

Lyapunov Analysis of Robot Motion

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1 Introduction

The practice of automatic control has its origins in antiquity [1,2]. It is only recently — within the middle decades of this century — that a body of scientific theory has been developed to inform and improve that practice. Control theorists tend to divide their history into two periods. A “classical” period, prior to the sixties witnessed the systematization of feedback techniques based upon frequency domain analysis dominated by applications to electronics and telephony. A “modern” period in the sixties and seventies was characterized by a growing concern with formal analytical techniques pursued within the time domain motivated by the more stringent constraints posed by space applications and the enhanced processing capability of digital technology [3]. The hallmark of control theory has been, by in large, a systematic exploitation of the properties of linear dynamical systems whether in the frequency or time domain. Its great success in applications is a remarkable tribute to the diverse range of physical phenomena for which such models are appropriate.

The field of robotics presents control theorists with a fascinating and novel domain. While numerically controlled kinematic chains with a few degrees of freedom have been available for over two decades, it is only within the last five years that mechanical systems with many degrees of freedom, each independently actuated, have been wedded to dedicated computational resources of considerable sophistication. To begin with, the dynamical behavior of such systems appears to depart dramatically from the familiar linear case. Further, typical robotic tasks involve complex interactions with diverse environments possessing, of themselves, kinematics and dynamics, which may change abruptly throughout the course of desired operations. Finally, the complexity of these tasks makes even their specification problematic for purposes of control. This article is concerned exclusively with control problems arising from the first of these considerations. Specifically, the intent is to show that a systematic application of Lyapunov Theory affords qualitative understanding of certain aspects of the input/output properties of a broad class of nonlinear systems (which includes all robots) analogous to that available for linear time invariant systems. For concreteness, the discussion is limited to robot arms — open kinematic chains with rigid links.

This first section concludes with a presentation of the central theoretical apparatus to be used throughout the sequel. The next section provides an elementary account of control in the context of a single degree of freedom kinematic chain, providing a means of introduction to the techniques of Lyapunov analysis by comparison to the presumably more familiar frequency domain methods. Section 3 constitutes a very brief derivation of the equations of motion arising from the dynamical properties of general kinematic chains. Section 4 presents an account of a robot planning and control methodology based upon unforced, pure feedback based control. Finally, Section 5 provides a brief look at several approaches to the general robot tracking problem.

1.1 Notation and Definitions

If $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ has continuous first partial derivatives, denote its $m \times n$ jacobian matrix as

$$Df \triangleq \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

If $m = 1$ then Df is a gradient; if $n = 1$ then Df is a tangent vector. It will often be necessary to obtain the jacobian of a matrix valued map — $m = p \times q$ — and the kronecker-stack notation presented in the Appendix is of great help in this regard. When we require only a subset of derivatives, e.g. when $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, and we desire the jacobian of f with respect to the variables $x_1 \in \mathbb{R}^{n_1}$, as x_2 is held fixed, we may write

$$D_{x_1}f \triangleq Df \begin{bmatrix} I_{n_1 \times n_1} \\ 0 \end{bmatrix}.$$

It will also be important to obtain bounds on the operator norm of matrix valued maps. If $A : J \rightarrow \mathbb{R}^{n \times n}$ is a smooth map taking matrix values then let

$$\mu_A \triangleq \sup_{q \in J} \sup_{\|x\|=1} |x^T A x|$$

and

$$\nu_A \triangleq \inf_{q \in J} \inf_{\|x\|=1} |x^T A x|.$$

If J is compact, or the entries of A are bounded then both ν_A, μ_A are non-negative real numbers. For any constant matrix, μ_A is the square root of the eigenvalue of greatest magnitude, while ν_A is the square root of the eigenvalue of least magnitude of $A^T A$, from which it follows that

$$\mu_A = \sup_{q \in J} \|A(q)\| \quad 1/\nu_A = \sup_{q \in J} \|A^{-1}(q)\|,$$

where $\|\cdot\|$ denotes the operator norm induced by the euclidean norm of \mathbb{R}^n .

Given a set \mathcal{S} , a smooth (possibly time varying) scalar valued map, $v : \mathbb{R} \times \mathcal{S} \rightarrow \mathbb{R}$ is said to be *positive definite at a point* $x \in \mathcal{S}$ if, for all t , $v(t, x) = 0$, and $v > 0$ in some open neighborhood of x . Given a smooth (possibly time varying) vector field, f , on some state space, \mathcal{S} , we shall say that, v , a positive definite map at $x_d \in \mathcal{S}$, constitutes a *Lyapunov function for f at x_d* if the time derivative along any motion of the vector field is non-positive,

$$\dot{v} = [D_x v]f + D_t v \leq 0,$$

in some neighborhood of x_d , for all t , and that it constitutes a *strict Lyapunov function for f* if the inequality is strict [4,5]. The *domain* of v with respect to x_d is the largest neighborhood around p which is free of additional critical points and upon which the derivative is still non-positive. A strict Lyapunov function will be called a *quadratic Lyapunov function for f on the domain, \mathcal{D}* if it is analytic and there exist three positive constants, $\alpha_1, \alpha_2, \alpha_3$, with the properties,

$$\alpha_1 \|x\|^2 \leq v(x, t) \leq \alpha_2 \|x\|^2 \text{ and } \dot{v}(x, t) \leq -\alpha_3 \|x\|^2 \quad (1)$$

for all t and $x \in \mathcal{D}$.

1.2 Some Results from Lyapunov Theory

The existence of a strict Lyapunov function at a point is a sufficient condition for asymptotic stability of that equilibrium state. If a strict Lyapunov function has not been found, asymptotic

stability may, nevertheless, be assured if a further condition on the possible limiting set holds. This is “LaSalle’s Invariance Principle” [5]. It is possible, as well, to draw conclusions about the tracking capability of a forced dynamical system in consequence of the stability properties of the unforced vector field at a particular equilibrium state. However, this seems to require the use of a strict Lyapunov function.

The central concern of this paper is with the application of two results from Lyapunov theory to the vector fields arising from the dynamics of kinematic chains. First, it has been known for quite some time that the total energy of a mechanical system may be interpreted as a Lyapunov function [6], and we will make extensive use of this fact in Sections 2.2 and 4. Unfortunately, this choice of Lyapunov function is never strict, and an appeal to LaSalle’s invariance principle is required. Second, we will make use of a strict Lyapunov function in Sections 2.3 and 5 which turns out to be quadratic (in the sense defined above). The tracking results follow as a standard consequence.

In order to introduce Lasalle’s Invariance principle, one further definition is required. A *positive invariant set* relative to some vector field, f , is a set in state space with the property that any trajectory originating there stays there for all future time.

Theorem 1 (LaSalle’s Invariance Principle [5]) *If v is a Lyapunov function for the time invariant vector field f on some pre-compact domain \mathcal{D} , then any trajectory originating in that domain approaches the largest positive invariant set contained within the subset of \mathcal{D} with the property that $\dot{v} = 0$.*

In order to make use of quadratic Lyapunov functions, the following technical result is required.

Lemma 1 *If*

$$\dot{v} \leq \phi(v),$$

and $u(t)$ is a maximal solution to the differential equation, $\dot{u} = \phi(u)$, and $v(t_0) \leq u(t_0)$, then

$$v(t) \leq u(t),$$

for all $t > t_0$.

Proof: This is a standard application of a differential inequality. For example, see the reference [7][Theorem III.4.1].

□

This useful fact leads to a variety of standard results (e.g. see [8]) involving transient and steady state behavior of disturbed dynamical systems. The following will prove particularly useful in the sequel.

Theorem 2 *Consider the disturbed dynamical system*

$$\dot{x} = A(x)x + d(t)$$

If v is a quadratic Lyapunov function at the point 0 for the undisturbed system (i.e. $d \equiv 0$) on some domain, \mathcal{D} , and there may be found a positive constant, α_4 , such that

$$|D_x v d| \leq \alpha_4 \|x\|$$

then the response of the disturbed system from any initial condition, $x(t_0) \in \mathcal{D}$, is bounded by

$$\|x(t)\| \leq e^{-\frac{1}{2}\rho t} \chi + \beta$$

where

$$\rho \triangleq \frac{\alpha_3}{\alpha_2}; \beta \triangleq \frac{\alpha_2 \alpha_4}{\alpha_1 \alpha_3}; \chi \triangleq \frac{\alpha_2}{\sqrt{\alpha_1}} \|x(t_0)\|^2,$$

and the $\alpha_i, i = 1, 3$, are defined by (1).

Proof: According to the hypothesis we have

$$\begin{aligned} \dot{v} &= D_x v [Ax + d] + D_t v \\ &\leq -\alpha_3 \|x\|^2 + \alpha_4 \|x\| \\ &\leq -\rho v + \frac{\alpha_4}{\sqrt{\alpha_1}} \sqrt{v}. \end{aligned}$$

Since

$$u(t) \triangleq \left(e^{-\frac{1}{2}\rho t} u(0) + \frac{\alpha_4}{\sqrt{\alpha_1}} \right)^2$$

is a maximal solution to the differential equation

$$\dot{u} = -\rho u + \frac{\alpha_4}{\sqrt{\alpha_1}} \sqrt{u},$$

it follows from Lemma 1 that

$$v(t) \leq \left(e^{-\frac{1}{2}\rho t} v(0) + \frac{\alpha_4}{\sqrt{\alpha_1}} \right)^2.$$

The result follows by noting $\alpha_1 \|x\|^2 \leq v$, according to (1).

□

2 A Single Degree of Freedom Robot Arm

The “frequency domain” techniques of classical control theory lie at the foundation of the discipline and offer design methods proven over the last sixty years in the context of a great variety of physical problems. Unfortunately, the class of dynamical systems represented by high performance robots with revolute arms is not amenable to a general rigorous analysis using these tools: strongly coupled nonlinear dynamics do not admit representation by transfer function.¹ For the purposes of tutorial exposition, in this section the central insights of classical control theory will be (roughly) translated into the language of lyapunov analysis. While frequency domain analysis appears to be less cumbersome, we will make some effort in this section to show that the results of the present analysis are quite similar. It will be seen subsequently (Sections 4 and 5) that much of this translation carries over to the more general robotic domain as well, where the classical analysis is unavailable.

In this section attention is limited to the case of a single degree of freedom mechanical control system — the actuated simple pendulum. While this system cannot convey the scope of sixty years of control research, the insights motivated by such second order linear time invariant systems pervade the field. At the same time, this system represents the simplest possible revolute robot arm. Section 2.1 introduces the notion of a dynamical model, and the need for control theory, hopefully making clear that the fundamental problem of control is not a consequence of limited power, but of limited information. Section 2.2 presents a sketch of linear feedback theory, and Section 2.3 provides a quick account of linear servo theory. Constraints of time and space have unfortunately precluded the addition of sections concerning many “modern” techniques such as optimal control, stochastic filtering, or learning theory as applied to this system. The reader may note that each subsection here anticipates a more specialized treatment of analogous material for the general robot arm in later sections of the article: compare Section 2.1 with Section 3; Section 2.2 with Section 4; and Section 2.3 with Section 5.

2.1 Dynamics: A Source of Delay and Uncertainty

A simple pendulum consists of a mass, M , attached via a rigid (massless) link to a joint which permits rotational motion limited to the plane on which the link lies. Perfect angular position and velocity sensors located at the joint deliver exact measurements, $\theta, \dot{\theta}$, respectively, continuously and instantaneously. A perfect actuator has been placed at the joint: this idealized device has no power limitations, hence can deliver arbitrarily large torques, τ , instantaneously. We assume that the plane of motion is horizontal so that there are no gravitational or other disturbance torques.²

The control problem may be rendered roughly as follows: design an algorithm which produces a time profile of torques, $\tau(t)$, so as to elicit some specified behavior of the simple pendulum. In the context of robotics, the following terminology (which is alien to conventional control theory) proves quite useful. The precise nature of the desired property determines what might be called the *task domain*, and the particular instance, an *encoded task specification*.

¹It is important to mention that some researchers, e.g. Horowitz [9], have successfully used modified frequency domain methods for restricted nonlinear control problems.

²Such artificial assumptions will be relaxed very soon, and serve here merely to underscore the insight that the fundamental problems of control arise from uncertain information and intrinsic delay rather than power constraints, as discussed in the introduction.

2.1.1 The Newtonian Dynamical Model

In order to think about control algorithms, we first require some understanding of the relationship between adjustments in τ and resulting changes in θ . This relationship is completely specified by Newton's law relating torque to angular acceleration, given as

$$M\ddot{\theta} = \tau.$$

This is a system with *memory*: changes in θ (and, hence, $\dot{\theta}$, itself) at any time, t , depend upon the past history of τ rather than simply its value at time t . The fact that physical systems give rise to dynamical rather than memoryless relationships necessitates the need for a theory of control as will be shown directly.

For the purposes of this article, it will prove more convenient to express such relationships, second order differential equations involving n variables, in the equivalent form of first order differential equations involving $2n$ variables. Defining

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \triangleq \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$$

to be the *state variable* expressed in *phase space*, emphasizes the fact obtaining from elementary properties of differential equations that the future behavior of the system, $x(t), t > t_0$, is entirely determined by its initial conditions, $x_{10} \triangleq \theta(t_0), x_{20} \triangleq \dot{\theta}(t_0)$, and future values of the *control input*, i.e. the torque, $u(t) \triangleq \tau(t), t > t_0$. For *time invariant systems* such as this, the behavior is independent of initial time, and it will be assumed in the sequel that $t_0 \triangleq 0$. These definitions are more carefully discussed in standard control texts [10,11,12].

The system may now be specified by phase space dynamics of the form

$$\dot{x} = f(x, u),$$

with the *vector field* [13], given as

$$f(x, u) = A_0 x + bu \tag{2}$$

where

$$A_0 \triangleq \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}; \quad b \triangleq \begin{bmatrix} 0 \\ 1/M \end{bmatrix} u.$$

According to the description of the system given above, the entire state variable is measured, thus the system *output*, y , takes the form

$$y(t) = x(t).$$

A traditional circumstance of linear control theory is that the system output — the set of available measurements — contains incomplete information from which it is required to reconstruct the entire state. In the context of robotics, this situation is most typically reversed: the system state (joint positions and velocities) are available, while the output (workspace positions, velocities, and forces) cannot easily be directly measured (*cf.* Section 3.1.2).

2.1.2 The Need for a Theory of Control

In the task domain of *set-point regulation* consider the following task specification: bring θ to some desired position θ_d and keep it there. Given the ideal context described above, it is quite easy to come up with ad hoc control algorithms.³ An obvious procedure which requires no great body of theory might be the following. Measure the present position and velocity, $x(0) = [x_{10}, x_{20}]^T$, and apply an impulse torque at the same instant,

$$u_{start}(t) \triangleq M(1 - x_{20})\delta_0(t).$$

An impulse has the effect of resetting initial conditions, thus equation (2) implies that a new constant angular velocity results,

$$x(t) = \begin{bmatrix} t + x_{10} \\ 1 \end{bmatrix}; \quad t > 0,$$

and the desired position, $x_1(t) = \theta_d$, is achieved at time $t^* \triangleq \theta_d - x_{10}$. If, at this instant, a second impulse,

$$u_{stop} \triangleq -M\delta_{t^*}(t).$$

is applied in the opposite direction of motion, then the new velocity is canceled exactly, the end-effector comes to rest in the desired position, and remains there for all $t > t^*$. The algorithm,

$$u = u_{start}(t) + u_{stop}(t),$$

requires à priori knowledge of M ; instantaneous measurement of and actuation energy exactly proportional to $x(0)$; and an exact timer for marking t^* . Note that despite our best efforts to idealize the capabilities of sensors and actuator, some finite time must necessarily elapse between the application of the control torque, and the desired result.⁴

Now suppose that the prior estimate of M , \hat{M} has some error (for instance, suppose the robot is holding an object whose mass is not known à priori). If the same control is applied, substituting \hat{M} for M , the true response of the system will be

$$x(t) = \begin{bmatrix} [\hat{M}/M + (1 - \hat{M}/M)x_{20}]t^* + (1 - \hat{M}/M)x_{20}t \\ (1 - \hat{M}/M)x_{20} \end{bmatrix}; \quad t > t^*.$$

Thus, finite and increasing error results from arbitrarily small inaccuracy in \hat{M} . It is easy to see that the same problems would result given any inaccuracy in the sensors, or magnitude of energy delivered by the actuator. Certainly, a subsequent check of true response at some future

³A *unit impulse* is modeled by the "Dirac delta function", $\delta_t(\zeta)$, defined by

$$\int_{-\infty}^{\infty} f(\zeta)\delta_t(\zeta)d\zeta = f(t)$$

— an infinitesimally rapid amount of infinitely large magnitude possessed of unit area (finite energy) which our ideal actuator is able to deliver.

