THE THERMAL DIELECTROPHORETIC FORCE ON A DIELECTRIC PARTICLE IN ELECTRIC AND TEMPERATURE FIELDS

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A DISSERTATION

in

Mechanical Engineering and Applied Mechanics

Presented to the Faculties of the University of Pennsylvania

in

Partial Fulfillment of the Requirements for the

Degree of Doctor of Philosophy

2015

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ACKNOWLEDGEMENT

First and foremost, I would like to thank my advisor, Dr. Howard H. Hu, for his support and guidance throughout this process. Although it took me a long time, he always persisted in helping me along the way. Truly without his help, I would not be where I am today. I am especially thankful for his advice in guiding the direction of my research. Whenever I didn't know what direction to go next, he always gave me several possibilities to continue moving forward. In addition to research, Dr. Hu has been a great help to me from being a student in his classes to being a teaching assistant for his classes. I also want to thank Dr. Haim H. Bau for serving on my committee and for his help since I began here at Penn. I will especially remember his insightful comments during presentations at our lab group meetings. Thanks also to Dr. Paulo E. Arratia for being so willing to serve on my committee and his helpful comments about my thesis. I am also grateful that I was able to take several classes from him, since his teaching always makes us students think about problems from a new perspective.

I would also like to thank both the current and former members of the Bau and Hu research lab, especially to Drs. Han-Sheng Chuang, Tong Gao, and Mohamed Sadik, and soon-to-be Drs. Sean Anderson, Mohammed Asaduzzaman, Michael Norton, Nicholas Schneider, Rebecca Winter, and Jinzhou Yuan, for their help with any questions I had as well as the fun and enlightening conversations in the lab. Truly without them, my time as a PhD student would not have been the same. Special thanks to Dr. Han-Sheng Chuang for his help conducting the DEP particle sorting experiments and for his guidance during my internship in Taiwan, in addition to his encouragement to learn Chinese. Special thanks also to Yinglong Guo for collaborating

with me on the theoretical analysis of the thermal DEP force.

Thanks also to all the faculty whose classes I took, students who were my classmates, and staff in the Department of Mechanical Engineering and Applied Mechanics. I especially want to thank Maryeileen, Susan, Olivia, and Desirae for their assistance with academic issues throughout the years. I also want to thank the Department of Education for the GAANN Fellowships, the National Institutes of Health (1R21HG004854-01A1), and the National Science Foundation for the IGERT (Grant #DGE-0221664) for their financial support.

Finally, I would like to thank my family for supporting me along the way. Thanks to my parents Raymond and Alice, for giving me the same thirst for knowledge that they have. Thanks to my sibilings and their spouses: Benyah and Qi, Bithyah, Berayah, Binayah and Caleb. They have always been there for me. This accomplishment would not have been possible without their support.

ABSTRACT

THE THERMAL DIELECTROPHORETIC FORCE ON A DIELECTRIC PARTICLE IN ELECTRIC AND TEMPERATURE FIELDS

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Many microfluidic devices use an applied electric field to control and manipulate particles immersed in a fluid through the electrostatic force caused by dielectrophoresis (DEP). Additionally, electrothermal flow in the fluid can be caused by the effects of nonuniform temperature and the temperature-dependent electrical permittivity and conductivity material properties. We examine the effects on a particle immersed in a fluid subjected simultaneously to an electric and a nonuniform temperature field and find that the particle experiences an electrostatic force given by not only classical dielectrophoresis, but also an additional force, which we term thermal DEP. Assuming the change in the background electric field across the particle is small, and the relative change of temperature-dependent electric properties across the particle is also small, we develop a linearized model to solve the electric field analytically and integrate the Maxwell stress tensor to find an expression for the thermal DEP force. This thermal DEP force is proportional to the temperature gradient, the square of the electric field strength, and the particle's volume. We solve two special cases, one where the electric field and temperature gradient are aligned, and a second case where they are perpendicular to each other. The general case for an arbitrary angle can be found simply by a superposition of these two cases. We compute the fully-coupled system in COMSOL to determine a range of validity for our linearized model and show a practical way to superimpose the classical DEP and thermal DEP forces to find the total electrostatic force on the particle relative to the fluid. Due to the high electrical conductivity of common biological buffers, the thermal DEP force can play an important role when an electric field is used to control and manipulate cells or bacteria. The thermal DEP force may also modify the heat transfer rates in nucleate boiling applications, where large temperature gradients are present.

