

BANK LENDING AND RELATIONSHIP CAPITAL

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Dedicated to my parents, Fouzia and Abdelaziz Boualam.

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ABSTRACT

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Mohammed Yasser Boualam

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I develop an equilibrium theory of bank lending relationships in an economy subject to search frictions and limited enforceability. The model features a dynamic contracting problem embedded within a directed search equilibrium with aggregate and bank-specific uncertainty. The interaction between search and agency frictions generates a slow accumulation of lending relationship capital and distorts the optimal allocation of credit along both intensive and extensive margins. A crisis characterized by a sizable destruction of lending relationships therefore leads to a significant contraction in credit and a slow recovery, consistent with the Great Recession. I calibrate the model to study aggregate and cross-sectional implications and analyze policies aimed at reviving bank lending.

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CHAPTER 1 : BANK LENDING AND RELATIONSHIP CAPITAL

1.1. Introduction

The recent financial crisis caused a severe disruption in bank credit markets. Limited access to bank financing impacted households and small and medium-sized enterprises (SMEs), with sharp economic consequences.¹ Banks' persistent reluctance to lend has been at the heart of policy debate and academic research, with recent macro-finance literature highlighting the critical role of deteriorated bank balance sheets and the scarcity of financial intermediary capital in amplifying the crisis and restricting bank lending.² Yet, despite the stabilization of the banking system, an improving economic outlook, and many policy interventions, the flow of business lending has remained markedly low.

This paper argues that two key factors contribute to the sluggish credit recovery: the severance of *bank lending relationships* during the crisis and the consequent process of *credit reallocation*. In particular, I show how an environment characterized by search frictions and long-term financing contracts subject to limited enforcement can generate slow recoveries, consistent with the Great Recession.

My analysis is based on two premises inherent to bank lending markets for SMEs. First, relationship lending matters because banks are a critical source of external finance and the repeated interaction between borrowers and lenders relaxes contracting distortions and gradually enhances credit flow.³ In this paper, these long-term financing contracts are subject to limited enforceability, which reflects the borrower's inability to commit to a given arrangement. This is notably relevant for small and private firms with limited or opaque collateral. Second, the process of credit reallocation is important because establishing these lending relationships can be costly and time-consuming in decentralized and imperfectly competitive environments. This is the case for credit markets in which both borrowers and

¹Ivashina and Scharfstein (2010), Chodorow-Reich (2014), Greenstone et al. (2014).

²Gertler and Kiyotaki (2010), Gertler and Karadi (2011), He and Krishnamurthy (2012), Brunnermeier and Sannikov (2014), Adrian and Boyarchenko (2012), Bigio (2014), Rampini and Viswanathan (2015).

³Petersen and Rajan (1995), Boot and Thakor (2000).

lenders often devote significant time and resources to locate the right matches.

A salient feature of my approach to bank lending is that it highlights the importance of the market structure and the contracting environment, and does not rely on fluctuating bank balance sheets or firm collateral values. I use the term “relationship capital” to describe a slow-moving form of intangible capital reflecting the banking sector’s aggregate capacity to funnel credit into existing lending relationships. Accumulating relationship capital is influenced by the joint effects of frictions hampering *both* the formation (extensive margin) and build-up (intensive margin) of bank-firm pairs. As a consequence, the reallocation of credit in the aftermath of a crisis can be very slow. The model uncovers a propagation mechanism relying on two distinct channels. The first channel (“credit relationship channel”) affects the dynamics of credit availability and pricing of existing lending relationships. The second channel (“credit origination channel”) operates through search and matching and impacts the bank’s decision to offer new credit opportunities as well as the contractual terms at origination.

In the first part of this paper, I develop and fully characterize a dynamic contracting problem embedded within a directed search equilibrium with aggregate and bank-specific uncertainty. The model has two interconnected building blocks. The first relates to the dynamic contracting problem with limited enforceability, as in Albuquerque and Hopenhayn (2004). The borrowing capacity of the firm endogenously emerges as part of the optimal contract solution. When firm value is initially low, the agency problem impedes the amount of credit available because the entrepreneur has the option of not repaying the debt and searching for a new financier after a temporary exclusion from credit markets. The optimal contract specifies credit terms that gradually improve over time. Intuitively, by backloading firm claims to future cash flows, the bank can minimize the contract distortions due to the participation constraint of the entrepreneur and therefore extend more credit throughout the lending relationship.

The second building block of the model describes the problem of credit origination preced-

ing the contracting stage. I consider a frictional meeting process modeled through directed search, as in Moen (1997), where heterogeneous banks compete for borrowers by posting long-term credit offers. Banks differ with respect to their funding costs and optimize over the offered contractual terms by taking into account the trade-off between loan profitability and the probability of attracting unfunded borrowers. The nature of this trade-off is endogenously determined through bank entry and the ratio of the number of credit opportunities to the number of applications. The introduction of search frictions delays the formation of lending relationships. More important, it endogenizes contractual terms at origination and firm outside option and characterizes the degree of competition in credit markets.

It is particularly interesting to analyze a dynamic framework in which both intensive and extensive margins are at work. The interaction between agency and search frictions induces credit market conditions to affect firm default incentives directly. It therefore shapes the dynamics of optimal contracts and the transmission of shocks across borrowers and lenders. The analysis exhibits differences between effects at both micro and macro levels. At the bank level, search frictions limit the access to credit for defaulting firms, ease the agency problem, and hence allow for larger credit availability. At the aggregate level, however, this slows down the creation of new lending relationships and can consequently lead to lower credit supply overall.

In the second part of the paper, I evaluate whether this mechanism is a meaningful source of persistence in credit markets. I consider two types of aggregate shocks: a productivity shock and a bank funding cost shock. I show that negative shocks to firm productivity and positive shocks to bank funding costs can cause a significant decline in credit supply along both intensive and extensive margins. The effect on the intensive margin is short-lived and is driven directly by the diminishing returns to production. Conversely, shocks are propagated along the extensive margin of credit through their negative impact on the stock of lending relationships and hence the number of producing firms in the economy.

In the cross-section, the model allows us to study how aggregate shocks impact the real

sector. In particular, the analysis reveals an asymmetric treatment between funded and unfunded firms. The banking sector provides insurance against aggregate shocks in the economy and helps smooth out the cash flow profiles of incumbent borrowers. The extent of the pass-through depends on the bank’s funding cost level and the duration of the lending relationship. However, unfunded firms are not shielded from shocks. As credit market tightness and bank competition decrease, they not only face a more limited access to lenders, but they also experience a sharp decline in credit availability and high interest rates once matched.

Finally, this paper has important policy implications. The model provides a better understanding of the credit reallocation process and is therefore particularly relevant when analyzing the effects of policies targeted toward business lending and banking regulations. Significantly, I show that a policy subsidizing the cost of credit origination can have redistributive consequences and, while being effective at incentivizing banks to expand their lending supply in the long-run, can in fact be counterproductive in the short-run.

The model integrates relationship banking into the macro-finance literature. To my knowledge, this paper is the first joint study of the macroeconomic implications of search frictions and long-term financing contracts. The standard paradigm in the literature studying aggregate implications of financial frictions — starting with Bernanke and Gertler (1989) and Kiyotaki and Moore (1997) — relies on simple one-period interactions between anonymous borrowers and lenders.⁴ My paper departs from this line of research along two key dimensions by constructing a dynamic equilibrium model that takes into account both the process of formation of bank-firm pairs and the long-term nature of financing contracts. Moreover, by considering repeated interactions between borrowers and lenders, the model allows for endogenous borrowing limits that depend on the history of the lending relationship and credit market conditions.

This novel approach to credit markets emphasizes the importance of relationship banking

⁴Brunnermeier et al. (2012) provide an excellent survey of this literature.

at the aggregate level.⁵ In particular, it is related to Allen and Gale (1997) and Berlin and Mester (1999), who highlight the role of banks as providers of intertemporal insurance for long-term borrowers. The paper is also connected to Bolton et al. (2016), who analyze the difference between relationship and transaction lending during normal and crisis times within a three-period setting.

From a modeling perspective, my paper builds on the literature of long-term financing contracts in which credit constraints emerge endogenously as a feature of the optimal contract design.⁶ Specifically, it draws on insights from Albuquerque and Hopenhayn (2004) and departs from existing literature by constructing a dynamic general equilibrium model that endogenizes the firm value at origination and its outside option and allows for aggregate and idiosyncratic shocks.⁷ The focus on the aggregate implications of long-term financing contracts is also shared with Cooley et al. (2004), Jermann and Quadrini (2007), and Monge-Naranjo (2008).⁸ Cooley et al. (2004) study a general equilibrium model with limited contract enforceability and analyze how aggregate shocks to technological innovation can be amplified in the absence of market exclusion. In a similar vein, Jermann and Quadrini (2007) investigate how these contracts shape the economy's response following a stock market boom or gains in productivity. Monge-Naranjo (2008) examines the effects of changes in interest rates, but takes the firm's outside option as exogenous. In contrast, my paper considers the joint aggregate implications of limited enforceability and search frictions in a general equilibrium setup, and examines how market conditions endogenously affect the dynamics of aggregate credit supply.

While a large literature has extensively studied the importance of agency problems in credit

⁵Petersen and Rajan (1995), Boot and Thakor (2000), and surveys in Boot (2000) and Degryse et al. (2009).

⁶Gertler (1992), Thomas and Worrall (1994), Albuquerque and Hopenhayn (2004), Quadrini (2004), Clementi and Hopenhayn (2006), De Marzo and Sannikov (2006), De Marzo and Fishman (2007), Biais et al. (2010).

⁷For examples of contractual relationships with limited commitment and endogenous outside option in a static setup, see Phelan (1995) and Krueger and Uhlig (2006).

⁸Other recent papers also related to the limited-commitment contracting model of Albuquerque and Hopenhayn (2004) include Schmid (2012), Rampini and Viswanathan (2013), and Li et al. (2016).

markets, little is known about the role of search frictions in this context. Previous work (Diamond (1990), Den Haan et al. (2003), Wasmer and Weil (2004), Becsi et al. (2005)) mainly considers static random search environments with simple contracts.⁹ My paper shares some insights with Den Haan et al. (2003), who highlight the lasting damage due to the joint effects of the destruction of credit relationships and coordination failure in investment decisions. In contrast, I provide a rich and dynamic setup by embedding long-term financing contracts within a directed search equilibrium. The property of block-recursivity characterizing this equilibrium provides a numerically tractable solution and allows for the introduction of aggregate shocks and the analysis of the economy’s transitional dynamics (Menzio and Shi (2011)).

The paper also belongs to the growing theoretical literature studying the interaction between search and agency frictions. These studies have focused mainly on labor and asset markets as in Rudanko (2009), Guerrieri et al. (2010), Moen and Rosén (2011), and Lamadon (2014). In this paper, I develop a credit markets model with limited contract enforceability and search frictions and show how the interaction between these frictions can provide novel insights into the dynamics of credit along both intensive and extensive margins and lending rates.

Finally, this paper complements the emerging theoretical research explaining credit market freezes motivated by the Great Recession. This includes Bebchuk and Goldstein (2011), who show how coordination failure among financial institutions can lead to self-fulfilling credit contractions. Diamond and Rajan (2011) argue that the reluctance to extend credit is related to banks’ fear of future fire sales. Benmelech and Bergman (2012) analyze the credit channel transmission of monetary policy and show that the interplay among financial frictions, market liquidity, and collateral values can give rise to credit traps. Philippon and Schnabl (2013) examine the problem of efficient recapitalization when banks are subject to

⁹Inderst and Müller (2004) and Silveira and Wright (2006) also analyze the role of search frictions within models of venture capital. See also Petrosky-Nadeau and Wasmer (2011) and Petrosky-Nadeau (2014), who introduce search frictions in multiple markets simultaneously.

debt overhang. In contrast, my paper provides a novel “flow-driven” theory focusing on frictions hindering the accumulation of relationship capital.

The rest of the paper is organized as follows. Section 2 describes the model. Section 3 characterizes the optimal contract, while section 4 analyzes the equilibrium properties of credit markets. Section 5 discusses comparative statics and the general equilibrium effects of search and limited enforceability. Sections 6 and 7 explore the quantitative properties of the model. Section 8 examines policy implications. Section 9 provides additional extensions and comments, and section 10 concludes.

1.2. Model

In this section, I develop a general equilibrium model of credit markets, featuring both search and matching frictions and dynamic contracting. The contracting problem relies on limited commitment as in Albuquerque and Hopenhayn (2004) and is embedded within a directed search equilibrium, with aggregate and bank-specific uncertainty, and where heterogeneous financiers compete for borrowers by posting long-term contract offers. Optimal contracts are history-dependent and specify both loan size and corresponding interest rate in each period of the credit relationship. The frictional meeting environment implies the coexistence of multiple credit submarkets in which borrowers and lenders match, and where the optimal contract offered by each bank trades off loan profitability with the probability of finding a borrower. In equilibrium, unfunded entrepreneurs are indifferent across all active submarkets, and the matching probability depends on the number of firms and banks present within each submarket. Proofs are presented either within this section or in the appendix.

1.2.1. Environment

The model is in discrete time with an infinite horizon. The economy is populated by two types of infinitely lived agents: entrepreneurs and banks. The mass of entrepreneurs is

normalized to one, while the mass of active banks is determined endogenously through entry and exit.

Agents and preferences

Both agents share the same discount factor $\beta \in (0, 1)$. Banks are risk neutral. Entrepreneurs, on the other hand, are risk averse. They operate their firms in order to maximize their expected lifetime utility of consumption $\mathbb{E}_0[\sum_{t=0}^{\infty} \beta^t u(d_t)]$, where the vector $\{d_t\}_{t=0}^{\infty}$ represents the net dividend payout consumed within each period and the flow utility $u : \mathbb{R} \rightarrow \mathbb{R}$ satisfies standard regularity conditions: $u' > 0$, $u'' < 0$, $\lim_{d \rightarrow 0} u'(d) = \infty$, and $\lim_{d \rightarrow \infty} u'(d) = 0$. The assumption of risk-averse entrepreneurs departs from the literature on dynamic debt contracting, which typically assumes risk neutrality.¹⁰ This is justified for firms whose managers derive consumption from their business venture without any ability to diversify firm-specific risk (Stulz (1984)).¹¹ In order to ensure the model remains tractable for both search and contracting problems, entrepreneurs are also assumed to be hand-to-mouth and without access to a storage technology.¹²

Technology and shocks

Each entrepreneur has access to a production technology subject to stochastic aggregate productivity shocks. However, she is initially cashless, and has to seek out external financing in order to start production. When funded with capital K , a project can generate output $F(z, K) = z f(K)$, where z is the aggregate productivity shock. The function f is differentiable, strictly increasing and strictly concave in capital, and satisfies $f(0) = 0$, $\lim_{k \rightarrow 0} f_k(k) = +\infty$, and $\lim_{k \rightarrow \infty} f_k(k) = 0$. The realization of the productivity shock $z \in \mathbb{Z} = \{z_1, z_2, \dots, z_{N_z}\}$ is publicly observed every period, and follows a Markov process with transition probability $\Gamma_z : \mathbb{Z} \times \mathbb{Z} \rightarrow [0, 1]$. For simplicity, the production function abstracts from labor and can be viewed as a profit function that already accounts for the optimal choice of labor input and associated wages. When unfunded, the entrepreneur has

¹⁰Exceptions include Marcet and Marimon (1992) and Thomas and Worrall (1994).

¹¹This is also the case for larger corporations actively engaged in risk management policies, as in Froot et al. (1993) or Rampini and Viswanathan (2010).

¹²See section 9 for a discussion.

access to a “garage” production technology generating a constant cash flow d_0 per period.

The repeated interaction between borrowers and lenders can help alleviate the agency friction in place. Banks arise in this economy because they can originate and commit to long-term financing relationships at a cost that is lower than that incurred by a repeated sequence of short-term interactions with direct monitoring. When matched, a given bank i acts as an intermediary channelling funds from depositors to entrepreneurs at a funding cost $r_d^i = \bar{r} + s^i$, where $\bar{r}$ corresponds to the aggregate state of the banking sector and s_i is a bank-specific spread. In this section, and to keep the theory exposition simple, I assume that $\bar{r}$ is constant while the bank-specific spread $\{s^i\}_i \in \mathbb{S} = \{s_1, s_2, \dots, s_{N_s}\}$ follows independent Markov processes with transition probability $\Gamma_s : \mathbb{S} \times \mathbb{S} \rightarrow [0, 1]$.¹³

Bank heterogeneity is motivated by differences in deposit technology, competition across deposit markets, bank size and economies of scale, too-big-to-fail subsidies, or the ability to access interbank lending and repo markets, consistent with evidence in Berlin and Mester (1999) and Gilchrist et al. (2013). Since the focus of the paper is on the credit allocation process and its impact on the real economy, banks’ liability structure is modeled in a parsimonious way, abstracting from potential feedback effects between asset quality and funding cost. Each bank offers a long-term credit contract and serves one entrepreneur at a time. Hence, at any given point in time, active banks can either be participating in a lending relationship or seeking a borrower.¹⁴

Credit markets

Credit markets are decentralized and subject to search and matching frictions.¹⁵ This assumption reflects the “localized” nature of bank lending markets (Agarwal and Hauswald

¹³To keep notations simple, I also define the set of bank funding shocks such that $\{r_d^i\}_i \in \mathbb{R}_d = \{\bar{r} + s_1, \bar{r} + s_2, \dots, \bar{r} + s_{N_s}\}$, with $\Gamma_{r_d} = \Gamma_s$. In the quantitative analysis section, I will allow for a more general version of the model allowing for $\bar{r}$ to be stochastic with $\bar{r} \in \mathbb{R} = \{r_1, r_2, \dots, r_{N_r}\}$ and transition probability $\Gamma_{\bar{r}} : \mathbb{R} \times \mathbb{R} \rightarrow [0, 1]$. I will also consider a more general interpretation of the cost r_d^i by mapping it to banks’ marginal operating cost, which accounts for both interest and non-interest expenses.

¹⁴Given that the loan origination technology is constant-return-to-scale, a bank can be thought of as a portfolio of lending relationships sharing the same underlying funding cost.

¹⁵The costs associated with search can be viewed as either explicit (e.g., transaction costs, time spent to prepare an application), or implicit (e.g., opportunity cost foregone when the firm remains unfunded).

(2010)), within which banks exert a certain degree of market power. Thus, this environment generates situations of bilateral monopoly between borrowers and lenders, where the degree of competition in credit markets is endogenously determined and depends on the aggregate state of the economy. This allows for a richer and more realistic setting, as opposed to the standard cases of perfect or monopolistic bank competition.

Credit markets are modeled using a competitive search framework in the spirit of Moen (1997), where each bank advertises its contractual terms and where each entrepreneur directs her search by applying to certain loan offers.¹⁶ Credit markets are organized as a continuum of submarkets or market segments consisting of a subset of borrowers and lenders and indexed by the firm value V specified by the contract. This value corresponds to the expected net present value of dividend payouts generated throughout the lending relationship and reflects the share of joint match surplus allocated to the firm. The origination process is costly and banks incur a cost c whenever they enter the credit market. This parameter captures fixed operating costs related to screening applications, loan officer wages, real estate, advertisement of bank services, or deposit creation.

The matching function $m(u, v)$, which takes as input the mass of unfunded entrepreneurs u and active banks v within a given submarket, implicitly accounts for congestion and coordination externalities among borrowers and lenders and can be viewed as a reduced-form way to capture banks' screening technology or lending standards. The function m is assumed to be continuous, concave, and homogeneous of degree one in both variables. I define credit market tightness $\theta = \frac{v}{u}$, the probability of getting access to credit $p = \frac{m(u, v)}{u} = m(1, \theta) = p(\theta)$, and the probability of a bank locating a borrower $q = \frac{m(u, v)}{v} = q(\theta)$. The functions p and q are assumed to be twice continuously differentiable, with p strictly increasing and concave, q strictly decreasing and convex, and $p \circ q^{-1}$ strictly concave. Moreover, when credit market tightness tends to zero, the probability of securing a loan and that of finding a borrower tend to 0 and 1, respectively: $\lim_{\theta \rightarrow 0} p(\theta) = 0$; $\lim_{\theta \rightarrow 0} q(\theta) = 1$. Conversely,

¹⁶Banks are assumed to commit to the initially posted offer and cannot bargain with the firm at the matching stage.

when credit market tightness tends to infinity, the probability of securing a loan and that of finding a borrower tend to 1 and 0, respectively: $\lim_{\theta \rightarrow \infty} p(\theta) = 1$; $\lim_{\theta \rightarrow \infty} q(\theta) = 0$. Without loss of generality and as it is standard in discrete-time search models, I assume a CES matching function with elasticity parameter γ_m throughout the paper.

Long-term credit contracts

Borrowers and lenders sign a state-contingent long-term credit contract upon matching. The value of the contract at origination is determined by the aggregate state of the economy and the characteristics of the credit search market. A lending relationship associated with aggregate productivity z , bank funding cost r_d , and capital K generates the following per-period match surplus:

$$S(z, r_d, K) = zf(K) - r_d K.$$

While lenders are assumed to fully commit to established long-term contracts, borrowers are subject to limited enforceability. A default event occurs whenever a borrower decides to walk away with a fraction η of the existing capital stock. The diverted amount of capital is assumed to be consumed in the same period and cannot be used for future production.¹⁷ A default also triggers the severance of the lending relationship, with defaulting borrowers only regaining access to credit markets with a constant “fresh start” probability $\xi \in [0, 1]$. The extreme cases are obtained for $\xi = 0$ (permanent exclusion), or $\xi = 1$ (no exclusion). Thus, this parameter governs the cost of bankruptcy and reflects legal and institutional systems in place.¹⁸

The value of contract repudiation, which represents the firm’s outside option, is given by

¹⁷This assumption can be relaxed at the expense of increasing the dimensionality of credit submarkets. Here, it greatly simplifies the exposition because defaulting and unfunded firms look ex-ante identical to banks and hence follow similar credit search strategies. Note also that the model does not allow entrepreneurs to directly switch lenders or have multiple lending relationships.

¹⁸Bebchuk (2000) relates this cost to the length of time spent on bankruptcy procedures. Efficient court ruling, fast liquidation procedures, and a short period of discharge can therefore provide entrepreneurs with the opportunity to move forward and potentially start up a new business venture faster (Peng et al. (2010)).

the sum of contract-specific and aggregate components: (i) the utility value obtained from diverting fraction η of existing capital stock, and (ii) the discounted expected firm value after default $\{H(z)\}_z$:

$$V^O(z, K; W) = u(\eta K) + \beta H(z) \quad (1.1)$$

$$H(z) = \xi \mathbb{E}_z[W(z')] + (1 - \xi)(u(d_0) + \beta \mathbb{E}_z[H(z')]), \quad (1.2)$$

where $\{W(z)\}_z$ represents the value of an unfunded firm in each aggregate state, as determined in the directed search equilibrium, and d_0 is the per-period profit derived from “garage” production.

In order for the contract to remain enforceable along the equilibrium path, the current firm value generated through the lending relationship should always be at least as high as the utility derived from repudiation. The outside option and the threat of credit market exclusion discipline the entrepreneur’s incentives and shape the dynamics of capital supplied to firms. Moreover, the firm value after default H introduces a feedback effect between contract dynamics and credit market conditions. The strength of such feedback is governed by parameter ξ . Note also that full enforceability is a special case obtained for $\xi = \eta = 0$.

Eventually, long-term contracts are subject to an exogenous separation shock. With probability σ , the lending relationship is terminated. In this case, the entrepreneur loses her firm as capital is liquidated, becomes unfunded, and starts searching for new sources of credit. The bank, on the other hand, receives 0 until it matches with a new borrower.¹⁹ Otherwise, the lending relationship continues with probability $1 - \sigma$.

Let $\omega_t^\tau = \{(z_t, r_{d,t}), (z_{t+1}, r_{d,t+1}), \dots, (z_\tau, r_{d,\tau})\}$ denote the history of shocks associated with an ongoing lending relationship starting from date t and state $\omega_t = (z_t, r_{d,t})$, and remaining in place up to period τ . A long-term contract is a set of policies for capital K_τ and dividend

¹⁹The bank does not receive any income once the lending relationship is severed. In addition, the search for a new borrower is costly, and would also deliver 0 profits ex-ante due to bank free-entry condition.

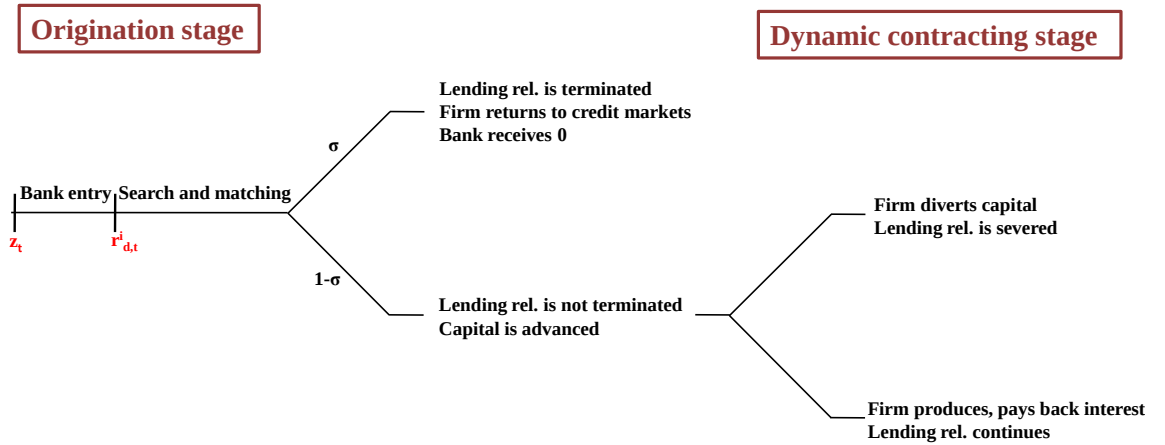
payout d_τ :

$$\mathcal{C}(\omega) = \{(K_\tau(\omega_t^\tau), d_\tau(\omega_t^\tau), \forall \omega^\tau, \tau = t, \dots, \infty \text{ s.t. } \omega = \omega_t)\}.$$

Timing

Figure 1 displays the model timeline, which comprises two main stages, each divided into two steps. First is the origination stage (extensive margin) involving (i) bank entry, and (ii) credit search and matching. Second is the dynamic contracting stage (intensive margin) involving (iii) capital intermediation, and (iv) firm production or default. The realization of aggregate productivity z , bank funding cost r_d , and separation shocks are public information observed by both borrowers and lenders.

