phi \mid \hat{\Sigma}, \mathbf{z}, \mathbf{u}\rangle \mathbf{du d\bar{u}} \tag{3.1$'$}$$

where Δ is again real and ϕ is primary of weight $(\frac{1}{2}, \frac{1}{2})$. The integral is over a supermanifold with boundary $|z| = 1$; θ is unrestricted. The integral $(3.1)'$ is over all SRS with two punctures which reduce to $(\hat{\Sigma}, \mathbf{z})$ when we forget the location of one puncture. As we have noted, this is ill-defined for spin (or Ramond) punctures since there is no canonical forgetful map which forgets a spin puncture.

We can now repeat the derivation leading to (3.4). Letting $\mathbf{z}' = (z - \epsilon z^{n+1} + \frac{1}{2}\alpha\theta z^{k+1/2}, \theta - \frac{1}{2}\alpha z^{k+1/2} - \frac{1}{2}\theta\epsilon(n+1)z^n)$ we define

$${}_0\langle \hat{\Sigma}, \mathbf{z}'| = {}_0\langle \hat{\Sigma}, \mathbf{z}|(\mathbf{1} + \epsilon L_n + \bar{\epsilon}\bar{L}_n + \tfrac{1}{2}\alpha G_k + \tfrac{1}{2}\bar{\alpha}\bar{G}_k)$$

[6] See *e.g.* [12][14][16][17].

$$ {}_{\Delta}\langle\hat{\Sigma},\mathbf{z}'| = {}_{\Delta}\langle\hat{\Sigma},\mathbf{z}|(\mathbf{1}+\epsilon(L_n+\Delta X_n)+\tfrac{1}{2}\alpha(G_k+\Delta Y_k)+\text{conj.}) \quad . \tag{3.2$'$} $$

Again we drop order ϵ^2, $\epsilon\alpha$, and Δ^2, but keep $\epsilon\Delta$, $\alpha\Delta$. Thus

$$ {}_0\langle\hat{\Sigma},\mathbf{z}|(\epsilon X_n+\tfrac{1}{2}\alpha Y_k+\text{conj.}) = \frac{1}{2\pi i}\int_{\hat{Q}\in D_{\mathbf{z}}\setminus D_{\mathbf{z}'}} {}_0\langle\hat{\Sigma},\mathbf{z},\mathbf{u}\mid\phi\rangle_{\hat{Q}}\ \mathbf{du}\mathbf{d}\bar{\mathbf{u}} \quad . \tag{3.3$'$} $$

We need to interpret the rhs. We do this formally by introducing a step function $\vartheta(x)$ and writing

$$ \frac{1}{2\pi i}\int_{D_{\mathbf{z}}}\vartheta(|z-\epsilon z^{n+1}+\tfrac{1}{2}\alpha\theta z^{k+1/2}|-1)\ {}_0\langle\hat{\Sigma},\mathbf{z},\mathbf{z}-\mathbf{z}(\hat{\theta})\mid\phi\rangle_{\hat{Q}}\ \mathbf{dz}(\hat{Q})\ \mathbf{d}\bar{\mathbf{z}}(\hat{Q}) \ . $$

in an evident notation.

Expand the step function as

$$ \vartheta(|z|-1)-\tfrac{1}{2}\delta(|z|-1)(\epsilon z^n-\tfrac{1}{2}\alpha\theta z^{k-1/2}+\text{c.c.}) $$

to read off

$$ X_n = \frac{1}{2\pi i}\oint_{|z|=1} i\mathbf{d}\bar{\mathbf{z}}\ \mathrm{d}\theta(-2i)(-\tfrac{1}{2})z^{n+1}\langle\mathbf{P}^{1|1}_{NS},\mathbf{z},\mathbf{z}^{-1},\mathbf{z}-\mathbf{z}(\hat{Q})|\phi\rangle_{\hat{Q}} $$

$$ Y_k = \frac{-1}{2\pi i}\oint_{|z|=1} i\mathbf{d}\bar{\mathbf{z}}(-2i)(-\tfrac{1}{2})z^{k+1/2}\langle\mathbf{P}^{1|1}_{NS},\mathbf{z},\mathbf{z}^{-1},\mathbf{z}-\mathbf{z}(\hat{Q})|\phi\rangle_{\hat{Q}}|_{\theta(\hat{Q})=0} \quad . $$

In the second equation the rules of Grassmann integration tell us to discard terms with θ.