⁴This, too, could be obviated by the further idealization of applying a "unit doublet" — the derivative of the delta function. We stop short of introducing such a degree of unreality since, unlike the impulse, the doublet cannot be even approximated by real actuators.

time, $x(t^* + \epsilon)$, would reveal whether or not such errors had occurred, and a similar course of action could be planned based upon the new observation. But there is no systematic procedure for performing such checks and re-adjustments: a “higher level” of control authority would be required to decide when they should be effected. More disturbing, it is not yet clear that any systematic application and re-application of this procedure exists that can guarantee subsequent improvement from one response to the next. This sort of error propagation is characteristic of *unstable* systems.

The origins of control theory, then, rest in the following observations. Dynamical systems give rise to delay which must be taken into account by any control strategy regardless of available actuator power and sensor accuracy. Moreover, information regarding the real world is inevitably uncertain and may have adverse effect upon performance no matter how small the uncertainty or powerful and accurate the apparatus.

2.2 Feedback Control: The Behavior of Error-Driven Systems

The difficulties described above suggest the desirability of control strategies which make systematic and continual use of actuator information in order to reduce successively the performance errors caused by initial uncertainty. This study — the discovery and elucidation of feedback algorithms — is arguably the most profound contribution of control theory to physical science. Here we present some of the basic results from a purely control theoretic perspective. In Part 4 suitable generalizations begin to suggest a unified approach to dynamically sound robot task encoding methodologies.

A feedback algorithm is essentially an error-driven control law. Presented with a linear system, it makes sense to investigate feedback laws which are linear in the errors as well. In the context of set-point regulation, the errors in question are the distance of the true angular position from the desired, and the true angular velocity from zero. The most general linear function of these two errors is

$$u \triangleq \omega^2(\theta_d - x_1) + 2\zeta\omega(0 - x_2). \quad (3)$$

defining a class of algorithms known as *PD* (“proportional and derivative”) control schemes. According to our model of system dynamics, (2), the resulting *closed loop* system is a homogeneous linear time invariant differential equation

$$\dot{e} = A_1 e \triangleq \begin{bmatrix} 0 & 1 \\ -\omega^2/M & -2\zeta\omega/M \end{bmatrix} e \quad (4)$$

in the translated “error” coordinate system, $e \triangleq \begin{bmatrix} x_1 - \theta_d \\ x_2 \end{bmatrix}$. The desired end position is an *equilibrium state* of this closed loop system - i.e. if the initial position and velocity of the arm were exactly at $(\theta_d, 0)$ to begin with, then the resulting future trajectory would remain there for all time. An equilibrium state of a dynamical system which has the property that solutions originating sufficiently near remain near, and asymptotically approach it in the future is called *asymptotically stable*. All those initial conditions which are near enough to asymptotically approach an asymptotically stable equilibrium state are said to lie within its *domain of attraction* [14].

2.2.1 Stability of the Closed Loop System

We seek to show that this algorithm produces an asymptotically stable closed loop equilibrium state (the desired end point) whose domain of attraction includes all positions and velocities. This desirable property may be shown to hold in a number of ways: the following demonstration is the only means which may be rigorously extended to the general case of revolute arms with many degrees of freedom as shown in Section 4.2.

It may be observed that the algorithm and consequent closed loop dynamics would be the intrinsic result of introducing a physical spring, with constant ω^2 , stretched between the desired and true position, along with a viscous damping mechanism opposing motion with force proportional to velocity, with constant $2\zeta\omega$. Accordingly, considerable insight into the asymptotic behavior of the resulting system may be obtained by studying its mechanical energy.

This is defined as the sum, $v = \kappa + \mu$ of kinetic energy, due to the velocity of the mass,

$$\kappa \triangleq \frac{1}{2}M\dot{e}_2^2 = \frac{1}{2}M\dot{\theta}^2,$$

and potential energy stored in the spring,

$$\mu \triangleq \frac{1}{2}\omega^2 e_1^2 = \frac{1}{2}\omega^2(\theta - \theta_d)^2.$$

The change in energy of the closed loop system is expressed as

$$\begin{aligned} \dot{v} &= \omega^2 e_1 \dot{e}_1 + M \dot{e}_2 \ddot{e}_2 \\ &= \omega^2 e_1 \dot{e}_2 - \omega^2 e_1 \dot{e}_2 - 2\zeta\omega \dot{e}_2^2 \\ &= -2\zeta\omega \dot{e}_2^2 \leq 0, \end{aligned} \tag{5}$$

hence, if $2\zeta\omega > 0$, v must decrease whenever the velocity is not zero. If M, ω^2 are positive as well, then v is positive, except at the desired state, $e = 0$, where $\theta = \theta_d, \dot{\theta} = 0$. It is intuitively clear (and may be rigorously demonstrated) that these conditions guarantee v will tend asymptotically toward zero, hence, that $e(t)$ approaches 0 as well.

This argument, employs the total energy as a *Lyapunov function* [14]. It demonstrates that the control algorithm succeeds, asymptotically, in accomplishing the desired task: *stabilizing* the system with respect to $\begin{bmatrix} \theta_d \\ 0 \end{bmatrix}$ solves the set point regulation problem for that end point. It has been already remarked that this stability property is *global* in the sense that any initial position and velocity of the robot arm will “decay” toward the desired equilibrium state. Moreover, no exact information regarding the particular value of M has been used to achieve the result, other than the assumption that it is positive. Since the choice of θ_d was arbitrary, it is also clear that the analogous feedback algorithm will stabilize the system around any other desired zero velocity state, $\begin{bmatrix} \theta'_d \\ 0 \end{bmatrix}$, as well, with no further re-adjustment of the “feedback gains”, $\omega^2, 2\zeta\omega$. The PD controller provides a general solution to the set point regulation problem.

2.2.2 Robust Properties of Stable Systems

The model proposed for the single degree of freedom arm (2) cannot be exactly accurate: there will be inevitable small disturbance forces and torques placed upon the shaft and arm varying in position and over time. Moreover, real actuators, even those possessed of ample power, are subject to imprecision in the profile of torques or forces output in response to any command. If

the cumulative effect of all these uncertainties is small, then equation (4) may be more accurately written in the form

$$\dot{e} = A_1 e + d(e, t)$$

where the scalar "noise" function is bounded, $\|d(e, t)\| < \delta_0$, by some small $\delta_0 > 0$. Another advantage of the feedback control algorithm developed above is that the inevitably resulting errors in performance remain strictly smaller than δ_0 , no matter what the form of the *noise* function, $d(e, t)$. This, again, may be demonstrated in a variety of ways: in keeping with the philosophy of exposition detailed in the beginning of this section, we appeal to a modified Lyapunov analysis.

Define the symmetric matrix

$$P \triangleq \begin{bmatrix} \gamma_0 \omega^2 & \zeta \omega \\ \zeta \omega & \gamma_0 M \end{bmatrix},$$

where γ_0 is some positive constant, and define the modified Lyapunov candidate,

$$v \triangleq \frac{1}{2} e^T P e = \frac{1}{2} [\gamma_0 \omega^2 e_1^2 + 2\zeta \omega e_1 e_2 + \gamma_0 M e_2^2].$$

This is a *positive definite* function (i.e. $e^T P e \geq 0$ with equality only for $e = 0$) if and only if $\gamma_0, \omega^2, M > 0$ and

$$\gamma_0 > \frac{\zeta^2}{\sqrt{M}}. \quad (6)$$

which condition clearly obtains for some sufficiently large choice of γ_0 . Taking the derivative, we have

$$\begin{aligned} \dot{v} &= e^T P A_1 e + e^T P d \\ &= -e^T Q e + e^T P d \end{aligned}$$

where

$$Q \triangleq \frac{1}{2} [P A_1 + A_1^T P] = \begin{bmatrix} \omega^2 & \omega \zeta \\ \omega \zeta & M \gamma_0' \end{bmatrix}$$

and we now write $\gamma_0 = (1 + \gamma_0')/2$. Note that Q is positive definite if and only if

$$\gamma_0' > \frac{\zeta^2}{M}. \quad (7)$$

Assuming γ_0' is chosen sufficiently large so that both (6) and (7) hold true, we have satisfied the conditions of (1) for a quadratic lyapunov function with

$$\begin{aligned} \alpha_1 &= \nu_P = \frac{1}{2} \left(\gamma_0 (M + \omega^2) - \sqrt{\gamma_0^2 (M - \omega^2)^2 + 4\omega^2 \zeta^2} \right) \\ \alpha_2 &= \mu_P = \frac{1}{2} \left(\gamma_0 (M + \omega^2) + \sqrt{\gamma_0^2 (M - \omega^2)^2 + 4\omega^2 \zeta^2} \right) \\ \alpha_3 &= \frac{\zeta \omega}{M} \cdot \nu_Q = \frac{\zeta \omega^2}{2M} \left((\gamma_0' M + \omega) - \sqrt{(\gamma_0' M - \omega^2)^2 + \omega^2 \zeta^2} \right), \end{aligned} \quad (8)$$

and the hypothesis of Theorem 2 with

$$\alpha_4 \triangleq \delta_0 \mu_P.$$

It follows, according to that result, that

$$\|e(t)\| \leq e^{-\frac{1}{2}\rho t} \chi + \beta$$

where

$$\rho = \frac{\omega \zeta \nu_Q}{M \mu_P}, \quad \beta = \frac{\delta \mu_P}{\rho \nu_P}.$$

This implies that $\|e\|$ decays exponentially toward a bounded magnitude regardless of the nature of d (as long as it is bounded).

2.2.3 Adjustment of Response: High Gain Feedback

It now seems worth comparing the results of the previous analysis with traditional frequency domain techniques. The transfer function from the disturbance vector to the output error may be written as

$$e_1(s) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}^T [sI - A_1]^{-1} d(s) = \frac{1}{Ms + 2\omega\zeta s + \omega^2} \begin{bmatrix} Ms + 2\omega\zeta \\ 1 \end{bmatrix}^T d(s), \quad (9)$$

with poles given by

$$-\frac{\omega}{M} \left(\zeta \pm \sqrt{\zeta^2 - M} \right).$$

As shown in Section 2.2.1, the system is stable if and only if M, ζ, ω are all positive numbers. This confirms, as well, the analysis of the immediately preceding section — i.e. bounded disturbances give rise to bounded errors.

Two additional important facts are evident from equation (9). First, the transient response is improved by increasing the magnitude of the poles, and this is simply achieved by increasing ω , assuming $\zeta < \sqrt{M}$. Second, assuming that d attains some steady state value,

$$\lim_{s \rightarrow 0} s d(s) = \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix}$$

we have

$$\lim_{t \rightarrow \infty} e_1(t) = \frac{1}{\omega^2} (\delta_2 + 2\omega\zeta\delta_1).$$

Thus, the ultimate bound on the steady state error may be decreased, as well, by increasing the magnitude of ω for any fixed value of ζ and M . Both of these adjustments may be made with no additional information concerning the nature of the disturbance, or the magnitude of the system parameter, M . If some further knowledge is available, then a choice of ζ slightly less than M will produce “nice” slightly underdamped transient.

The benefits of “high gain” feedback are well understood in the linear systems literature. There are some well known liabilities as well. The presence of higher order dynamics than modeled in (2) will virtually guarantee that gain increases past a certain magnitude result in deleterious performance, and, ultimately, destabilize the closed loop system. Moreover, given the inevitable power constraints of the real world, there will be some upper limit on the magnitude of $\omega^2, 2\zeta\omega$ which may be implemented, and, hence, on the rate of convergence toward and error bound around the desired goal which may be attained. Nevertheless, the utility of suitably

tuned "PD" controllers is underscored by their ubiquitous presence in world of real actuators and sensors. It seems well worth the effort to gain similar understanding of the more general class of mechanical systems.

In order to suggest how these insights might be extended, we examine the bounding ratios, ρ, β , introduced in the previous section. For ease of the present exposition, assume that a lower bound for M is available and that we have chosen $\zeta < M$, accordingly (underdamped response). This guarantees that the choice of $\gamma_0 = \gamma'_0 = 1$ will still satisfy conditions (6), (7) with the result $P = Q$. In this case, the results of the previous section indicate that bounds on transient response rate and steady state error are given by

$$\rho = \frac{\omega \zeta \nu_P}{M \mu_P}, \quad \beta = \delta_0 \left(\frac{\mu_P}{\nu_P} \right)^2.$$

The ratio of minimum to maximum eigenvalues of P may be written

$$\eta(u) \triangleq \frac{\mu_P}{\nu_P} = \frac{1 + \sqrt{1-u}}{1 - \sqrt{1-u}}$$

where

$$u \triangleq \frac{M\gamma_0 - \zeta^2}{\gamma_0} \cdot \frac{4\omega^2}{(M + \omega^2)^2},$$

is the ratio of the determinant of P to the squared trace of P . It is clear that η is monotone decreasing in u . Since

$$\frac{du}{d\omega} = \frac{M\gamma_0 - \zeta^2}{\gamma_0} \cdot \frac{2\omega}{(M + \omega^2)^3} \cdot (M - \omega^2),$$

under the assumption that (6) holds, increasing the magnitude of ω up until the magnitude \sqrt{M} results in increasing values of u , and, hence, decreasing values of η . According to our analysis, the exponential time constant, ρ , is inversely proportional to η , while the bound on the steady state error, β , is proportional to η^2 .

The foregoing arguments show that a slightly more complicated analysis of the performance bounds due to lyapunov analysis yields insights analogous to those obtained from frequency domain techniques, albeit under considerably more constrained assumptions regarding relative magnitudes. In the sequel we will find that additional information concerning the structure of d will afford even better correspondence between the two techniques.

Of course, the exact characteristics of a closed loop system resulting from linear state feedback, (3), are completely determined by the *poles* — eigenvalues of the resulting system matrix, A_1 in (4) — and, hence, by the roots of a second order polynomial whose coefficients are exactly specified by the second row of that array [15,3,16]. Thus, performance concerns may be precisely addressed only if full information regarding the parameters of the system is available. Namely, if M is known, the feedback strategy

$$u = M(k_1\theta + k_2\dot{\theta}) \tag{10}$$

results in a closed loop system with vector field

$$A_2x \triangleq \begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix},$$

whose eigenvalues,

$$-\frac{1}{2}(k_2 \pm \sqrt{k_2^2 - 4k_1}).$$

may be exactly assigned.

2.3 The Servo Problem: Tracking

The most general task domain of classical control theory is the servo problem. Task specification is by means of a *reference trajectory* — a time varying signal, $x_d(t)$, which it is desired that the system output, $x(t)$, should repeat, or *track* as closely as possible. Typically, the control law takes the form

$$u_{serv} = u_{fb} + u_{pc}$$

where u_{fb} is some stabilizing feedback algorithm, e.g. (3) or (10), and

$$u_{pc} \triangleq \Gamma[x_d]$$

is some appropriately conditioned or *pre-compensated* form of the reference signal. A thorough treatment of these ideas can be found in standard classical texts [15,3,16]. This section introduces the narrower point of view which will predominate throughout the sequel: a tracking problem is a “disturbed set point problem” — a problem requiring motion toward the “new set point” at a particular instant, $x_d(t)$, in the presence of “disturbances” caused by nonzero derivatives, $\dot{x}_d(t)$.

2.3.1 The Forced Response of Linear Systems

Linear time invariant systems constitute an important exception to the general rule that the output of a forced dynamical system has no closed form expression involving elementary functions. Consider a general example of such a system,

$$\dot{x} = Ax + bu. \quad (11)$$

Let the initial position and velocity be specified by $x = [\theta_d, \dot{\theta}_d]^T$, and define

$$\exp\{tA\} \triangleq \sum_{k=0}^{\infty} (tA)^k / k!$$

Recall that this sum converges for all matrices, A , and real values, t , and that its (operator) norm decays with increasing values of t if and only if the eigenvalues of A have negative real part. It may be verified by direct computation that

$$x(t) = \exp\{tA\}[x_0 + \int_0^t \exp\{-\tau A\}bu'(\tau)d\tau], \quad (12)$$

satisfies the closed loop dynamical equations, (4). Of course, the availability of this exact “input/output description” is the basis for the powerful frequency domain methods of classical control theory [15,3,16].