Figure 1: Timeline



- *Bank entry*: following the realization of aggregate shocks, banks pay cost c in order to enter credit markets.
- *Credit search and matching*: following the realization of their idiosyncratic shock, banks decide to become active by posting loan offers or to stay idle. Unfunded en-

entrepreneurs search and apply for credit opportunities within the active submarkets. Upon approval of the loan application, a bank-firm pair is formed and a long-term financing contract specifying the allocation of surplus between the two agents (and, hence, the evolution of contractual terms) is signed.

- *Capital intermediation:* Capital is intermediated within ongoing bank-firm pairs that did not experience an exogenous separation shock, as prescribed by the financing contract.
- *Firm production/default:* In case of default, entrepreneurs divert and consume fraction η of capital K , and face a per-period probability of exclusion $1 - \xi$ from credit markets following that. Otherwise, they use capital for production and consume dividend payout d (or, equivalently, pay back interest rate r to the bank), as prescribed by the financing contract.

1.3. Optimal contracts

In this section, I characterize the optimal long-term financing contract in a partial equilibrium setting, taking as given both firm value V at origination and the vector of unfunded firm values $\{W(z)\}_z$ for each aggregate state. These objects will be endogenously determined when I characterize the credit search market.

1.3.1. Intuition

Let us develop the general intuition behind this problem before moving to the analytical characterization. With full commitment on both sides, capital allocation is always first-best, meaning that it always maximizes the per-period match surplus independent of any constraint. However, the first-best allocation of capital cannot be generally achieved because the entrepreneur retains the option of not repaying the debt, being temporarily excluded from the credit market, and, eventually, searching for a new financier. The entrepreneurs'

participation constraint limits the amount of capital that can be lent. This limit is endogenous because the value to the entrepreneur of remaining in the relationship depends on the future terms of trade and the value of leaving the relationship depends on the aggregate credit market conditions.

As in Albuquerque and Hopenhayn (2004), these distortions are minimized within an optimal default-free contract that features — abstracting from aggregate and idiosyncratic shocks — terms of trade that become more and more favorable to the entrepreneur. Intuitively, by offering a contract with terms of trade that improve over time rather than remain constant, the bank can relax the entrepreneur’s participation constraint in the future (and, hence, lend more capital) without affecting the current participation constraint. The limit to backloading comes from the fact that the entrepreneur is averse to risk and, hence, dislikes consumption paths that are too steep.

Most important, by taking into account the repeated interaction between banks and entrepreneurs, the model endogenizes borrowing limits. These limits are not an exogenous and constant fraction of the value of the entrepreneur’s collateral, but are an endogenous variable that depends on credit market conditions at origination, the history of the relationship between bank and entrepreneur, and the state of the economy.

1.3.2. Contracting problem

The optimal contract maximizes the expected discounted payments to the bank, subject to the promise-keeping, participation, and limited liability constraints. I first write this contractual problem in its recursive form using the firm value V as a state variable in the spirit of Spear and Srivastava (1987) and Abreu et al. (1990).

To simplify notations, the value of the contract to the bank is written as $B(z, r_d, V) = B(z, r_d, V; W)$ taking the equilibrium object W as given, and the dependence of the continuation values $\{V'\}_{z', r_d'}$ on (z', r_d') is implicitly considered. The recursive formulation of the problem is as follows:

$$\begin{aligned}
B(z, r_d, V) &= \max_{K, d, \{V'\}} z f(K) - d - r_d K + \beta \mathbb{E}_{z, r_d} [(1 - \sigma) B(z', r'_d, V')] \\
&\text{subject to} \\
V &= u(d) + \beta \mathbb{E}_{z, r_d} [(1 - \sigma) V' + \sigma W(z')] , \quad (\text{Promise-Keeping}) \\
V^O(z, K; W) &\leq V, \quad (\text{Participation}) \\
d &\geq 0. \quad (\text{Limited Liability})
\end{aligned} \tag{1.3}$$

The control variables are capital K , firm profits d , and vector of continuation values $\{V'\}_{z', r'_d}$. The promise-keeping constraint represents the bank's full commitment to deliver value V , which accounts for today's utility from payout $u(d)$, and the discounted promised value $\beta \mathbb{E}_{z, r_d} [(1 - \sigma) V' + \sigma W(z')]$. The second inequality is the participation or enforcement constraint. In order for the contract to be self-enforcing, this constraint requires that the value of staying V is always at least as high as the firm's outside option. Eventually, the contract assumes the firm's payout to be non-negative, reflecting the entrepreneur's limited liability. As in Cooley et al. (2004), this assumption is justified by the fact that entrepreneurial consumption cannot be negative when the bank is the only source of financing and all of the entrepreneur's assets are inside the firm.²⁰

1.3.3. *Recursive multiplier formulation*

The forward-looking nature of the enforcement constraint and the existence of persistent aggregate and idiosyncratic shocks make this problem difficult to solve using standard dynamic programming techniques. Thus, I adapt the methodology developed in Marcet and Marimon (2011) and transform the problem into a saddle-point dynamic program. Here, instead of solving the problem in the value space, the problem is rewritten in its recursive Lagrangian form, where λ is the Lagrange multiplier associated with the current enforcement constraint and Λ is the cumulative Lagrangian (i.e., the sum of past Lagrange multipliers)

²⁰This is, for example, the case when outside investment opportunities are assumed to yield lower returns and hence are not held in equilibrium.

representing the relative weight associated with firm value V within the planner's problem. By introducing the expectational constraint directly into the objective function, this formulation circumvents the time-inconsistency issue present in the original maximization problem, since it imposes a certain law of motion for Λ .²¹

Given model assumptions, Theorems 1 and 2 in Marcet and Marimon (2011) justify the equivalence between the original problem and its saddle-point transformation when separation is exogenous. This formulation will be extremely useful in order to characterize the main properties of the optimal contract and compute a numerical solution.

Proposition 1. *The maximization problem (1.3) is equivalent to the saddle-point problem:*

$$\begin{aligned}
P(z, r_d, \Lambda) &= \inf_{\lambda} \sup_{K, d} \quad z f(K) - d - r_d K - \lambda [u(\eta K) + \beta H(z)] \\
&\quad + (\Lambda + \lambda) [u(d) + \beta \sigma \mathbb{E}_z [W(z')]] \\
&\quad + \beta (1 - \sigma) \mathbb{E}_{z, r_d} [P(z', r'_d, \Lambda + \lambda)] \\
&\text{subject to} \\
&\quad d \geq 0, \quad \lambda \geq 0.
\end{aligned} \tag{1.4}$$

The value of the cumulative Lagrangian Λ is strictly increasing — independent of shock realizations — as long as the firm's participation constraint remains binding (i.e., $\lambda > 0$). Its law of motion is given by $\Lambda' = \Lambda + \lambda$, and can be interpreted as an additional condition that the planner imposes in order for the contract to follow the optimal path. From the envelope condition, it is straightforward to show that the marginal cost to the bank of a one-unit increase in firm value is equalized across all possible continuation states (z, r_d) :

$$\frac{\partial B(z, r_d, V(z, r_d))}{\partial V} = -\Lambda. \tag{1.5}$$

²¹Cooley et al. (2004) also use this technique to solve a similar problem.

1.3.4. Characterization

Capital policy

The optimal choice of capital K^* is static. Indeed, K^* solves the following intermediate problem, which depends only on firm value V and current shock realizations:

$$\begin{aligned} \pi(z, r_d, V) &= \max_K z f(K) - r_d K \\ \text{subject to} \quad & u(\eta K) + \beta H(z) \leq V. \end{aligned} \tag{1.6}$$

This problem generates a constrained region where the enforcement condition binds and the level of working capital is suboptimal, and an unconstrained region where the first-best level of capital K_{FB} is defined for each state (z, r_d) as:

$$K_{FB}(z, r_d) = \arg \max_K z f(K) - r_d K.$$

Let us also define $\bar{V}(z, r_d)$ as the lowest continuation value associated with a non-binding participation constraint:

$$\bar{V}(z, r_d) = u(\eta K_{FB}(z, r_d)) + \beta H(z),$$

and the constrained level of capital $\bar{K}(z, V)$, satisfying $V^O(z, \bar{K}; W) = V$ for $V < \bar{V}(z, r_d)$.²²

Proposition 2. *The firm's optimal capital K^* is characterized by:*

$$K^*(z, r_d, V) = \begin{cases} \bar{K}(z, V), & \text{if } V < \bar{V}(z, r_d), \\ K_{FB}(z, r_d), & \text{if } V \geq \bar{V}(z, r_d). \end{cases}$$

²²Note that for each state (z, r_d) , the upper bound of the cumulative Lagrangian $\bar{\Lambda}(z, r_d)$ is reached whenever the firm value is $\bar{V}(z, r_d)$. It is given by: $\frac{\partial B(z, r_d, \bar{V}(z, r_d))}{\partial V} = -\bar{\Lambda}(z, r_d)$.

Corollary 1. *K^* is decreasing in ξ , η and W in the constrained region, all else equal.*

The above corollary is intuitive, since the limited commitment problem becomes more severe as the outside option increases. Given that an increase in the share of diverted capital η , the probability of fresh start ξ , or the unfunded firm value W implies a larger outside option, the financier lowers the amount of capital advanced to prevent the entrepreneur from running away. The links among the firm's equity value, capital allocation, and market conditions are apparent from the above expression. The higher the firm value V , the higher its capital allocation K . However, a higher W distorts the contract further and leads to a lower allocation.²³

Dividend payout policy

The following proposition shows that the dividend payout d exhibits downward rigidity; in other words, d is never decreasing, and either increases whenever the participation constraint is binding ($\lambda > 0$) or stays constant once the unconstrained region is reached.

Proposition 3. *Dividend payout increases over time whenever the firm's participation constraint is binding.*

This result is driven by the following envelope condition, which exhibits the positive relation between d and V since $u' > 0$, and their link to the cumulative Lagrangian Λ' :

$$\frac{1}{u'(d)} = -\frac{\partial B(z', r'_d, V_{z', r'_d})}{\partial V} = \Lambda'. \quad (1.7)$$

The dividend payout increases in Λ' . A higher degree of risk aversion is accompanied by a relatively smoother path of dividend payouts.²⁴ This contrasts with the risk-neutral case,

²³Note also that two firms with same value V , but linked to banks with different funding costs such that $r_{d,1} < r_{d,2}$, receive the same amount of credit as long as $V < \bar{V}(z, r_{d,2})$.

²⁴The backloaded dividend dynamics are analogous to the ones obtained in standard dynamic contracting problems with limited commitment as in Harris and Holmstrom (1982), in which a risk-neutral financier is indeed providing insurance to risk-averse borrowers. The difference vis-a-vis labor models with limited com-

wherein the entrepreneur is indifferent about the timing of consumption, and in which it is always optimal to reinvest all proceeds in the firm in order to grow out of the constrained region faster. In the continuation region, the law of motion for dividends is given by:

$$\frac{1}{u'(d')} = \frac{1}{u'(d)} + \lambda.$$

Effective intra-temporal interest rate

The contract implicitly specifies an effective intra-temporal interest rate charged to borrowers as a function of the optimal capital and dividend payout policies. It is given by:²⁵

$$r^*(z, r_d, V) = \frac{zf(K^*(z, r_d, V)) - d^*(z, r_d, V)}{K^*(z, r_d, V)}.$$

Firm value dynamics

Now that we have characterized the relationship among equity value, capital, and dividend payout, we can turn to the analysis of the dynamic aspect of the contract. I show that, first, in the absence of shocks, firm value is always increasing in the constrained region. This generalizes the result shown in Albuquerque and Hopenhayn (2004) for the risk-neutral case and confirms that the incentives to save in order to outgrow the borrowing constraints are still present despite the consumption smoothing motive.²⁶ Second, the introduction of shocks implies that firm value can now follow a non-monotonic pattern.

Proposition 4. *Fix (z, r_d) . For a given firm value V , the continuation value V' increases over time whenever $V < \bar{V}(z, r_d)$.*

mitment (Rudanko (2009)) comes from the agency problem faced by borrowers, which generates additional savings incentives and different payout dynamics.

²⁵Note, however, that the implementation of the contract is not unique.

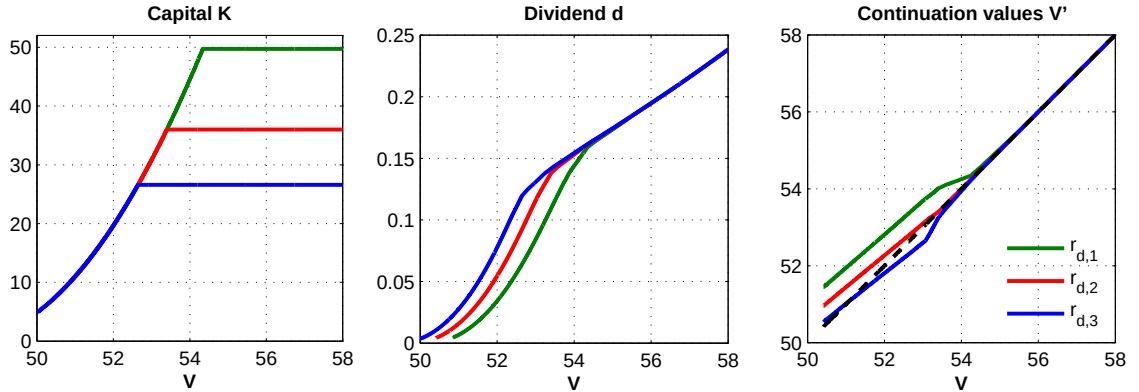
²⁶Marcet and Marimon (1992) also feature a dynamic contracting problem with risk-averse agents and limited commitment in a different context.

Keeping aggregate and idiosyncratic shocks constant, the cumulative Lagrangian (and, hence, firm value) increases whenever the participation constraint is binding. Hence, $V' > V$ because B is a decreasing function of V . This result follows directly from the one for dividend payout (since V is the net present value of future dividends) and can be directly generalized to the expectation of promised values when the economy is subject to shocks.

Proposition 5. *Fix z . For a given firm value V and funding cost r_d , the continuation value $V'(z, r'_d)$ decreases in r'_d whenever $V < \bar{V}(z, r_d)$.*

This proposition relates to firm dynamics in the cross-section. It states that banks with relatively low funding costs offer higher continuation values, all else equal. This is intuitive, since lending relationships established with low-funding-cost banks generate a larger match surplus, given that the marginal product of capital is lower, and as a consequence allow firms to receive larger capital allocations and dividends. Figure 2 illustrates how capital, dividends, and continuation values $\{V'\}$ depend on current firm value V and funding costs, holding aggregate shocks constant and taking both firm value at origination and the vector associated with unfunded firm value $\{W(z)\}_z$ in each aggregate state as given.²⁷

Figure 2: Contract policies



Notes. Contract policies as a function of firm value V , for low (green), intermediate (red), and high (blue) levels of funding cost: capital K , firm payout d , continuation values V' .

²⁷Note that continuation values $\{V'\}$ depend on both current and future funding costs. For simplicity, the continuation value profiles shown in the right-hand panel correspond to the case in which the current funding cost is at the intermediate level.

The dynamics of the firm's capital, dividend payout, and value can eventually be summarized as follows. Upon matching, the newly formed lending relationship starts with an initial firm value V_0 , which depends on credit market conditions determined in general equilibrium. The firm typically starts with a limited borrowing capacity and operates at a suboptimal scale as long as its participation constraint is binding.

Throughout time, the participation constraint gets relaxed and the firm gets allocated more capital as its value increases. In this setup, entrepreneurs care about the time allocation of dividend payments because they are risk averse. The optimal contract therefore allows for positive dividend payouts even in the constrained region in order to generate a smooth consumption profile. The lending relationship eventually matures whenever V becomes sufficiently large to sustain the first-best level of capital across all possible future states.

The dynamics of the contractual terms and the speed at which the firm reaches the unconstrained region are in particular shaped by the curvature of the entrepreneur's utility function and the general equilibrium forces impacting the firm's outside option.

1.4. Directed search equilibrium

I now introduce the general equilibrium version of the model. I first characterize the credit search environment, and then address the ways in which aggregate market conditions endogenously impact contract dynamics. The interaction between agency and search frictions affects the firm value in both funded and unfunded stages and generates novel implications. Moreover, directed search provides a framework that is tractable and amenable to the introduction of heterogeneous agents and persistent shocks, thanks to the property of block-recursivity of the equilibrium. Let us start by analyzing the search behavior of borrowers and lenders separately.

1.4.1. Problem of unfunded firms

In order to resume production, entrepreneurs search for credit opportunities whenever they are unfunded. Let $W = \{W(z)\}_z$ denote the vector associated with the unfunded firm value (or reservation value) for each aggregate state z . The Bellman equation of $W(z)$ satisfies:

$$W(z) = u(d_0) + \beta\rho(z) + \beta\mathbb{E}_z[W(z')] \quad (1.8)$$

$$\rho(z) = p(\theta(z, V))(V - \mathbb{E}_z[W(z')]), \quad \forall V \quad (1.9)$$

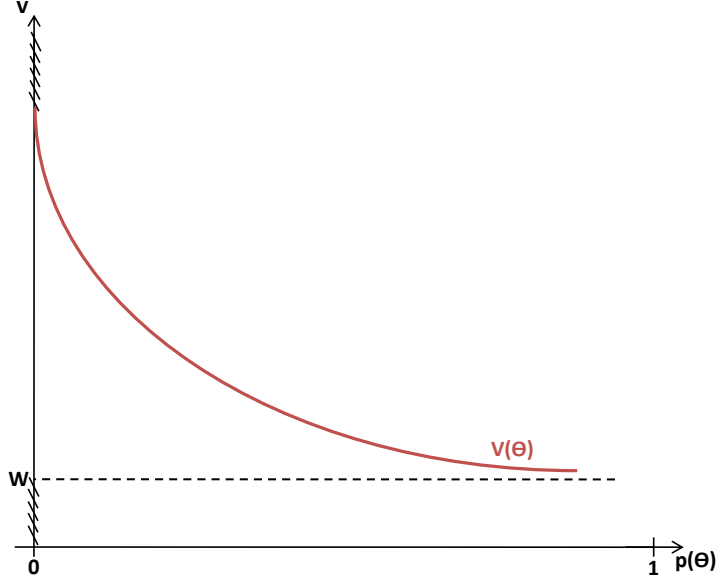
where $\rho(z)$ denotes the added firm value derived from forming a lending relationship times the probability of a match, and $\theta(z, V)$ is the credit market tightness associated with the submarket delivering firm value V . We can rearrange the two expressions above to establish the link between market tightness and firm value as follows:

$$p(\theta(z, V)) = \frac{W(z) - u(d_0) - \beta\mathbb{E}_z[W(z')]}{\beta(V - \mathbb{E}_z[W(z')])}. \quad (1.10)$$

This expression determines a bijective mapping between firm value V and credit market tightness $\theta(z, V)$ within each submarket. Given that the function p is strictly increasing in θ , it follows that θ is strictly decreasing in V . This means that credit submarkets where banks offer contracts with relatively low firm value have higher matching probability. Thus, they are more liquid in the sense that they feature higher approval rates and faster relationship creation. In contrast, submarkets where banks offer contracts with relatively high firm value have lower credit market tightness and lower approval rates.

Eventually, for each aggregate state z , any credit submarket offering a contract with firm value below the reservation value $W(z)$ cannot attract borrowers and is therefore inactive. Figure 3 shows borrowers' indifference curve across active submarkets in the $(p(\theta), V)$ space.

Figure 3: Relationship between approval rates and firm value



1.4.2. Problem of banks

Banks decide whether to enter credit markets after the realization of the aggregate shock, but before observing the idiosyncratic component of their funding cost. If the funding cost is not sufficiently low to warrant gains from trade, the bank stays idle and does not offer any lending opportunity. This is the case when $B(z, r_d, W(z); W) \leq 0$, which means the lending contract cannot simultaneously deliver a firm value above its reservation W and positive bank profits. Otherwise, each active bank optimally chooses the submarket that maximizes its expected profits, taking into account the probability of finding a borrower, given the reservation value W :

$$V^*(z, r_d; W) = \arg \max_V q(\theta(z, V))B(z, r_d, V; W). \quad (1.11)$$

Let us define the compact interval $\mathcal{S}_0 = [\underline{S}, \bar{S}]$, where $\underline{S} = \frac{u(d_0)}{1-\beta}$, and $\bar{S}$ is the maximum value obtained by the entrepreneur when the joint match surplus is kept entirely inside the firm.

Lemma 1. *The solution $V^*(z, r_d; W)$ to the maximization problem (1.11) exists and is unique for each $(z, r_d, W) \in \mathbb{Z} \times \mathbb{R}_d \times \mathcal{S}_0$.*

As in Moen (1997), the solution $V^*(z, r_d; W)$ corresponds to the tangency point between firms' indifference condition and each active bank's objective function.

1.4.3. Free entry

Because a loan offer is posted only if it provides positive profits to the bank, we can define the expected bank profits prior to entry as:

$$B^*(z; W) = \mathbb{E} \left[\left(q(\theta(z, V^*(z, r_d; W))) B(z, r_d, V^*(z, r_d; W); W) \right)_+ \right], \quad (1.12)$$

where the expectation is taken over all possible bank funding states.

The conditions for free entry and complementary slackness are as follows:

$$c = B^*(z; W), \quad (1.13)$$

$$0 = \theta(z, V)[B^*(z; W) - c], \quad \forall V. \quad (1.14)$$

The free-entry condition (1.13) states that banks keep posting new loan offers as long as the ex-ante profits from credit origination are at least equal to the cost c . Condition (1.14) is the standard complementary slackness condition, which specifies the set of active ($\theta > 0$) and inactive ($\theta = 0$) submarkets.

As is common in this class of search models, the free-entry condition, combined with positive bank entry in each state of the economy, is critical to ensure that the equilibrium is block-recursive and tractable for the analysis of transitional dynamics. Without such conditions, agents in this economy would need to forecast the evolution of the distribution of bank-firm pairs in order to determine the dynamics of credit market tightness in each submarket, in the spirit of Krusell and Smith (1998). Here, this allows the equilibrium market tightness

to be independent of the distribution of banks, since agents already anticipate that credit relationships are formed until the free-entry condition is satisfied with equality.

1.4.4. *Equilibrium*

Characterization of credit markets

The following lemma states that each bank type chooses its optimal submarket, indexed by firm value V , that maximizes its expected profits from posting a loan offer. Whenever it is active, the optimal submarket associated with each bank funding level is unique.

Lemma 2.

- (i) *For a sufficiently small cost c , a solution W satisfying the free entry condition (equation 1.13) exists and is unique.*
- (ii) *The added value obtained from searching for a loan contract, $\rho(z)$ is equalized across all active submarkets. There is an optimal firm value $V^*(z, r_d) \forall (z, r_d) \in \mathbb{Z} \times \mathbb{R}_d$ such that:*
 - (a) *Each active submarket $V^*(z, r_d)$ satisfies $\theta(z, V^*(z, r_d)) > 0$, with*

$$p(\theta(z, V^*(z, r_d)))(V^*(z, r_d) - \mathbb{E}_z[W(z')]) = \rho(z).$$

- (b) *For a given r_d and expected bank profits*

$$\bar{B}(z, r_d, V^*(z, r_d)) = q(\theta(z, V^*(z, r_d)))B(z, r_d, V^*(z, r_d)),$$

we have:

$$\theta(z, V^*(z, r_d)) = \begin{cases} 0 & , \text{ if } \bar{B}(z, r_d, V^*(z, r_d)) \leq 0, \\ q^{-1}\left(\frac{\bar{B}(z, r_d, V^*(z, r_d))}{B(z, r_d, V^*(z, r_d))}\right) & , \text{ if } \bar{B}(z, r_d, V^*(z, r_d)) > 0. \end{cases}$$

In equilibrium, all active submarkets offer the same ex-ante value $\rho(z)$ and borrowers are indifferent across all of them. Indeed, unfunded entrepreneurs keep entering a given sub-

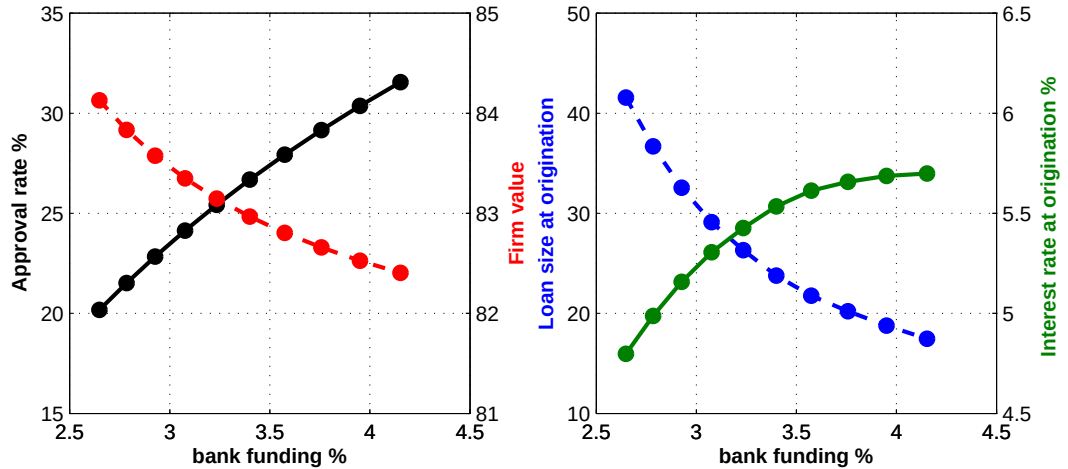
market until the probability of finding a credit opportunity becomes so low that they would eventually prefer applying and accepting less generous contractual terms from other banks.