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CHAPTER 1 : Introduction

The field of microfluidics has been growing rapidly in recent years, especially with regard to microfluidic total analysis systems. These systems provide a way to analyze particles, from polymer or magnetic beads to blood cells and bacteria, at the micron level. The reduced size makes it possible to perform virtually all the analysis of a full-scale laboratory on a small credit card-sized chip, known as lab-on-a-chip technology. These devices are much cheaper to manufacture than regular-sized flow devices, but complex flow designs can still be built with ease. Such devices are desirable because they can be used to provide testing at the point of care, with quick diagnosis within minutes or hours, and because of their low cost compared to traditional lab work.

For lab-on-a-chip devices, the samples to be analyzed, such as blood or saliva samples, generally contain dilute concentrations of various kinds of cells and bacteria in a very diverse mixture. Thus, a key component in analyzing these samples is the ability to manipulate particles and sort them into bins of particles sharing common properties, such as size, shape, surface charge, or thermal response, in order to improve the sample quality by isolating and concentrating the desired particles for analysis.

Sorting techniques can be broadly categorized into two types: those that use a specialized geometry for the fluid microchannel, such as an array of aligned micropillars or a narrow channel segment, to take advantage of hydrodynamic effects to sort particles passively, and those that actively use an externally applied field, such as an electric or temperature field, to effect sorting. For example, Davis et al. [9] used the geometric constraints of an array of posts to separate red and white blood cells, while Yamada et al. [61, 63] developed a method called pinched flow fractionation, which utilizes a narrow segment of microchannel to effect size-based particle sorting. On the other

hand, the field flow fractionation sorting techniques that use an externally applied field were first developed by Giddings et al. [14, 15], and Han et al. [18, 19] used an external electric field with the dielectrophoretic force to sort red and white blood cells as well as various sizes of polystyrene particles.

For hydrodynamic-based sorting devices, the geometry of the channel can be customized specifically to sort particle samples with well-known properties and little variation between samples. The second category of sorting methods is more widely adaptable to different particle samples, since the externally applied field can be modified more easily without the need to develop another device with a different geometry to sort a different set of particles. Field flow fractionation takes advantage of particle response to various fields depending on their material properties and uses motion such as electrophoresis, dielectrophoresis, magnetophoresis, thermophoresis, or chemotaxis to force particles to migrate relative to the fluid. Furthermore, techniques that use continuous sorting allow for higher sample throughput over batch sorting, where sorting must be stopped to collect the sorted samples. I will now briefly discuss in more detail two sorting techniques, one hydrodynamic and one using an external field, for sized-based particle separation.

1.1. Pinched Flow Fractionation

The sorting technique of pinched flow fractionation (PFF), first developed by Yamada et al. [63], is conducive to performing continuous size-based particle separation (Figure 1). In PFF two streams, one a sample stream containing particles and the other a buffer stream without particles, are set up to meet at a junction, and then flow collinearly through a pinched segment of microchannel. The buffer stream, at a much higher flow rate, forces the sample stream to be squeezed along the side wall of the pinched



Figure 1: Pinched flow fractionation separates particles according to size by aligning them along the wall of a pinched channel segment by means of a strong buffer flow and then spreading the flow into separate outlets. Image from Takagi et al. [61].

segment. The particles in the stream are thus forced to move across fluid streamlines and become aligned to the side wall, so that particles of different sizes have their centers of mass aligned at different distances from the wall. When the channel broadens as it leaves the pinched segment, the particles follow the diverging fluid streamlines, causing the minor differences in particle position to be greatly amplified, and the particles flow through different outlet branches depending on their size.

Due to the small length scales in microfluidic flow, a typical Reynolds number for a PFF device is around 0.58 when using the width of the pinched segment w for a length scale and the properties of water for the fluid medium. Thus, we can assume that we are safely within the laminar flow regime of Stokes' flow. Assuming a fully-developed Poiseuille flow in a 2D channel, we can use the parabolic velocity profile \vec{v} to calculate the flow rate ratio needed to position the dividing streamline between the sample and buffer streams a set distance d from the wall (Figure 2). The particle stream with flow rate Q_1 and the buffer stream with flow rate Q_2 combine as they enter the pinched segment and establish a parabolic velocity profile. For a dividing streamline



Figure 2: The particle sample stream and the buffer stream, with flow rates Q_1 and Q_2 , respectively, combine as they enter the pinched segment of width w. The dividing streamline, a distance d from the top wall, denotes the streamline where these two streams meet for a fully-developed parabolic velocity profile \vec{v} .

distance d and pinched segment width w, the required flow rate ratio

$$\frac{Q_2}{Q_1} = \frac{1}{3(d/w)^2 - 2(d/w)^3} - 1.$$
(1.1)

For d/w = 0.5, the flow ratio ratio $Q_2/Q_1 = 1$ as expected. For d/w = 0.25, the required flow rate ratio is only 5.4. However, because the flow is slower near the walls than in the center of the channel, there are diminishing returns as the distance d is decreased. A flow rate ratio of 8.6 is needed for d/w = 0.2, while a ratio of 34.7 is required to achieve d/w = 0.1.