To underscore this point, it is worth obtaining an explicit representation of resulting errors, in effect the time domain version of (9), by recourse to direct integration. Thus, given a specified

trajectory, $x_d(t) = [\theta_d(t), \dot{\theta}_d(t)]^T$, it will be possible to provide a complete account of the efficacy of any particular control input by examining the resulting error,⁵

$$e(t) \triangleq x(t) - x_d(t).$$

Assume a control of the form $u \triangleq u_{fb} + u_{pc}$, as above. The closed loop plant equations are now of the form (11),

$$\dot{x} = A_1 x + b u_{pc} \quad (13)$$

with $A = A_1$, and $u' = u_{pc} = \Gamma[x_d]$. Integrating the second term in $x(t)$ of (12) by parts twice affords the expression⁶

$$e(t) = x_d(t) - A_1^{-2} [\exp\{tA_1\}a_0 - A_1 b u_{pc}(t) - b \dot{u}_{pc}(t) - \int_0^t \exp\{(t-\tau)A_1\} b \ddot{u}_{pc}(\tau) d\tau] \quad (14)$$

where we define for convenience the controlled "initial acceleration",

$$a_0 \triangleq A_1^2 x(0) + A_1 b u_{pc}(0) + b \dot{u}_{pc}(0).$$

2.3.2 Robust Tracking via High Gain Feedback

In Section 2.2.2 it was seen that the effect upon the steady state response of bounded noise perturbations could be made arbitrarily small through the use of high gain feedback. Subsequently, in Section 2.2.3, it was shown that high gain feedback increases the rate at which the system tends toward its steady state. Here, we will combine these insights, and attempt to track $x_d(t)$ as if it were a moving set point through the continued exploitation of high gain feedback. Intuitively, we hope that the resulting tendency to steady state will be "faster" than the rate of change of the set point. As usual, the advantage attending the reliance upon intrinsic stability properties of the system will be the reduced need for a priori information.

Let the feedback control be chosen using the "high gain" philosophy as in equation (3) If the pre-compensating function is simply set to be proportional to the desired position,

$$\Gamma_p[x_d] = \omega^2 \theta_d,$$

then, according to equation (14) of Section 2.3.1,

$$e(t) = \begin{bmatrix} 2\dot{\theta}_d \zeta / \omega \\ 0 \end{bmatrix} - A_1^{-2} \exp\{tA_1\} [a_0 + \int_0^t \exp\{-\tau A_1\} b \ddot{\theta}_d(\tau) d\tau],$$

and, not surprisingly, there are terms which contribute error in proportion to the desired velocity and "time averaged" desired acceleration.

⁵Note that this reduces to the earlier definition of error,

$$e = \begin{bmatrix} x_1 - \theta_d \\ x_2 \end{bmatrix}$$

when $\dot{\theta}_d \equiv 0$.

⁶Since the gains, $\omega^2, 2\zeta\omega$, have been chosen strictly greater than zero to stabilize the desired equilibrium position, there is no question as to the invertibility of A_1

Let us now attempt the analogous investigation by appeal to a Lyapunov argument after making the further assumption that there is some (in general unknown) bound on the desired speed,

$$|\dot{\theta}_d| \leq \delta_0.$$

Defining the modified error coordinates,

$$\tilde{e} \triangleq \begin{bmatrix} \theta_d - \theta \\ \dot{\theta} \end{bmatrix},$$

the closed loop system takes the form

$$\dot{\tilde{e}} = A_1 \tilde{e} - d$$

where the disturbance, $d \triangleq \begin{bmatrix} \dot{\theta}_d \\ 0 \end{bmatrix}$ is due to the non-zero reference derivative. The quadratic lyapunov function,

$$v(\tilde{e}) \triangleq \frac{1}{2} \tilde{e}^T P \tilde{e},$$

may be used just as in Section 2.2.2 to show that the error is bounded, and both the transient response as well as the steady state bound may be improved by high gain feedback.

2.3.3 Inverse Dynamics

If it is desired that the true response of the closed loop system track the reference signal exactly, then the most obvious recourse is to “inverse dynamics”. Suppose M is exactly known, and u_{fb} has been chosen in the form (10) to achieve some specified set of poles. Assume that the output of the stabilized system (4) is exactly the desired signal, $\theta(t) \equiv \theta_d(t)$: it follows that all derivatives are equivalent as well, $\dot{\theta} \equiv \dot{\theta}_d$; $\ddot{\theta} \equiv \ddot{\theta}_d$; and, hence, solving for u in the second line of (13), that

$$u_{id}(t) \equiv M([\ddot{\theta}_d + k^T \begin{bmatrix} \theta_d \\ \dot{\theta}_d \end{bmatrix}])$$

In terms of the framework above, this corresponds to a choice for Γ of the form

$$\Gamma_{id}[x_d] \triangleq \frac{1}{b^T b} b^T [\dot{x}_d - A_2 x_d].$$

Using frequency domain analysis, it is easy to see that this control input is a copy of the reference signal fed through dynamics whose transfer function is the reciprocal of the feedback stabilized plant. From the point of view of time domain analysis, the resulting closed loop system expressed in error coordinates takes the form

$$\begin{aligned} \dot{e} &= \dot{x}_d - A_2 x - \frac{1}{b^T b} b b^T [\dot{x}_d - A_2 x_d] \\ &= A_2 e + [I - \frac{1}{b^T b} b b^T] [\dot{x}_d - A_2 x_d] \\ &= A_2 e, \end{aligned}$$

since $[I - \frac{1}{b^T b} b b^T]$ is a projection onto the subspace of \mathbb{R}^2 orthogonal to b , while $\dot{x}_d - A_2 x_d$ lies entirely in the image of b . Thus, appealing to a Lyapunov analysis once more, if $v \triangleq \frac{1}{2} e_2^2 + \frac{k_1}{2} e_1^2$

then $\dot{v} = -k_2 e_2^2$, which implies that the error is non-increasing and from which it can be deduced as well that (see Theorem 3) e tends toward zero asymptotically. For purposes of comparison it is worth displaying the actual form of the closed loop error, (which may be computed from the exact input/output map (14))

$$\begin{aligned}
 e(t) &= -A_1^{-2} [\exp\{tA_1\}a_0 - \ddot{x}_d + \int_0^t \exp\{(t-\tau)A_1\}b\ddot{u}_{id}(\tau)d\tau] \\
 &\quad (\text{since } \ddot{x}_d = A_2^2 x_d + A_2 b u_{id} + b \dot{u}_{id}) \\
 &= -A_1^{-2} \exp\{tA_1\} [a_0 - \ddot{x}_d(0) + \int_0^t \frac{d}{d\tau} \exp\{-\tau A_1\} \ddot{x}_d(\tau) d\tau + \int_0^t \exp\{-\tau A_1\} b \ddot{u}_{id}(\tau) d\tau] \\
 &= -A_1^{-2} \exp\{tA_1\} [a_0 - \ddot{x}_d(0) + \int_0^t \exp\{-\tau A_1\} \frac{d^2}{d\tau^2} (A_1 x_d + b u_{id} - \dot{x}_d) d\tau] \\
 &= -A_1^{-2} \exp\{tA_1\} [a_0 - \ddot{x}_d(0)].
 \end{aligned}$$

The only source of error is due to initial conditions, x_0, a_0 , which may be cancelled exactly by appropriate choice of u_0, \dot{u}_0 , since $b, A_1 b$ are linearly independent (the system is *controllable*). If not cancelled exactly, the term must decay asymptotically under the assumption that k_1, k_2 are stabilizing gains. Tracking is perfect or at least asymptotically perfect for any arbitrary input.

This strategy resembles the first open loop control scheme introduced in Section 2.1.2 in that it assumes perfect information regarding the plant dynamics. Any uncertainty in the model, its parameters, or the presence of noise, will invalidate the result. Moreover, since it requires derivatives of the reference trajectory, \dot{x}_d , the scheme would be practicable only in cases where the entire reference trajectory is known in advance: differentiating unknown signals online generally results in unacceptably large noise amplitudes.

2.3.4 Adaptive Control

The final approach to the linear time invariant servo problem to be considered in this article might be said to marshall the power of schemes such as pole placement (Section 2.2.3) or inverse dynamics (Section 2.3.3), without requiring exact a priori knowledge of the system parameters. The field of adaptive control is relatively new, since the first convergence results for general linear time invariant systems were reported only in 1980 [17]. The method described here falls within the class of "model reference" schemes [18,19].

Suppose a desired trajectory, x_d , has been specified along with a precompensating feedforward law, $u_{pc} \triangleq \Gamma[x_d]$, which forces a known model

$$\dot{x}_m = A_m x_m + b_m u_{pc}$$

to track x_d acceptably. Suppose, moreover, that the true system to be controlled,

$$\dot{x} = Ax + bu,$$

satisfies the condition $\gamma b = b_m$ for some (unknown) positive scalar, $\gamma > 0$, and is known to admit a feedback law, $u_{fb} \triangleq [\alpha, \beta]x$ such that the closed loop system yields the model behavior, $A + b[\alpha, \beta] = A_m$, for some (unknown) set of gains, $\alpha, \beta \in \mathbb{R}$. The adaptive control law takes the form,

$$u_{ad} \triangleq \hat{k}^T(t) \begin{bmatrix} x \\ u_{pc} \end{bmatrix},$$

where $\hat{k}(t)^T = [\hat{\alpha}(t), \hat{\beta}(t), \hat{\gamma}(t)]$ denotes a set of gain estimates which will be continuously adjusted on the basis of observed performance. Let $k^T = [\alpha, \beta, \gamma]$ denote unknown “true” vector of gains. The closed loop system resulting from u_{ad} may be written in the form

$$\dot{x} = A_m x + b_m u_{pc} + [\hat{k}(t) - k]^T \begin{bmatrix} x \\ u_{pc} \end{bmatrix},$$

hence, defining the state error coordinates, $e \triangleq x_m - x$, and the “parameter error” coordinates, $\tilde{k} \triangleq k - \hat{k}$, yields the system

$$\dot{e} = A_m e - \frac{1}{\gamma} b_m \tilde{k}^T \begin{bmatrix} x \\ u_{pc} \end{bmatrix}.$$

Notice, further, that $\dot{\tilde{k}} = -\dot{\hat{k}}$, hence, adjustments in \hat{k} afford exact adjustments of the opposite sign in the parameter error vector. The question remains, then, concerning the choice of an “adaptive law”, $\dot{\hat{k}} = f(\hat{k}, e, u_{pc})$ that will make the complete error system converge.

At the very least, we may assume that A_m defines an asymptotically stable closed loop system, since the model is capable of tracking x_d in the first place. According to the theory of Lyapunov [5], it is therefore guaranteed that a positive definite symmetric matrix, P_m , exists such that $x^T P_m A_m x \leq 0$ with equality only for $x = 0$. Find such a P_m , and set the adaptive law as

$$\dot{\hat{k}} = - \begin{bmatrix} x \\ u_{pc} \end{bmatrix} e^T P_m b_m.$$

To show that the resulting closed loop error equations converge, consider the extended Lyapunov candidate $v(e, \tilde{k}) \triangleq \frac{1}{2} [e^T P_m e + \frac{1}{\gamma} \tilde{k}^T \tilde{k}]$. Since

$$\begin{aligned} \dot{v} &= e^T P_m A_m e - \frac{1}{\gamma} e^T P_m b_m \tilde{k}^T \begin{bmatrix} x \\ u_{pc} \end{bmatrix} + \frac{1}{\gamma} \tilde{k}^T \begin{bmatrix} x \\ u_{pc} \end{bmatrix} e^T P_m b_m \\ &= e^T P_m A_m e \leq 0, \end{aligned}$$

it is guaranteed that v , and therefore, e, \tilde{k} remain bounded for all time. A further technical argument based upon the assumption that u is bounded for all time may be used to finally show that e actually converges to zero as well [20].

Now apply this general method to the particular system representing the one degree of freedom robot, (2), using the model resulting from the pole placement feedback scheme of Section 2.2.3,

$$A_m \triangleq \begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix}; \quad b_m \triangleq \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

In this particular context the required assumptions outlined above are easily seen to hold. The “true” gain settings which make the closed loop system behave like the model are given as $\alpha = -Mk_1$, $\beta = -Mk_2$, $\gamma = M$, and the assumption that $M > 0$ is unexceptionable. Note that

$$P_m \triangleq \begin{bmatrix} k_1 & k_2/2 \\ k_2/2 & 1 \end{bmatrix}$$

satisfies the conditions listed above for the Lyapunov matrix — it is positive definite (as long as $4k_1 > k_2^2$); and its product with A_m defines a negative definite symmetric matrix. Thus, the full adaptive controller takes the form

$$u_{ad} \triangleq \hat{\alpha}(t)\theta + \hat{\beta}(t)\dot{\theta} + \hat{\gamma}(t)u_{pc}$$

where

$$\frac{d}{dt} \begin{bmatrix} \hat{\alpha}(t) \\ \hat{\beta}(t) \\ \hat{\gamma}(t) \end{bmatrix} = \begin{bmatrix} \hat{\alpha}(t) \\ \hat{\beta}(t) \\ \hat{\gamma}(t) \end{bmatrix} [(\theta - \theta_m)k_2/2 + (\dot{\theta} - \dot{\theta}_m)].$$

3 General Robot Arm Dynamics

This section introduces an important case of a mechanical nonlinear system — multi-jointed open kinematic chains. In Section 3.1 a brief but fairly general treatment of kinematics affords quick derivation of the general rigid body model of robot arm dynamics. In reality, this model is a simplification of empirically observed phenomena. Depending upon what class of robot manipulator one studies, additional nonlinearities and dynamics cannot be ignored: Section 3.2 will sketch briefly several examples of such phenomena.

3.1 Rigid Body Model: Lagrangian Formulation of Newton's Laws

Contemporary robots are built to be rigid, and models of their idealized behavior are based upon the geometry of rigid transformations. After reviewing some elementary facts leading to a useful algebraic formalism for manipulating objects which obey this “extrinsic” geometry in Section 3.1.1, we shall investigate robot kinematics — the “intrinsic” geometry of robots — in Section 3.1.2, and use both geometric domains to understand the dynamical properties of robot motion in Section 3.1.3.

3.1.1 Rigid Transformations and Frames of Reference

For purposes of this article, the physical world is an *affine space*, \mathbf{A}^3 : each element is a point described by three real numbers, however there is no pre-defined origin [21]. By taking differences between elements of the affine space, $a, b \in \mathbf{A}^3$, we obtain elements of *Euclidean Vector Space*, $\overrightarrow{ab} \in \mathbf{E}^3$, with vector addition, scalar multiplication, inner product and norm [21,22]. Define a *rigid transformation* to be a continuous transformation of affine 3-space which preserves distance:

$$\mathcal{T} \triangleq \left\{ \tau \in C^0[\mathbf{A}^3, \mathbf{A}^3] : \|\overrightarrow{\tau(a)\tau(b)}\| = \|\overrightarrow{ab}\| \text{ for all } a, b \in \mathbf{A}^3 \right\}$$

Examples of rigid transformations include the vector translations,

$$\mathcal{V} \triangleq \left\{ \tau_v \in \mathcal{T} : \text{for some } v \in \mathbf{E}^3, \tau_v(a) = b \implies \overrightarrow{ab} = v, \text{ for all } a, b \in \mathbf{A}^3 \right\},$$

and rotations around fixed points

$$\begin{aligned} \mathcal{R} \triangleq \{ & \tau_{(R,o)} \in \mathcal{T} : \text{for some } R \in SO(3), o \in \mathbf{A}^3, \tau_{(R,o)}(o) = o, \\ & \text{and } \overrightarrow{o\tau_{(R,o)}(b)} = R\overrightarrow{ab}, \text{ for all } b \in \mathbf{A}^3 \}, \end{aligned}$$

where $SO(3)$ is the set of orthogonal linear operators on \mathbf{E}^3 with positive determinant. In fact, it can be shown [22,23] that these examples constitute the entirety of \mathcal{R} : that is, for any arbitrarily chosen “origin”, $o \in \mathbf{A}^3$, every rigid transformation of \mathbf{A}^3 may be uniquely expressed as the composition of a translation with a rotation around o : $\tau = \tau_v \circ \tau_{(R,o)}$. Thus, once a fixed point or “origin”, o , has been chosen, the set of rigid transformations may be put into one-to-one correspondence with the set of translations and rotations: $\mathcal{T} \approx \mathcal{W} \triangleq \mathbb{R}^3 \times SO(3)$. Since $\tau_v^{-1} = \tau_{-v}$ and $\tau_R^{-1} = \tau_{R^{-1}} = \tau_{R^T}$ it follows that $\tau^{-1} = \tau_{R^T} \circ \tau_{-v}$ is always defined, and hence \mathcal{T} is a group.