Proposition 6. *Credit markets in the cross-section*

- $V^*(z, r_d)$ and capital level at origination $K_0(z, r_d)$ decrease with r_d .
- $p(\theta(z, V^*(z, r_d)))$ increases with r_d .

Figure 4 summarizes the main equilibrium properties of these markets. The introduction of search frictions provides a natural and realistic equilibrium setting that allows for the coexistence of multiple loan offers with different contractual terms. It also characterizes the link between contractual terms and the extensive margin of credit. In particular, approval rates are inversely related to both application rates and offered firm values. Banks with low funding costs have a higher opportunity cost if they cannot locate borrowers within a given period. Thus, they offer higher firm value (and, hence, more generous contractual terms) to attract borrowers faster. As a result, firms applying to these banks face a larger number of competing applicants and thus a lower approval rate relative to high-funding banks.

Figure 4: Approval rates and contractual terms in the cross-section



Notes. Approval rates and contractual terms (i.e., loan size and interest rate) at origination, as a function of bank funding cost.

As a corollary, when credit markets are liquid and the degree of bank competition is high (i.e., unfunded firm reservation value W is high), banks with high funding costs may not find lending profitable and, hence, they are deterred from entering credit markets.

I now turn to the definition of the directed search equilibrium and establish its existence.

Definition 1. *Directed search equilibrium*

A directed search equilibrium of the economy consists of: the value for unfunded $\{W(z)\}_z$ and funded $\{V(z, r_d)\}_{z, r_d}$ firms, market tightness $\{\theta(z, V)\}_z$, and loan contract policies $\{(K^*, d^*, V'^*)\}_{z, r_d}$, such that:

- a. Credit search strategy of unfunded firms is optimal. That is, the added value provided by a lending contract is consistent with equation (1.8), and the relationship between this value and market tightness satisfies equation (1.10).
- b. Bank lending policy is optimal. Given $\{W(z)\}_z$, and for all z and r_d , banks maximize their profits by solving problem (1.3).
- c. Bank entry is consistent with free entry condition (1.13), and is strictly positive for all aggregate states z .
- d. Measure of unfunded firms in the economy evolves according to:

$$v_t = v_{t-1} \left(1 - (1 - \sigma) \sum_{r_d} \Gamma_{r_d}^0(r_d) p(\theta(z, V_{r_d})) \right) + (1 - v_{t-1}) \sigma$$

where $\Gamma_{r_d}^0$ is the unconditional distribution of the idiosyncratic funding cost at entry, and $V_{r_d} = V^*(z, r_d)$ is the optimal submarket associated with bank type r_d .

Proposition 7. *A directed search equilibrium exists for a sufficiently small cost c .*

This proposition establishes the existence of a solution consistent with the definition of a directed search equilibrium. Note that this equilibrium solution is well defined — in the sense that it allows for block-recursivity — only when bank entry is strictly positive across

all possible histories of the states of the economy.

1.4.5. Efficiency

Let us introduce the social planner's problem in order to analyze the efficiency of the directed search equilibrium defined above. The social planner maximizes the discounted sum of utilities derived by banks and firms across all incumbent credit relationships, in addition to the utility derived by unfunded entrepreneurs, minus total origination costs. The problem is subject to the dynamics of existing contracts represented by the function f_c (which depends only on $(V_{t-1}, r_{d,t-1}, r_{d,t})$), and the laws of motion of the mass of unfunded firms v_t and the distribution of credit relationships g_t .

To simplify notations, and without loss of generality, I abstract from aggregate shocks. Also, the social planner faces the same contracting frictions as each individual bank. Hence, I can immediately replace the original problem with the corresponding solution to its Lagrange multiplier formulation, by taking the optimal weights associated with the firm value to be $\Lambda_{t+1} = \frac{1}{u'(d_t)}$. The social planner therefore maximizes the following objective function:

$$\begin{aligned}
& \max_{v_t, g_t, \theta_V, J_t, V_t} \quad \mathbb{E} \sum_t \beta^t \left[\sum_{r_{d,t}, V_t} g_t(r_{d,t}, V_t) [S(r_{d,t}, V_t) - d(V_t) + \Lambda_{t+1}(r_{d,t}, V_t) u(d(V_t))] \right. \\
& \quad \left. - cJ_t + v_t u(d_0) \right] \\
& s.t. \quad \forall (t, z^t) \\
\Lambda_{t+1}(r_{d,t}, V_t) &= \frac{1}{u'(d(V_t))}, & \forall (r_{d,t}, V_t) \\
V_t &= f_c(V_{t-1}, r_{t-1}, r_t), & \forall (V_{t-1}, r_{t-1}, r_t) \\
v_t &= v_{t-1} \left(1 - (1 - \sigma) \sum_r \Gamma_{r_d}^0(r) p(\theta_{V_r}) \right) + (1 - v_{t-1}) \sigma \\
g_t(r, V) &= \sum_{V_{t-1} | V_t = V} (1 - \sigma) g_{t-1}(r_{t-1}, V_{t-1}) \Gamma_{r_d}(r_{t-1}, r) + J_t q(\theta_{V_r}) \Gamma_{r_d}^0(r) \mathbb{1}_{V_r = V}, \forall (r, V),
\end{aligned}$$

where Γ_{r_d} is the transition probability, $\Gamma_{r_d}^0$ is the unconditional distribution of the idiosyncratic funding cost at entry, and V_{r_d} is the optimal submarket chosen by the bank with funding cost r_d .

Proposition 8. *Whenever it exists, the directed search equilibrium is constrained inefficient when entrepreneurs are risk averse. When entrepreneurs are risk neutral, the directed search equilibrium is unique and delivers the efficient allocation.*

Proposition 8 shows the existence and uniqueness of a solution to the planner’s problem and establishes that the corresponding allocation coincides with that of the directed search equilibrium only if entrepreneurs are risk neutral. The equilibrium inefficiency is due to the combination of risk-averse entrepreneurs and search frictions in credit markets, and generalizes the results obtained by Acemoglu and Shimer (1999) and Golosov et al. (2013) in the context of labor markets. In particular, it states that credit-rationed entrepreneurs are inefficiently “impatient” and choose to apply for loans in markets that offer low firm values but high approval rates. Thus, banks offer insurance to these risk-averse and unfunded borrowers by supplying such loan contracts. As a result, capital is allocated at an inefficiently fast rate and the market share associated with high-funding-cost banks is too high. This result is interesting because it provides new grounds for a theory of inefficient lending relying on credit search frictions and leaves room for novel forms of policy intervention.²⁸

1.5. General equilibrium effects of search and limited enforceability

This section illustrates the general equilibrium effects generated by the key parameters governing contract dynamics (i.e., share of divertible capital η and probability of fresh start ξ) and credit origination (i.e., matching elasticity γ_m and origination costs c). In particular, I investigate how the interaction between search and agency frictions operates through both intensive and extensive margins of credit and explore its implications at the aggregate level.

First, the contracting parameters impact the extensive margin of credit in that a tighter participation constraint (i.e., higher η and ξ) slows down firms’ ability to outgrow their borrowing constraint. Thus, it limits the total match surplus and negatively impacts bank

²⁸A further theoretical investigation of this result, and its consequences on bank lending markets and the real economy, is left for future research.

entry, credit market tightness, matching probabilities, and firm value at origination. Second, search parameters not only affect the extensive margin of credit, but also shape the dynamics of incumbent lending relationships due to the feedback effect operating through the outside option. For example, highly liquid credit markets (i.e., markets with low search frictions, with low c or high γ_m) can generate further distortions in the contract by tightening the borrowing constraint more and slowing down capital intermediation.

1.5.1. Comparative statics

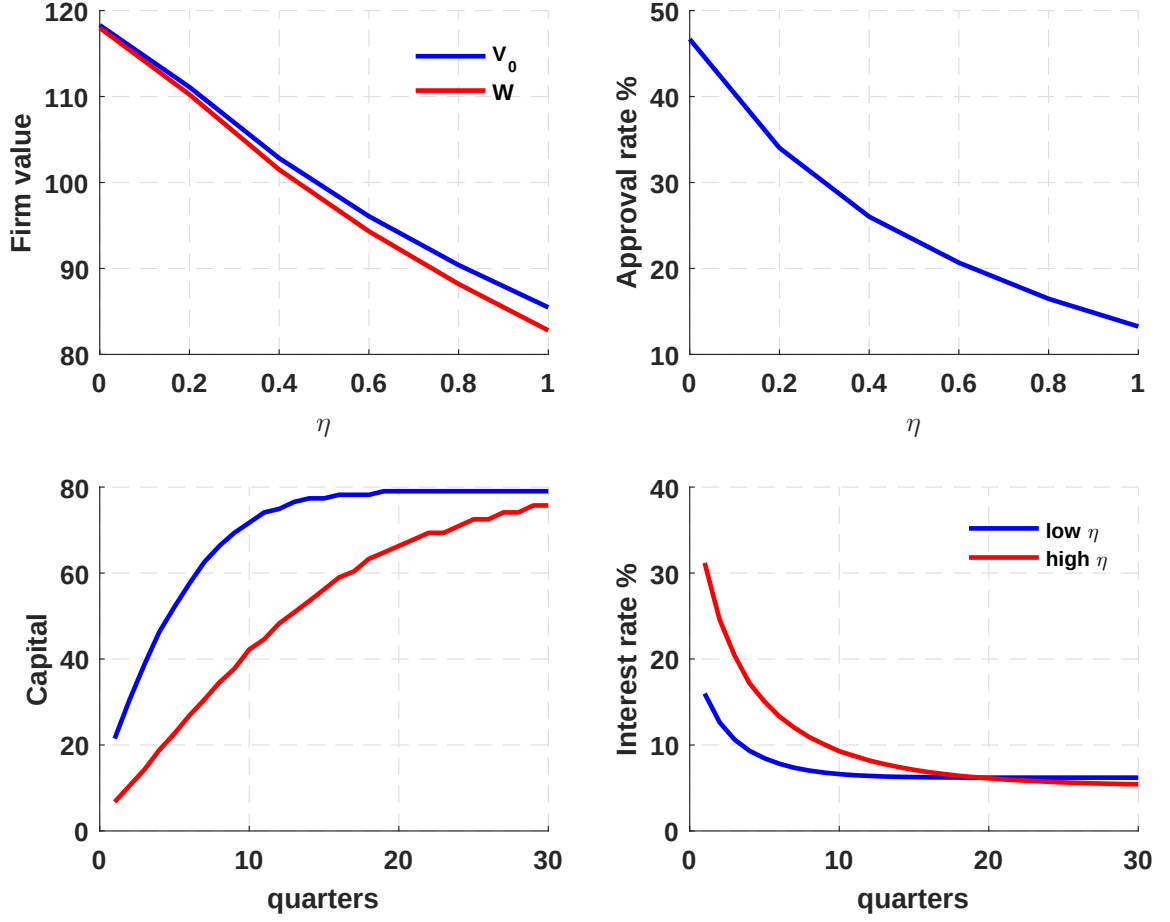
Contract parameters η and ξ

Parameters η and ξ enter directly in the firm's outside option and have qualitatively similar effects on both extensive and intensive margins. These parameters, however, govern two distinct aspects: η affects the relationship-specific component of the outside option (i.e., growth rate of firm as a function of their current level of capital), whereas ξ governs the sensitivity of credit growth rate to aggregate shocks.

Share of divertible assets η . The share of divertible capital η captures the entrepreneur's default incentives. A higher diversion rate amplifies the agency problem and severely distorts the contract. As a consequence, the total surplus extracted from the match and the firm value at origination are also lower.

Figure 5 shows how η impacts several variables of interest. The top left-hand-side panel shows the firm value both at contract origination V_0 and when unfunded W . Both variables decrease with η , as the agency problem becomes more severe and the joint match surplus decreases. The wedge between V_0 and W depends on the approval rates shown in the top right-hand-side panel, and widens with η as firms' ability to access credit declines. The bottom panels compare the evolution of contractual terms for low and high η . Given that they offer a higher firm value at origination, environments with limited agency frictions (i.e., low η) allow firms to start at higher levels of capital and lower interest rates. As a result, the first-best allocation of capital is also attained at a faster rate.

Figure 5: Comparative statics - η



Fresh start probability ξ . The fresh start probability ξ reflects the entrepreneurs' ability to access credit markets after default. The inverse of ξ is the average period of exclusion and can be viewed as a measure of the degree of leniency of the bankruptcy code toward entrepreneurs. A higher fresh-start probability allows for a shorter exclusion from credit markets, raises the outside option of the borrower, and further distorts the contracting problem.²⁹ This parameter also captures the quality of legal institutions and the speed at which bankruptcy procedures are dealt with. The comparative statics are consistent with the studies of La Porta et al. (1997) and La Porta et al. (1998) which show that economies with more entrepreneur-friendly bankruptcy laws typically exhibit lower access to credit.

²⁹The comparative statics for ξ are reported in Figure 15 in the appendix.

Search parameters γ_m and c

I now turn to exploring the effects of parameters linked to search in credit markets. Higher matching elasticity γ_m and lower origination costs c yield the same qualitative features of credit markets, namely, more liquid markets with a higher approval rate and firm value. In this context, the degree of competition in credit markets is captured by the equilibrium market tightness θ , which is the ratio of loan offers to applications.³⁰ The model thus provides interesting dynamics for contractual terms, whereas increased bank competition actually implies lower credit availability and higher interest rates at origination. However, such effects get reversed throughout the lending relationship as the agency problem becomes less severe and the feedback effect weakens.

Matching elasticity γ_m . The matching elasticity parameter governs the degree of matching frictions in credit markets. The following comparative statics show that markets with a high degree of competition typically generate higher approval rates and better access to credit. This is consistent with evidence from the Senior Loan Officer Opinion Survey (SLOOS) conducted by the Federal Reserve Board, which reports that competition among lenders is often highlighted as one of the major reasons for easing lending standards.³¹

The bottom right panel of Figure 6, which highlights how interest rates vary throughout a relationship for both low and high degrees of competition, is also qualitatively consistent with the empirical results in Petersen and Rajan (1994) and Santos and Winton (2013).³²

Origination cost c . Decreasing c provides similar qualitative results. From the free-entry condition, we can see that a lower value of c is accompanied by more bank entry, producing easier access to credit and larger unfunded firm value W in equilibrium.³³

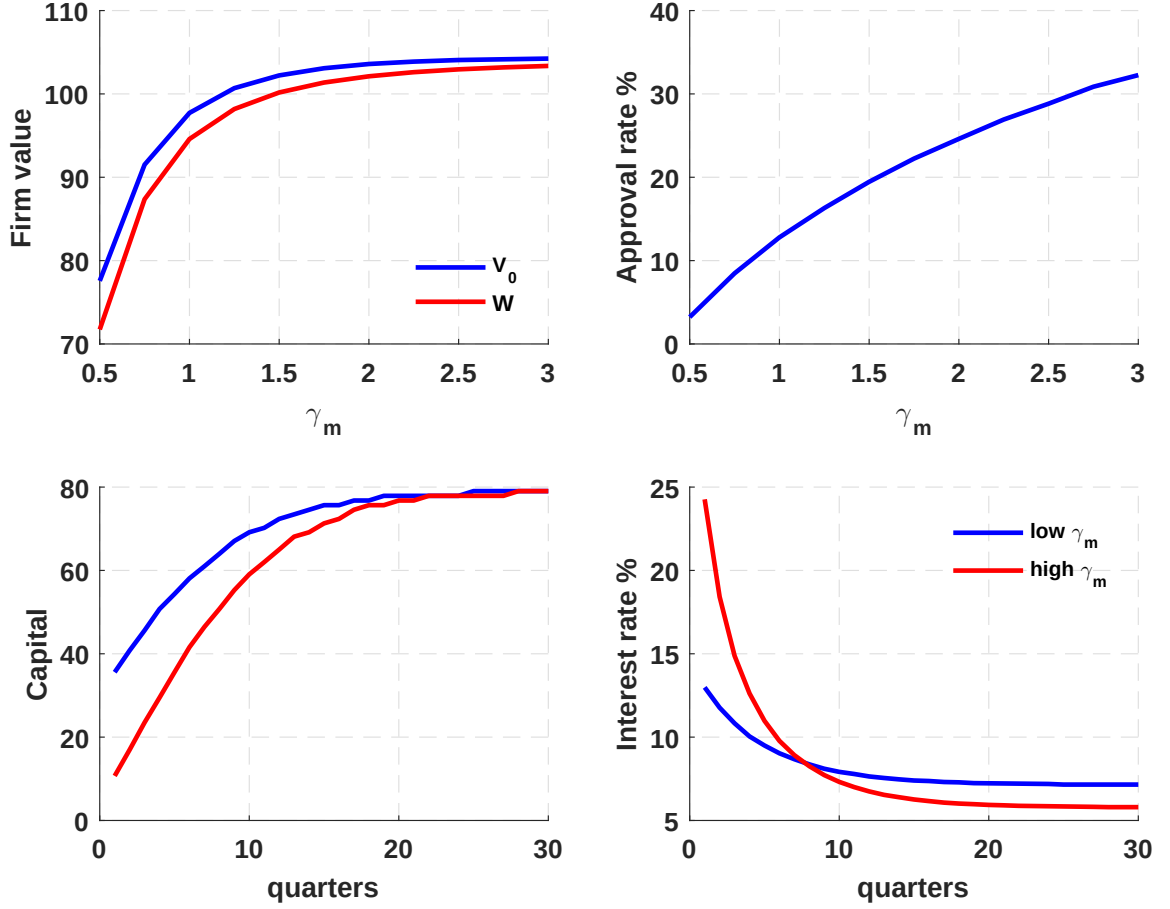
³⁰Inderst and Müller (2004) provide a similar interpretation.

³¹“Among domestic respondents that reported having eased either standards or terms on C&I loans over the past three months, the majority of banks cited more-aggressive competition from other banks or nonbank lenders as an important reason for having done so.” - Senior Loan Officer Opinion Survey on Bank Lending Practices, January 2014: www.federalreserve.gov/boarddocs/SnloanSurvey/201402/.

³²Note that Petersen and Rajan (1994) look at firm age instead of relationship length.

³³The comparative statics for c are reported in Figure 16 in the appendix.

Figure 6: Comparative statics - γ_m



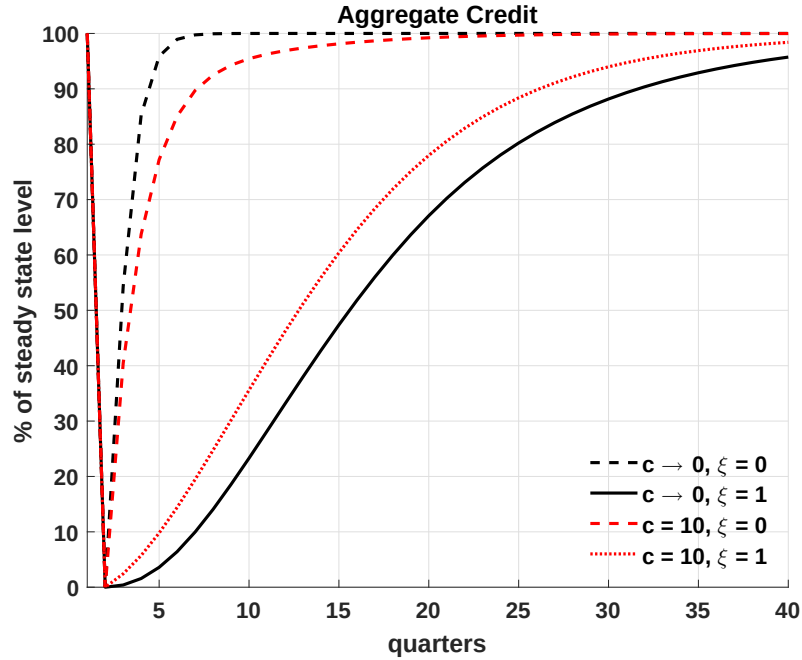
The origination cost c can be interpreted in several ways. First, it can be viewed as an initial sunk investment needed to start up the firm's project. The above result can therefore provide grounds for potentially explaining why entrepreneurs seeking lower levels of financing have relatively higher approval rates. Second, it can be associated with screening costs at the bank level. In this case, the model predicts that lower screening costs materialized, for example by online lending, technological progress, or the adoption of credit scoring, would thus imply a faster creation of lending relationships but a potentially slower credit growth at the bank-firm level.

1.5.2. Aggregate implications: trade-off between extensive and intensive margins and the speed of credit allocation

The model implications at the lending relationship level do not necessarily apply at the macroeconomic level. Indeed, the interaction between search and agency frictions implies a clear trade-off between intensive and extensive margins of credit. On the one hand, a high degree of credit market liquidity or bank competition would indeed lead to higher approval rates and more creation of lending relationships. On the other hand, this would also generate a higher outside option for the borrower, and hence further contract distortions and lower initial capital allocation at the bank-firm level.

Let us illustrate how both extensive and intensive margins affect the aggregate dynamics of bank credit and the speed of credit allocation through the following experiment. Figure 7 shows the evolution of aggregate credit following the destruction of 100% of lending relationships, for two economies with different degrees of search frictions (the case where c tends to 0 refers to low search and perfect competition) and fresh-start probability ξ .

Figure 7: General equilibrium effects and credit allocation



Parameter ξ governs the strength of the feedback effect from credit markets to individual loan contracts and turns out to be an important determinant of the speed of credit allocation. Indeed, as ξ tends to 0, the feedback effect becomes muted, and both agency and search frictions combine in a positive way to generate very slow recoveries. This is the case where the extensive margin effect due to search frictions dominates. Conversely, when ξ is sufficiently high, the feedback effect becomes strong enough so that search frictions actually help mitigate agency frictions. This ultimately generates faster capital allocations as the outside option becomes small. This is the case where the intensive margin effect dominates.

In light of this result, it is therefore important to recognize that policies aimed at increasing bank competition in the aftermath of financial crises can actually slow down credit recovery instead of boosting it. This is particularly relevant when the general equilibrium forces described above are strong, as may be the case for economies with relatively weak investor protection or bankruptcy laws that are “too” entrepreneur-friendly.

1.5.3. Empirical predictions

The model provides theoretical insights at both micro and macro levels, which can be tested empirically. These model predictions relate mainly to the dynamics of lending relationships, credit markets and bank competition, and institutional environment.

Lending relationships

- Credit availability and pricing improve with the length of the lending relationship.
- Credit availability is higher and increases at a relatively faster rate for firms matched with low-funding banks.

Credit markets

- *Cross-section.* Banks with low funding costs have higher application rates and lower approval rates. They also offer better contractual terms, since they provide greater

access to credit and charge lower interest rates throughout the lending relationship.

- *Business cycle.* Access to credit is more difficult during downturns. The degree of bank competition and approval rates are countercyclical.

Bank competition

- Bank competition generates higher approval rates and increases the creation rate of lending relationships (i.e., less credit rationing at the extensive margin), but decreases the amount of credit available at origination (i.e., more credit rationing at the intensive margin).
- Controlling for bank type, bank competition increases the dispersion of interest rates across borrowers. Controlling for the length of the lending relationship, bank competition decreases the dispersion of interest rates offered across banks.

Legal environment

- Economies with more entrepreneur-friendly bankruptcy laws have higher levels of credit rationing at both intensive and extensive margins.
- Credit availability and interest rates improve at slower rates in these economies.

1.6. Quantitative analysis

I now move to the analysis of the quantitative properties of the model and its application to commercial lending. I first specify the functional forms associated with the model and calibrate its parameters, and then evaluate its steady-state and business-cycle properties. Next, I examine the response of the economy to aggregate bank funding and productivity shocks, to show how the propagation mechanism operates through the extensive margin of credit, and highlight the asymmetric impact across borrowers. Finally, I run policy

experiments and analyze the effects of subsidies to origination costs and bank funding.

1.6.1. Model specification

I specify the firm-level profit function to be of the general form $F(z, K) = zK^\alpha - k_f$, with decreasing returns to scale parameter α and fixed cost k_f .³⁴ I set the period utility of entrepreneurs to be CRRA of the form $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$ with coefficient of relative risk aversion γ . I also assume a CES matching function with elasticity γ_m . This function generates the following meeting probabilities:

$$\begin{aligned} p(\theta) &= \theta(1 + \theta^{\gamma_m})^{-\frac{1}{\gamma_m}}, \\ q(\theta) &= (1 + \theta^{\gamma_m})^{-\frac{1}{\gamma_m}}. \end{aligned}$$

In this numerical exercise, I now generalize the theoretical model specified in the previous section and allow for two aggregate shocks, namely productivity z and common component of bank funding $\bar{r}$, in addition to the idiosyncratic component of bank funding s . The aggregate log-productivity follows an AR(1) process:

$$\log(z_{t+1}) = (1 - \rho_z) \log(\bar{z}) + \rho_z \log(z_t) + \sigma_z \epsilon_{t+1}, \quad \epsilon \sim \mathcal{N}(0, 1).$$

Similarly, the aggregate and idiosyncratic components of the bank funding cost follow AR(1) processes:

$$\begin{aligned} \bar{r}_{t+1} &= (1 - \rho_r) \bar{r}_0 + \rho_r \bar{r}_t + \sigma_r \nu_{t+1}^r, & \nu^r &\sim \mathcal{N}(0, 1), \\ s_{t+1} &= \rho_s s_t + \sigma_s \nu_{t+1}^s, & \nu^s &\sim \mathcal{N}(0, 1). \end{aligned}$$

The process innovations $\{\epsilon_t\}_t$ and $\{\nu_t^r\}_t$ are assumed to be uncorrelated. Similarly, the innovations for idiosyncratic shocks $\{\nu_{i,t}^s\}_{i,t}$ are uncorrelated with respect to aggregate coun-

³⁴Introducing a fixed cost does not affect any of the theoretical results.

terparts and are independent across banks. Eventually, the distribution of bank-specific cost shocks at entry $\Gamma_{r_d}^0$ is uniform over the interval $[-\sigma_{s,entry}, \sigma_{s,entry}]$.