Since we derive this estimate (1.1) using the fully-developed fluid velocity profile in a 2D channel geometry, it does not exactly correspond to the 3D flow profile. Often, the height of the pinched segment is close to its width, i.e., the pinched segment is nearly square. Thus, the top and bottom walls of the channel could also modify the flow profile. However, a good rule of thumb is to have the distance d approximately equal to the radius of a particle to be sorted. This will ensure that the particle is forced

to migrate across fluid streamlines to the dividing streamline or even to the buffer stream. For example, a flow rate ratio of 20 with a pinched segment width of 40 μ m places the dividing streamline at 5.28 μ m from the wall, meaning that a particle with a diameter of approximately 10 μ m (5 μ m radius) should be able to be separated from the sample stream in this device. The chosen width of the pinched segment is directly proportional to the particle size that can be sorted with the device at a given flow rate ratio. A narrower pinched segment allows for the sorting of smaller particles, but can lead to problems with clogging if a particle that is too large or a clump of particles passes through the device. A wider pinched segment limits the ability to sort smaller sizes of particles, but reduces chances of clogging and allows for higher throughput.

Notice also that for a particle whose diameter is the width of the pinched segment, its center of mass will be located on the centerline of the pinched segment. Using only the inlet flow ratio parameter to control sorting will result in half of the outlet channels remaining unused. Therefore, a second parameter dealing with the outlet flow is necessary to take full advantage of all the outlet branches. By specifying the outlet flow for the last outlet (Branch 5 in Figure 1), we can draw the excess buffer flow into this outlet. This will result in more divergence in the fluid streamlines as they exit the pinched segment and provide greater separation between particles. Adjusting this outlet flow control also allows us to move the streamline that separates between two different outlet channels, so that the sizes of particles carried to each outlet can be adjusted.

Thus, pinched flow fractionation requires two parameters for effective control of the sorting. The inlet flow rate parameter ensures that the particles are squeezed against the wall of the pinched segment, regardless of their initial positions in the channel. The outlet parameter controls where the flow is divided between successive outlet channels.



Figure 3: Overlay of particle pathlines for 5 μ m (green) and 10 μ m (blue) diameter particles introduced at top left inlet shows complete separation achieved using pinched flow fractionation. Dimensionless flow rates for the sample and buffer streams are 1 and 20, respectively, while the dimensionless flow rate is 15.9 for the bottom right outlet. Channel dimensions are in μ m.

To verify that these two parameters can sort particles by size, let's examine the 2D numerical simulation results depicted in Figure 3. Here, particles with diameters 5 and 10 μ m are introduced from the top left inlet channel with a dimensionless flow rate of 1, and a buffer stream of dimensionless flow rate 20 flows in from the bottom left inlet. (The middle inlet is not used for this simulation.) These two streams combine and pass through a pinched segment of width 40 μ m, before spreading into the five outlet channels on the right. The dimensionless flow rate of the bottom right outlet is set at 15.9. This flow setup is solved numerically using the discretized 2D Navier–Stokes equations with two-way coupling between the particle and the fluid. The arbitrary Lagrangian–Eulerian finite element description is used so that the mesh near the particle can move along with it as we iterate through time. To account for the low particle concentration, each simulation is run with only a single particle to remove any possible particle-particle interactions. The results of these simulations are overlaid to show the particle pathlines of Figure 3. Note that there is complete separation



Figure 4: Schematic depicting the origin of dielectrophoresis for a particle located between two electrodes. An electric field $\bar{\mathbf{E}}$ polarizes both the dielectric particle and the surrounding fluid medium. Due to differences in dielectric properties, an imbalance of charge develops around the particle, inducing a net dipole moment $\bar{\mathbf{p}}$. This dipole interacts with the nonuniform electric field to cause a net motion of the particle. (Note: Figure not drawn to scale.)

between the 5 and 10 μ m particles for these flow parameters. All the smaller particles flow into the first outlet channel, and all the larger particles into the second outlet channel, independent of their initial positions.