The position and orientation of any rigid body may be precisely described by fixing four points of \mathbf{A}^3 called a *frame of reference*

$$\mathcal{F} \triangleq \{a, b, c, o\}$$

where

$$\mathcal{B} = \{x, y, z\} \triangleq \{\overrightarrow{oa}, \overrightarrow{ob}, \overrightarrow{oc}\}$$

is a “right handed”⁷ orthonormal basis of \mathbf{E}^3 . Note that the image of a frame under a rigid transformation, $\tau(\mathcal{F}_1) = \{\tau(a_1), \tau(b_1), \tau(c_1), \tau(o_1)\}$ is also a frame since, e.g. $\langle \overrightarrow{\tau(o)\tau(a)} | \overrightarrow{\tau(o)\tau(b)} \rangle = \langle \overrightarrow{oa} | \overrightarrow{ob} \rangle = 0$ in consequence of the parallelogram rule relating norms and inner products. On the other hand, if $\mathcal{F}_1, \mathcal{F}_2$ are two frames then there exists an orthogonal transformation with positive determinant, $R \in SO(3)$ such that $x_2 = Rx_1, y_2 = Ry_1, z_2 = Rz_1$ since both sets of vectors define a right hand orthonormal basis. Thus, defining $\tau_{12} \triangleq \tau_{\overrightarrow{o_1o_2}} \circ \tau_{(R, o_1)}$, we have $\tau_{12}(o_1) = \tau_{\overrightarrow{o_1o_2}}(o_1) = o_2$, and, for instance,

$$\begin{aligned} \overrightarrow{a_2\tau_{12}(a_1)} &= \overrightarrow{o_2\tau_{12}(a_1)} - \overrightarrow{o_2a_2} \\ &= \overrightarrow{\tau_{12}(o_1)\tau_{12}(a_1)} - x_2 \\ &= \overrightarrow{o_1\tau_{R, o_1}(a_1)} - x_2 \\ &= 0 \end{aligned}$$

so that $\mathcal{F}_2 = \tau_{12}(\mathcal{F}_1)$. Thus, an exact description of the change in position and orientation of a given rigid body is provided equivalently by fixing a frame in the body and specifying either a rigid transformation or a second frame. Alternatively, given any two rigid bodies, an exact description of their relative position and orientation is provided by fixing a frame in each body. Assuming the existence of some “base frame”, \mathcal{F}_0 , we now have a model of any other rigid position and orientation in terms of $\mathcal{W} \triangleq \mathbf{R}^3 \times SO(3)$ which we will call “workspace”.

It has become a tradition in robotics to use *homogeneous coordinates* in the representation of objects in \mathbf{A}^3 . Since rigid transformations (which, of course, are affine rather than linear) admit a matrix representation in these coordinates, the resulting simplicity in notation seems worth the slight conceptual complication.⁸ Thus, the *matrix representation of a point, $p \in \mathbf{A}^3$ with respect to the any frame, $\mathcal{F}_j = \{o_j, a_j, b_j, c_j\}$* , is denoted

$${}_j p = \begin{bmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \end{bmatrix}$$

and is understood to represent the geometric relation

$$\pi_4 \overrightarrow{o_j p} = \pi_1 x_j + \pi_2 y_j + \pi_3 z_j.$$

Note that two arrays in \mathbf{R}^4 represent the same point with respect to the same frame if and only if they are scalar multiples. The attending conceptual complications arise because it is useful,

⁷That is, the direction of z on the line orthogonal to the xy plane of \mathbf{E}^3 obeys a right hand rule with respect to rotations on that plane.

⁸Homogeneous coordinates result in a more complicated representation from the numerical point of view as well.

at the same time, to employ matrix representations of vectors in \mathbf{E}^3 . Confusion between the previous arrays and these may be avoided by treating the latter as constituting "ideal points" of \mathbf{A}^3 which, together, comprise projective space, \mathbf{P}^3 [22]. Thus, if $v \in \mathbf{E}^3$, then

$${}^j v = \begin{bmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \\ 0 \end{bmatrix}$$

represents the geometric relation

$$v = \nu_1 x_j + \nu_2 y_j + \nu_3 z_j.$$

It follows that for any $q, p \in \mathbf{A}^3$, if ${}^j q = [\xi_1, \xi_2, \xi_3, \xi_4]^T$, ${}^j p = [\pi_1, \pi_2, \pi_3, \pi_4]^T$ then ${}^j \overrightarrow{pq} = \frac{1}{\xi_4} {}^j q - \frac{1}{\pi_4} {}^j p$. Finally, define the *matrix representation of a frame*, \mathcal{F}_i with respect to \mathcal{F}_j to be the 4×4 array whose columns are

$${}^j F_i \triangleq [{}^j x_i, {}^j y_i, {}^j z_i, {}^j o_i \frac{1}{\omega_4}]$$

where ω_4 is the fourth component of ${}^j o_i$. According to the discussion of the previous paragraph, this definition is interchangeable with the following: the *matrix representation of a rigid transformation*, $\tau \in \mathcal{W}$ with respect to frame \mathcal{F}_j is given by ${}^j F_i$ where $\mathcal{F}_i = \tau(\mathcal{F}_j)$.

3.1.2 Kinematics

Although we are most interested in the actions of a robot in workspace, \mathcal{W} , commands are effected through a set of joints which constrain the relative motion of a sequence of rigid links — a *kinematic chain*. Kinematics, the study of spatial relationships between such a configuration of mechanically interconnected and constrained rigid bodies, is a very old discipline, and it is not possible to provide more than a superficial account of its role in robot control. A much more thorough treatment of these considerations is provided in the general robotics literature [24,25].

The position of a kinematic chain possessed of $n - k$ prismatic and k revolute joints may be described as a point, q , in *joint space*, the cross product of \mathbb{R}^{n-k} with a k dimensional torus. There are two circumstances under which J may be accurately modelled as a subset of \mathbb{R}^n . If full revolution is mechanically possible, but joint sensors are available which transduce angular displacement with respect to some absolute position - e.g., from which a revolution and a half displacement is read as 540 degrees rather than 180 degrees - then this is clearly the case. If each revolute joint is mechanically constrained to prevent a full 360 degree revolution, then any reasonable set of position sensors return a bounded linear measurement of joint displacement. We will assume that one of these situations prevails, thereby assuring that $J \subset \mathbb{R}^n$.

A *joint transformation* is a map from \mathbb{R} into \mathcal{W} which relates a coordinate system fixed in link $i - 1$ to one fixed in link i through the action of the i^{th} joint. According to the standard conventions, the joint transformation depends upon three parameters which describe the link and one variable — the i^{th} coordinate of J . Since the link is rigid, these frames are related by a rigid transformation whose matrix representation may be written in the form

$${}^{i-1} \mathcal{F}_i = \begin{bmatrix} \cos \theta_i & -\cos \alpha_i \sin \theta_i & -\sin \alpha_i \sin \theta_i & a_i \cos \theta_i \\ \sin \theta_i & \cos \alpha_i \cos \theta_i & \sin \alpha_i \cos \theta_i & a_i \sin \theta_i \\ 0 & \sin \alpha_i & \cos \alpha_i & \delta_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where either θ_i or δ_i is the joint variable depending upon whether the joint is revolute or prismatic, respectively, and the other kinematic parameters are defined in the link body, e.g. as in [24].

More generally, a *kinematic transformation* is a map

$$g : J \rightarrow \mathcal{W}$$

which is the group product of n joint transformations, representing the rigid transformation required to align the “base frame” with the “end effector frame”:

$$g(q) = {}^0F_1(q_1) {}^1F_2(q_2) \dots {}^{n-1}F_n(q_n).$$

3.1.3 Dynamics

The rigid body model of robot arm dynamics may be most quickly derived by appeal to the lagrangian formulation of Newton's Equations. If a scalar function, termed a *lagrangian*, $\lambda = \kappa - v$, is defined as the difference between total kinetic energy, κ , and total potential energy, v , in a system, then the equations of motion obtain from

$$\frac{d}{dt} D_{\dot{q}} \lambda - D_q \lambda = \tau^T,$$

where τ is a vector of external torques and forces [21,26].

First consider the kinetic energy contributed by a small volume of mass δm_i at position p in link \mathcal{L}_i ,

$$\delta \kappa_i = \frac{1}{2} {}^0\dot{p}_i^T {}^0\dot{p}_i \delta m_i$$

where ${}^0p_i = {}^0F_i {}^i p$ is the matrix representation of the position p in the base frame of reference, 0F_i is the matrix representation of the frame of reference of link \mathcal{L}_i in the base frame, and ${}^i p$ is the matrix representation of the point in the link frame of reference, and, hence,⁹

$$\dot{p}_i = \dot{F}_i {}^i p,$$

since the position in the body is independent of the generalized coordinates. The total kinetic energy contributed by this link may now be written

$$\begin{aligned} \kappa_i &= \int_{\mathcal{L}_i} \frac{1}{2} \left[\dot{F}_i {}^i p \right]^T \dot{F}_i {}^i p dm_i \\ &= \int_{\mathcal{L}_i} \frac{1}{2} \text{trace} \{ \dot{F}_i {}^i p \left[\dot{F}_i {}^i p \right]^T \} dm_i, \\ &= \frac{1}{2} \text{trace} \{ \dot{F}_i \int_{\mathcal{L}_i} p^i p^T dm_i \left[\dot{F}_i \right]^T \} \\ &= \frac{1}{2} \text{trace} \{ \dot{F}_i \bar{P}_i \dot{F}_i^T \}, \end{aligned}$$

(since the frame matrix is constant over the integration), where \bar{P}_i is a symmetric matrix of *dynamical parameters* for the link. Explicitly, if the link has mass, $\bar{\mu}_i$, center of gravity (in the local link coordinate system) \bar{p}_i , and inertia matrix, \bar{J}_i , then

$$\bar{P}_i \triangleq \begin{bmatrix} \bar{J}_i & \bar{\mu}_i \bar{p}_i \\ \bar{\mu}_i \bar{p}_i^T & \bar{\mu}_i \end{bmatrix}.$$

⁹We will omit the prior superscript, 0, when it is clear the the coordinate system of reference is the base

Passing to the *stack representation* (refer to Appendix ??)

$$\begin{aligned}
 2\kappa_i &= \text{trace} \{ \dot{F}_i \bar{P}_i \dot{F}_i^T \} \\
 &= \left[(\dot{F}_i \bar{P}_i)^S \right]^T \dot{F}_i^S \\
 &= \left[(\bar{P}_i \otimes I)^T \dot{F}_i^S \right]^T \dot{F}_i^S \\
 &= \left[\dot{F}_i^S \right]^T \tilde{P}_i \dot{F}_i^S \\
 &= \left[(D_q F_i^S) \dot{q} \right]^T \tilde{P}_i (D_q F_i^S) \dot{q} \\
 &= \dot{q}^T M_i \dot{q},
 \end{aligned}$$

where we have implicitly defined

$$M_i(q) \triangleq \left[D_q F_i^S \right]^T \tilde{P}_i D_q F_i^S; \quad \tilde{P}_i \triangleq \bar{P}_i^T \otimes I.$$

It follows that the total kinetic energy of the entire chain is given as

$$\kappa = \frac{1}{2} \dot{q}^T M(q) \dot{q}; \quad M(q) \triangleq \sum_{i=1}^n M_i(q).$$

The potential energy contributed by δm_i in \mathcal{L}_i is

$$\delta v_i = z_0^T F_i^i p g \delta m_i$$

where g is the acceleration of gravity, hence the potential energy contributed by the entire link is

$$v_i = z_0^T F_i \int_{\lambda_i}^i p g d m_i = z_0^T F_i \bar{p}_i g,$$

and $v = \sum_{i=1}^n z_0^T F_i \bar{p}_i g$.¹⁰

To proceed with the computation, note that $D_{\dot{q}} \lambda = D_{\dot{q}} \kappa = \dot{q}^T M(q)$, hence,

$$\frac{d}{dt} D_{\dot{q}} \lambda = \dot{q}^T M(q) + \dot{q}^T \dot{M}(q).$$

Moreover,

$$\begin{aligned}
 D_q \kappa &= \frac{1}{2} \dot{q}^T D_q [M(q) \dot{q}] \\
 &= \frac{1}{2} \dot{q}^T [\dot{q}^T \otimes I] D_q M^S,
 \end{aligned}$$

hence, if all terms from Lagrange's equation involving the generalized velocity are collected, we may express them in the form $\dot{q}^T B^T$, where

$$B(q, \dot{q})^T \triangleq \dot{M}(q) - \frac{1}{2} [\dot{q}^T \otimes I] D_q M^S.$$

Finally, by defining $k(q) \triangleq [D_q v]^T$, Lagrange's equation may be written in the form,

$$M(q) \ddot{q} + B(q, \dot{q}) \dot{q} + k(q) = \tau. \quad (15)$$

¹⁰Assume that z_0 "points up" in a direction opposing the gravitational field.

M , called the “inertia” matrix, may be shown to be positive definite over the entire workspace as well as bounded from above since it contains only polynomials involving transcendental functions of q . B contains terms arising from “coriolis” and “centripetal” forces, hence is linear in \dot{q} (these forces are quadratic in the generalized velocities), and bounded in q , since it involves only polynomials of transcendental functions in the generalized position. Finally, k arises from gravitational forces, is bounded, and may be observed to have much simpler structure (still polynomial in transcendental terms involving q) than the other expressions. An important study of the form of these terms was conducted by Bejczy [27].

To bring this into the state space form discussed in Section 2, let $x \triangleq (q, \dot{q})^T$, $u \triangleq \tau$, to get

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -M^{-1}(x_1)[B(x_1, x_2)x_2 + k(x_1) - u] \end{aligned} \quad (16)$$

with output map provided by the kinematics,

$$w = g(x).$$

3.2 Omissions in the Rigid Body Model

In fact, most commercially available robots deviate from the model derived above quite dramatically. Flexibility in the links, slippage in transmissions, and backlash in gear trains introduce stiction, hysteresis, and other nonlinear effects. Harmonic drives and compliance at the joints, introduce extra dynamics [28]. De-magnetization and potential damage to the windings place limits upon the maximum permissible armature current and, therefore, output torque, of a dc servo. Moreover, while it is traditional in the control community to model electric motors as if they were first order lags [29], it is not impossible to find commercial robot arms employing dc servos whose mechanical and electrical time constants of similar magnitude, and which, in consequence, have second order dynamics, not uncommonly *oscillatory* [30]. Thus, not only may the model introduced in (24) have missing functional terms in practice, but its dimensionality, $2n$, may too low by at least again as much as the number of actuators, n .

Unfortunately, it does not seem likely that a better general model will be available in the near future. There is no generally accepted understanding of which dynamical effects are significant and which may be ignored beyond the rigid body model (24) (which, itself, is not universally acknowledged to be of greatest importance [28]). In part this is due to the great diversity of kinematic, actuator, and sensory arrangements which may be found on the commercial market. In part, it is a reflection of the relative novelty of the field. Similarly, there is insufficient understanding of the disturbances resulting from digital implementation of control algorithms - quantization and roundoff errors - to admit of any reliable model for these effects.¹¹

In this section, we will introduce models for some of the important dynamical and nonlinear disturbances not captured in (24). The relative importance of any of these discrepancies can only be determined by empirical investigation in the context of a specific mechanical apparatus. Nevertheless, for purposes of exposition, it is evidently necessary to choose some dynamical model of the system under examination. If there is any sign of convergence in the design of commercially available arms it would seem to be toward the class of revolute direct drive arms

¹¹Even in the control community itself, researchers are only beginning to come to terms with quantization problems [31].

[32]. Thus, after considering modifications in the model resulting from local nonlinearities at the joints in Section 3.2.1, we will somewhat artificially arrive at a modified model resulting from additional local linear dynamics in Section 3.2.2 on the theory that direct drive arms remove much of the hysteresis and nonlinear damping phenomena characteristic of transmissions, while flexibility introduced by joint and link compliance will always remain. The modified model for this class of arms is summarized by equation (18).