All processes are approximated using a finite grid with five shock realizations for aggregate productivity ($N_z = 5$) and seven shock realizations for both aggregate ($N_{\bar{r}} = 7$) and bank-specific ($N_s = 7$) funding components following the discretization methodology of Tauchen and Hussey (1991).

1.6.2. Calibration

The calibration exercise relies on the simulated method of moments (SMM). I set the length of a period in the model to one quarter. I need to assign values to the following set of model parameters $\{\beta, \alpha, \gamma, \sigma, \xi, \eta, c, \gamma_m, d_0, k_f\}$, in addition to the parameters governing productivity and bank funding shocks $\{\bar{z}, \rho_z, \sigma_z, \bar{r}_0, \rho_r, \sigma_r, \rho_s, \sigma_s, \sigma_{s,entry}\}$.

In the baseline estimation, I pre-calibrate four model parameters. The discount rate β is set to 0.9875, which corresponds to an annual real interest rate of about 5%. The coefficient of relative risk aversion γ is set to 0.5. I also set the decreasing returns to scale parameter α to 0.43, which is close to the lower bound of estimates provided in the literature. This parameter governs the firm's optimal scale and the dispersion of firm size distribution, and also the sensitivity of firm capital and output to productivity and bank funding costs. The probability of exogenous separation is set to 4.4% per quarter to match its empirical counterpart.³⁵

I also use the data to determine the parameters behind the two processes governing bank funding. To this end, I use information contained in banks' quarterly Consolidated Reports of Condition and Income and define the process of real funding costs in the model as the ratio of total operating expenses over total assets for the period 1984 through 2014, adjusted

³⁵See appendix for details about the construction of lending relationship flows based on DealScan data. This estimate is consistent with an average duration of lending relationship of about six years, which is in line with results in Ongena and Smith (2001). Note also that this is more likely to be a lower bound for the separation rate of small and medium-sized businesses, given that the DealScan database covers relatively large firms.

with the GDP deflator.³⁶ The cross-sectional average of the quarterly funding costs is given by $\bar{r}_0 = 1.15\%$. I also estimate the autocorrelation and standard deviation of this process to be $\rho_r = 0.94$ and $\sigma_r = 0.25\%$ at the quarterly frequency. Finally, I extract the bank-specific component of the funding cost by subtracting the cross-sectional average evaluated within each quarter and computing the autocorrelation and standard deviation for each individual bank time series available throughout the sample period. The tabulated cross-sectional averages obtained from this exercise are $\rho_s = 0.84$ and $\sigma_s = 0.1\%$.

The remaining ten parameters, namely origination costs c , share of divertible capital η , probability of fresh start ξ , fixed cost k_f , garage production d_0 , matching elasticity γ_m , productivity process parameters $(\bar{z}, \rho_z, \sigma_z)$, and upper bound of the distribution of idiosyncratic shocks at entry $\sigma_{s,entry}$, are calibrated using a particle swarm optimization algorithm minimizing the relative squared distances between empirical and simulated moments.

I target (i) a yearly real net return-on-assets of 1.2% for Commercial and Industrial loans, consistent with Boualam (2014). This moment is matched to the cross-sectional average net return per unit of capital lent, which takes into account the interest rate charged to borrowers minus bank operating costs (interest and non-interest costs). This moment helps identify origination cost c because it captures the expected bank profits realized throughout a relationship.

I target (ii) a yearly cross-sectional average investment rate of 14.5% (Gomes (2001)), calibrated to match the corresponding moments associated with credit growth rate given by $\frac{\Delta K}{K}$, and (iii) a fraction of constrained firms of 40% as in Cooley et al. (2004). These targets are particularly helpful in determining η and ξ , the parameters associated with the borrower's outside option. In particular, η governs the speed at which firms reach the unconstrained region (as η tends to 0, the outside option becomes independent of K and firms reach the first-best level almost immediately). On the other hand, because ξ is linked

³⁶I use here total operating costs, which include both interest and non-interest expenses rather than interest expenses alone, to better account for heterogeneity in marginal costs incurred by banks. For simplicity and to stay close to the exposition of the model, I will keep referring to these costs as funding costs.

to the firm value when unfunded, it relates the volatility of credit growth rate to that of the aggregate shocks in the economy.

I define (iv) the credit origination rate as the volume of newly issued credit (to both new and incumbent borrowers) divided by total stock of credit in the economy, and target a quarterly rate of 6.8%, as obtained for Commercial and Industrial loans over the period 2000-2014.³⁷ This moment is helpful in determining the average level of the productivity process $\bar{z}$.

I target a (v) market-to-book value of 2.1, consistent with the estimate for bank-dependent firms reported in Chava and Purnanandam (2011), and map it to its model counterpart at the firm level defined as $\frac{V+B}{K}$, and where the net present value of the contract to the bank B is interpreted as total debt of the firm. This measure captures the degree of distortion in the contract (and, hence, reflects the allocation of surplus across borrowers and lenders) and will be helpful in identifying garage production d_0 .

The fixed cost k_f also impacts the joint match surplus and, hence, the search behavior of agents in the economy. I seek to pin down this parameter by setting (vi) the average quarterly approval rate $p(\theta)$ to 28%, which corresponds to an annual application success rate of about 75% and an expected credit search period of about three quarters.³⁸

I also identify the matching elasticity parameter γ_m by targeting (vii) a slope coefficient of

³⁷Given an annual credit growth rate of 14.5% and a creation rate of relationships set at 17.6% in the steady state, this moment is a 1-1 map with the relative loan size offered to new borrowers through the following identity:

$$\begin{aligned} \text{credit origination rate} &= \frac{\text{new relationships} \times \bar{K} + \text{incumbent relationships} \times \Delta K}{\text{incumbent relationships} \times K} \\ &= 0.176 \times \frac{\bar{K}}{K} + 0.145 = 0.272, \end{aligned}$$

with $\bar{K}$ the average loan size at origination.

³⁸Eurostat provides data for the U.K. showing an annual success rates for firms seeking credit in 2007 and 2010 at 88% and 65%, respectively. See epp.eurostat.ec.europa.eu/portal/page/portal/european_business/special_sbs_topics/access_to_finance. Similarly, the 2014 Small Business Credit Survey (SBCS) shows that 44% (56%) of firms received all or most (at least some) of their financing needs during the first half of 2014.

14% for the relationship between approval and interest rates in the cross-section of banks.³⁹

The parameter $\sigma_{s,entry}$ governing the support of the distribution of bank-specific shocks at entry is calibrated to match (viii) an average dispersion in bank operating costs of 1.77%.

I finally use the autocorrelation and standard deviation of log-detrended output in order to determine the parameters associated with the aggregate productivity process. In particular, I use real quarterly log-GDP data (seasonally adjusted, and detrended using the HP filter with parameter 1600) obtained from the Bureau of Economic Analysis for the period Q1 1947 - Q4 2013, and tabulate the following targets: (ix) autocorrelation of 0.84 and (x) standard deviation of 0.017.

Table 1 reports all model parameter values. Table 2 reports calibration targets and model counterparts.

1.6.3. Model properties and validation

The model generates a two-dimensional stationary distribution of bank-firm relationships g , which depends on bank funding cost and current firm value. In the steady-state, unfunded firms account for 14% of the population, while the fraction of unconstrained firms is 45%. The remaining 41% are firms matched to banks, but with constrained levels of credit.

The model calibration also leads to an average credit growth rate of about 13% per year, which is consistent with borrowers being able to reach unconstrained levels of financing after five to six years. Similarly, interest rates charged to borrowers decrease by an average of 25 basis points per relationship year. This is in line with results in Hubbard et al. (2002) that suggest a decline of three to nine basis points per lending relationship year after controlling for size effects.⁴⁰

³⁹In the cross-section, this means that a 10% increase in approval rates is accompanied by a 1.4% increase in interest rates charged to borrowers.

⁴⁰Note that size and lending relationship effects are tightly linked in the model, suggesting that the relationship-year effect is biased upward.

Credit application rates, approval rates, and contractual terms in the cross-section

To test the validity of the calibrated model, I start by exploring the cross-sectional properties of credit markets, and in particular the relationships among credit application rates, approval rates, and interest rates offered to borrowers. As described in the theoretical section, the directed search setup predicts (i) a negative relationship between application and approval rates (Figure 8a) and, similarly, (ii) a negative relationship between approval rates and interest rates charged to borrowers (Figure 8b).

The empirical counterpart of these moments is tabulated based on information contained in the 2014 joint Small Business Credit Survey (SBCS) report.⁴¹ In particular, I use data on application and approval rates across different types of financial intermediaries to construct model-equivalent moments.⁴² Because of the data classification and focus of the paper, I consider only information available for large (classified as “large” and “regional” in the SBCS) and small (classified as “small regional or community”) banks. I then solve the model, considering two idiosyncratic bank shocks, and finally map low-funding and high-funding types to large and small banks, respectively.⁴³

In the same vein, Figure 8b displays both empirical and model-generated moments for interest rates across different types of banks. In particular, I construct the corresponding interest rate spreads using the effective weighted-average interest rates with minimal risk

⁴¹This survey is jointly conducted by the Federal Reserve Banks of New York, Atlanta, Cleveland, and Philadelphia, with the purpose of measuring and reporting information about the functioning of the small business credit market.

⁴²The appendix provides further details about data and assumptions related to firm credit search used to construct model-equivalent moments.

⁴³Note that with the assumption that borrowers can apply only once per period, the application rate, defined as the fraction of unfunded borrowers u_i applying to a given bank i , is given by:

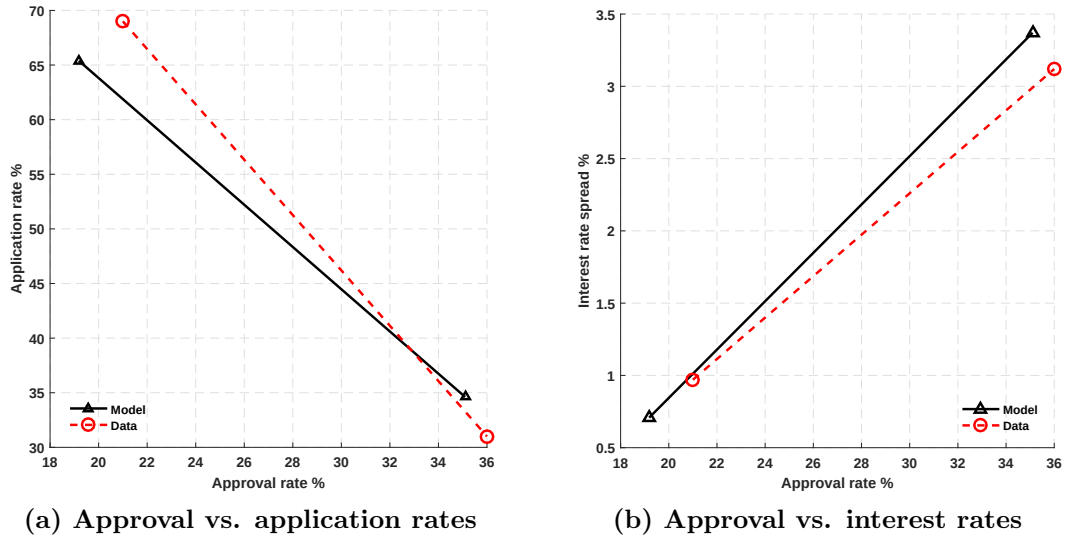
$$\text{application rate}_i = \frac{u_i}{\sum_i u_i} = \frac{\frac{1}{\theta_i}}{\sum_i \frac{1}{\theta_i}}.$$

We can then solve a simple system of two equations and two unknowns to determine the equivalent theoretical approval rates and infer the matching elasticity γ_m .

adjusted by the 1-year Treasury constant maturity rate, for the period Q2 1997 - Q1 2014.⁴⁴

Since contractual terms are not adjusted as frequently in practice, I map the empirical interest rates to the model-generated average interest rate spreads (interest rate charged minus risk-free rate) incurred throughout a 10-year lending relationship for each type of lender.⁴⁵ The relationships between approval and application rates, and between approval and interest rates, are both satisfied overall by the model-generated moments for both level and slope.

Figure 8: Credit application, approval, and interest rates in the cross-section



Credit relationship flows

I tabulate measures of creation and destruction rates of lending relationships from LPC's DealScan database over the period Q1 1995 - Q1 2013.⁴⁶ DealScan's coverage is biased

⁴⁴These time series are obtained from FRED (EEANXSLNQ, EEANXSSNQ). I report here the corresponding interest rates across a larger sample period, so as to compare the model-generated moments to the data. Details about data construction are available in the appendix.

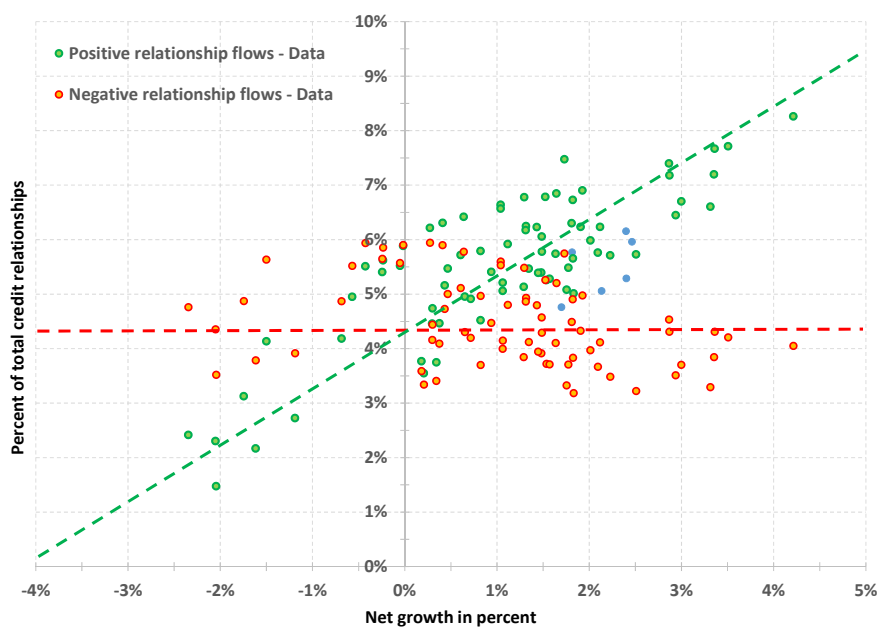
⁴⁵Note that the mapping between the data and model moments is not straightforward here and is subject to caveats. In particular, the data are at the aggregate level and, hence, do not contain information about characteristics of bank-firm pairs. Similarly, the model interest rate spreads are sensitive to the length of relationships considered.

⁴⁶Although the period covered by the database starts considerably earlier, my sample begins in 1995. Imposing this date ensures that the database is already well populated, thus limiting the mechanical bias

toward relatively large firms and may not be representative of the whole economy. Yet, this is, to my knowledge, the only comprehensive and publicly available source of information that can be used to construct proxies for these flows. While it is unclear whether positive flows are higher or lower when considering all firms in the economy, it is reasonable to assume that destruction rates are a priori understated, since smaller firms are potentially more likely to default or to not have their loans rolled over, given their relatively high propensity to violate loan covenants.

Figure 9 displays the quarterly gross rates of creation and destruction of credit relationships against the net growth rate (i.e., creation minus destruction rate). The model generates a one-to-one map between positive and net flows by construction, since the model assumes a constant destruction rate σ . This assumption seems reasonable as a first pass.⁴⁷

Figure 9: Positive and negative flows of lending relationships



Notes. Positive and negative flows of lending relationships as a function of net flows (theoretical moments are shown in dashed lines).

in positive flows induced by an increase in data coverage. See empirical appendix for details about data construction.

⁴⁷Introducing endogenous separation in the contracting problem would be a natural extension to improve the model fit for destruction rates during downturns, and is left for future research.

More important, Table 4 in the appendix shows that positive flows are more volatile and more sensitive to the business cycle relative to negative flows. This suggests that the adjustment of the stock of lending relationships weighs more heavily on the process of origination and entry rather than on destruction. In particular, the relative rigidity in destruction rates may reflect the fact that loan agreements are typically fixed-term and banks may not always have the flexibility to cut lending until contract maturity.

Funding cost and interest rate dispersion

The dispersions in funding costs and interest rates charged to borrowers mirror the degree of credit frictions in the economy. Indeed, a perfectly competitive environment with no friction should lead to the existence of a unique bank supplying credit to all borrowers at a fixed and unique interest rate. This framework generates dispersion in interest rates through two channels related to agency and search frictions.

First, the introduction of agency frictions generates interest rate dispersion among firms within the same bank, depending on the maturity of lending relationships. Second, the introduction of search also leads to a non-trivial distribution of contractual terms at the origination stage. The tensions created through these frictions and their evolution over the business cycle are thus reflected through the dynamics of interest rate dispersion.

The construction of measures of dispersion for bank operating costs (ratio of the sum of interest and non-interest expenses over total assets) and net interest rates on loans (interest income minus net charge-offs over total C&I loans) is described in the appendix.⁴⁸ These measures are then mapped to their model counterparts.⁴⁹

Data in Table 3 exhibit an increase in the dispersion of interest rates charged during recessions, while the dispersion in operating costs does not appear to comove with the business cycle. Since I have calibrated the model to fit the dispersion in bank funding costs in the

⁴⁸Figure 21 exhibits the corresponding time series.

⁴⁹Note that these proxies do not account for several heterogeneity dimensions absent in the model. For example, they abstract from differences in banks' business models or the collateral coverage ratio of loans.

steady state, I can now examine the amount of dispersion in interest rates explained by the model. The model-generated dispersion in interest rates accounts for about 50% of its empirical counterpart. While positively correlated, this moment is also relatively more sensitive to the business cycle when compared to the data.

1.7. Effects of aggregate shocks

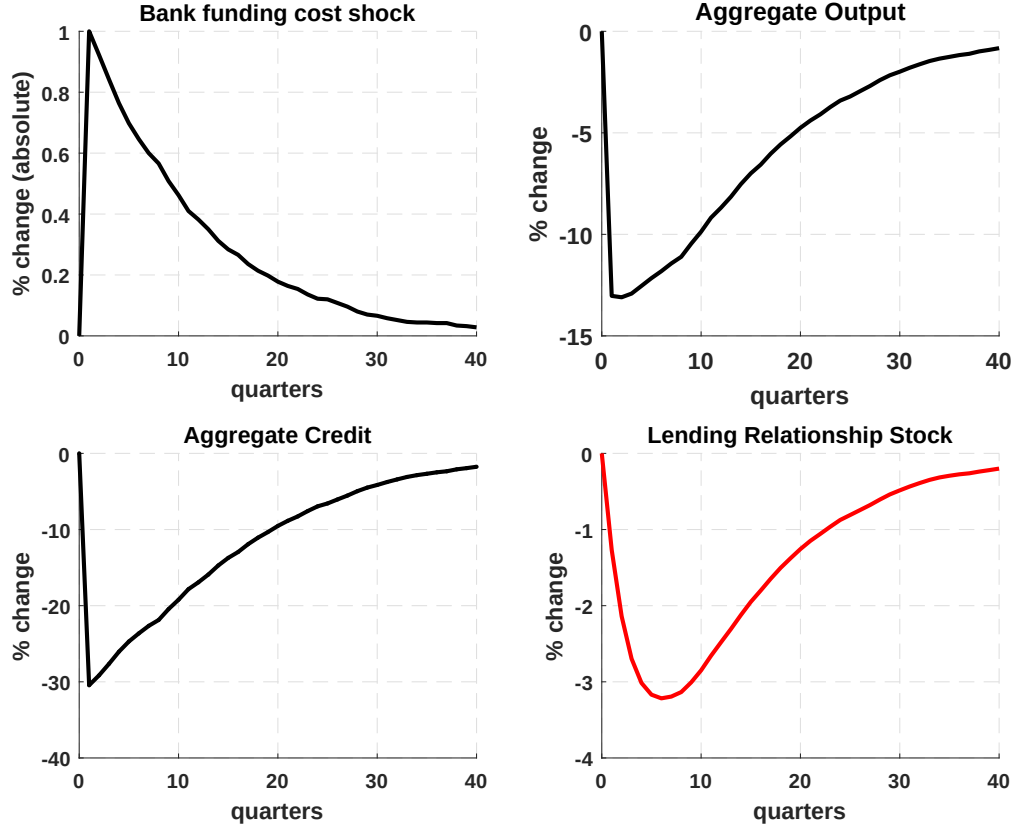
In this section, I study the effects of aggregate shocks related to bank funding and firm productivity. I examine the response of the economy along both extensive and intensive margins, and analyze how these shocks affect credit origination and contractual terms for both incumbent and new borrowers.

1.7.1. *Bank funding shocks*

I first examine the implications of a 1% upward shift in aggregate bank funding cost (the level of funding reverts back to the steady state with persistence $\rho_{\bar{r}}$). Both intensive and extensive margins of credit are at play. Along the intensive margin, funded firms (both constrained and unconstrained) scale down their production as the optimal level of capital decreases. In this case, both firm size and credit availability adjust immediately, since there are no adjustment costs or capital accumulation. Such effect is short-lived and is mapped one-to-one with the dynamics of the funding cost.

More important, the propagation mechanism in the model operates through the extensive margin. Indeed, a bad shock to banks severely impacts access to financing and contractual terms for unfunded firms. As the joint surplus declines, it becomes less profitable for banks to establish new lending relationships. Moreover, only banks with sufficiently low funding costs are able to extend loan offers. Thus, credit market tightness plummets and approval rates decline, due to the joint effect of a drop in the number of offers and a rise in the mass of unfunded firms. This is consistent with empirical evidence in Khwaja and Mian (2008), Jiménez and Ongena (2012), or Bonaccorsi di Patti and Sette (2012) who show that the

Figure 10: Impulse response: bank funding cost - aggregate variables



Notes. Impulse response: 1% upward shift in bank funding cost.

probability of extending loans to new borrowers declines after negative bank shocks.

The extensive margin effect therefore generates an overall decline in the number of relationships in the economy whenever the creation rate falls below the fixed rate of destruction. This effect remains at play until the funding cost reverts back to a sufficiently low level, allowing for a positive net creation of lending relationships, and therefore generates sustained periods of low credit and output, as shown in Figure 10. Thus, the more persistent the aggregate shock, the more amplified the decline in the stock of relationships and the more sluggish the recovery.

In contrast to other macroeconomic models, it is worth highlighting that the aggregate credit response is endogenously determined and depends on the distribution of lending

relationships in place. Thus, the economy’s resilience to bad shocks is dependent on the number and strength of bank-firm connections, rather than being driven by exogenous fluctuations in firm collateral value.

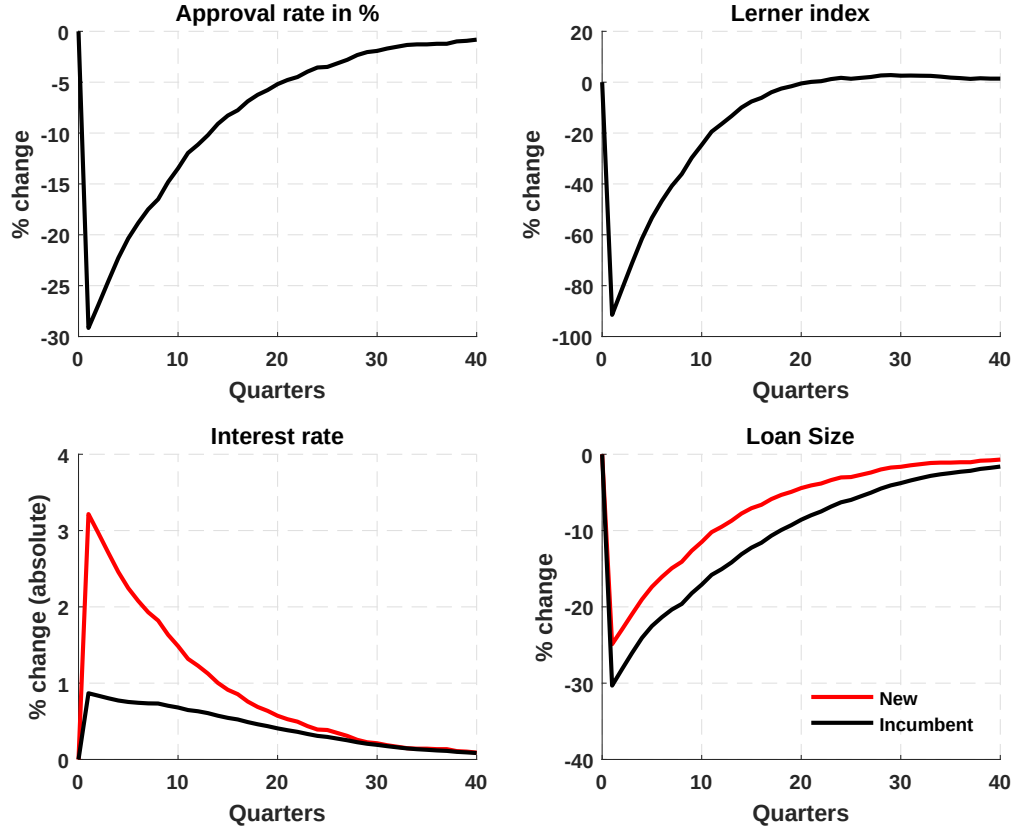
This theoretical insight highlighting the critical role of lending relationships and their aggregate implications in downturns is shared with Den Haan et al. (2003), and finds its empirical grounds in seminal work by Bernanke (1983) and more recent studies such as Chodorow-Reich (2014), Vickery (2005), Kandrac (2014), and Hansen and Ziebarth (2016). Moreover, by endogenizing firm financing uncertainty through a search-theoretic approach, this setup can be viewed as an alternative to models introducing additional forms of credit shocks, as shown in Jermann and Quadrini (2012) or Khan and Thomas (2013).