1.2. Dielectrophoresis

The motion of dielectric particles due to dielectrophoresis (DEP) is a field flow fractionation sorting technique that has been widely used to separate [12, 18, 19, 39, 40] and position [8, 24, 52] particles, cells, viruses, macromolecules [49], bubbles [30], and even animals [7]. DEP particle sorting is based on an externally applied nonuniform electric field and the relative dielectric material properties between the particles and the surrounding fluid medium. For particles placed into a nonuniform AC electric field, the net charge of the particle is unimportant. However, either neutral particles or particles with a net electric charge, along with the surrounding fluid medium, will become polarized by the presence of the AC electric field. Due to the material properties of electrical permittivity ϵ and conductivity σ , bound and free charges, respectively, develop at the interface between particle and fluid medium (Figure 4). The difference in polarizability between the particle and the medium causes an imbalance of charge around the particle, inducing a net dipole moment $\bar{\mathbf{p}}$. Finally, the interaction between this induced dipole moment and the nonuniform electric field gives rise to a net DEP force

$$\mathbf{F}_{\text{DEP}} = \text{Re}\big[\bar{\mathbf{p}} \cdot \nabla \bar{\mathbf{E}}^*\big],\tag{1.2}$$

where Re[...] denotes the real portion of the inner complex expression, overbars denote complex quantities, and the superscript * denotes the complex conjugate.

The time-averaged, classical DEP force acting on a particle of volume V_p suspended in a fluid in the presence of an external background AC electric field $\widetilde{\mathbf{E}}^o(\mathbf{x}, t) =$ $\mathbf{E}^o(\mathbf{x}) \exp(j\omega t)$ is often estimated with the dipole moment approximation [29, 37, 47]

$$\mathbf{F}_{\text{DEP}} = \lambda \epsilon_m V_p \operatorname{Re}(\bar{K}_E) (\mathbf{E}_c^o \cdot \nabla \mathbf{E}_c^o).$$
(1.3)

The subscripts p and m identify the particle and the suspending medium. The superscript o denotes the outer background field that exists in the suspending medium in the particle's absence, and the subscript c indicates that the corresponding quantity is evaluated at the position of the particle's center. In equation (1.3) λ is a numerical coefficient of O(1) and $\operatorname{Re}(\bar{K}_E)$ is the real part of the frequency-dependent Clausius– Mossotti factor that describes the relative polarization of the particle. When the particle is a sphere, $\lambda = 3/2$ and

$$\bar{K}_E = \frac{\bar{\epsilon}_p - \bar{\epsilon}_m}{\bar{\epsilon}_p + 2\bar{\epsilon}_m},\tag{1.4}$$

where $\bar{\epsilon} = \epsilon - j\sigma/\omega$ is the complex electrical permittivity that accounts for the effects of a lossy dielectric in an AC field, and $j = \sqrt{-1}$. AC electric fields are

often used in applications to shunt the electric double layer capacitance and minimize electrochemical reactions at electrode surfaces. The DEP force in a DC field can be found by simply taking the limit as the AC frequency $\omega \to \infty$.

There are several important things to note from the classical DEP force expression (1.3). First, the force on the particle is directly proportional to its volume, making the DEP force well-suited for size-based particle sorting, since the force scales as the cube of the particle radius $(F \sim a^3)$. Secondly, DEP requires the background electric field to be *nonuniform* due to the presence of the electric field gradient in the force expression. In other words, the electric field that exists in the absence of any particle must be nonuniform to cause a net DEP force. Finally, the direction of the force is determined by the Clausius–Mossotti factor \bar{K}_E , which is a function of the AC frequency ω and the material properties, and can vary between -0.5 and 1. When the particle is more polarizable than the medium $(\bar{\epsilon}_p > \bar{\epsilon}_m)$, the particle experiences a positive DEP force moving it toward regions of high electric field. Conversely, when the particle is *less* polarizable than the medium $(\bar{\epsilon}_p < \bar{\epsilon}_m)$, the particle experiences a negative DEP force moving it away from high electric field regions. Positive DEP causes particles to collect in regions of high field strength, such as near the edges of electrodes, requiring a sorting technique with batch processing that will enable the collected particles to be removed. However, negative DEP forces particles away from the electrodes and thus can be easily used in a continuous flow sorter without particles affixing to the electrodes.