It is worth remarking here that the utility of much of the theoretical work to be presented in subsequent sections will require empirical verification. Clearly, in the absence of trustworthy models, proofs (and even simulations) alone are not terribly convincing.

3.2.1 Local Nonlinearities

As well as (at least) doubling the dimensionality of the underlying dynamics, the presence of a mechanical transmission introduces a variety of memoryless nonlinearities — backlash, hysteresis, etc. — which depend very much on the nature of the mechanism. Returning to the direct drive arm, probably the two most significant sources of nonlinearities distinct from those due the rigid body equations (23) are friction and saturation.

A solid object in contact with any surface is subject to a variety of frictional forces which may be observed, in general, to vary with its velocity relative to that surface. The simplest of these is *viscous damping* a force exerted in the opposite direction of motion in direct proportion, β_v , to the velocity magnitude. Further, it may be observed that at low speeds, the magnitude of these opposing forces ceases to diminish beyond a certain level, hence, a constant term, β_c , called *coulomb* friction must be added to the viscous term. Finally, the force, β_s , required to bring a motionless object to some non-zero velocity typically exceeds that needed overcome the friction forces at non-zero velocity: this is termed *stiction*. Thus, an appropriate model for actually observed frictional forces might be given as ¹²

$$\tau_{frict}(q, \dot{q}) = \beta_c + \beta_v \dot{q} + \beta_s \delta_0(\dot{q}).$$

The second source of additional nonlinearities unmodeled in the previous section is a consequence of the fact that all real devices can deliver only finite power. In this light, the admissible set of controls must be modified to include magnitude constraints, most easily modeled in the form of a saturation nonlinearity on the command input at the j^{th} joint,

$$s_i(v_j) \triangleq \begin{cases} \bar{s}_i; & v_j \geq \bar{s}_i \\ v_j; & v_j \in (\underline{s}_i, \bar{s}_i) \\ \underline{s}_i; & v_j \leq \underline{s}_i. \end{cases}$$

These forces act on each joint independent of the motion of the others (except, of course, through dynamical coupling of velocities), and are termed “local disturbances” for that reason. Let B_c, B_v, B_s , be diagonal matrices containing, respectively, the coulomb, viscous, and stiction coefficients for each joint, let $d(\dot{q}) = [\delta_0(\dot{q}_1), \dots, \delta_0(\dot{q}_n)]^T$ be the vector of delta functions in each generalized velocity. and let $s(v) \triangleq [s_1(v_1), \dots, s_n(v_n)]^T$ be the vector of saturation nonlinearities. Combining these within the vector of external disturbances, τ in equation (17), yields a new

¹²Recall that δ_0 denotes the delta function introduced in Section 2.1.2.

version of the state space dynamics of the form

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -[M(x_1) + M_m(x_1)]^{-1}[(B(x_1, x_2) + B_m + B_v)x_2 + B_s d(\dot{q}) + k(x_1) + B_c - V_m s(u)].\end{aligned}$$

3.2.2 Additional Dynamics

In the rigid body model developed above, the control input appears as a set of torques, τ , injected at each joint independently. In reality, all robotic actuators which deliver torque or force to a joint are themselves commanded by a reference voltage computed by the controller. For a significant class of actuators, the torque or force output is not a simple function of this reference voltage, but may itself involve a dynamical relationship. This may be seen most easily through a specific example.

A typical actuator for a revolute joint is the dc servo. A dc motor converts electrical to mechanical power through the exchange of energy in two sets of *windings* via electro magnetic forces. [29]. For a typical motor, the torque delivered to the output shaft is roughly proportional to the current in the “armature windings” — $\tau_g = K_\tau I_a$ — and this is exactly balanced by the d’Alembert torque due the angular acceleration of motor inertia, $\tau_m = J\ddot{\theta}$, as well as the external load torque, τ_l , placed upon the motor [29],

$$\tau_m + \tau_l = \tau_g.$$

On the other hand, the current in the armature winding results from the application of some command voltage, v_a , through the armature resistance and - inductance, R_a, L_a , opposed by the back generated voltage $v_b = K_b \dot{\theta}$,

$$L_a \frac{di_a}{dt} + R_a i_a + v_b = v_a.$$

There results the second order linear differential equation [29]

$$\begin{aligned}\frac{d\dot{\theta}}{dt} &= \kappa_m \dot{\theta} + \gamma_m i_a + \xi_m \tau_l \\ \frac{di_a}{dt} &= \kappa_e i_a + \gamma_e \dot{\theta} + \xi_e v_a\end{aligned}$$

If the “electric time constant”, κ_e , is a very large negative number in comparison to the magnitude of the “mechanical time constant”, κ_m , then the second equation is really algebraic,

$$R_a i_a + v_b = v_a,$$

and the relationship between command voltage and generated torque is given by

$$\mu \frac{d\dot{\theta}}{dt} = \beta \dot{\theta} + \phi v_a - \tau_l$$

$$\mu \triangleq J; \quad \beta \triangleq \frac{K_\tau K_b}{R_a}; \quad \phi \triangleq \frac{K_\tau}{R_a}.$$

This is typically the case for common dc servo motors [29], although important exceptions have been noted in the literature [30].

First suppose that we are interested in a direct drive arm, n such actuators are mounted directly at each joint being controlled. It follows that $\tau_j = \tau + j$ — the load torque seen by the j^{th} actuator, is exactly the j^{th} component of the rigid body external torque vector, τ , of equation (23). Let

$$M_m \triangleq \begin{bmatrix} \mu_1 & & 0 \\ & \ddots & \\ 0 & & \mu_n \end{bmatrix}; V_m \triangleq \begin{bmatrix} \phi_1 & & 0 \\ & \ddots & \\ 0 & & \phi_n \end{bmatrix}; B_m \triangleq \begin{bmatrix} \beta_1 & & 0 \\ & \ddots & \\ 0 & & \beta_n \end{bmatrix}$$

be the diagonal coefficient matrices corresponding to the n decoupled motors, and let $v \triangleq [v_{a1}, \dots, v_{an}]^T$ be the vector of command voltages at each joint. Then according to the reasoning of Section 3.1.3 we have

$$[M(q) + M_m]\ddot{q} + (B(q, \dot{q}) + B_m)\dot{q} + k(q) = V_m v, \quad (17)$$

producing a state space system with very similar characteristics to the original (24). It is worth noting that most of the theoretical results discussed in Sections 4 and 5 are still valid in this case.

Suppose, on the other hand, that an actuator is mechanically coupled to each joint j via some mechanical transmission — a chain, a harmonic drive, a gear train — as is the case with most commercially available robots. In this case, τ_j , the load torque seen by the motor is that delivered from the joint along the transmission. As a first order approximation, we will model the transmission as a passive intermediate inertial load, μ_t , lumped at the joint end of the transmission, on a shaft with torsional spring constant, κ_t , and torsional damping constant, β_t . Letting ρ_j denote the position of the rotor, and θ_j the position of the joint this assumption may be written as

$$\tau_t \triangleq \kappa_t(\rho_j - \theta_j) + \beta_t(\dot{\rho}_j - \dot{\theta}_j),$$

with $\tau_t \triangleq \mu_t \ddot{\theta}_j$ comprising the d'Alembert torque due to acceleration of the transmission inertia. The latter is balanced by the transmitted motor torque, τ_t , and the nonlinear coupling torques, τ_j , thus the complete torque equations are

$$\tau_g = \tau_m + \tau_j; \quad \tau_t = \tau_t + \tau_j.$$

Now define the transmission diagonal arrays, M_t, B_t , in terms of the inertial and viscous damping coefficients, respectively, as before. Define the motor shaft angle vector $r \triangleq [\rho_1, \dots, \rho_n]^T$. The torque balance equations take the vector form

$$-B_m \dot{r} + V_m v = M_m \ddot{r} + B_t(\dot{r} - \dot{q}) + K_t(r - q)$$

$$B_t(\dot{r} - \dot{q}) + K_t(r - q) = (M(q) + M_t)\ddot{q} + B(q, \dot{q})\dot{q} + k(q)$$

Writing this in state space notation yields a much more complex dynamical system. Letting

$$z = \begin{bmatrix} q \\ r \\ \dot{q} \\ \dot{r} \end{bmatrix} \triangleq \begin{bmatrix} z_{11} \\ z_{12} \\ z_{21} \\ z_{22} \end{bmatrix}$$

we have

$$\begin{aligned}
 \dot{z}_{11} &= z_{21} \\
 \dot{z}_{12} &= z_{22} \\
 \dot{z}_{21} &= -[M + M_t]^{-1}[(B + B_t)z_{21} + K_t z_{11} - B_t z_{22} - K_t z_{12} + k(z_{11})] \\
 \dot{z}_{22} &= -M_m^{-1}[(B_m + B_t)z_{22} + K_t z_{12} - B_t z_{21} - K_t z_{11} + V_m v].
 \end{aligned} \tag{18}$$

4 Feedback Control of a General Robot Arm

Section 2.2 presented an account of the behavior of classical feedback controllers in the context of set point regulation. Here, the attempt is made to generalize that account in two rather different directions. First, in Section 4.1, the breadth of task domain is considerably widened beyond set point regulation and we explore the possibility of formalizing a task encoding methodology based upon feedback control structures which generalize the error driven characteristics of the PD controller. Second, the underlying dynamics of the system to be controlled are specified by the n degree of freedom rigid body model, (24), of which the simple pendulum, (2), was a particularly easy example.

Section 4.1 presents three successively more generalized methods of task encoding beyond the set point error introduced in Section 2.2: specification in terms of the extrema of objective functions; in terms of the “fall lines” of gradient vector fields arising from objective functions; in terms of general first order dynamics. By interpreting an objective function as potential energy, its gradient is shown to determine a stabilizing feedback control structure for a general robot arm in Section 4.2. The implications of this result for achieving the tasks specified by the various encoding methods of Section 4.1 are discussed.

4.1 A Generalized Robot Task Encoding Methodology

By a “task encoding methodology” is meant any procedure through which an abstract goal is translated into robot control strategies resulting in its achievement. Here, we explore techniques which do not involve task specification via reference trajectory. In part, the exploration is motivated the difficulties involved in generalizing classical servo theory to the rigid body dynamics (24), as will be explored in Part 5. In part, it is motivated by the large set of task domains within which determination of an appropriate reference trajectory may involve unnecessary work, or even be effectively impossible. Typical tracking algorithms require the availability of velocity and acceleration reference information: in practice, differentiating noisy signals is impossible; such schemes are not applicable to tracking unknown time varying signals. Moreover, evidence mounts that the computational effort required to encode typical static tasks (such as moving in a cluttered space) in terms of exact reference trajectories may be prohibitive [33], [34]. Finally, since many tasks involve interacting forces and motions between the robot and its environment a task encoding methodology limited to the production of reference trajectories may be entirely inoperable. In summary, encoding a task in dynamical terms unrelated to those characterizing the robot or its environment may not be viable.

For the purposes of this article, an *objective function*, $\epsilon : \mathcal{W} \rightarrow \overline{R^+}$ is a non-negative scalar valued map on \mathcal{W} which has isolated critical points. Its associated *gradient vector field* is given by the system

$$\dot{w} = -D_w \epsilon^T(w),$$

and the resulting trajectory through any initial condition will be called a *fall line* of the system. It can be shown that fall lines are perpendicular to the level surfaces of ϵ [4]. The equilibrium states of the gradient system are exactly the critical points — i.e. the extrema — of ϵ ; and, since the linearized vector field is symmetric, an equilibrium state of the gradient system is either a source, a sink, or a saddle depending upon whether it is a local maximum, minimum, or saddle point of the objective function, ϵ . Thus, gradient systems display very simple dynamical behavior. In Section 4.2 it will be shown that certain gradient behavior may be duplicated, at

least asymptotically, by appropriately compensated Hamiltonian systems, and, hence, that such gradients are a particularly convenient feedback structure for robot control.

4.1.1 Task Encoding Via Objective Functions

Evidently, tasks within the domain of set point regulation — reaching and remaining at some desired end-point, w_d , — may be encoded as the objective of minimizing

$$\varepsilon \triangleq [w - w_d]^T [w - w_d]. \quad (19)$$

The desired end-point is the globally asymptotically stable unique equilibrium state of the associated gradient system.

Conversely, by designing a cost function with an isolated global maximum at some undesired cartesian position, a gradient system may be constructed whose fall lines, from any initial condition different from that point, define motion away from it. This specifies the task of avoiding the undesired position. In the event that there are several obstacles in the workspace, each of relatively small physical extent, cost functions attaining an isolated global maximum at the centroid of each obstacle may be summed and the resulting vector field will specify motions which avoid all of them. A plausible form for such cost functions is the familiar Newtonian Potential which varies in the inverse square of the distance from the obstacles. This and related methodologies have been suggested independently by workers in Japan, [35], the Soviet Union [36], and the United States [37]. In particular, Khatib has developed a rather general procedure for defining obstacle avoidance potentials for arbitrary rigid bodies [37]. It is not clear, however, that the computational complexity of this procedure makes it any more attractive than the algorithms developed for generating reference trajectories which solve piano mover type problems [34].

4.1.2 Task Encoding Via Gradient Dynamics

Task domains involving curve tracing may be specified by fall lines of gradient vector fields. Suppose it is desired to reach w_d via some parametrized curve,

$$c(\zeta) \triangleq \begin{bmatrix} \zeta \\ c_2(\zeta) \\ c_3(\zeta) \end{bmatrix}$$

where $w_d = c(0)$.¹³ Then the *shaping function*

$$\varepsilon \triangleq w_1^2 + \alpha_2 [w_2 - c_2(w_1)]^2 + \alpha_3 [w_3 - c_3(w_1)]^2$$

gives rise to a gradient system for which w_d is again the globally asymptotically stable unique equilibrium state, and whose fall lines “hug” the curve c , more or less sharply depending upon the magnitudes of α_1, α_2 .

Fundamental work by Hogan [38] advances persuasive arguments for encoding general manipulation tasks in the form of “impedances”. Impedances and admittances are formal relationships

¹³Assume, to avoid technical details, that we are only looking at the cartesian position components: errors in orientation may be measured similarly, although the justification requires more discussion.

between the force exerted on the world at some cartesian position and the motion variables - displacement, velocity, acceleration, etc. - at that position with respect to some reference point (or “virtual position” in Hogan’s terminology [38]). He argues that for purposes of modeling manipulation tasks, the kinematic and dynamical properties of a robot’s contacted environment must be understood as admittances - systems for which the relationship operates as a function describing a specified displacement for any input force. Arguing, further, that physical systems may only be coupled via port relationships which match admittances to impedances, and that robots can violate physics no more than any other objects with mass, he arrives at the conclusion that the most general model of manipulation is the specification of an impedance - a system which returns force as a function of motion. By construing motions relative to a virtual position as defining tangent vectors at that position, Hogan notes that an impedance may be defined in terms of a scalar valued function on the cross product of two copies of the tangent space at each virtual position whose gradient co-vector determines the relationship between motion and force. Thus, an impedance may be re-interpreted as the gradient co-vector field of an “objective function”, whose fall lines specify the desired dynamical response of the robot end-effector in response to infinitesimal motions imposed by the world. In this context, unlike the other gradient vector field task definitions, it is intended a priori that the dynamics be second order - i.e. define changes of velocity (force) rather than changes of position.

To conclude this brief discussion of task encoding via objective functions or their gradients it is worth noting that the gradient is a linear operator, hence combinations of tasks already encoded by means of objective functions are easily specified by appropriately weighted sums. For instance, if it is desired to shape an arm motion around a specified curve while simultaneously avoiding a set of known obstacles then the shaping function may be summed with the avoidance cost functions for each particular obstacle, and the gradient of the sum will preserve the local properties of each. In such cases, however, depending upon nature of the individual objective functions, their sum may define a gradient with unintended stable or partially stable critical points. Thus, globally, the fall lines of complex gradients may specify “stall” behavior at undesired equilibrium states.

4.1.3 More General Feedback Structures

It is certainly possible to imagine the desirability of “first order dynamical behavior” more complex than can be encoded via the gradient vector field of an objective function. For instance, it has been proposed within the neurobiology community to model phenomena such as animal gait in terms of a dynamical system possessed of a stable limit cycle [39]. It would seem equally attractive to use such models as task encoding feedback structures for robot activities which require repetitive motion. Unfortunately, while “useful” gradient vector field behavior may be embedded “asymptotically” in dissipative Hamiltonian systems, as will be demonstrated in the next section, it is not clear how to do the same for more general dynamics.