Let us now focus on the response of the contractual terms, namely credit availability and pricing. The model generates an interesting stark asymmetric transmission of shocks in the cross-section. Indeed, strong long-term relationships partially insure borrowers against negative shocks. For example, banks shield their incumbent borrowers by marginally increasing their interest rates by about 90 basis points during the crisis. Contractual terms for new borrowers respond more significantly and interest rates charged soar by over 3%. Conversely, incumbent and new borrowers face similar relative declines in the amount of credit supplied to them.⁵⁰

The asymmetry in the treatment between these two types of borrowers is due in part to the implicit insurance mechanism provided by long-term contracts, and the decline in bank competition during downturns. On the one hand, banks end up subsidizing their long-term borrowers during recessions (at the expense of their own profitability) in order to smooth out borrowers’ dividends whenever the match surplus is low. Interestingly, the decline in credit competition actually favors incumbent borrowers. Due to the general equilibrium effect, their outside option declines, which mitigates contract distortions and helps dampen

⁵⁰Note that the garage production parameter d_0 is constant and thus partly mitigates the sensitivity of the unfunded firm value to aggregate shocks.

Figure 11: Impulse response: bank funding cost - contractual terms



Notes. Impulse response: 1% upward shift in bank funding cost. The Lerner index is defined here as the ratio of banks' net return on assets (i.e., the difference between interest rates and total bank costs) over interest rates.

the drop in credit availability.⁵¹

On the other hand, as the bad shock hits the economy, credit market tightness declines and so does bank competition. Thus, the shift of the relative bargaining power toward banks extending credit offers means that they get to extract a larger share of the surplus, further impacting new borrowers. Therefore, the contractual terms offered to unfunded firms are adjusted unfavourably during recessions and reflect a sharp decline in credit availability in addition to higher interest rates. This result relates directly to the empirical literature on the bank lending channel and is particularly consistent with findings in Santos and Winton (2013). That work shows that the degree of bank bargaining power can be a critical factor

⁵¹The dampening effect of relationship banking is also present in Bolton et al. (2016) through a different channel.

governing loan terms following a deterioration of borrowers' cash flows, and suggests that part of credit tightening and interest rate increases may relate to a drop in bank competition during downturns.

1.7.2. Productivity shocks

Let us now consider the response of the economy following a one-standard-deviation decrease in aggregate productivity (aggregate productivity returns to its steady-state level with persistence ρ_z).⁵² This shock generates the same qualitative responses as a positive bank funding shock. Overall, however, responses are relatively less sensitive.⁵³ Two main reasons explain this difference. First, productivity and bank funding costs enter into the surplus function differently. Second and more important, aggregate productivity is less volatile relative to bank funding costs in this calibration.

1.8. Policy experiments

This section explores potential policies aimed at reviving bank lending. The model is particularly helpful at comparing blanket policies that impact all banks and firms with policies that are targeted toward the creation of new lending relationships, such as subsidies to origination costs. I analyze here how such policies can lead to different outcomes in the context of the model.

1.8.1. A simple policy targeting the extensive margin of credit

I analyze the effects of a policy targeted specifically toward increasing credit origination. The goal of this policy is broadly similar to that of the Small Business Lending Fund (SBLF) proposed in the U.S. as part of the 2010 Small Business Jobs Act. It is also related to the Funding for Lending Scheme implemented in the U.K. and the European Central Bank's

⁵²Figure 17 in the appendix displays this response.

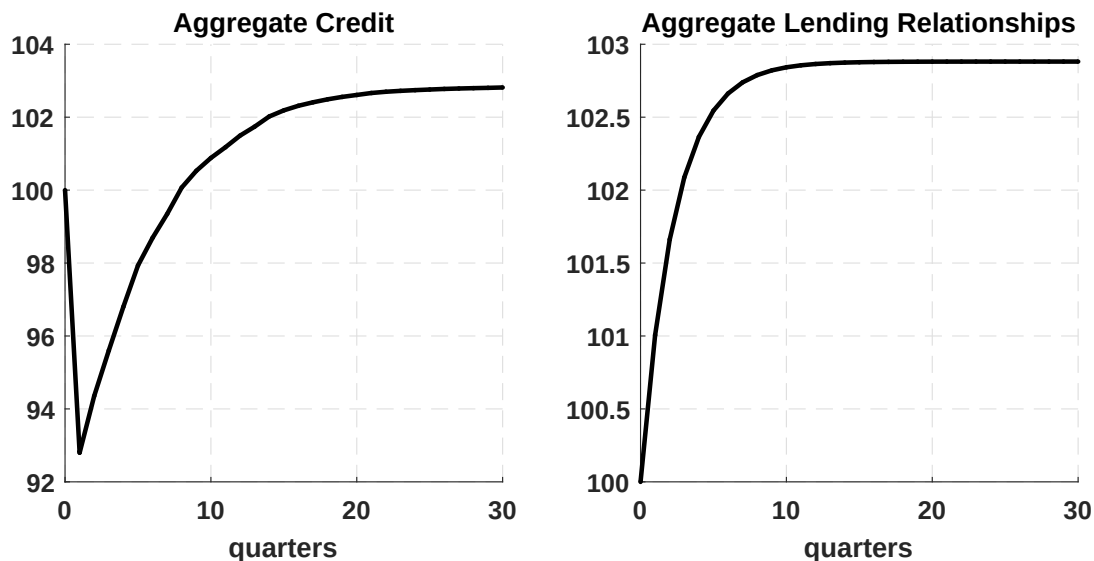
⁵³The response of aggregate output is again overstated given that the model does not account for the accumulation of capital stock.

Targeted Long-Term Refinancing Operation (T-LTRO) program.⁵⁴ While the SBLF only targets relatively small banks, the other programs are broader in scope and subsidize bank funding whenever certain lending criteria are met.

I run a simple policy experiment in which the government subsidizes part of the origination costs c .⁵⁵ This closely resembles the policies mentioned above, in the sense that it is a subsidy contingent on the origination of new credit, but with one main difference being that this is a one-time lump-sum subsidy transferred at origination.

Figures 12 and 13 display the economy's transitional path following an unanticipated implementation of the policy at date 1 (here subsidizing 20% of the origination costs). Although the aim here is not to analyze welfare implications, this experiment can already help us gauge the policy's short- and long-run effects on credit allocation and contractual terms across new and incumbent borrowers.

Figure 12: Response to credit origination subsidies - aggregate variables



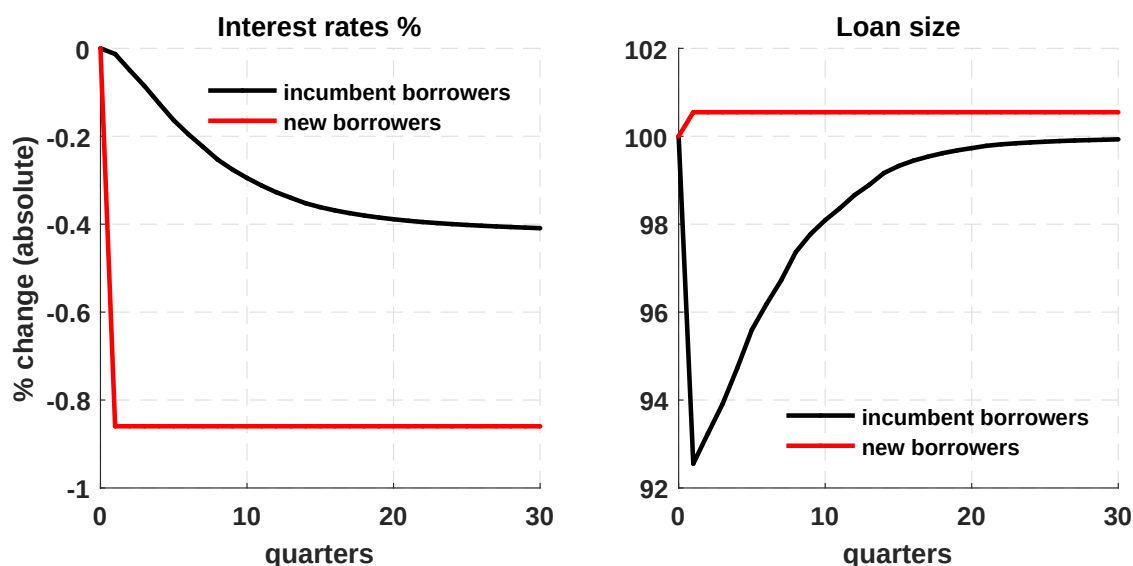
⁵⁴See www.bankofengland.co.uk/markets/Pages/FLS/default.aspx for more details.

⁵⁵Note that this parameter not only reflects screening costs and non-interest expenses incurred by banks during origination, but can be more generally interpreted as the initial sunk investment or long-term debt of the entrepreneur. Lowering c can therefore also be viewed as a subsidy to entrepreneurs, as in, for example, the case of the Small Business Administration (SBA) loan guarantee program.

While it naturally delivers an increase in the number of lending relationships and credit supplied in the economy in the long run, the policy may actually appear initially counter-productive. More important, by directly affecting credit market conditions, the policy has important redistributive consequences in the short run, as it has a different impact across borrowers.

On the one hand, it is beneficial to unfunded firms because it improves their access to banks and overall credit availability. On the other hand, it may negatively impact currently funded but constrained borrowers. As origination costs decrease, bank entry and competition increase as well, which improves liquidity in credit markets. This, however, also increases incumbent borrowers' outside options and leads banks to adjust their credit supply downward to prevent any default. As incumbent borrowers gradually grow out of their borrowing constraints, this adverse effect dissipates, allowing for the aggregate credit supply to eventually increase.⁵⁶

Figure 13: Response to credit origination subsidies - contractual terms



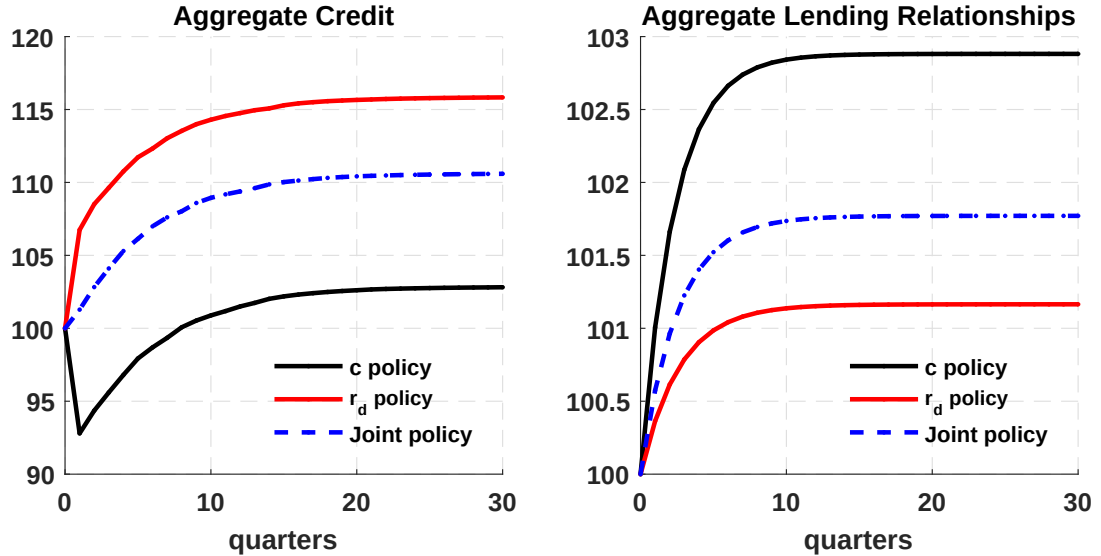
⁵⁶ Although not accounted for in the model, this origination policy may decrease the lenders' incentives to screen borrowers. Decreased screening would result in more firm defaults and loan charge-offs. The policy may therefore need to be accompanied by further monitoring by regulators to restore such incentives.

1.8.2. Joint policy intervention

How can we mitigate these adverse short-term effects? One potential solution is to combine this origination subsidy policy (“ c policy”) with a decline in bank funding costs (“ r_d policy”), keeping total intervention cost constant.⁵⁷ This “joint policy” can appear to be the most cost-effective way to spur credit along both intensive and extensive margins, with the benefit of muting any adverse feedback effects in the short term.

The overall benefits clearly depend on aggregate credit market variables, such as the mass of unfunded firms and banks offering credit, and the institutional parameters affecting the strength of the feedback effect.⁵⁸

Figure 14: Policy comparison - aggregate variables



⁵⁷While a simple decline in bank funding costs would also generate an immediate increase in aggregate credit, its effects on lending relationships are relatively moderate. This policy also comes at a very high cost, since it is non-targeted and also benefits incumbent borrowers. Moreover, because the model features a lag in the transmission of monetary policy from the financial sector to the real economy, this also leads to a substantial rise in bank profits in the short term.

⁵⁸A potential alternative to breaking the general equilibrium feedback effect would be to limit entrepreneurs' ability to get a fresh start after default (i.e., decrease parameter ξ). Thus, stronger creditor rights and more strict bankruptcy laws may also have a positive effect on aggregate credit supply.

1.9. Extensions and comments

The framework analyzed in this paper can be extended along many directions, which are left for future research. In this section, I describe three possible extensions and explain their general implications.

1.9.1. *Endogenous separation*

I consider the possibility of endogenous separation between borrowers and lenders. In this case, the bank also takes into account the firm's discrete decision rule with regard to separation. Following the realization of aggregate and bank-specific shocks, the firm's choice is therefore driven by $\max\{V_{z,r_d}, W(z)\}$, which balances the current value of the contract V_{z,r_d} if the relationship continues and the firm's value following separation $W(z)$.

Because the contract specifies state-contingent continuation values $\{V_{z',r'_d}\}_{z',r'_d}$ before shocks are realized, this simply translates into a probability of termination in each future state, given by:

$$\sigma_{z',r'_d} = \sigma(z', r'_d, V_{z',r'_d}) = \begin{cases} \sigma & \text{if } V_{z',r'_d} \geq W(z'), \\ 1 & \text{otherwise.} \end{cases}$$

with σ being an exogenous destruction rate. In equilibrium, the bank can never promise a value that is below the unfunded firm value W . Otherwise, the firm will always walk away in order to search for a better lending opportunity. However, in certain states where there are no gains from trade, both agents may be better off with separation. In this case the bank receives 0 (the bank derives no income stream when unmatched), and the firm becomes unfunded with value W .

Since the bank is fully committed to deliver promised value V , the promised values offered in the continuation states are higher relative to the case where separation is not allowed.

In this case, the saddle-point problem is slightly modified as follows:

$$\begin{aligned}
P(z, r_d, \Lambda) &= \inf_{\lambda} \sup_{K, d} \quad z f(K) - d - r_d K - \lambda [u(\eta K) + \beta H(z)] \\
&\quad + (\Lambda + \lambda) [u(d) + \beta \sigma \mathbb{E}_z[W(z')]] \\
&\quad + \beta(1 - \sigma) \mathbb{E}_{z, r_d} [\max((\Lambda + \lambda)W(z'), P(z', r'_d, \Lambda + \lambda))] \\
&\text{subject to} \\
&\quad d \geq 0, \quad \lambda \geq 0.
\end{aligned}$$

Introducing endogenous separation allows credit market conditions to affect the contract dynamics through a third channel. In this case, aggregate shocks may be further amplified, since they affect both the creation and destruction of lending relationships and hence may generate even larger credit declines and more sluggish recoveries.

1.9.2. Capital accumulation

The model is tractable enough to allow for the introduction of capital accumulation or entrepreneurial savings. In this case, the firm capital stock becomes an additional state variable affecting contract dynamics and the distribution of firms in the economy. From a qualitative standpoint, this will not impact the shape of the contracts as long as firms' capital is perfectly observable by the lender. However, the ability to accumulate capital also means that entrepreneurs are in general better off — relative to the baseline model — whenever they become unfunded. Thus, the level of accumulated capital creates an additional layer of heterogeneity in credit markets and has additional aggregate implications due to its effect on the search behavior of unfunded firms.

1.9.3. Credit markets for startups vs. established firms

In a similar vein, the model can also be augmented by allowing for market segmentation among unfunded borrowers and distinguishing between newly created firms (“startups”), and “established” firms (i.e., previously funded firms currently searching for new financiers).

One dimension in which these two markets may differ is related to search and origination costs. Indeed, startups may be viewed as more opaque with a more costly and screening-intensive credit origination process as opposed to more established firms with publicly observable track records. Such difference in search costs creates environments with distinct credit market conditions and can, for example, help justify differences observed in the level of interest rate dispersion between small and large firms (Cerqueiro et al. (2011)). This feature is also easy to implement and would generate additional predictions for contractual terms and the dynamics of credit allocation across firm types.

1.10. Conclusion

This paper develops and characterizes a novel dynamic equilibrium theory of bank relationship capital in an economy subject to search frictions and limited enforceability. The model features a dynamic contracting problem within a directed search equilibrium, with aggregate and bank-specific uncertainty, and where heterogeneous financiers compete for borrowers by posting long-term credit offers. The interaction between these two frictions generates a slow accumulation of lending relationship capital and distorts the optimal allocation of credit along both intensive and extensive margins.

This research sheds light on the process of credit relationship formation and its macroeconomic consequences. Significantly, it highlights a new propagation mechanism stemming from the destruction of credit relationships and their slow build-up after an adverse aggregate shock. Crises characterized by a sizable destruction of this relationship capital can therefore generate slow subsequent recoveries.

The model provides a framework that captures multiple dimensions of the credit reallocation process and is particularly relevant when analyzing the effects of policies targeted toward reviving business lending after a crisis. In particular, the paper shows that policies directly subsidizing the cost of originating new credit relationships are effective at boosting the

aggregate credit supply in the long run but can also lead to adverse effects in the short run.

Further empirical analysis focusing on the dynamics of the extensive margin of credit and the process of loan origination and matching between banks and firms is a fruitful area for future work. More generally, this paper is a first investigation of the notion of relationship capital with an application to credit markets. The theory and methodology developed here can be further extended to examining other contexts in which relationships matter, e.g., those between firms and managers or between suppliers and customers in production networks. These potential applications are left for future research.

APPENDIX

A.1. Tables

Table 1: Parameter values

Parameter	Value	Description
calibrated		
β	0.9875	Discount factor
γ	0.5	Coefficient of relative risk aversion
α	0.43	Decreasing returns to scale parameter
σ	0.044	Exogenous separation probability
$\bar{r}_0$	0.0115	Average funding cost
ρ_r	0.94	Persistence of aggregate funding shock
σ_r	0.0025	Standard deviation of aggregate funding shock
ρ_s	0.84	Persistence of idiosyncratic funding shock
σ_s	0.001	Standard deviation of idiosyncratic funding shock
estimated		
ξ	0.29	Probability of fresh start
η	0.37	Share of divertible assets
c	7.03	Origination cost
γ_m	2.28	Matching elasticity
d_0	0.127	Garage production
k_f	0.19	Fixed cost
$\bar{z}$	0.311	Average aggregate productivity
σ_z	0.0135	Standard deviation of aggregate productivity shock
ρ_z	0.85	Persistence of aggregate productivity shock
$\sigma_{s,entry}$	0.006	Upper bound of the idiosyncratic funding shock at entry

Table 2: Targeted moments

	Data	Model
Return on Assets	0.012	0.012
Credit growth rate $\Delta K/K$	0.145	0.136
Fraction of constrained firms	0.40	0.41
Credit origination rate	0.068	0.054
Market-to-Book ratio	2.1	2.0
Approval rate $p(\theta)$	0.28	0.27
Slope ($p(\theta)$, interest rate)	0.14	0.17
Dispersion in funding costs	0.018	0.017
$\rho(\log(\text{output}))$	0.84	0.76
$\sigma(\log(\text{output}))$	0.017	0.016

Notes. This table reports empirical and simulated moments. Log(output) is detrended using HP filter parameter 1600, and autocorrelations and standard deviations are computed based on log-deviations from trend.

Table 3: Dispersion statistics - funding costs and loan interest rates

	Data		Model	
	average	cor(X,GDP)	average	cor(X,GDP)
Dispersion in funding costs	1.77	0.07	1.65	0
Dispersion in loan rates	3.82	0.34	1.92	0.73

Notes. Sample period: 1984 - 2014. Quarterly correlations with GDP are tabulated between cyclical components of HP-filtered logged time series; HP filter parameter: 1600.