Thus we find $L^{(\Delta)}_n = L_n+\Delta X_n$; $G^{(\Delta)}_k = G_k+\Delta Y_k$ with

$$ X_n = -\frac{1}{2\pi i}\oint_{|z(\hat{Q})|=1}\mathbf{d}\bar{\mathbf{z}}\ \mathrm{d}\theta z^{n+1}\Phi_{\mathbf{z},\bar{\mathbf{z}}}(\hat{Q}) \tag{3.4$'$} $$

$$ Y_k = \frac{1}{2\pi i}\oint_{|z(\hat{Q})|=1}\mathbf{d}\bar{\mathbf{z}}\ z^{k+1/2}\Phi_{\mathbf{z}\bar{\mathbf{z}}}(\hat{Q})|_{\theta(\hat{Q})=0} \quad . \tag{3.7} $$

These formulas look even more coordinate-dependent than (3.4), but once again we know that they must be well-defined and satisfy the NS algebra simply because they were derived from a perturbation (3.1)$'$ which preserves the superconformal structure of the theory.

The super stress tensor is now

$$ T^{(\Delta)}_{\mathbf{zz}}(\mathbf{z}) = \sum_k z^{-n-3/2}\tfrac{1}{2}(G_k+\Delta Y_k)+\theta\sum_n z^{-n-2}(L_n+\Delta X_n) \quad . \tag{3.5$'$} $$

Again it is holomorphic by fiat and obeys the appropriate Ward identity.

One can readily expand (3.4)$'$, (3.7) in components to recover the explicit formulas of Ovrut and Rama [5]. This requires some superficial modifications for the heterotic case.

3.3. Heterotic case

In the heterotic case we have no $\bar{\theta}$ nor $\bar{\alpha}$. Thus for example one term is missing from (3.6) on the rhs. We can deform using a state ϕ annihilated by $L_n,\ \bar{L}_n,\ G_{n-\frac{1}{2}},\ n>0$ and of weight $(\frac{1}{2},1)$. Also in contrast to the previous subsection we take the overall parity of ϕ to be odd. Then letting

$$ {}_{\Delta}\langle\Sigma,\mathbf{z}| = {}_{0}\langle\Sigma,\mathbf{z}| + \frac{\Delta}{2\pi i}\int_{\hat{Q}\in\Sigma\backslash D_{\mathbf{z}}}[\mathrm{d}u\mathrm{d}\bar{u}|\mathrm{d}\theta]\ {}_{0}\langle\Sigma,\mathbf{z},\mathbf{u}|\phi\rangle_{\hat{Q}} \tag{3.1$''$} $$

we find that the rhs is coordinate invariant; more generally for a field ψ of weight $(h,\bar{h})$ the expectation

$$ \langle\psi(P)\rangle_{\hat{\Sigma}} \equiv \langle\hat{\Sigma},\mathbf{z}\mid\psi\rangle(\mathbf{dz})^{2h}(\mathrm{d}\bar{z})^{\bar{h}} \tag{2.8$'$} $$

is invariant. Recalling that odd variables like α now anticommute with the measure $\mathbf{dz}\mathrm{d}\bar{z} \equiv [\mathrm{d}z\mathrm{d}\bar{z}|\mathrm{d}\theta]$ we again find the changes

$$ X_n = -\frac{1}{2\pi i}\oint_{|z(\hat{Q})|=1}[\mathrm{d}\bar{z}|\mathrm{d}\theta]z^{n+1}\Phi_{\mathbf{z}\bar{z}}(\hat{Q}) \tag{3.4$''$} $$

$$ Y_k = -\frac{1}{2\pi i}\oint_{|z(\hat{Q})|=1}\mathrm{d}\bar{z}\ z^{k+1/2}\Phi_{\mathbf{z}\bar{z}}(\hat{Q})|_{\theta(\hat{Q})=0} \quad . \tag{3.7$'$} $$

Substituting various states ϕ of weight $(\frac{1}{2},1)$ we recover the formulas of [5].

4. Conclusion

We set out to find a formula for the change of the stress tensor under a small perturbation of a generic CFT, *i.e.* one whose state space $\mathcal{H}$ doesn't change suddenly in structure. Eqns. (3.4), (3.5), and their super generalizations, are the correct answer to our problem, but they may still seem distasteful. The point is that we cannot expect manifestly holomorphic formulas when the space of theories is itself not a complex manifold (for example for $c=1$ theories it has one real dimension [18]). In particular we see that Φ, being of dimension (1,1), cannot be analytic; *locality* has taken precedence over analyticity.

Throughout this paper we have touched only on first-order perturbations. It is well known that second-order changes pose new problems, essentially related to the divergences of string perturbation theory [18];[7] there may be no natural choice of framing for the

[7] In this connection the ideas of [19] may be helpful.

bundle $\mathcal{H}$ of state spaces due to some sort of curvature. But we have suggested that even the first-order formula (3.1) has some important information about the structure group of $\mathcal{H}$. It would be very interesting to uncover this additional structure.

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