A second failing of the objective function methodology is its intrinsic time invariant character. Even if $\epsilon(w, t)$ defines a good objective for each t , it is not clear how to proceed, except in the quadratic case. For example, let $w_d(t)$ describe the trajectory of a fly we’d like the robot to swat as encoded via the objective

$$\epsilon(w, t) \triangleq [w - w_d]^T [w - w_d].$$

Then the “gradient system”

$$\dot{w} = -w + w_d(t)$$

is a forced, exponentially stable, linear system, and further information concerning the nature of w_d may afford statements concerning the “steady state error”. In the special case that w_d is, itself, the output of some linear system, then according to the internal model principle, a higher order linear dynamical compensator may be added to the gradient system with the assurance of asymptotic tracking. The question remains, again, as to how to transplant the internal model principle to the second order Hamiltonian setting.

4.2 Gradient Vector Fields and Hamiltonian Systems

It is well known that Lagrangian mechanics may be placed within the more general framework of Hamiltonian dynamics [21]. To form the Hamiltonian we return to the scalar energy functions, λ, κ, v of Section 3.1.3, and define the *generalized momenta*,

$$p \triangleq D_q \lambda^T$$

and the *Hamiltonian*

$$\eta \triangleq p^T \dot{q} - \lambda.$$

It is not hard to show that the *Hamiltonian dynamical system*

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} [D\eta]^T$$

is equivalent to the rigid body model, (24), with all external forces and torques, set to zero, $\tau \equiv 0$ [21]. Moreover, it is easy to see that any trajectory of this system satisfies $\eta \equiv \eta_0$, a constant since

$$\dot{\eta} = D\eta \begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = D\eta \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} [D\eta]^T \equiv 0.$$

Finally, note that when the potential energy is free of generalized velocity then $D_q \lambda = D_q \kappa$, and when, additionally, the kinetic energy is quadratic in velocity, then

$$p^T \dot{q} = D_q \kappa \dot{q} = 2\kappa$$

and, hence, the Hamiltonian represents the total energy of the system,

$$\eta = 2\kappa - \kappa + v = \kappa + v.$$

It should be apparent from the derivation of Section 3.1.3 that the robot energy terms satisfy these conditions.

By taking this slightly more general perspective we are able to again use the total energy as a Lyapunov function obtaining a rather simple generalization of the stabilizing PD controller from Section 2.2.

4.2.1 Stability of Dissipative Hamiltonian Systems

The central result presented in this section has been known for at least a century: Lagrange demonstrated the stability of motion around the equilibrium state of a conservative system in 1788 [40]; asymptotic stability resulting from the introduction of dissipative forces to a conservative system was discussed by Lord Kelvin in 1886 [6]. Over the years, these ideas seem to have been re-discovered several times by different engineering communities. For instance, a similar set of observations was made in the context of satellite control in 1966 [41]. Credit for first introducing these ideas to the general robotics literature would appear to be due Arimoto and colleagues [42]. Similar independent work has appeared more recently by Van der Schaft [43] and this author [44].

It has been shown above that for a broad range of mechanical systems, including actuated kinematic chains, the Hamiltonian is an exact expression for total energy. In a conservative force field this scalar function is a constant (defines a first integral of the equations of motion) and, in the presence of the proper dissipative terms, it must decay [45]. By replacing the gravitational potential term in the energy function with the objective function which defines a task, and construing the resulting total energy as a Lyapunov Function for the closed loop robot, set point regulation may be achieved as follows. Let the input be defined as

$$u \triangleq k(q) + K_2 x_2 + K_1(x_1 - q_d). \quad (20)$$

Theorem 3 *Let J be a simply connected subset of \mathbb{R}^n . The closed loop system of equation (24), under the state feedback algorithm (20),*

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -M^{-1}[(B + K_2)x_2 + K_1(x_1 - q_d)] \end{aligned}$$

is globally asymptotically stable with respect to the state $(q_d, 0)$ for any positive definite symmetric matrices, K_1, K_2 .

Proof: The Lyapunov Function

$$v \triangleq \frac{1}{2}[x_1^T K_1 x_1 + x_2^T M(x_1) x_2]$$

has time derivative

$$\dot{v} = x_1^T K_1 x_2 - x_2^T [(B + K_2)x_2 + K_1 x_1] + \frac{1}{2} x_2^T \dot{M} x_2$$

and since $x_2^T [\frac{1}{2} \dot{M} - B] x_2 \equiv 0$ as shown in Corollary 3 in Section 5.1, this evaluates to

$$\dot{v} = -x_2^T K_2 x_2 \leq 0.$$

According to LaSalle's invariance principle, the attracting set is the largest invariant set contained in $\{(x_1, x_2) \in \mathcal{P} : \dot{v} \equiv 0\}$, which, evidently, is the origin since the vector field is oriented away from $\{x_2 \equiv 0\}$ everywhere else on that hyperplane.

□

Note that this control law requires the exact cancellation of any gravitational disturbance. While $k(q)$ has a much simpler structure than the moment of inertia matrix, $M(q)$, or the coriolis matrix, $B(q, \dot{q})$, exact knowledge of the plant and load dynamical parameters would still be required, in general, to permit its computation. Since the dynamical parameters enter linearly in k , some progress has been made in the design of “adaptive gravity cancellation” algorithms [46] as will be discussed in Section 5.3.2. A successful adaptive version of this algorithm would remove the need for any *a priori* information concerning the dynamical parameters.

Note, as well, that with diagonal feedback gains, K_1, K_2 , and in the absence of gravitational cancellation, the feedback algorithm (20) is exactly identical to n decoupled PD controllers operating at each joint independently — the procedure employed by almost all commercially marketed arms.

4.2.2 Integrating Gradient Systems by Means of Dissipative Hamiltonian Systems

The real utility of dissipative Hamiltonian systems in the present context arises from the possibility of embedding a first order gradient system - the task definition - in the second order robot arm dynamics with no change in limiting behavior. Let the objective function $\varepsilon : \mathcal{W} \rightarrow \mathbb{R}$ be defined according to some description in task space, as described in Section 4.1, and let $\tilde{\varepsilon} \triangleq \varepsilon \circ g$ be the composition of this objective with the kinematics map. The encoded description now takes the form of a gradient system over joint space,

$$\dot{q} = -D\tilde{\varepsilon}(q)^T, \quad (21)$$

among whose equilibrium states are desired end-points, and whose fall lines define a desirable spatial curve or mechanical response function. The desired performance might be simulated on any analog computer with programmable first order integrators. Instead, it is appealing (and correct) to think of “solving” this gradient system on the “programmable” second order integrators defined by the intrinsic dynamics of a robot arm.

Among the extrema of ε ,

$$\mathcal{E} \triangleq \{w \in \mathcal{W} : D_w \varepsilon = 0\}$$

are the desired task space positions - \mathcal{S} , the optimal points of the objective function. The task is accomplished if joint space variables tend toward the inverse image of this set

$$\tilde{\mathcal{S}} \triangleq \{q \in J : g(q) \in \mathcal{S}\},$$

along the trajectories determined by the fall lines of ε . Now define the feedback control structure

$$u \triangleq k(q) - K_2 \dot{q} - D_q \tilde{\varepsilon}(q)^T.$$

To fix the idea, consider the simple example introduced in Section 4.1.1: end-point control defined in task space. Based upon the error minimizing objective defined in (19), the required feedback control law is

$$u = k(q) + K_2 \dot{q} + Dg^T[g(q) - y_d], \quad (22)$$

resulting in convergence from any initial position and velocity toward critical points of $\tilde{\varepsilon}$. Explicitly, the closed loop system of equation (24), under the state feedback algorithm (22),

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -M^{-1}[(B + K_2)x_2 + D_{x_1} \varepsilon(x_1)^T], \end{aligned}$$

has a globally attracting set defined by the critical points

$$\tilde{\mathcal{E}} = \{(q, 0) \in \mathcal{P} : D_q \tilde{\epsilon}(q)^T = 0\}$$

for any positive definite symmetric matrices, K_1, K_2 .

This may be seen as follows. The non-negative Lyapunov Function

$$v \triangleq \tilde{\epsilon}(x_1) + \frac{1}{2} x_2^T M(x_1) x_2$$

has time derivative

$$\dot{v} = (D_{x_1} \tilde{\epsilon}) x_2 - x_2^T [(B + K_2) x_2 + D_{x_1} \tilde{\epsilon}^T] = -x_2^T K_2 x_2 \leq 0.$$

as in the proof of Theorem 1. According to LaSalle's invariance principle, the attracting set is the largest invariant set contained in $\{(x_1, x_2) \in \mathcal{P} : \dot{v} \equiv 0\}$, which, evidently, is the set of equilibria, $\tilde{\mathcal{E}}$, as claimed.

This result shows that local convergence and global boundedness are assured but that a characterization of the stability properties of individual equilibrium points may be complicated. Namely, if $\tilde{\mathcal{E}}$ has critical points outside of $\tilde{\mathcal{S}}$ then it must be shown that these are not locally attracting equilibrium states of the closed loop system in order to guarantee global convergence to the desired optima. Expressing the objective function gradient as $D_q \tilde{\epsilon} = D_y \epsilon D_q g$ affords the equivalent formulation of the set of equilibria of the closed loop

$$\tilde{\mathcal{E}} = \{(q, 0) \in \mathcal{P} : D_y \epsilon^T \in \ker D_q g^T\}.$$

This makes clear the two distinct causes of such stall points: local extrema and saddle points of the task space objective, ϵ ; and critical points of the output map, g , i.e., the set of kinematic singularities.

As discussed in Section 4.1, stall behavior due to excess critical points of ϵ is an artifact of the task encoding methodology. Sophisticated tasks encoded as summed gradients almost inevitably give rise stall points in \mathcal{W} . While it is possible that a more careful construction of ϵ from constituent objectives might mitigate the problem, this is probably an intrinsic limitation in the "global intelligence" of feedback controllers. Research addressing the interplay between higher planning levels and lower control levels should result in guidelines for the degree of supervision required to assure global convergence. For all presently available commercial robots, kinematic singularities may be found in the interior of the workspace, thus the second problem is more intrinsic to an arm, and, potentially, of considerable practical concern. For a lucky choice of ϵ it might well turn out that $\tilde{\mathcal{E}}$ consists only of the points in the solution set, however this would be unlikely. Some aspects of these questions have been addressed in a recent paper by this author [47].

Of equal pragmatic importance and theoretical interest is a characterization of transient behavior obtaining from those "useful" feedback structures described above. The tasks encoded in terms of gradient dynamics in Section 4.1.2 are not achieved merely by asymptotic approach to an extremum of $\tilde{\epsilon}$. While the theory of linear time invariant controllers includes excellent analytical and graphical methods for determining the appropriate magnitude of damping in relation to position gains, as discussed in Section 2.2.3, no such theory is available for nonlinear Hamiltonian Systems. However effective the methodology is in command of steady state behavior, a gripper which oscillates wildly toward the specified end-point is of no practical use.

Thus one of the most important aspects of controls research in this project is the study of K_2 relative to K_1 , and their nonlinear analogues in equation (22). More generally, it would be of great interest to know how to choose a damping function so that the motion of the second order system projected down onto the zero velocity plane of \mathcal{P} follows the fall lines of the original gradient system as closely as possible: i.e. what is the analogy to a critically damped linear time invariant system?

5 Servo Control of a General Robot Arm

We return here to the paradigm of the servomechanism, wherein tasks are encoded by means of a reference trajectory. As has been remarked above, for linear time invariant dynamics such tracking problems comprise the central arena for the classical control theory. Here, we show that the Lyapunov analysis developed in Section 2.3 carries over to the general case where classical analysis fails.

5.1 Robust Tracking via High Gain Feedback

We return to the rigid body model of robot dynamics,

$$M[q]\ddot{q} + B[\dot{q}, q]\dot{q} + k(q) = \tau \quad (23)$$

where the generalized positions take values in a configuration space, $q \in J$, which is a simply connected subset of \mathbb{R}^n . We have seen in Section 3 that M is a positive definite invertible symmetric matrix for all $q \in J$, and, along with B , the “coriolis and centrifugal” terms, and k , the gravitational disturbance vector, varies in q by polynomials of transcendental functions. It follows that $\nu_M > 0$ and $\mu_M < \infty$.

This system may be rewritten in the form

$$\begin{aligned} \dot{q}_1 &= q_2 \\ \dot{q}_2 &= M^{-1}[Bq_2 + k - \tau] \end{aligned} \quad (24)$$

where the generalized positions and velocities take values $p \triangleq \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \in \mathcal{P} \triangleq TJ$ in *phase space* — the tangent bundle over J .

While M, k are always bounded, the coriolis and centripetal forces are quadratic in the velocity — i.e. B is linear in \dot{q} — and, therefore, may become unbounded. It is, however, bounded with respect to q , as the following technical result shows.

Lemma 2 *For any $k \in \mathbb{R}^n$,*

$$k^T B q_2 = q_2^T \hat{M}(k) q_2$$

where

$$\hat{M}(k) \triangleq \begin{bmatrix} k^T D_{q_{11}} M \\ \vdots \\ k^T D_{q_{1n}} M \end{bmatrix} - \frac{1}{2} \sum_{i=1}^n \kappa_i D_{q_{1i}} M.$$

is bounded above.

Proof:

$$B q_2 = (q_2 \otimes I)^T D_q M^S q_2 - \frac{1}{2} \left[(q_2 \otimes I)^T D_q M^S \right]^T q_2$$

and, hence,

$$\begin{aligned}
 q_2^T B^T k &= q_2^T \left[D_q M^S \right]^T (q_2 \otimes I) k - \frac{1}{2} q_2^T (q_2 \otimes I)^T D_q M^S k \\
 &= q_2^T \left[D_q M^S \right]^T \left(k q_2^T \right)^S - \frac{1}{2} q_2^T (q_2 \otimes I)^T D_q M^S k \\
 &= q_2^T \left(\begin{bmatrix} k^T D_{q_{11}} M \\ \vdots \\ k^T D_{q_{1n}} M \end{bmatrix} - \frac{1}{2} \sum_{i=1}^n \kappa_i D_{q_{1i}} M \right) q_2.
 \end{aligned}$$

Since M contains transcendental functions in q , all of its derivatives in q must be bounded.

□

It follows that for some $\hat{\mu} < \infty$,

$$\|\hat{M}(k)\| \leq \hat{\mu} \|k\|. \quad (25)$$

Corollary 3 For all $p \in \mathcal{P}$,

$$q_2^T \left[\frac{1}{2} \dot{M} - B \right] q_2 \equiv 0.$$

Proof:

$$\begin{aligned}
 e_2^T \left[\frac{1}{2} \dot{M} - B \right] e_2 &= e_2^T \left[\frac{1}{2} \dot{M} - \hat{M}(e_2) \right] e_2 \\
 &= \frac{1}{2} q_2^T \left[\left[D_q M^S \right]^T (q_2 \otimes I) - (q_2 \otimes I)^T D_q M^S \right] q_2 \\
 &\equiv 0.
 \end{aligned}$$

□

5.1.1 A Quadratic Lyapunov Function for Nonlinear Mechanical Systems

The following technical lemma will be of use in the main result, below.

Lemma 4 For $M(q)$ as in (29) and any positive scalars, $\omega, \varsigma, \gamma_0 \in \mathbb{R}^+$,

$$\mu_{\hat{P}} \|e\|^2 \geq e^T \begin{bmatrix} \gamma_0 \omega^2 I & \begin{pmatrix} \beta I \\ \beta I \end{pmatrix} \\ \begin{pmatrix} \beta I \\ \beta I \end{pmatrix} & \gamma M(q) \end{bmatrix} e \geq \nu_{\hat{P}} \|e\|^2$$

for all $q \in J$, where

$$\hat{P} \triangleq \begin{bmatrix} \gamma_0 \omega^2 & \omega \varsigma \\ \omega \varsigma & \gamma_0 \mu_M \end{bmatrix} \quad \check{P} \triangleq \begin{bmatrix} \gamma_0 \omega^2 & \omega \varsigma \\ \omega \varsigma & \gamma_0 \nu_M \end{bmatrix}.$$

Proof: Since

$$\hat{P} \otimes I > \begin{bmatrix} \gamma_0 \omega^2 I & \omega \varsigma I \\ \omega \varsigma I & \gamma_0 \mu_M \end{bmatrix} > \begin{bmatrix} \gamma_0 \omega^2 I & \omega \varsigma I \\ \omega \varsigma I & \gamma_0 M(q) \end{bmatrix} > \begin{bmatrix} \gamma_0 \omega^2 I & \omega \varsigma I \\ \omega \varsigma I & \gamma_0 \nu_M \end{bmatrix} = \check{P} \otimes I,$$

it will suffice to show that

$$\nu_{\hat{P} \otimes I} = \nu_{\hat{P}} \text{ and } \mu_{\hat{P} \otimes I} = \mu_{\hat{P}}.$$

This follows since all eigenvalues of $K \otimes I$ are eigenvalues of K , according to Lemma 11 in the appendix.