A.2. Figures

Figure 15: Comparative statics - ξ

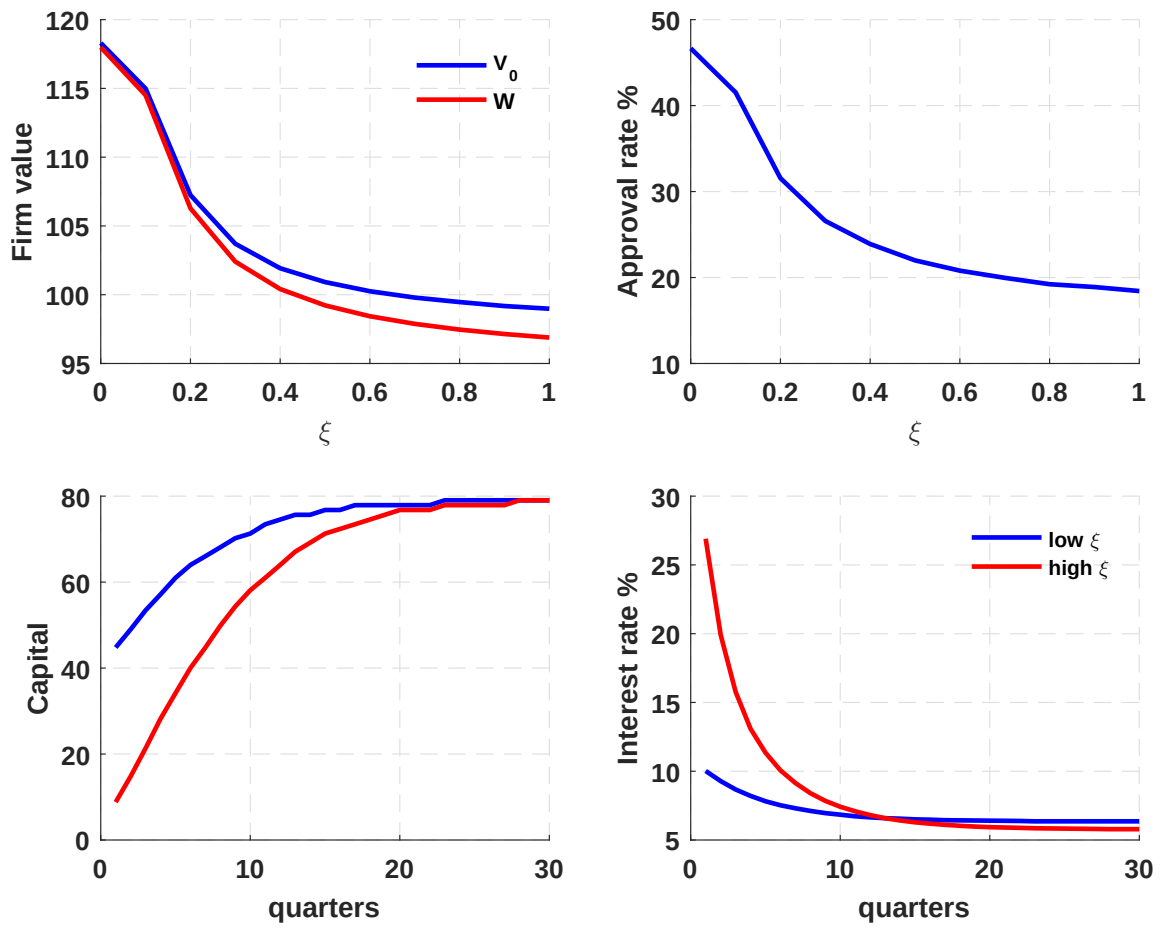


Figure 16: Comparative statics - c

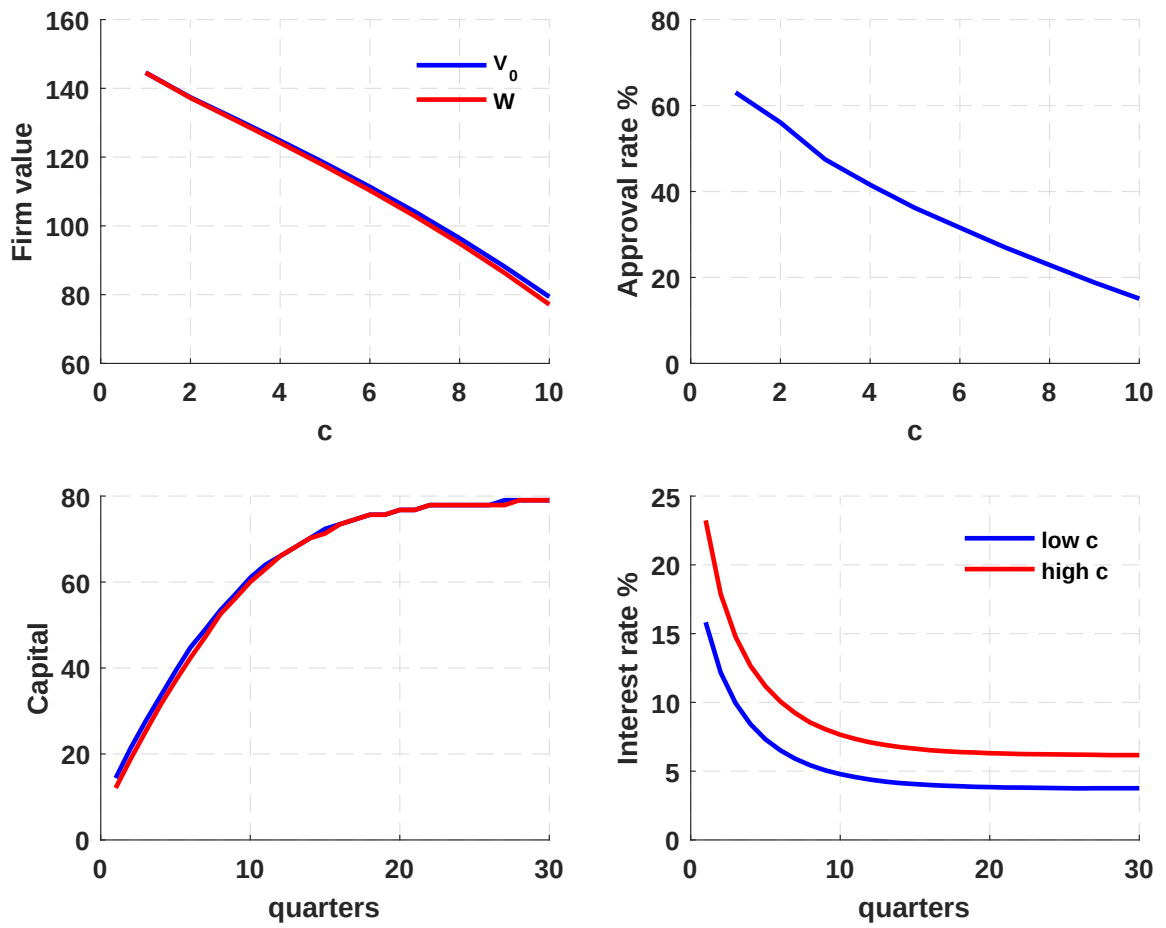
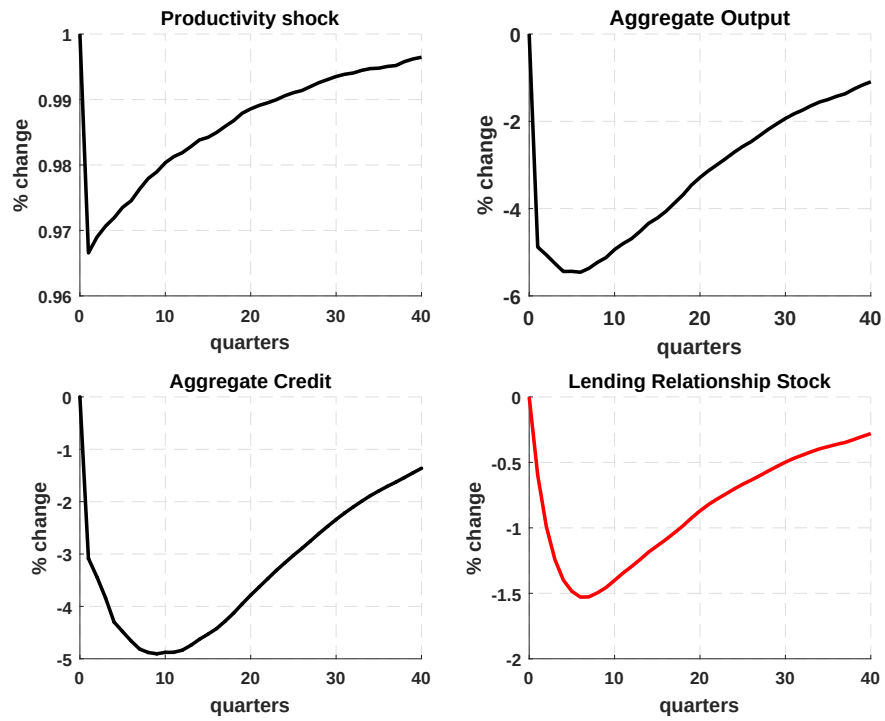
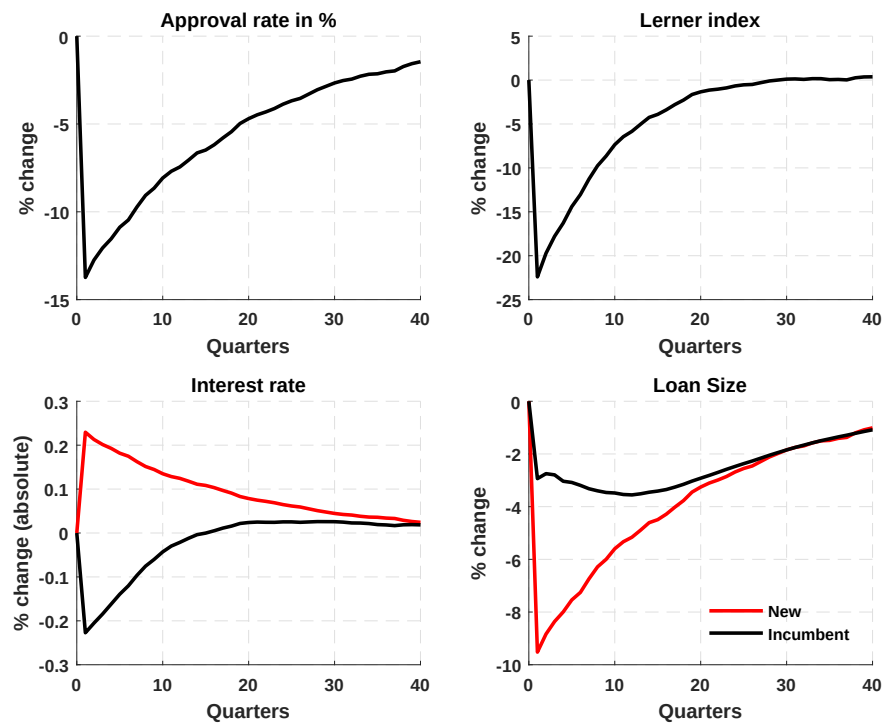


Figure 17: Impulse response - aggregate productivity



(a) Aggregate variables



(b) Contractual terms

A.3. Data construction

This section provides details about data sources and the construction of time series used in the quantitative analysis section of the paper.

A.3.1. *Bank lending relationships*

I document in this section the methodology behind the construction of time series for the rates of creation and destruction of credit relationships. To that end, I use the Loan Pricing Corporation DealScan database (LPC) and focus on U.S. loan syndications denominated in U.S. Dollars. I construct time series for positive, negative, and net relationship flows based on a sample covering the period from January 1986 through March 2016.

In order to measure these flows, I need to determine the date of inception and termination of each bank-firm pair forming a lending relationship. First, since LPC comprises mainly syndicated loans with potentially many bank participants, I consider only the lead arranger and/or main agent for each loan package. When a given loan package involves more than one lead arranger, I consider the ensuing lending relationship for each lender separately.

For a given bank-firm pair, I define $\bar{m}_t$ as the maximum maturity date recorded across all loan agreements made up to date t . For any given date t , a bank-firm pair is considered *inactive* if it has never been matched up to this date, or if it has been matched in the past but no new transaction took place in the three years following the maximum maturity date of all previous deals, i.e., $\bar{m}_t < t - 3Y$. In the former case, the bank-firm pair is considered inactive starting from date $\bar{m}_t$. Otherwise, it is considered active between the corresponding dates of inception and termination. The date of inception is defined as the date in which an inactive bank-firm pair is formed, while the date of termination is given by the date in which an existing and active bank-firm pair becomes inactive.¹

In the benchmark case, I do not control for bank mergers. That said, I also consider a

¹Exploring the determinants behind the destruction of a match is important to fully uncover the economics of bank-firm relationships. This is left for future research.

more general definition aggregating all lenders matched to a given firm. This allows me to control for potential bank switching or bank mergers (it therefore mitigates potential churning effects). The results remain qualitatively similar in this case.

I track the dates of inception and termination of each lending relationship and construct the aggregate portfolio of lending relationships for all active banks.²

Finally, I construct time series for the stock of lending relationships based on the cumulated net flows calculated over the period 1986 through 1995. I then use the stock value beginning in January 1995, as my reference point to define creation and destruction rates. This is a reasonable assumption, since the average duration of a lending relationship is about six years. The time series for creation and destruction rates are considered for the period Q1 1995 through Q1 2013. I chose to start in 1995 to ensure that the database is already well-populated and hence avoid spurious changes in entry rates due to improved firm coverage in LPC. Similarly, the data series stops in 2013 given that the tabulation of destruction rates is forward-looking.

Figure 18 plots the time series associated with positive (creation), negative (destruction), and net lending relationship flows. Figures 19 and 20 show the evolution of the stock of lending relationships and C&I loans and their cyclical properties.

Table 4: Properties of lending relationship flows

	Net flows	Positive flows	Negative flows
mean (%)	1.05	5.52	4.47
std(%)	1.37	1.34	0.79
corr(x,GDP)	0.40	0.30	-0.18

Notes. Sample period: Q1 1995 - Q1 2013. Data are at the quarterly frequency. GDP refers to the cyclical component of detrended log GDP; HP filter parameter: 1600.

²Note that bank name entries are not recorded uniformly. Thus, I manually consolidate bank names reported in the database to prevent other spurious creations or destructions of lending relationships. Additional details and robustness checks related to (i) multiple lending relationships, (ii) inactivity periods, and (iii) flow decomposition by firm size are also available upon request.

A.3.2. Measures of application and approval rates

I report below aggregated information provided in the 2014 Small Business Credit Survey. The SCBS classifies credit sources into four categories: Large banks, Regional banks, Small regional or community banks, and Online lenders. Table 5 shows the results for all polled firms as reported on page 15 of the survey.

Table 5: Credit application and approval rates

	Application rate	Approval rate
Large bank	35%	31%
Regional bank	41%	45%
Small regional or community bank	34%	59%
Online lenders	18%	38%

Notes. Source: Small Business Credit Survey 2014.

Next, I describe the methodology used to tabulate empirical moments associated with the model-generated credit application and approval rates. To map the above information to the model moments, I need to make the following set of assumptions:

- Online lenders are excluded, with the assumption that their business model differs significantly from that of a traditional bank.
- Large and Regional banks are combined into one category: “Large and regional”.
- Application rates are normalized to sum up to 1 to be consistent with the model’s assumption of one application per bank type and per period. Such an assumption is also used to convert semiannual approval rates provided by the SBCS in Table 6 into quarterly frequency.

Adjusted application and approval rates at the quarterly frequency are shown in Table 7.³

³An alternative source of information comes from the loan broker Biz2credit.com, which reports approval rates of small and large banks on a monthly basis. Assuming one application per quarter, the quarterly approval rates reported for June 2014 are 18% and 52% for large and small banks, respectively.

Table 6: Credit application outcomes

	Received all	Received most	Received some	Received none
Credit application outcome	33%	9%	12%	44%

Notes. Credit applicants' response to the question: "How much of the financing your business applied for in the first half of 2014 was approved?" "Received most" refers to receiving more than 50%; "Received some" refers to receiving less than 50% of the financing need. Source: Small Business Credit Survey 2014.

Table 7: Model-equivalent credit application and approval rates

	Application rate (adjusted)	Approval rate (adjusted)
Large & regional bank	69%	21%
Small regional or community bank	31%	36%

Notes. Source: Small Business Credit Survey 2014.

A.3.3. Other time series

Bank marginal operating costs. I define bank marginal operating costs as the sum of interest and non-interest expenses over total assets as follows:

$$\begin{aligned}
 \text{Marginal operating costs} &= \frac{\text{Interest expenses} + \text{Non-interest expenses}}{\text{Total assets}} \\
 &= \frac{\text{RIAD4073} + \text{RIAD4093}}{\text{RCFD2170}}
 \end{aligned}$$

Data are obtained from the Reports of Condition and Income for commercial banks, from 1984 through 2014. These operating costs are then converted into real terms using the GDP deflator.

Net loan returns. I construct the net real returns for C&I loans to proxy for the interest rate offered to risk-free commercial borrowers. In order to control for heterogeneity in the quality and riskiness of borrowers, I define this return as the real yield on C&I loans adjusted

for both charge-offs and recoveries, as follows:

$$\begin{aligned}\text{Net loan returns} &= \frac{\text{Interest income} - \text{Charge-offs} + \text{Recoveries}}{\text{Total C\&I loans}} \\ &= \frac{\text{RIAD4012} - \text{RIAD4638} + \text{RIAD4608}}{\text{RCFD1766}}\end{aligned}$$

Data are obtained from the Reports of Condition and Income for commercial banks, from 1984 through 2014. Similarly, net loan returns are converted into real terms using the GDP deflator. Item RCFD1766 refers to C&I loans excluding bank acceptances.

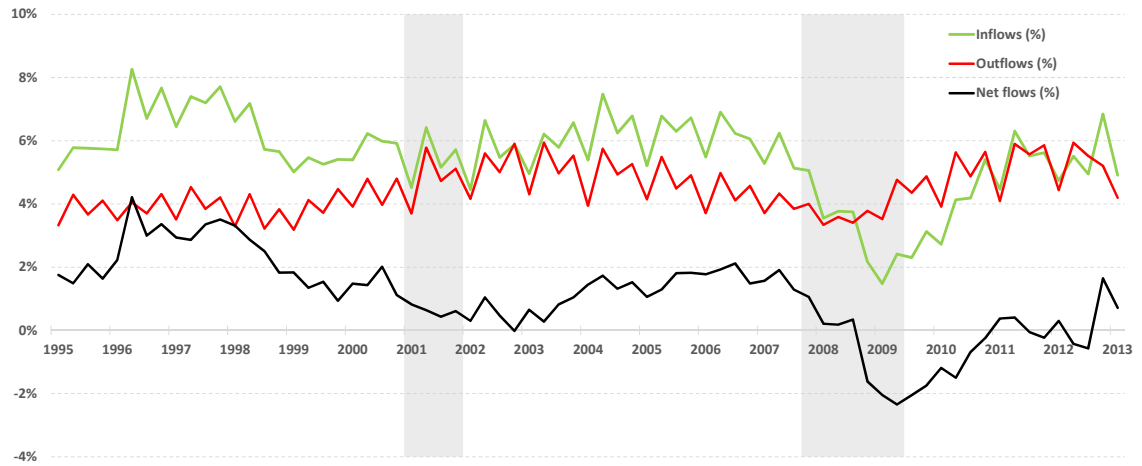
Note that the time series and moments associated with marginal operating costs and net loan returns are constructed as follows:

- I first winsorize all variables mentioned above (except total assets and C&I loans) at the 1% level in order to mitigate the effect of outliers.
- Cross-sectional averages and dispersion moments are value-weighted.
- To determine the parameters governing the bank-specific costs, I first determine the autocorrelation and standard deviation of the AR(1) process at the bank-level. The estimated parameters are then winsorized at the 1% level before computing cross-sectional averages.

Credit origination rate. It is defined as the ratio of new commercial credit origination, obtained from the Survey of Terms of Business Lending (STBL) over total volume of C&I loans (FRED:BUSLOANS). Data are available from 2000 through 2014.

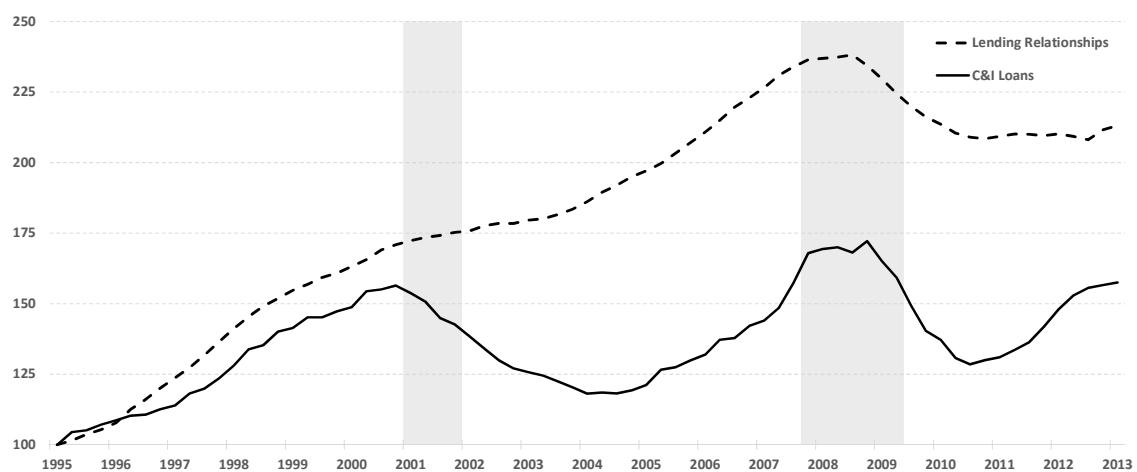
Interest rate spreads. Interest rate spreads are computed as the difference between the effective weighted average interest rates with minimal risk borrowers offered by Large and Small banks, provided by the STBL (FRED: EEANXSLNQ, EEANXSSNQ), and the 1-year Treasury constant maturity rate (FRED: DGS1), over the period Q2 1997 - Q1 2014.

Figure 18: Deconstructing lending relationships flows



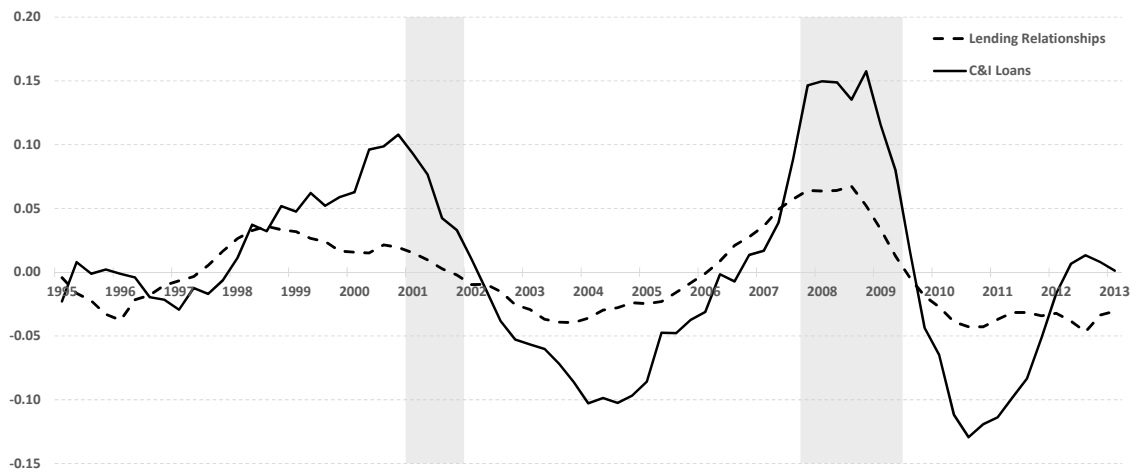
Notes. Sample period: Q1 1995 - Q1 2013. Author's calculations based on DealScan data.

Figure 19: C&I loans and lending relationship stock



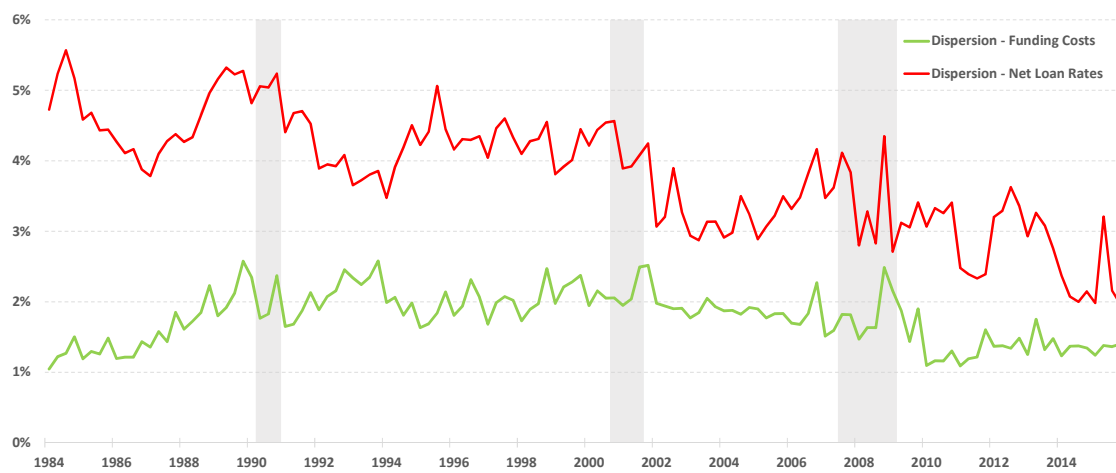
Notes. Sample period: Q1 1995 - Q1 2013. Time series are normalized to 100 in Q1 1995. C&I loans are taken from the Fed Board's H.8 release of Assets and Liabilities of Commercial Banks and deflated using the GDP deflator. The stock of lending relationships is tabulated from DealScan data.

Figure 20: Cyclical properties of C&I loans and lending relationships



Notes. Sample period: Q1 1995 - Q1 2013. Cyclical components of HP-filtered logged time series for C&I loan and lending relationship stocks, HP filter parameter: 1600.

Figure 21: Dispersion rates for bank funding costs and net C&I loan rates



Notes. Sample period: Q1 1984 - Q4 2014. Time series tabulated based on the Reports of Condition and Income for commercial banks.

A.4. Computational methodology

This section summarizes the numerical procedure used to solve the dynamic contracting problem and the competitive search equilibrium. Solving the full model is equivalent to solving a fixed-point problem in the vector of unfunded firm values $\{W(z)\}_z$.

- Initialize $\{W(z)\}_z$, with $W(z)$ increasing in aggregate state z .
- Loop over the following steps until entry condition $c = B^*(z; W)$ is satisfied for all aggregate states jointly:
 1. Given $\{W(z)\}_z$, solve recursively the contracting problem formulated in the Lagrange multiplier space as described to get $P(z, r_d, \Lambda)$, and determine the corresponding contract value to the bank in the promised utility space $B(z, r_d, V)$.
 2. Determine the indifference condition for unfunded firms and compute $\{\rho(z)\}_z$ based on the following identity:

$$\begin{aligned}\rho(z) &= p(\theta(z, V))(V - \mathbb{E}_z[W(z')]) \\ &= \frac{W(z) - u(d_0) - \beta \mathbb{E}_z[W(z')]}{\beta}.\end{aligned}$$

3. For each idiosyncratic state r_d , solve for the contract value V optimizing expected bank profits:

$$V^*(z, r_d; W) = \arg \max_V q(\theta(z, V))B(z, r_d, V; W),$$

and compute

$$B^*(z; W) = \mathbb{E}_{r_d} [[q(\theta(z, V^*(z, r_d; W)))B(z, r_d, V^*(z, r_d; W); W)]_+].$$

4. For each z , update $W(z)$ either upward if $c < B^*(z; W)$, or downward otherwise.

A.5. Theory, derivations, and proofs

A.5.1. Contracting problem

I rewrite below the general formulation of the contracting problem for the bank, and the derivation of the recursive multiplier formulation for the exogenous separation case.⁴

$$\begin{aligned}
B(z, r_d, V) &= \max_{K, d, \{V'\}} z f(K) - d - r_d K + \beta \mathbb{E}_{z, r_d} [(1 - \sigma) B(z', r'_d, V')] \\
&\text{subject to} \\
V &= u(d) + \beta \mathbb{E}_{z, r_d} [(1 - \sigma) V' + \sigma W(z')] , \quad (\text{Promise-Keeping}) \\
V^O(z, K; W) &\leq V, \quad (\text{Participation}) \\
d &\geq 0. \quad (\text{Limited Liability})
\end{aligned}$$

Notations

V	: current firm value
$\{V_{z', r'_d}\}_{z', r'_d}$	: state-contingent continuation values
$\{W(z)\}_z$	: firm value when unfunded
$\{H(z)\}_z$	: firm value after default
K	: capital
d	: dividend payout
z	: productivity shock
r_d	: bank funding shock
η	: share of divertible assets
ξ	: probability of access to credit markets after default

⁴Note that the endogenous separation case discussed in section 9 introduces an additional difficulty because it features a kink in the bank value function.

Proof. **Proposition 1. Lagrangian formulation.**

The problem above is not immediately solved using standard dynamic programming techniques because of the forward-looking nature of the participation constraint. The saddle-point problem methodology developed in Marcet and Marimon (2011) allows for a more tractable approach and provides a recursive formulation to the problem.

The following derivations show how I adapt this methodology to my problem.

First, let us denote Λ the cumulative Lagrange multiplier associated with the borrower's participation constraint and define the Pareto problem $P(z, r_d, \Lambda) = \sup_V B(z, r_d, V) + \Lambda V$ as follows:

$$\begin{aligned}
P(z, r_d, \Lambda) &= \sup_{V, K, d, \{V_{z', r'_d}\}} z f(K) - d - r_d K + \beta \mathbb{E}_{z, r_d}[(1 - \sigma)B(z', r'_d, V_{z', r'_d})] + \Lambda V \\
&\text{s.t.} \\
V &\geq u(\eta K) + \beta H(z) \\
V &= u(d) + \beta \mathbb{E}_{z, r_d} \left[(1 - \sigma)V_{z', r'_d} + \sigma W(z') \right].
\end{aligned}$$

Replacing V in the equation above yields:

$$\begin{aligned}
P(z, r_d, \Lambda) &= \sup_{V, K, d, \{V_{z', r'_d}\}} z f(K) - d - r_d K + \beta \mathbb{E}_{z, r_d}[(1 - \sigma)B(z', r'_d, V_{z', r'_d})] \\
&\quad + \Lambda \left[u(d) + \beta \mathbb{E}_{z, r_d}[\sigma W(z') + (1 - \sigma)V_{z', r'_d}] \right] \\
&\text{s.t.} \\
u(d) + \beta \mathbb{E}_{z, r_d} \left[(1 - \sigma)V_{z', r'_d} + \sigma W(z') \right] &\geq u(\eta K) + \beta H(z).
\end{aligned}$$

We can now include the participation constraint with weight λ and rearrange terms to get

the saddle-point problem:

$$\begin{aligned}
P(z, r_d, \Lambda) = \inf_{\lambda} \sup_{K, d} & \quad z f(K) - d - r_d K + (\Lambda + \lambda) [u(d) + \beta \sigma \mathbb{E}_z[W(z')]] \\
& \quad - \lambda [u(\eta K) + \beta H(z)] \\
& \quad + \beta(1 - \sigma) \left[\underbrace{\mathbb{E}_{z, r_d} \left[\sup_{V_{z', r'_d}} B(z', r'_d, V_{z, r_d}) + (\Lambda + \lambda) V_{z', r'_d} \right]}_{P(z', r'_d, \Lambda + \lambda)} \right].
\end{aligned}$$

Equivalently, with $\Lambda' = \Lambda + \lambda$, we have:

$$\begin{aligned}
P(z, r_d, \Lambda) = \inf_{\Lambda' \geq \Lambda} \sup_{K, d} & \quad z f(K) - d - r_d K + \Lambda' [u(d) + \beta \sigma \mathbb{E}_z[W(z')]] \\
& \quad + (\Lambda - \Lambda') [u(\eta K) + \beta H(z)] \\
& \quad + \beta(1 - \sigma) \mathbb{E}_{z, r_d} [P(z', r'_d, \Lambda')].
\end{aligned}$$

Eventually, we can easily check that all the standard assumptions and regularity conditions needed for the application of theorems 1 and 2 in Marcet and Marimon (2011) are verified in order to justify that a solution to the saddle-point problem is indeed equivalent to that of the original maximization problem. $\square$

Once the saddle-point problem is solved, we can eventually recover the bank profit and firm values thanks to the definition of the Pareto problem as follows:

$$\begin{aligned}
V(z, r_d, \Lambda) &= \frac{\partial P}{\partial \Lambda}(z, r_d, \Lambda) \\
B(z, r_d, V) &= P(z, r_d, \Lambda^*(z, r_d, V)) - \Lambda^*(z, r_d, V)V.
\end{aligned}$$

Given the cumulative Lagrange multiplier Λ' , the state-contingent continuation values

$\{V_{z',r'_d}\}_{z',r'_d}$ can be obtained from the following first-order condition:

$$\frac{\partial B}{\partial V}(z', r'_d, V_{z',r'_d}) = -\Lambda'.$$

A.5.2. Properties of the cumulative Lagrange multiplier Λ

Conditional on continuation, the solution to the optimal contract verifies the following first-order conditions linking the cumulative Lagrange multiplier Λ to the optimal policies for capital K and dividends d :

$$z \frac{\partial f(K)}{\partial K} = r_d - (\Lambda - \Lambda') \eta u'(\eta K) \quad (\text{A.1})$$

$$\frac{1}{\Lambda'} = u'(d). \quad (\text{A.2})$$

Equation (A.1) determines the optimal level of capital as a function of Lagrange multipliers (Λ, Λ') . When $\Lambda = \Lambda'$ (or, equivalently, $\lambda = 0$), the participation constraint is never binding and the firm is unconstrained. In such cases, capital is at the first-best level by K_{FB} .

A.5.3. Properties of B

The following lemmas establish a series of properties of B that are useful for the remaining proofs in this section.

Auxiliary Lemma. *$B(z, r_d, V)$ is strictly increasing in z and decreasing in r_d .*

Proof. This result is straightforward and follows from the fact that the function $\pi(., ., V)$ defined in the intermediate problem 1.6 is strictly increasing in z and decreasing in r_d . $\square$

Auxiliary Lemma. *$B(., ., V)$ is strictly decreasing and concave in V in the continuation region, with a slope in $[-\frac{1}{w'(d)}, 0]$.*

Proof. This result stems from the observation that an increase in promised value V is always costly to the lender. The lower bound of the slope follows from equation (1.7). For example, when $u(d) = \frac{d^{1-\gamma}}{1-\gamma}$, and for z and r_d constant, this lower bound is given by $-\bar{d}^\gamma$. As in Lemma 3 in Albuquerque and Hopenhayn (2004), the concavity of $B(.,., V)$ follows directly from the concavity of function $\pi(.,., V)$ in equation 1.6 in this paper and Theorem 9.8 in Stokey and Lucas (1989). $\square$

Auxiliary Lemma. Fix z and W . $B(r_d, V) = B(z, r_d, V; W)$ is submodular in r_d and V .

Proof. Let us fix aggregate shock z and unfunded firm value W without loss of generality and write $B(r_d, V) = T(B)(r_d, V)$, where T is the operator mapping the set of continuous functions defined over $[W, V] \times [r_0, r_N]$ into itself.

Let us first define the surplus function

$$S(z, r_d, V) = \begin{cases} zf(K_{cons}(z, V)) - r_d K_{cons}(z, V), & \text{if } V < \underline{V}(z, r_d), \\ zf(K_{FB}(z, r_d)) - r_d K_{FB}(z, r_d), & \text{if } V \geq \underline{V}(z, r_d). \end{cases}$$

It is straightforward to show that S is continuous and differentiable in both r_d and V .

Moreover, it is submodular in V and r_d . Indeed, we have:

$$\frac{\partial^2 S}{\partial r_d \partial V} = \begin{cases} - \underbrace{\frac{\partial K_{cons}(z, V)}{\partial V}}_{\leq 0}, & \text{if } V < \underline{V}(z, r_d), \\ 0 & \text{if } V \geq \underline{V}(z, r_d). \end{cases}$$

Let us now consider a function B to be submodular in (r_d, V) , and let us write the cross-

derivative of $T(B)$ with respect to both r_d and V :

$$\begin{aligned}\frac{\partial^2 T(B)}{\partial r_d \partial V}(r_d, V) &= \frac{\partial S}{\partial r_d \partial V}(r_d, V) + \beta(1 - \sigma)\mathbb{E}\left[\frac{\partial^2 B}{\partial r_d \partial V}(r_d, V)\right] \\ &\leq 0.\end{aligned}$$

The operator T therefore maps the space of submodular functions into itself and the unique fixed point is also submodular. $\square$

A.5.4. Main proofs

Proof. **Corollary 1.**

Notice that $K_{cons}(z, V)$ satisfies the following expression in the constrained region:

$$u(\eta K_{cons}(z, V)) = V - \beta H(z). \quad (\text{A.3})$$

The results follow immediately, given that both utility and production functions are strictly increasing in K . Note, however, that this result is valid in partial equilibrium, as both ξ and η can affect (negatively) W in the general equilibrium version of the model. $\square$

Proof. **Proposition 2.** In section 2. $\square$

Proof. **Proposition 3.** In section 2. $\square$

Proof. **Proposition 4.**

Let us first look at the case in which the borrower is risk neutral as in Albuquerque and Hopenhayn (2004). In this case, both agents are indifferent about the timing of consumption. Thus, it is always efficient to postpone dividend payouts, to allow for faster increase in firm value until the unconstrained region is reached. Here, the promise-keeping constraint yields $V' = \frac{V}{\beta}$. The firm value is therefore always increasing in the constrained region at

rate $\frac{1}{\beta}$.

Let us now look at the generalization of this result to risk-averse borrowers. In this case, agents are faced with two counteracting motives, namely consumption smoothing and higher savings incentives. The incentive for higher savings, however, does not dominate the agent's willingness to grow out of the borrowing constraint. Indeed, assume by contradiction that $V' \leq V$, then $V < \frac{u(d)}{1-\beta}$ from the promise-keeping constraint. But the firm payout is strictly increasing whenever the participation constraint is binding. Therefore, firm value V must be at least greater than $\frac{u(d)}{1-\beta}$, and V' must be strictly greater than V . $\square$

Proof. Proposition 5.

To prove this result, I write down the following equality derived from the envelope condition and the first order condition on V' :

$$\frac{\partial B(z, r_{d,1}, V_{z',r_{d,1}})}{\partial V} = \frac{\partial B(z, r_{d,0}, V_{z',r_{d,0}})}{\partial V} = -\Lambda',$$

for all states $r_{d,0} \leq r_{d,1}$. I then proceed by using the property of submodularity of B with respect to r_d and V derived in Lemma A.5.3, which gives us the immediate result that if $r_{d,0} \leq r_{d,1}$, then necessarily $V_{z',r_{d,1}} \leq V_{z',r_{d,0}}$.

Fix z and r_d . Let us now define $\bar{B}(z, r_d, V; W) = q(\theta(z, V; W))B(z, r_d, V; W)$ over the compact interval $\mathcal{S} = [W, \bar{S}]$, where $\bar{S}$ is the maximum value obtained by the entrepreneur when the joint match surplus is kept entirely by the firm. $\square$

Proof. Lemma 1. Existence of an interior solution V^* .

$\bar{B}$ is continuous in V over $\mathcal{S}$ as a product of two continuous functions in V over $\mathcal{S}$. The problem is therefore well defined and the solution to the maximization problem must also be in $\mathcal{S}$. But $\bar{B}(z, r_d, W, W) = \bar{B}(z, r_d, \bar{S}; W) = 0$, because $q(\theta(z, W; W)) = B(z, r_d, \bar{S}, W) = 0$ and the supremum of $\bar{B}$ must be strictly positive (for at least some r_d) to warrant bank

entry in the first place; hence, for z, r_d , and W given, the solution maximizing $\bar{B}^V(V) = \bar{B}(z, r_d, V, W)$ must be in $(W, \bar{S})$.

Moreover, we can show that V^* is unique if $\bar{B}(V)$ is indeed strictly concave in V over $\mathcal{S}$. To that end, let us define the function $\phi(V) = q \circ p^{-1}(\frac{\rho(z)}{V - \mathbb{E}_z[W(z')]})) = q(\theta(z, V))$. The function ϕ is strictly increasing and strictly concave in V , thanks to the regularity properties of q and $q \circ p^{-1}$ (in particular the assumption that $q \circ p^{-1}$ is strictly decreasing and concave). Moreover, we can differentiate $\bar{B}^V(V)$ twice with respect to V , to get:

$$\begin{aligned} \bar{B}^{V''} &= \underbrace{\phi''}_{<0} \underbrace{B}_{\geq 0} + 2 \underbrace{\phi'}_{>0} \underbrace{B'}_{<0} + \underbrace{\phi}_{\geq 0} \underbrace{B''}_{<0} \\ &< 0. \end{aligned}$$

$\bar{B}$ is therefore strictly concave in V and has a unique supremum in $(W, \bar{S})$. $\square$

Auxiliary Lemma. *B is decreasing in W .*

Proof. Since $\bar{B}(., ., ., W)$ is continuous and decreasing in W , by the envelope theorem, its maximum over $\mathcal{S} = [W, \bar{S}]$ must also be decreasing and continuous over the same interval $\mathcal{S}$. Therefore $B(., ., ., W)$ defined in the dynamic program (A.1) is also decreasing in W over $\mathcal{S}$ whenever $B(., ., ., W) > 0$. $\square$

Proof. Lemma 2.

(i) We know from above that $\bar{B}$ is continuous and decreasing in W . Moreover, it is strictly decreasing in W whenever it is positive. If no entrant bank posts a loan offer, then expected bank profits cannot be positive. Hence, the entry condition implies the existence of at least one solution. A solution exists for sufficiently small c . $B(z, r_d, W, W)$ is strictly positive, since $f'(0) = \infty$ and $\pi > 0$. When c is sufficiently small, the intermediate value theorem justifies the existence of a solution given that $B(., ., ., W) > 0$ (i.e., there exists a non-

empty interval for r_d such that $\bar{B}(z, r_d, W, W) > 0$), $B(., ., ., W)$ is strictly decreasing in W , and $\lim_{W \rightarrow \bar{S}} B(., ., ., W) = 0$. Moreover, when it exists, the solution is unique given that $B(., ., ., W)$ is strictly monotonic in W .

(ii) Straightforward from equations (1.9), (1.13) and (1.14). $\square$

Proof. Proposition 6. Credit markets in the cross-section.

Let us first show that $\bar{B}(z, r_d, V, W)$ is submodular in V and r_d .

The submodularity of $\bar{B}$ with respect to V and r_d is a direct consequence of the submodularity of B and the convexity of q . Indeed, we have:

$$\begin{aligned} \frac{\partial^2 \bar{B}}{\partial V \partial r_d} &= \underbrace{\frac{\partial q(V)}{\partial V}}_{\geq 0} \underbrace{\frac{\partial V}{\partial B \partial r_d}}_{\leq 0} + \underbrace{q(V)}_{\geq 0} \underbrace{\frac{\partial^2 T(B)}{\partial V \partial r_d}}_{\leq 0} \\ &\leq 0. \end{aligned}$$

(i) Let us first show that V^* is decreasing in r_d . Let us fix z and W without loss of generality. To simplify notations, let us also denote $B(r_d, V) = B(z, r_d, V, W)$ and define $V_0^* = \arg \max_V \bar{B}(r_{d,0}, V)$. From the submodularity property of $\bar{B}$, we have:

$$\begin{aligned} 0 &= \frac{\partial q}{\partial V} B(r_{d,0}, V_0) + q(V_0) \frac{\partial B}{\partial V}(r_{d,0}, V_0) \\ &\geq \frac{\partial q}{\partial V} B(r_{d,1}, V_0) + q(V_0) \frac{\partial B}{\partial V}(r_{d,1}, V_0). \end{aligned}$$

Eventually, since $\bar{B}$ is strictly concave in V , then if $V_1^* = \arg \max_V \bar{B}(r_{d,1}, V)$ exists such that $\frac{\partial q}{\partial V} B(r_{d,1}, V_1) + q(V_1) \frac{\partial B}{\partial V}(r_{d,1}, V_1) = 0$, it must be that $V_1^* < V_0^*$.

Eventually, the properties of capital level at origination $K_0(r_d)$ and approval rate $p(\theta(V^*(r_d)))$ follow immediately from the properties of capital policy (Proposition 2) and matching probability p . $\square$

Proof. **Proposition 7. Existence of a directed search equilibrium.**

I prove the existence of an equilibrium using Schauder's fixed point Theorem as stated in Stokey and Lucas (1989) – Theorem 17.4 and following the general exposition in Menzio and Shi (2010) and Schaal (2015).

Let us first define the set of functions $\mathcal{P} : \mathbb{Z} \times \mathbb{R}_s \times \mathcal{S} \rightarrow \mathbb{R}$ such that $\forall B \in \mathcal{P}$, B is: (i) bounded, (ii) decreasing and concave in V , (iii) continuous and bi-Lipschitz in V . In order to apply Schauder's theorem, I proceed by showing the following properties: (a) equilibrium objects W , θ , p , and q are well defined and continuous; the operator T defined by the dynamic program (1.3) (b-0) maps $\mathcal{P}$ into itself; (b-1) is continuous over $\mathcal{P}$; and (c) the family of functions $T(B)$ is equicontinuous.

(a) *Existence, uniqueness, and boundedness of $W_B(z)$, given $B \in \mathcal{P}$.*

First, for a given $B \in \mathcal{P}$, Lemma 2 gives us the existence and uniqueness of W_B (assuming c is sufficiently small). The boundedness is immediate since W must lie in the compact set $\mathcal{S}_0 = [\underline{S}, \bar{S}]$.

Let us define $\mathcal{A} = (W, \frac{\bar{S}}{1-\beta}]$. The complementary slackness condition (equation 1.14) tells us that either $\theta(z, V) = 0$ or $\exists a > 0$, such that $q(\theta(z, V))B(z, r_d, V; W) = a$. For $V \notin \mathcal{A}$, such a doesn't exist, and $\theta = 0$ in this region; otherwise, for $V \in \mathcal{A}$, the above expression has a unique solution given by: $\theta(z, V) = q^{-1}(\frac{a}{B(z, r_d, V; W)})$:

$$\theta(z, V) = \begin{cases} 0, & \text{if } V \in \mathcal{A}, \\ q^{-1}(\frac{a}{B(z, r_d, V; W)}), & \text{if } V \notin \mathcal{A}. \end{cases}$$

Eventually, the existence and uniqueness of p and q follows immediately from the above results and equations (1.8)-(1.10).

(b-0) *The operator T is well defined and maps $\mathcal{P}$ into itself.*

Let us consider $B \in \mathcal{P}$, and define $T_B = T(B)$.

1. T_B is continuous and concave in V . This is true since T_B is a linear combination of the auxiliary functions π and B , which are both continuous and concave in V .
2. From the property above, T_B is differentiable (almost) everywhere and we can use the envelope theorem to show that the first-order derivative verifying $\frac{\partial T_B(z, r_d, V)}{\partial V} = -\frac{1}{u'(d(z, r_d, V))}$. We have already established that for a given pair $(z, r_d) \in \mathbb{Z} \times \mathbb{R}$, the dividend payout policy is bounded. The derivative of T_B is therefore also bounded on both sides and is strictly negative. It is therefore also the case for T_B given that V is bounded.

Eventually, T_B is decreasing in V and the bi-Lipschitz continuity property follows directly, given that T_B is differentiable with first-order derivative bounded on both sides.

This concludes the proof of $B \in \mathcal{P} \Rightarrow T_B \in \mathcal{P}$.

(b-1) *The operator T is continuous over $\mathcal{P}$.*

Let us introduce the infinite norm $\|\cdot\|$ such that $\|B\| = \sup_{z, r_d, V \in \mathbb{Z} \times \mathbb{R}_d \times \mathbb{V}} B(z, r_d, V)$. Let us also fix z, r_d, V and consider two functions $B_1, B_2 \in \mathcal{P}^2$ and their respective images $\hat{B}_1 = T\hat{B}_1$, and $\hat{B}_2 = T\hat{B}_2$. In order to establish continuity over $\mathcal{P}$, I need to show that $\forall l > 0$, such that $\|B_1 - B_2\| < l$, $\exists \epsilon > 0$ s.t. $\|\hat{B}_1 - \hat{B}_2\| < \epsilon$.

Let $\Phi_1 = (d_1, K_1, \{V'_1\})$ and $\Phi_2 = (d_2, K_2, \{V'_2\})$ be the optimal policies maximizing the bank's contracting problem associated with B_1 and B_2 . Let us also consider the suboptimal policy $\tilde{\Phi}_2 = (\tilde{d}_2, \tilde{K}_2, \{V'_1\})$, where the vector of continuation values $\{V'_1\}$ is exactly the same as for policy Φ_1 and where $\tilde{d}_2$ and $\tilde{K}_2$ satisfy the corresponding promise-keeping and participation constraints for B_2 .

$$\begin{aligned}
\|T_{B_1}(z, r_d, V) - T_{B_2}(z, r_d, V)\| &= \|B_1(z, r_d, V, \Phi_1) - B_2(z, r_d, V, \Phi_2)\| \\
&\leq \|B_1(z, r_d, V, \Phi_1) - B_2(z, r_d, V, \tilde{\Phi}_2)\| \\
&\leq \|\pi_1(\Phi_1) - \pi_2(\tilde{\Phi}_2) + \beta(1 - \sigma)\mathbb{E}[B_1 - B_2]\| \\
&\leq \|\pi_1(\Phi_1) - \pi_2(\tilde{\Phi}_2)\| + \beta(1 - \sigma)\|B_1 - B_2\|.
\end{aligned}$$

We now need to show that there exists a finite upper bound α_T , such that the first component of the right-hand-side $\|\pi_1(\Phi_1) - \pi_2(\tilde{\Phi}_2)\|$ is bounded above by $\alpha_T\|B_1 - B_2\|$. First, notice that:

$$\|\pi_1(\Phi_1) - \pi_2(\tilde{\Phi}_2)\| \leq \|d_1 - \tilde{d}_2\| + \|zf(K_1) - r_d K_1 - zf(\tilde{K}_2) + r_d \tilde{K}_2\|. \quad (\text{A.4})$$

Technical assumption: f is bi-Lipshitz continuous in K , such that there exist upper and lower bounds $(\underline{\alpha}_f, \bar{\alpha}_f)$:

$$\underline{\alpha}_f |K_2 - K_1| < |f(K_2) - f(K_1)| < \bar{\alpha}_f |K_2 - K_1| \quad \forall (K_1, K_2).$$

Let us now show the following auxiliary lemma, which will be useful for establishing the bounds of the right-hand-side of the expression above.

Auxiliary Lemma. *For $B_1, B_2 \in \mathcal{P}^2$, we have:*

1. $\|\theta_1 - \theta_2\| < \alpha_\theta \|B_1 - B_2\|$
2. $\|p_1 - p_2\| < \alpha_p \|B_1 - B_2\|$
3. $\|W_1 - W_2\| < \alpha_W \|B_1 - B_2\|$

Proof. Consider (z, r_d) given.

1. Let us assume market V is open for both B_1 and B_2 . We have:

$$\begin{aligned} 0 &= B_1(V_1)q(\theta_1) - B_2(V_2)q(\theta_2) \\ &= [B_1(V_1) - B_2(V_2)]q(\theta_1) + B_2(V_2)[q(\theta_1) - q(\theta_2)] \\ &\leq [B_1(V_1) - B_2(V_2)] + \|B_2\|[q(\theta_1) - q(\theta_2)]. \end{aligned}$$

But q is a convex function, hence we can write:

$$q(\theta_1) - q(\theta_2) \leq q'(\max(\theta_1, \theta_2))(\theta_1 - \theta_2),$$

and

$$-\|B_2\|[q(\theta_1) - q(\theta_2)] \leq [B_1(V_1) - B_2(V_2)],$$

and

$$-\|B_2\|q'(\max(\theta_1, \theta_2))(\theta_1 - \theta_2) \leq [B_1(V_1) - B_2(V_2)].$$

Using the definition of $\bar{B}$, we have by construction $c \geq \Gamma_{r_d}^0(r_d)\bar{B}(z, r_d, V)$ (where $\Gamma_{r_d}^0(r_d)$ is the unconditional probability assigned to state r_d at entry). We can then define $\bar{c} = \max_{r_d \in \mathbb{R}_d} \left\{ \frac{c}{\Gamma_{r_d}^0(r_d)} \right\}$, and eventually write the following inequality, using $\|B_2\| \leq \bar{c}$, and define $\alpha_\theta = \frac{1}{\bar{c}q'(\max(\theta_1, \theta_2))}$:

$$|\theta_1 - \theta_2| \leq \frac{1}{\bar{c}q'(\max(\theta_1, \theta_2))} \|B_1 - B_2\|.$$

2. Fix V . By definition, we have:

$$p_1(z, r_d, V) - p_2(z, r_d, V) = p(\theta_1(z, r_d, V)) - p(\theta_2(z, r_d, V)).$$

We can therefore write the following inequality thanks to the concavity of p :

$$\begin{aligned} |p_1(z, r_d, V) - p_2(z, r_d, V)| &\leq p'(0)|\theta_1(z, r_d, V) - \theta_2(z, r_d, V)| \\ &\leq p'(0)\alpha_\theta \|B_1(V_1) - B_2(V_2)\|. \end{aligned}$$

3. We have:

$$\begin{aligned}
W_1(z) - W_2(z) &= \beta [\mathbb{E}[W_1(z') - W_2(z')] + p_1(\theta_1)(V_1 - W_1) - p_2(\theta_2)(V_2 - W_2)] \\
&\leq \beta [\mathbb{E}[W_1(z') - W_2(z')] + p_1(\theta_1)(V_1 - W_1) - p_2(\theta_1)(V_1 - W_2)] \\
&\leq \beta [\mathbb{E}[W_1(z') - W_2(z')] + V_1(p_1(\theta_1) - p_2(\theta_1)) - p_1(\theta_1)W_1 + p_2(\theta_1)W_2] \\
&\leq \beta [(1 - p_1(\theta_1))\|W_1 - W_2\| + (V_1 - W_2)(p_2(\theta_1) - p_1(\theta_1))].
\end{aligned}$$

We can eventually proceed by re-arranging terms to obtain the following inequality:

$$\begin{aligned}
\|W_1 - W_2\| &\leq \frac{\beta}{1 - \beta} (\max(|V_1|) + \max(|W_2|)) |p_2(\theta_1) - p_1(\theta_1)| \\
&\leq \underbrace{\frac{\beta}{1 - \beta} (\max(V_1) + \max(W_2)) p'(0) \alpha_\theta}_{\alpha_W} \|B_1 - B_2\|.
\end{aligned}$$

This completes the proof of the lemma. $\square$

We can now go back to inequality (A.4) to establish our continuity result:

$$\|\pi_1(\Phi_1) - \pi_2(\tilde{\Phi}_2)\| \leq \|d_1 - \tilde{d}_2\| + \|zf(\tilde{K}_1) - r_d \tilde{K}_1 - zf(\tilde{K}_2) + r_d \tilde{K}_2\|.$$

Let us look at each component in the right-hand-side separately. First, we want to bound $\|d_1 - \tilde{d}_2\|$. To establish this result, let us first notice that:

$$\min\{u'\} \|d_1 - \tilde{d}_2\| \leq \|u(d_1) - u(\tilde{d}_2)\|,$$

and

$$\begin{aligned}
\|u(d_1) - u(\tilde{d}_2)\| &= \|V - \beta \mathbb{E}[\sigma W_1(z') + (1 - \sigma)V_1'] - [V - \beta \mathbb{E}[\sigma W_2(z') + (1 - \sigma)V_1']\| \\
&\leq \beta \|W_1 - W_2\| \\
&\leq \beta \alpha_W \|B_1 - B_2\|.
\end{aligned}$$

Second, we want to bound $\|zf(K_1) - r_d K_1 - zf(\tilde{K}_2) + r_d \tilde{K}_2\|$.