□

Proposition 5 *For all $p_d \in \mathcal{P}$ and $\omega, \zeta > 0$, given any bounded set, $\mathcal{B} \subset \mathcal{P}$, containing p_d there exists a scalar $\gamma_0 > 0$ such that*

$$v(e) \triangleq \frac{1}{2} e^T \tilde{P}(q) e = \frac{1}{2} e^T \begin{bmatrix} \omega^2 \gamma_0 I & \omega \zeta I \\ \omega \zeta I & \gamma_0 M(q) \end{bmatrix} e$$

is a quadratic Lyapunov Function for the “undisturbed”

$$\dot{e} = \begin{bmatrix} 0 & I \\ -M^{-1}(q)\omega^2 & -2M^{-1}(q)\omega\zeta \end{bmatrix} e \triangleq A(t)e$$

with the bounding constants

$$\alpha_1 \triangleq \nu_{\tilde{P}}, \alpha_2 \triangleq \mu_{\tilde{P}}, \alpha_3 \triangleq \frac{\omega \zeta}{\mu_M} \nu_Q,$$

where

$$Q \triangleq \begin{bmatrix} \omega^2 & \omega \zeta \\ \omega \zeta & \gamma_0 \nu_M \end{bmatrix},$$

on the domain \mathcal{B} .

Proof: Letting

$$\beta \triangleq \sup_{e \in \mathcal{B}} \|e\|,$$

find some γ_0 satisfying

$$\gamma_0 > \max \left\{ \frac{\zeta}{\sqrt{\nu_M}}, \frac{\zeta^2}{\nu_M}, \frac{\mu_{\tilde{M}} \beta}{\nu_M} + 1 \right\}. \quad (26)$$

According to Lemma 4 and the inequality involving the first entry of the inferior set in (26), it follows that $\alpha_1 \triangleq \nu_{\tilde{P}}, \alpha_2 \triangleq \mu_{\tilde{P}}$ are positive constants with the property

$$\alpha_1 \|e\|^2 \leq v \leq \alpha_2 \|e\|^2.$$

Taking time derivatives along the solutions of system (??), we have

$$\dot{v} = \frac{1}{2} e^T [\tilde{P}A + A^T \tilde{P} + \dot{\tilde{P}}] e,$$

which may be expanded as

$$\begin{aligned} \dot{v} = & -\omega \zeta e^T \begin{bmatrix} \omega^2 M^{-1} & \omega \zeta M^{-1} \\ \omega \zeta M^{-1} & \gamma_0 I \end{bmatrix} e \\ & -\omega \zeta (\gamma_0 - 1) e_2^T e_2 - \omega \zeta e_1^T M^{-1} B e_2 \\ & + \gamma_0 e_2^T \left[\frac{1}{2} \dot{M} - B \right] e_2. \end{aligned}$$

The term in the last line vanishes according to Corollary 3. Moreover, the block matrix in the first line is positive definite according to the inequality (26) and the result of Lemma 4 since

$$\begin{bmatrix} \omega^2 M^{-1} & \omega \zeta M^{-1} \\ \omega \zeta M^{-1} & \gamma_0 I \end{bmatrix} = \begin{bmatrix} M^{-\frac{1}{2}} & 0 \\ 0 & M^{-\frac{1}{2}} \end{bmatrix} \begin{bmatrix} \omega^2 I & \omega \zeta I \\ \omega \zeta I & \gamma_0 M \end{bmatrix} \begin{bmatrix} M^{-\frac{1}{2}} & 0 \\ 0 & M^{-\frac{1}{2}} \end{bmatrix}.$$

Finally, according to Lemma 2, the term in the middle may be rewritten as

$$\omega \zeta [(\gamma_0 - 1)e_2^T e_2 + e_1^T M^{-1} B e_2] = \omega \zeta e_2^T [(\gamma_0 - 1)I + \hat{M}(k)] e_2 > 0,$$

where $k \triangleq M^{-1} e_1$, and the inequality immediately above follows from the inequality involving the last entry of the set in (26).

We may now write

$$\begin{aligned} \dot{v} &\leq -\omega \zeta e^T \begin{bmatrix} M^{-\frac{1}{2}} & 0 \\ 0 & M^{-\frac{1}{2}} \end{bmatrix} \begin{bmatrix} \omega^2 I & \omega I \\ \omega I & \gamma_0 M \end{bmatrix} \begin{bmatrix} M^{-\frac{1}{2}} & 0 \\ 0 & M^{-\frac{1}{2}} \end{bmatrix} e \\ &\leq -\omega \zeta \left(\left[I_{2 \times 2} \otimes M^{-\frac{1}{2}} \right] e \right)^T [Q \otimes I] \left[I_{2 \times 2} \otimes M^{-\frac{1}{2}} \right] e \\ &\leq -\omega \zeta \left(\frac{1}{\mu_{M^{\frac{1}{2}}}}^2 \|e\| \right)^2 \nu_Q \\ &= -\frac{\omega \zeta}{\mu_M} \nu_P \|e\|^2, \end{aligned}$$

and the result follows.

□

5.1.2 Consequences for Tracking Unknown Reference Signals

Now consider the decoupled "PD" compensated system forced by a continuously differentiable reference signal, $q_d(t)$,

$$\tau = k(q) - \omega^2 [q_d(t) - q] - 2\omega \zeta \dot{q}. \quad (27)$$

Assume that the reference trajectory is "unpredictable" — i.e. its first and second derivatives are unknown — but there is available an à priori bound on the maximum rate of change,

$$\|\dot{q}_d\| \leq \delta_0.$$

Notice that the forced closed loop system may be written in the same error coordinates as above,

$$\dot{e} = A[\dot{q}, q]e + d, \quad (28)$$

where $d \triangleq \begin{bmatrix} \dot{q}_d(t) \\ 0 \end{bmatrix}$, is a "disturbance" input due to the unknown but non-zero reference derivative.

Theorem 4 *The closed loop "disturbed" error system has trajectories which are bounded in magnitude by*

$$\|e\| \leq e^{-\frac{\alpha_3}{2\alpha_2} t} \frac{\alpha_2}{\sqrt{\alpha_1}} \|e(0)\| + \frac{\alpha_4 \alpha_2}{\alpha_3 \alpha_1},$$

where α_i , ($i = 1, 3$) are defined in Proposition 5, and

$$\alpha_4 \triangleq \delta_0 \omega \sqrt{\gamma_0^2 \omega^2 + \zeta^2}.$$

5.2 Exact Linearization by Coordinate Transformation

In the last decade a significant body of work has developed within the field of nonlinear systems theory concerning the question of when a specified control system has a dynamical structure which is intrinsically linear, or at least, “linearizable”. More precisely, presented with a system,

$$\dot{x} = f(x, u),$$

it would be of considerable interest to know whether there exists an invertible (memoryless) change of coordinates,

$$\begin{bmatrix} z \\ v \end{bmatrix} = T(x, u),$$

under which the resulting dynamics are linear time invariant,

$$\dot{z} = Az + Bv.$$

For, this being the case, the well understood servo techniques of classical control theory sketched in Section 2.3 could be applied to the reference input expressed in the new coordinate system, and the resulting control, v_d , translated through the inverse coordinate transformation, T^{-1} , would result in an effective control, u_d to be applied to the original system. Early discussion of this question is provided in [48,49], while more recent results have been presented in [50]. More general discussion of this literature may be found in the recent monograph of Isidori [51] or the text by Casti [52].

It will be observed that this policy amounts to exact cancellation of the underlying dynamics of the original system. Thus, as has been remarked, such schemes represent a generalization of the method of pole placement presented in Section 2.2.3, and necessitate exact knowledge of all kinematic and dynamic parameters (including those of the load); à priori knowledge of the reference trajectory; and the ability to compute exactly and implement through a set of actuators the entire dynamics in real time. Work by a number of researchers, most notably Hollerbach [53], has persuasively demonstrated that such computation is possible in real time, and computational architectures have already been designed to do so [54,55]. Recent empirical results [56] suggest that this methodology may achieve good results when the requisite à priori information concerning dynamical parameters and reference trajectory is available.

5.2.1 The Computed Torque Algorithm

In context of the robot equations (24) these ideas lead to the technique of “computed torque”, which has been proposed independently under a variety of names by several different researchers over the last five years [57,58,59]. It seems most instructive to present the variations in the computed torque algorithm as particular examples of the following exact linearization scheme. Let $h : \mathcal{J} \rightarrow \mathbb{R}^n$ be a local diffeomorphism. Under the change of coordinates, defined by $T : \mathcal{P} \times \mathcal{U} \rightarrow \mathbb{R}^{3n}$,

$$\begin{bmatrix} z_1 \\ z_2 \\ v \end{bmatrix} = T(x_1, x_2, u) \triangleq \begin{bmatrix} h(x_1) \\ Dh \ x_2 \\ Dh \ x_2 - Dh \ M^{-1}[Bx_2 + k(x_1) - u] \end{bmatrix} \quad (29)$$

system (24) has linear time invariant dynamics given by

$$\begin{aligned}\dot{z}_1 &= z_2 \\ \dot{z}_2 &= v\end{aligned}\tag{30}$$

with output map

$$w = g \circ h^{-1}(z_1).$$

This may be seen according to the definition of z_1, z_2 by applying the chain rule,

$$\dot{z}_1 = Dh \ x_2 = z_2,$$

and by noting,

$$\dot{z}_2 = Dh \ x_2 - Dh \ M^{-1}[Bx_2 + k - u].$$

This is not the most general class of transformations that might be used to linearize (24), according to the nonlinear systems literature cited above, e.g. [50], but it includes methods commonly encountered in the field of robotics. In particular, for non-redundant kinematics, if we identify h , the first component of T with the kinematic map,

$$h(x_1) \triangleq g(x_1),$$

then, locally, T not only linearizes (24), but dynamically decouples each input and output pair, e.g., as reported in [57] or [59], since $w = z_1$.

As suggested above, the advantage of this approach is that the servo design problem may now be addressed by the classical methods introduced in Section 2, shifting the problem of task specification to lie within the domain of some independent "higher level" process. Namely, suppose such a higher level algorithm produces a desired trajectory in workspace, $w_d(t)$, which it is required that the robot reproduce. Defining $z_d \triangleq [h \circ g^{-1}(w_d), Dh \ Dg^{-1}\dot{w}_d]^T$, it is quite straightforward to choose some linear feedback compensator, $v_{fb} \triangleq Kz$, and feedforward pre-compensator, $v_{pc} \triangleq \Gamma[z_d]$, which determine a "classical" linear control law,

$$v_d \triangleq -Kz + \Gamma(z_d),\tag{31}$$

under whose action the output of (30) behaves in a desired fashion with respect to the reference input, z_d . The particular choice of linear control scheme determines the nature of overall performance along the lines explored in Section 2. Since the relationship between (x, u) and (z, v) has no dynamics (is "memoryless") the input to the robot (24) defined by the inverse coordinate transformation for u under T (obtained by solving for u in the last row of (29)),

$$u_d \triangleq Bx_2 + k(x_1) + MDh^{-1}[v_d - Dh \ x_2],\tag{32}$$

forces the output $w(t) \equiv g[h^{-1}(z_1)]$.

Perhaps the best known example of this approach in the robotics literature is provided by the "resolved acceleration" method of [58]. A control law, $\Gamma[z_d]$, is chosen for the linearized system using the "inverse filter" method described in Section 2.3,

$$\Gamma_{id}(z_d) \triangleq Kz_d + \ddot{z}_{d1} = K \begin{bmatrix} h \circ g^{-1}(w_d) \\ Dh \ Dg^{-1}\dot{w}_d \end{bmatrix} + \left[\frac{d}{dt} Dh \ Dg^{-1} \right] \dot{w}_d + Dh \ Dg^{-1} \ddot{w}_d$$

that is, the inverse of the filter specified by the equivalent closed loop linear time invariant system (30). The control law, u_d , to be applied to the robot is then given by (32). Note that this choice, $h \equiv g$ satisfies the conditions for a local diffeomorphism almost everywhere in J : the condition fails at the “kinematic singularities”,

$$\mathcal{C} \triangleq \{q \in J : \text{rank}(Dg) < \dim \mathcal{W}\},$$

the critical points of g . Most realistic robots have kinematic singularities whose image under g is in the interior of \mathcal{W} and which may not be easily located, hence such a transformation may be impracticable.

As an alternative example, if the task is specified as a trajectory in joint space, $x_d \triangleq [q_d, \dot{q}_d]^T$, this methodology corresponds to a trivial change of coordinates under the identity map, $h \equiv I$, and the control law (32) reduces to

$$u_d \triangleq Bx_2 + k(x_1) + Mv_d.$$

The experimental results of exact linearization schemes in robotics reported to date [56] have employed this version of the algorithm.

5.2.2 Other Coordinate Transformation Schemes

Other choices for h might be imagined: if the kinematics and dynamical parameters which give rise to system (23) define a moment of inertia matrix whose square root is the jacobian of some map, then a much simpler coordinate transformation, T , results. This has been explored by the author in [60]. For ease of discussion, define the set of *square roots* of a smooth positive definite symmetric matrix valued function, $M(q)$, as

$$\mathcal{N}(M) \triangleq \{N \in C^\infty[J, \mathbb{R}^{n \times n}] : NN^T = M\}$$

Note that since M is assumed to be positive definite, $\mathcal{N}(M)$ is not empty, and any h satisfying the hypothesis is an immersion.

Suppose there exists a smooth map, $h : J \rightarrow \mathbb{R}^n$, such that $Dh^T = N \in \mathcal{N}(M)$. Then under the change of coordinates, defined by $T : \mathcal{P} \times \mathcal{U} \rightarrow \mathbb{R}^{3n}$,

$$\begin{bmatrix} z_1 \\ z_2 \\ v \end{bmatrix} = T(x_1, x_2, u) \triangleq \begin{bmatrix} h(x_1) \\ N^T x_2 \\ N^{-1}(u - k(x_1)) \end{bmatrix}, \quad (33)$$

the system has linear time invariant dynamics given by (30). To show this, note that $z_2 \triangleq Dh \ x_2 = \dot{z}_1$. Moreover,

$$\begin{aligned} \dot{z}_2 &= D\dot{h} \ x_2 + Dh \ \dot{x}_2 \\ &= \dot{N}^T x_2 - N^T [NN^T]^{-1} [Bx_2 + k(x_1) - u] \quad \text{from (24)} \\ &= [\dot{N}^T - N^{-1}B]x_2 + N^{-1}u, \end{aligned}$$

and it remains to show that $[\dot{N}^T - N^{-1}B] = 0$. To see this, recall, from Section 3.1.3,

$$\begin{aligned} Bx_2 &\triangleq \dot{M}x_2 - \frac{1}{2} \left[D_q z_2^T z_2 \right]^T \\ &= [\dot{N}N^T + N\dot{N}^T]x_2 - \left[D_q \frac{d}{dt} z_1 \right]^T z_2 \\ &= [\dot{N}N^T + N\dot{N}^T]x_2 - \left[\frac{d}{dt} D_q z_1 \right]^T z_2 \\ &= N\dot{N}^T x_2, \end{aligned}$$

from which the result follows. Note that the exchanged order of differentiation in the third line is justified since z_1 is continuously differentiable in both q and t .

Some of the advantages of a coordinate transformation based upon the square root of the moment of inertia matrix are immediately evident. Given the choice of classical controller, v_d , from equation (31), the inverse transformation for u in terms of z, v is considerably simplified,

$$u_d \triangleq Dh^T v_d + k(x_1)$$

in comparison to (32). Moreover, since h is an immersion, T may be computed everywhere on J . The conditions for the existence of such a map, h , whose jacobian is in $\mathcal{N}(M)$ were given by Riemann in 1854 [61], and amount to the question of when an apparently non-euclidean metric is "flat" — i.e. gives rise to a space of zero curvature.¹⁴ The transformation in question is a particular instance of a local *isometry*, and a more general problem of some interest concerns the existence of other isometries which simplify the dynamics (24). Research exploring whether any useful class of robot arms gives rise to a flat metric and whether more general isometries might be helpful for purposes of control continues [60].