Let us fix (z, r_d, V) . We have:

$$u(\eta K_1) - u(\eta \tilde{K}_2) = \beta \mathbb{E}_z[H_1(z') - H_2(z')].$$

But, we also have:

$$H_1(z) - H_2(z) = \xi \mathbb{E}_z[W_1(z') - W_2(z')] + (1 - \xi) \beta \mathbb{E}_z[H_1(z') - H_2(z')],$$

and, from the definition of H given by equation (1.2)

$$\begin{aligned} |H_1 - H_2| &\leq \frac{\xi}{1 - \beta(1 - \xi)} |W_1 - W_2| \\ &\leq \frac{\xi}{1 - \beta(1 - \xi)} \alpha_W \|B_1 - B_2\|. \end{aligned}$$

Eventually, we use the property of concavity of u to get:

$$|K_1 - \tilde{K}_2| \leq \frac{1}{\eta u'(\eta \max(K))} [u(\eta K_1) - u(\eta \tilde{K}_2)],$$

and

$$|K_1 - \tilde{K}_2| \leq \beta \alpha_k \|B_1 - B_2\|,$$

with $\alpha_k = \frac{\xi}{1 - \beta(1 - \xi)} \frac{\alpha_W}{\eta u'(\eta \bar{K})}$, and $\bar{K} = \max \{K_{FB}(z, r_d)\}_{(z, r_d) \in \mathbb{Z} \times \mathbb{R}_d}$.

The result above also goes through for any concave function of K (adjusting the multiplicative term by $\frac{1}{f'(\bar{K})}$), and we can finally write:

$$\|T_{B_1}(z, r_d, V) - T_{B_2}(z, r_d, V)\| \leq \beta \left(1 + \alpha_W + \frac{\alpha_k}{z f'(\bar{K})}\right) \|B_1 - B_2\|.$$

This completes the proof of (b-1).

(c) Equicontinuity of $T(\mathcal{P})$.

Let us show that $\forall \epsilon > 0, \exists \delta > 0$, such that for all $\nu_i = (z_i, r_{d,i}, V_i)$, $i = 1, 2$

$$\|\xi_1 - \xi_2\| < \delta \Rightarrow TB(\nu_1) - TB(\nu_2) < \epsilon, \forall B \in \mathcal{P}.$$

Fix $\epsilon > 0$, and pick $\delta < \min \left(\min_{(z_1, z_2) \in \mathbb{Z}} |z_1 - z_2|, \min_{(r_{d,1}, r_{d,2}) \in \mathbb{R}_d} |r_{d,1} - r_{d,2}|, \frac{\epsilon}{\alpha_V} \right)$.

For ξ_1, ξ_2 such that $\|\xi_1 - \xi_2\| < \delta$, we have $z_1 = z_2$, and $r_{d,1} = r_{d,2}$.

We can therefore conclude that:

$$\|TB(\xi_1) - TB(\xi_2)\| \leq \alpha_v |V_1 - V_2| \leq \alpha_v \|\xi_1 - \xi_2\| < \epsilon.$$

Now that we have shown that assumptions (i), (ii), and (iii) are verified, Schauder's fixed point theorem applies and there exists a fixed point $B^* \in \mathcal{P}$ such that $T(B^*) = B^*$. Eventually, all the remaining equilibrium objects (W^*, ρ^*, θ^*) and policy functions associated with the optimal contract are also well defined.

This concludes the proof. □

Proof. **Proposition 8.** Social Planner's problem.

Let us start by simplifying some of the notations of the model before formalizing the social planner's problem. To keep notations simple, let us also abstract from the variables' dependence on aggregate shocks and bank-firm characteristics $(r_{d,t-1}, V_{t-1})$ carried from period $t - 1$. Let us also denote $\theta_V = \theta(V)$, the market tightness associated with firm value V , V_r^0 the optimal firm value offered by banks with funding cost r , and the dividend policy $d(V_r) = d(r, V_r)$ and the joint surplus $S(r, V_r)$ associated with funding cost r and firm value V_r .

The social planner maximizes the discounted sum of utilities derived by banks and firms

for incumbent lending relationships, utility derived by rationed entrepreneurs, minus total origination costs. The problem is subject to the dynamics of the existing lending contracts represented by the function f_c (which depends only on (V_{t-1}, r_{t-1}, r_t)), the laws of motion for credit rationing v_t , and the distribution of lending relationships g_t .

In order to keep this proof reasonably tractable, notice that the social planner faces the same contracting frictions as each individual bank; hence, we can immediately replace the original problem with the corresponding solution to its Lagrange multiplier formulation and by taking the optimal weight on firm value to be $\Lambda_{t+1} = \frac{1}{u'(d_t)}$.

The social planner therefore maximizes the following objective function:

$$\begin{aligned}
& \max_{v_t, g_t, \theta_V, J_t, V_t} \mathbb{E} \sum_t \beta^t \left[\sum_{r_d, t, V_t} g_t(r_d, t, V_t) [S(r_d, t, V_t) - d(V_t) + \Lambda_{t+1}(r_d, t, V_t)u(d(V_t))] \right. \\
& \quad \left. - cJ_t + v_t u(d_0) \right] \\
& s.t. \forall (t, z^t) \\
\Lambda_{t+1}(r_d, t, V_t) &= \frac{1}{u'(d(V_t))}, & \forall (r_d, t, V_t) \\
V_t &= f_c(V_{t-1}, r_{t-1}, r_t), & \forall (V_{t-1}, r_{t-1}, r_t) \\
v_t &= v_{t-1} \left(1 - (1 - \sigma) \sum_r \Gamma_{r_d}^0(r) p(\theta_{V_r}) \right) + (1 - v_{t-1}) \sigma \\
g_t(r, V) &= \sum_{V_{t-1} | V_t = V} (1 - \sigma) g_{t-1}(r_{t-1}, V_{t-1}) \Gamma_{r_d}(r_{t-1}, r) + J_t q(\theta_{V_r}) \Gamma_{r_d}^0(r) \mathbb{1}_{V_r = V}, \forall (r, V),
\end{aligned}$$

where Γ_{r_d} is the transition probability and $\Gamma_{r_d}^0$ is the unconditional entry distribution of the idiosyncratic funding cost.

The planner's problem is also constrained by the credit market clearing conditions, which imply that the total number of funded entrepreneurs equals the total number of loans originated within each active submarket. In the context of the model, this is simply given by the following standard condition:

$$v_{t-1} p(\theta_{V_{r_t}}) = q(\theta_{V_{r_t}}) J_t, \quad \forall \theta > 0.$$

Note also that the mass of firms in the economy satisfies the following identity:

$$v_{t-1} + \sum_{r,V} g_{t-1}(r, V) = 1.$$

To further characterize this problem, let us now denote μ the multiplier associated with the law of motion of v , and $\{\zeta_{\theta_V}\}_V$ the set of multipliers associated with the market clearing condition for each active submarket. We can write the following generalized expression:

$$\begin{aligned} \max_{v_t, g_t, \theta_V, J_t, V_t} \quad & \mathbb{E} \sum_t \beta^t \left(\sum_{r_d, t, r_{d,t+1}, V_{t-1}} (1 - \sigma) g_{t-1}(r_{d,t-1}, V_{t-1}) \Gamma_{r_d}(r_{d,t-1}, r_{d,t}) [S(r_{d,t}, V_t) \right. \\ & - d(V_t) + \frac{1}{u'(d(V_t))} u(d(V_t))] \\ & + J_t \left(\sum_r \Gamma_{r_d}^0(r) q(\theta_{V_{r_t}}) \left[[S(r_t, V_{r_t}) - d(V_{r_t}) + \frac{1}{u'(d(V_{r_t}))} u(d(V_{r_t}))] - c \right] \right. \\ & - \sum_{V, \theta_V > 0} \zeta_{\theta_V, t} \sum_{r, V_r = V} \Gamma_{r_d}^0(r) [J_t q(\theta_V) - p(\theta_V) v_{t-1}] \\ & \left. \left. + v_t u(d_0) - \mu_t \left(v_t - v_{t-1} (1 - (1 - \sigma) \sum_r \Gamma_{r_d}^0(r) p(\theta_{V_r})) - \sigma \sum_{r,V} g_{t-1}(r, V) \right) \right) \right). \end{aligned}$$

Replacing $\varphi_V = q(\theta_V)$ in the objective function allows us to have a well-defined and strictly concave problem in all its maximands over a convex set. This in turn is sufficient to establish the existence and uniqueness of an optimal solution to the social planner's problem.

We can eventually decompose the above expression into three auxiliary and independent problems for (i) incumbent banks, (ii) entrant banks, and (iii) unfunded firms, which are all independent from g_{t-1} , the distribution of lending relationships at $t - 1$:

(i) *Incumbent banks*:

$$\mathbb{E} \left[\sum_t \beta^t \Gamma_{r_d}(r_{d,t-1}, r_{d,t}) \left[(1 - \sigma) [S(r_{d,t}, V_t) - d(V_t) + \frac{1}{u'(d(V_t))} u(d(V_t))] + \sigma \mu_t \right] \right]$$

(ii) *Entrant banks:*

$$\max_{\theta_V, \{V_r\}_r, J_t} J_t \mathbb{E} \sum_r \Gamma_{r_d}^0(r) q(\theta_{V_r}) \left[\sum_{t' \geq t} \beta^{t'-t} \left[(1-\sigma) [S(r_{t'}, V_{r_{t'}}) - d(V_{r_{t'}}) + \frac{1}{u'(d(V_{r_{t'}}))} u(d(V_{r_{t'}}))] + \sigma \mu_{t'} \right] \right] - J_t [c + \sum_r \zeta_{\theta_{V_r}} \Gamma_{r_d}^0(r) q(\theta_{V_r})]$$

(iii) *Unfunded firms:*

$$\max_{v_t} \sum_t \beta^t \left[v_t u(d_0) - \mu_t (v_t - v_{t-1} (1 - \sum_r \Gamma_{r_d}^0(r) p(\theta_{V_r}))) + v_{t-1} \sum_r \Gamma_{r_d}^0(r) \zeta_{\theta_{V_r}, t} p(\theta_{V_r}) \right].$$

Let us now compare the above problems with those obtained in the competitive equilibrium. First, let us identify $\mu_t = W_t$. As a result, the problem for incumbent banks is exactly the same as the saddle-point problem obtained in the competitive case.

Turning to the problem of entrant banks, we can rewrite the maximization problem at date t , associated with each entrant bank of type r as:

$$\max_{V_r} \sum_{t' \geq t} \beta^{t'-t} \left[(1-\sigma) \left[S(r_{t'}, V_{r_{t'}}) - d(V_{r_{t'}}) + \frac{1}{u'(d(V_{r_{t'}}))} u(d(V_{r_{t'}})) \right] + \sigma \mu_{t'} \right] - \zeta_{\theta_{V_r}, t}.$$

Thus, the above problem is similar to the entrants' problem in the competitive equilibrium if:

$$\zeta_{\theta_{V_r}, t} = \sum_{t' \geq t} \beta^{t'-t} \left[(1-\sigma) \frac{u(d(V_{r_{t'}}))}{u'(d(V_{r_{t'}}))} + \sigma \mu_{t'} \right].$$

Finally, turning to the unfunded firm problem, the first order condition on v_t yields:

$$0 = u(d_0) - \mu_t + \beta \mu_{t+1} \left[1 - \sum_r \Gamma_{r_d}^0(r) p(\theta_{V_r}) \right] + \beta \sum_r \zeta_{\theta_{V_r}, t} \Gamma_{r_d}^0(r) p(\theta_{V_r, t+1}).$$

Using $\mu_t = W_t$, we can re-arrange terms of the above condition to obtain the following

identity, satisfied within each active submarket:

$$\begin{aligned}
W_t &= u(d_0) + \beta \left[(1 - p(\theta_{V_r}))W_{t+1} + p(\theta_{V_r})\zeta_{\theta_{V_r},t} \right] \\
&= u(d_0) + \beta \left[(1 - p(\theta_{V_r}))W_{t+1} + p(\theta_{V_r}) \sum_{t' \geq t} \beta^{t'-t} \left[(1 - \sigma) \frac{u(d(V_{r_{t'}}))}{u'(d(V_{r_{t'}}))} + \sigma W_{t'} \right] \right].
\end{aligned}$$

The above expression resembles the one obtained in the competitive search equilibrium (equations (1.8) - (1.9)), with one major difference. Indeed, the equivalence between the two expressions is only true if $\sum_{t' \geq t} \beta^{t'-t} [(1 - \sigma) \frac{u(d(V_{r_{t'}}))}{u'(d(V_{r_{t'}}))} + \sigma W_{t'}] = V$. However, this is the case only when entrepreneurs are risk neutral and $u' = 1$. More generally, the above result establishes that the obtained competitive search equilibrium is always constrained inefficient whenever entrepreneurs are risk averse.

This concludes the proof. □

BIBLIOGRAPHY

- Abreu, D., D. Pearce, and E. Stacchetti. 1990. Toward a theory of discounted repeated games with imperfect monitoring. *Econometrica* 58:1041–1063.
- Acemoglu, D., and R. Shimer. 1999. Efficient unemployment insurance. *Journal of Political Economy* 107:893–928.
- Adrian, T., and N. Boyarchenko. 2012. Intermediary leverage cycles and financial stability. Staff Report, Federal Reserve Bank of New York.
- Agarwal, S., and R. Hauswald. 2010. Distance and private information in lending. *Review of Financial Studies* 23:2757–2788.
- Albuquerque, R., and H. A. Hopenhayn. 2004. Optimal lending contracts and firm dynamics. *Review of Economic Studies* 71:285–315.
- Allen, F., and D. Gale. 1997. Financial markets, intermediaries, and intertemporal smoothing. *Journal of Political Economy* 105:523–546.
- Bebchuk, L. A. 2000. Using options to divide value in corporate bankruptcy. *European Economic Review* 44:829–843.
- Bebchuk, L. A., and I. Goldstein. 2011. Self-fulfilling credit market freezes. *Review of Financial Studies* 24:3519–3555.
- Becsi, Z., V. E. Li, and P. Wang. 2005. Heterogeneous borrowers, liquidity, and the search for credit. *Journal of Economic Dynamics and Control* 29:1331–1360.
- Benmelech, E., and N. K. Bergman. 2012. Credit traps. *American Economic Review* 102:3004–3032.
- Berlin, M., and L. J. Mester. 1999. Deposits and relationship lending. *Review of Financial Studies* 12:579–607.
- Bernanke, B. 1983. Nonmonetary effects of the financial crisis in the propagation of the Great Depression. *American Economic Review* 73:257–276.
- Bernanke, B., and M. Gertler. 1989. Agency costs, net worth, and business fluctuations. *American Economic Review* 79:14–31.
- Biais, B., T. Mariotti, J.-C. Rochet, and S. Villeneuve. 2010. Large risks, limited liability, and dynamic moral hazard. *Econometrica* 78:73–118.
- Bigio, S. 2014. Financial risk capacity. Working paper.
- Bolton, P., X. Freixas, L. Gambacorta, and P. E. Mistrulli. 2016. Relationship and transaction lending in a crisis. *Review of Financial Studies* - Forthcoming.

- Bonaccorsi di Patti, E., and E. Sette. 2012. Bank balance sheets and the transmission of financial shocks to borrowers: evidence from the 2007-2008 crisis. *Bank of Italy Temi di Discussione Working Paper* No 848.
- Boot, A. W. 2000. Relationship banking: What do we know? *Journal of Financial Intermediation* 9:7–25.
- Boot, A. W., and A. V. Thakor. 2000. Can relationship banking survive competition? *Journal of Finance* 55:679–713.
- Boualam, Y. 2014. Bank dynamics and credit market competition. Working paper.
- Brunnermeier, M. K., T. M. Eisenbach, and Y. Sannikov. 2012. Macroeconomics with financial frictions: A survey. National Bureau of Economic Research.
- Brunnermeier, M. K., and Y. Sannikov. 2014. A macroeconomic model with a financial sector. *American Economic Review* 104:379–421.
- Cerqueiro, G., H. Degryse, and S. Ongena. 2011. Rules versus discretion in loan rate setting. *Journal of Financial Intermediation* 20:503–529.
- Chava, S., and A. Purnanandam. 2011. The effect of banking crisis on bank-dependent borrowers. *Journal of Financial Economics* 99:116–135.
- Chodorow-Reich, G. 2014. The employment effects of credit market disruptions: Firm-level evidence from the 2008–9 financial crisis. *Quarterly Journal of Economics* 129:1–59.
- Clementi, G. L., and H. A. Hopenhayn. 2006. A theory of financing constraints and firm dynamics. *Quarterly Journal of Economics* 121:229–265.
- Cooley, T., R. Marimon, and V. Quadrini. 2004. Aggregate consequences of limited contract enforceability. *Journal of Political Economy* 112:817–847.
- De Marzo, P. M., and M. J. Fishman. 2007. Optimal long-term financial contracting. *Review of Financial Studies* 20:2079–2128.
- De Marzo, P. M., and Y. Sannikov. 2006. Optimal security design and dynamic capital structure in a continuous-time agency model. *Journal of Finance* 61:2681–2724.
- Degryse, H., M. Kim, and S. Ongena. 2009. *Microeconometrics of banking: methods, applications, and results*. Oxford University Press.
- Den Haan, W. J., G. Ramey, and J. Watson. 2003. Liquidity flows and fragility of business enterprises. *Journal of Monetary Economics* 50:1215–1241.
- Diamond, D. W., and R. G. Rajan. 2011. Fear of fire sales, illiquidity seeking, and credit freezes. *Quarterly Journal of Economics* 126:557–591.

- Diamond, P. 1990. Pairwise credit in search equilibrium. *Quarterly Journal of Economics* 105:285–319.
- Froot, K. A., D. S. Scharfstein, and J. C. Stein. 1993. Risk management: Coordinating corporate investment and financing policies. *Journal of Finance* 48:1629–1658.
- Gertler, M. 1992. Financial capacity and output fluctuations in an economy with multi-period financial relationships. *Review of Economic Studies* 59:455–472.
- Gertler, M., and P. Karadi. 2011. A model of unconventional monetary policy. *Journal of Monetary Economics* 58:17–34.
- Gertler, M., and N. Kiyotaki. 2010. Financial intermediation and credit policy in business cycle analysis. *Handbook of Monetary Economics* 3:547–599.
- Gilchrist, S., J. W. Sim, and E. Zakrajšek. 2013. Misallocation and financial market frictions: Some direct evidence from the dispersion in borrowing costs. *Review of Economic Dynamics* 16:159–176.
- Golosov, M., P. Maziero, and G. Menzio. 2013. Taxation and redistribution of residual income inequality. *Journal of Political Economy* 121:1160–1204.
- Gomes, J. F. 2001. Financing investment. *American Economic Review* 91:1263–1285.
- Greenstone, M., A. Mas, and H.-L. Nguyen. 2014. Do credit market shocks affect the real economy? Quasi-experimental evidence from the Great Recession and normal economic times. National Bureau of Economic Research Working paper.
- Guerrieri, V., R. Shimer, and R. Wright. 2010. Adverse selection in competitive search equilibrium. *Econometrica* 78:1823–1862.
- Hansen, M. E., and N. L. Ziebarth. 2016. Credit relationships and business bankruptcy during the Great Depression. Working paper.
- Harris, M., and B. Holmstrom. 1982. A theory of wage dynamics. *Review of Economic Studies* 49:315–333.
- He, Z., and A. Krishnamurthy. 2012. A model of capital and crises. *Review of Economic Studies* 79:735–777.
- Hubbard, R. G., K. N. Kuttner, and D. N. Palia. 2002. Are there bank effects in borrowers' costs of funds? Evidence from a matched sample of borrowers and banks. *Journal of Business* 75:559–581.
- Inderst, R., and H. M. Müller. 2004. The effect of capital market characteristics on the value of start-up firms. *Journal of Financial Economics* 72:319–356.

- Ivashina, V., and D. Scharfstein. 2010. Bank lending during the financial crisis of 2008. *Journal of Financial Economics* 97:319–338.
- Jermann, U. J., and V. Quadrini. 2007. Stock market boom and the productivity gains of the 1990s. *Journal of Monetary Economics* 54:413–432.
- Jermann, U. J., and V. Quadrini. 2012. Macroeconomic effects of financial shocks. *American Economic Review* 102:238–271.
- Jiménez, G., and S. Ongena. 2012. Credit supply and monetary policy: Identifying the bank balance-sheet channel with loan applications. *American Economic Review* 102:2301–2326.
- Kandrac, J. 2014. Bank failure, relationship lending, and local economic performance. Finance and Economics Discussion Series 2014-41. Board of Governors of the Federal Reserve System (U.S.).
- Khan, A., and J. K. Thomas. 2013. Credit shocks and aggregate fluctuations in an economy with production heterogeneity. *Journal of Political Economy* 121:1055–1107.
- Khwaja, A. I., and A. Mian. 2008. Tracing the impact of bank liquidity shocks: Evidence from an emerging market. *American Economic Review* 98:1413–1442.
- Kiyotaki, N., and J. Moore. 1997. Credit cycles. *Journal of Political Economy* 105:211–248.
- Krueger, D., and H. Uhlig. 2006. Competitive risk sharing contracts with one-sided commitment. *Journal of Monetary Economics* 53:1661–1691.
- Krusell, P., and A. A. Smith, Jr. 1998. Income and wealth heterogeneity in the macroeconomy. *Journal of Political Economy* 106:867–896.
- La Porta, R., F. Lopez-De-Silanes, A. Shleifer, and R. W. Vishny. 1997. Legal determinants of external finance. *Journal of Finance* 52:1131–1150.
- La Porta, R., F. Lopez-de Silanes, A. Shleifer, and R. W. Vishny. 1998. Law and finance. *Journal of Political Economy* 106:1113–1155.
- Lamadon, T. 2014. Productivity shocks, dynamic contracts and income uncertainty. Working paper.
- Li, S., T. M. Whited, and Y. Wu. 2016. Collateral, taxes, and leverage. *Review of Financial Studies* - Forthcoming.
- Marcet, A., and R. Marimon. 1992. Communication, commitment, and growth. *Journal of Economic Theory* 58:219–249.
- Marcet, A., and R. Marimon. 2011. Recursive contracts. Working paper.

- Menzio, G., and S. Shi. 2010. Block recursive equilibria for stochastic models of search on the job. *Journal of Economic Theory* 145:1453–1494.
- Menzio, G., and S. Shi. 2011. Efficient search on the job and the business cycle. *Journal of Political Economy* 119:468–510.
- Moen, E. R. 1997. Competitive search equilibrium. *Journal of Political Economy* 105:385–411.
- Moen, E. R., and Å. Rosén. 2011. Incentives in competitive search equilibrium. *Review of Economic Studies* 78:733–761.
- Monge-Naranjo, A. 2008. Limited commitment, firm dynamics and aggregate fluctuations. Working paper.
- Ongena, S., and D. C. Smith. 2001. The duration of bank relationships. *Journal of Financial Economics* 61:449–475.
- Peng, M. W., Y. Yamakawa, and S.-H. Lee. 2010. Bankruptcy laws and entrepreneur-friendliness. *Entrepreneurship Theory and Practice* 34:517–530.
- Petersen, M. A., and R. G. Rajan. 1994. The benefits of lending relationships: Evidence from small business data. *Journal of Finance* 49:3–37.
- Petersen, M. A., and R. G. Rajan. 1995. The effect of credit market competition on lending relationships. *Quarterly Journal of Economics* 110:407–443.
- Petrosky-Nadeau, N. 2014. Credit, vacancies and unemployment fluctuations. *Review of Economic Dynamics* 17:191–205.
- Petrosky-Nadeau, N., and E. Wasmer. 2011. Macroeconomic dynamics in a model of goods, labor and credit market frictions. Working paper.
- Phelan, C. 1995. Repeated moral hazard and one-sided commitment. *Journal of Economic Theory* 66:488–506.
- Philippon, T., and P. Schnabl. 2013. Efficient recapitalization. *Journal of Finance* 68:1–42.
- Quadrini, V. 2004. Investment and liquidation in renegotiation-proof contracts with moral hazard. *Journal of Monetary Economics* 51:713–751.
- Rampini, A. A., and S. Viswanathan. 2010. Collateral, risk management, and the distribution of debt capacity. *Journal of Finance* 65:2293–2322.
- Rampini, A. A., and S. Viswanathan. 2013. Collateral and capital structure. *Journal of Financial Economics* 109:466–492.

- Rampini, A. A., and S. Viswanathan. 2015. Financial intermediary capital. Available at SSRN 1785877.
- Rudanko, L. 2009. Labor market dynamics under long-term wage contracting. *Journal of Monetary Economics* 56:170–183.
- Santos, J. A., and A. Winton. 2013. Bank capital, borrower power, and loan rates. Working paper.
- Schaal, E. 2015. Uncertainty and unemployment. Working paper.
- Schmid, L. 2012. A quantitative dynamic agency model of financing constraints. Working paper.
- Silviera, R., and R. Wright. 2006. The venture capital cycle. Working paper.
- Spear, S. E., and S. Srivastava. 1987. On repeated moral hazard with discounting. *Review of Economic Studies* 54:599–617.
- Stokey, N., and R. Lucas, with E. Prescott. 1989. *Recursive methods in economic dynamics*. Harvard University Press.
- Stulz, R. M. 1984. Optimal hedging policies. *Journal of Financial and Quantitative Analysis* 19:127–140.
- Tauchen, G., and R. Hussey. 1991. Quadrature-based methods for obtaining approximate solutions to nonlinear asset pricing models. *Econometrica* 59:371–396.
- Thomas, J., and T. Worrall. 1994. Foreign direct investment and the risk of expropriation. *Review of Economic Studies* 61:81–108.
- Vickery, J. 2005. Banking relationships and the credit cycle: evidence from the Asian financial crisis. Federal Reserve Bank of New York Working Paper.
- Wasmer, E., and P. Weil. 2004. The Macroeconomics of labor and credit market imperfections. *American Economic Review* 94:944–963.