5.3 Global Adaptive Controllers

It has been mentioned in Section 2.3.4 that a rigorous theory of adaptive control for linear time invariant systems is a relatively recent development. Accordingly, the prospects for theoretically sound adaptation algorithms for general nonlinear systems seems quite remote in the near future. Fortunately, in contrast, the possibility of developing practicable globally stable adaptive robot control algorithms within the next few years seems quite bright. This optimism is grounded upon the following observations.

The rigid body model (23) presented in Section 3.1.3 is highly nonlinear in the state, x , and kinematic parameters, but linear in the dynamical parameters (as will be verified shortly). Future robots will probably be built using the direct drive technology [32]. This implies that the omissions in rigid body model take the form presented in Section 3.2.2 which is still linear in (an augmented set of) the dynamical parameters in contrast to the additional nonlinearities presented in Section 3.2.1. Adaptive problems with linear parameter dependence are much more tractable than general nonlinear problems, as will be seen below.

To see that system (23) is linear in the dynamical parameters note that

$$M(q) = \sum_{i=1}^n \left[\tilde{H}_i(q) \right]^T \tilde{P}_i \tilde{H}_i(q)$$

¹⁴The author is indebted to Professors R. Szczerba and G. Zuckermann, Mathematics Department, Yale University, for illuminating discussions concerning Riemannian Curvature.

where $\tilde{H}_i(q) \triangleq D_q F_i^s$ depends entirely on joint positions and kinematic parameters, and

$$\tilde{P}_i \triangleq \begin{bmatrix} \bar{J}_i & \bar{\mu}_i \bar{p}_i \\ \bar{\mu}_i \bar{p}_i^T & \bar{\mu}_i \end{bmatrix} \otimes I,$$

depends on the dynamical parameters, as shown in Section 3.1.3. Since

$$\left([\tilde{H}_i(q)]^T \tilde{P}_i \tilde{H}_i(q) \right)^s = ([\tilde{H}_i(q)]^T \otimes [\tilde{H}_i(q)]^T) \tilde{P}_i^s$$

(refer to Appendix ?? for material concerning the “stack representation”), defining

$$\tilde{H}(q) \triangleq \begin{bmatrix} ([\tilde{H}_1(q)]^T \otimes [\tilde{H}_1(q)]^T) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & ([\tilde{H}_n(q)]^T \otimes [\tilde{H}_n(q)]^T) \end{bmatrix}; \quad p \triangleq \begin{bmatrix} \tilde{P}_1^s \\ \vdots \\ \tilde{P}_n^s \end{bmatrix},$$

yields $M(q)^s = H(q)p$. Since M is linear in p , its derivatives must be as well, and, hence, $B(q, \dot{q})\dot{q} = H'p$. It is clear from the derivation in Section 3.1.3 that k is linear in p , $k(q) \triangleq H'(q)p$.

5.3.1 Adaptive Computed Torque

Now consider the general problem of adaptive control for the robot servo problem. Suppose a desired trajectory, q_d , is given, along with a linear precompensating scheme, $u_{pc} \triangleq \Gamma[q_d]$ which makes x_{m1} , the output of a forced model reference dynamical system,

$$\dot{x}_m = \begin{bmatrix} 0 & I \\ -K_1 & -K_2 \end{bmatrix} x_m + \begin{bmatrix} 0 \\ I \end{bmatrix} u_{pc},$$

track q_d in an acceptable fashion. Assume, moreover, that the structure of the robot dynamics, H, H', H'' is entirely known although the dynamical parameters, p , are not.

If p were known then the appropriate control strategy would be a generalization of the pole placement scheme presented in Section 2.2.3,

$$\begin{aligned} u_d &\triangleq k(q) + B(q, \dot{q})\dot{q} + M[-K_1 q - K_2 \dot{q} + u_{pc}] \\ &= H''(q)p + H'p + ([-K_1 q - K_2 \dot{q} + u_{pc}]^T \otimes I)Hp \\ &= \bar{H}(q, \dot{q}, u_{pc})p, \end{aligned}$$

since this “linearizes” the robot dynamics in the sense of Section 5.2 and places the poles such that the closed loop system has the dynamics of the reference model. Thus, a generalization of the reasoning in Section 2.3.4 suggests that the appropriate adaptive control take the form

$$u_{ad} \triangleq \bar{H}\hat{p} = u_d + \bar{H}[\hat{p} - p]$$

where \hat{p} , the parameter estimate, will be continuously adjusted over the course of the robot’s motion. The closed loop error equations, $e \triangleq x_m - x$, under this control take the form

$$\dot{e} = \begin{bmatrix} 0 & I \\ -K_1 & -K_2 \end{bmatrix} e + \begin{bmatrix} 0 \\ I \end{bmatrix} (M^{-1}\bar{H}[\hat{p} - p]).$$

Again generalizing from the earlier section on adaptive control, the adaptive law should depend upon a Lyapunov function for the reference plant. A convenient choice is $v \triangleq x_m^T P_m x_m$ where

$$P_m \triangleq \begin{bmatrix} K_1 & K_2/2 \\ K_2/2 & I \end{bmatrix}$$

may be shown to be positive definite as long as the reference system is chosen such that $4K_1 - K_2^2$ is positive definite. Note that $\dot{v} = -x_m^T (K_2 \otimes I) P_m x_m$ along the motion of the reference system, and this is easily seen to be negative definite under the further assumption that K_1, K_2 commute. Given these assumptions, the choice of adaptive law consonant with the reasoning of Section 2.3.4 should be

$$\dot{\hat{p}} = \bar{H}^T M^{-1} \left(\frac{1}{2} K_2 e_1 + e_2 \right).$$

Unfortunately, this law is entirely impracticable since by involving M explicitly, it presupposes the availability of the very information which necessitated an adaptive approach in the first place.

It is quite appealing to consider instead the adaptive law,

$$\dot{\hat{p}} = \bar{H}^T \hat{M}^{-1} \left(\frac{1}{2} K_2 e_1 + e_2 \right),$$

where \hat{M} is computed according to the recipe for M using the current value of the parameter estimate, $\hat{H}\hat{p}$. Unfortunately, while M is known to be positive definite and, therefore, invertible over all $q \in \mathcal{J}$, the estimate at any given instant, \hat{M} , does not enjoy such a guarantee. Even if this condition could be assured, it is no longer obvious how to demonstrate convergence of the overall scheme.

Interesting recent work by Craig, Hsu and Sastry, [62] presents an approach to this problem based upon the inverse dynamics pre-compensator,

$$u_{pc} \triangleq \ddot{q}_d + K_2 \dot{q}_d + K_1 q_d.$$

Their analysis examines a "reverse causal" precompensator-robot forward path system — i.e., the reference model driven with inverse dynamics involving the true robot position and velocity. This leads to error equations of the form

$$\dot{e} = \begin{bmatrix} 0 & I \\ -K_1 & -K_2 \end{bmatrix} e + \begin{bmatrix} 0 \\ I \end{bmatrix} (\hat{M}^{-1} \bar{H}^* [\hat{p} - p]).$$

and an adaptive law of the form

$$\dot{\hat{p}} = \bar{H}^{*T} \hat{M}^{-1} \left(\frac{1}{2} K_2 e_1 + e_2 \right),$$

where \hat{M} is prevented from becoming singular or unbounded by explicitly arresting the adaptation when \hat{p} leaves a pre-determined compact region in the positive orthant of parameter space. Convergence obtains after a finite number of "adaptation resets". Unfortunately, as \bar{H} contains reference trajectory acceleration terms, so does \bar{H}^* — a portion of the adaptive law to be synthesized on-line — contain true response acceleration terms, \ddot{q} which would require either use of accelerometers or instantaneous differentiation of real-time signals in any physical implementation.

5.3.2 Adaptive Gravity Cancellation for a PD Controller

Since the complete paradigm of adaptive control does not easily translate into the nonlinear robotic setting, it is sensible to consider schemes where adaptation plays a reduced role. Assume that the desired task is achieved by a natural control law of the kind examined in Section 4, and consider the problem of adaptive cancellation of the gravity term mentioned in Section 4.2.1.

Specifically, it is required to cancel the destabilizing portion of the vector field,

$$k(q) = H\tilde{p}.$$

Following the ideas of Section 2.3.4, the appropriate control input is given as

$$u_{ad} \triangleq H\hat{p} - K_2x_2 - K_1x_1,$$

with $x_1 \triangleq q_d - q$, and \hat{p} , the present estimate of the unknown gains which will be adjusted continuously during the robot's motion. According to the earlier discussion of adaptation, the construction of the adaptive law requires use of an extended Lyapunov function. Unfortunately, the only presently available candidate is the total energy function,

$$v \triangleq \frac{1}{2}[x_2^T M x_2 + x_1^T K_1 x_1],$$

whose time derivative along trajectories of the (perfectly gravitationally canceled) closed loop system was shown to be negative semi-definite rather than negative definite. Proceeding anyway, in analogy to that discussion, set the adaptive law to be

$$\dot{\hat{p}} = -H^T x_2.$$

Notice that this is a practicable procedure, since all explicit dependence upon p is cancelled. The closed loop behavior is then governed by the equation

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -M^{-1}[(B + K_2)x_2 + K_1(x_1) + H\tilde{p}] \\ \dot{\tilde{p}} &= H^T x_2. \end{aligned} \tag{34}$$

It is shown in [46] that this system has a stable origin, and gives rise to bounded solutions whose limit set is contained in the subspace

$$\mathcal{L} \triangleq \left\{ \begin{bmatrix} x \\ \tilde{p} \end{bmatrix} : x_2 = 0 \right\},$$

thus, each physical trajectory will converge to some spatial position $q_0 \in J$, and the parameter estimate, \hat{p} , will converge to some constant $\hat{p}_0 \in \mathbb{R}^{10n}$. Unfortunately, the result says nothing about the relation of these constants to their desired values. In fact, the most likely result of this procedure would be entirely unsatisfactory. For all those positions $q_d \in J$ at which $H(q_d)$ has full rank, the origin of system (34) lies in the interior of a smooth submanifold of \mathcal{L} specified by

$$\mathcal{M} \triangleq \left\{ \begin{bmatrix} q \\ 0 \\ \tilde{p} \end{bmatrix} : \tilde{p} \in H^{-1}K_1(q_d - q) \right\},$$

which is a set of equilibrium states. Thus, not only is the origin non-attracting, but solutions will converge to constants in \mathcal{M} however distant from the origin that manifold extends. Physically, this corresponds to a command torque based upon a spatial error whose corruption by the parameter error exactly balances the gravitational force vector at a particular point in \mathcal{W} . Research attempting to improve this result is currently in progress.

A The Stack Representation

If $A \in \mathbb{R}^{n \times m}$, the “stack” representation of $A \in \mathbb{R}^{nm}$ formed by stacking each column below the previous will be denoted A^S [63].

If $B \in \mathbb{R}^{p \times q}$, and A is as above then the *kroncker product* of A and B is

$$A \otimes B \triangleq \begin{bmatrix} a_{11}B & \dots & a_{1m}B \\ a_{21}B & \dots & a_{2m}B \\ \vdots & & \vdots \\ a_{n1}B & \dots & a_{nm}B \end{bmatrix} \in \mathbb{R}^{np \times mq}.$$

The kroncker product is not, in general, commutative. Note that while the transpose “distributes” over kroncker products,

$$(A \otimes B)^T = (A^T \otimes B^T),$$

the stack operator, in general, does not.

Lemma 6 *If $A \in \mathbb{R}^{n \times m}$ then there exists a nonsingular linear transformation of \mathbb{R}^{nm} , T , such that*

$$(A^T)^S = T A^S$$

Proof: For $p = nm$, let $\mathcal{B} \triangleq \{b_1, \dots, b_p\}$ denote the canonical basis of \mathbb{R}^p — i.e., b_i is a column of p entries with a single entry, 1, in position i , and the other $p - 1$ entries set equal to zero. The transpose operator is a reordering of the canonical basis elements, hence may be represented by the elementary matrix,

$$T \triangleq [b_1, b_{n+1}, b_{2n+1}, \dots, b_{(m-1)n+1}, b_2, b_{n+2}, b_{2n+2}, \dots, b_{(m-1)n+2}, \dots, b_n, b_{2n}, b_{3n}, \dots, b_{mn}].$$

□

For $n = m$, if we define $P_+ \triangleq I + T$, $P_- \triangleq I - T$ then both operators are projections onto the set of “skew-symmetric”, “symmetric” operators of \mathbb{R}^n , respectively, since $P_\pm^2 = P_\pm$. Note that $\text{Ker } P_\pm = \text{Im } P_\mp$.

The kroncker product does “distribute” over ordinary matrix multiplication in the appropriate fashion.

Lemma 7 *If $A \in \mathbb{R}^{n \times m}$, $B \in \mathbb{R}^{p \times q}$, $C \in \mathbb{R}^{m \times k}$, $D \in \mathbb{R}^{q \times l}$ then*

$$(A \otimes B)(C \otimes D) = (AC \otimes BD).$$

Lemma 8 ([63]) *If $B \in \mathbb{R}^{m \times p}$, $A \in \mathbb{R}^{n \times m}$, and $C \in \mathbb{R}^{p \times q}$ then*

$$[ABC]^S = (C^T \otimes A)B^S.$$

Noting that for any column, $c \in \mathbb{R}^{p \times 1}$, we have

$$c^S = [(c)^T]^S = c,$$

there follows the corollary

Corollary 9 *If $B \in \mathbb{R}^{m \times p}$, $c \in \mathbb{R}^p$ then*

$$\begin{aligned} Bc &= Bc^{\mathbf{S}} = (c^{\mathbf{T}} \otimes I) B^{\mathbf{S}} \\ &= ([Bc]^{\mathbf{T}})^{\mathbf{S}} = (c^{\mathbf{T}} B^{\mathbf{T}})^{\mathbf{S}} = (I \otimes c^{\mathbf{T}}) (B^{\mathbf{T}})^{\mathbf{S}}. \end{aligned}$$

Noting, moreover, that

$$\text{tr } \{A\} = (I^{\mathbf{S}})^{\mathbf{T}} A^{\mathbf{S}},$$

there follows the additional result

Corollary 10 *If $A \in \mathbb{R}^{n \times m}$, $B \in \mathbb{R}^{p \times m}$ then*

$$\text{tr } \{AB^{\mathbf{T}}\} = (A^{\mathbf{S}})^{\mathbf{T}} B^{\mathbf{S}}.$$

Proof:

$$\begin{aligned} \text{tr } \{AB^{\mathbf{T}}\} &= (I^{\mathbf{S}})^{\mathbf{T}} (AB^{\mathbf{T}})^{\mathbf{S}} \\ &= (I^{\mathbf{S}})^{\mathbf{T}} (B \otimes I) A^{\mathbf{S}} \\ &= (A^{\mathbf{S}})^{\mathbf{T}} (B^{\mathbf{T}} \otimes I) I^{\mathbf{S}} \\ &= (A^{\mathbf{S}})^{\mathbf{T}} B^{\mathbf{S}}. \end{aligned}$$

□

Lemma 11 *For any square array, $A \in \mathbb{R}^{n \times n}$, if I_m is the identity on \mathbb{R}^m then the spectrum of $(A \otimes I_m)$ is contained in the spectrum of A .*

Proof: Suppose λ is an eigenvalue of $(A \otimes I_m)$. There must be some non-zero vector, $x \in \mathbb{R}^{mn}$ in the kernel of $\lambda(I_n \otimes I_m) - (A \otimes I)$. Since $x = X^{\mathbf{S}} \in \mathbb{R}^{n \times m}$, it follows that

$$\begin{aligned} 0 &= [\lambda(I_n \otimes I_m) - (A \otimes I)]x \\ &= [\lambda X - XA^{\mathbf{T}}]^{\mathbf{S}} \\ &= [X(\lambda I_n - A^{\mathbf{T}})]^{\mathbf{S}}. \end{aligned}$$

This implies that $\text{Im } X^{\mathbf{T}} \subset \text{Ker } \lambda I_n - A$, and since the former subspace has dimension at least 1 (according to the assumption that $X \neq 0$), the latter must as well. Thus, λ is an eigenvalue of A .

□

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