CHAPTER 2 : A Microfluidic Sorting Device

A microfluidic sorting device is not restricted to using only a single sorting technique in isolation. Rather, several of these can be combined in series to achieve greater separation or finer sorting. For example, in sorting a cell sample one may first need to remove the very large particles that might clog the system, next isolate a certain size of particles from the remaining sample, and finally determine which of these cells are alive or dead. Multiple sorting stages also permits the sorting of a wider range of particles in the same device, such as a size-based sorter with each stage optimized for a different range of particles sizes.

2.1. Size-Based Sorting Experiment

In the previous section I discussed the two main categories of particle sorting devices, passive hydrodynamic-based sorters and active sorters using an external field. Specifically, both PFF and DEP are well-suited to size-based particle sorting. Hence, let's examine a microfluidic device that uses these two sorting methods in series in order to sort particles by size (Figure 5, bottom left). First, in the PFF portion of the device, just like the PFF sorting described in Section 1.1, the particle sample stream and buffer stream combine to flow through a 40 μ m-wide pinched segment (Figure 5, right). At the exit from the pinched segment, the channel expands to a width of 400 μ m, but, unlike the previous example, before the channel splits into the five outlets, it first passes over an interdigitated array of four gold electrodes embedded at the bottom of the 400 μ m-wide section of microfluidic channel. This portion of the device provides a DEP force on the particles as they are carried downstream by the drag force from the fluid. The flow channels all have a height of 50 μ m and are made of PDMS. The inlet flow ratio of the buffer to sample streams is set by the two syringe pumps that



Figure 5: Top left: Setup of experimental apparatus. Bottom left: Picture of microfluidic chip. Right: Particle migration is due to the competing effects of the DEP force and the drag force. (Images courtesy of Han-Sheng Chuang.)

drive the flow, and a function generator with power amplifier is used to supply a voltage with frequency $f = \omega/2\pi = 2.2$ MHz for the electrode array (Figure 5, top left). The microfluidic chip is placed in an optical microscope, and the particle sorting is recorded by a CCD camera.

In this experiment only the single parameter of the inlet flow ratio is used to control the PFF portion. Furthermore, the buffer and particle streams have flow rates of 0.1 and 0.01 mL/hr, respectively, so that the inlet flow ratio is lowered to 10:1. This effectively doubles the throughput rate of the sample stream compared to the numerical PFF example from Section 1.1. However, in this case the dividing streamline for a parabolic

flow profile is now located at approximately 7.44 μ m from the channel wall, which means that a particle with radius *a* smaller than this may not be separated adequately by PFF alone. One of the benefits of including multiple sorting methods is increasing the range of usefulness for the device. The PFF portion can be used to sort particles in the range of 15 μ m up to the pinched segment width of 40 μ m, while DEP is used to sort particles sizes from 15 μ m down to approximately 1 μ m. Below this range, the DEP force greatly decreases and Brownian motion must be considered. (See Section 6.3 for more information about Brownian motion and the limiting size below which DEP sorting becomes impossible.)

The DEP portion of the device is controlled by the voltage difference between adjacent electrodes, which alternate between a nonzero voltage V_0 and ground. Since the DEP force in Equation (1.3) scales with the square of the potential V_0 , the direction of the force does not change whether the voltage difference between two consecutive electrodes is positive or negative, but only depends on the material properties. For polystyrene particles (permittivity $\epsilon_p = 2.5\epsilon_0$ [51], conductivity $\sigma_p = \frac{5 \times 10^{-9} \text{ S}}{a}$ [45]) in de-ionized water at 25°C ($\epsilon_m = 80\epsilon_0$ [36], $\sigma_m = 0.015 \text{ S/m}$ [50]) at an AC frequency f = 2.2 MHz, the real portion of the Clausius–Mossotti factor $\text{Re}(\bar{K}_E) = -0.426$. Thus, this experimental setup uses negative DEP, which forces the particles toward the top of the channel — away from the highest electric field regions, which are located in the gaps between the electrodes on the bottom of the channel. The electrodes each have a width of 65 μ m and a 25 μ m gap between them, providing an electric field on the order of 10⁶ V/m, and are oriented at a 45° angle from the flow so that the direction of the DEP force will deflect the particles across streamlines.

The drag force and the DEP force together act on the particle within the channel. When a particle encounters the electrode array, it experiences a negative DEP force that pushes it toward the top wall of the channel. This portion of the DEP force is not counteracted by any drag force from the 1-D flow. Near the channel wall, the flow is slower, resulting in a decreased drag force on the particle. Thus, in this region the DEP force can "capture" the particle when it experiences a DEP force over the electrode gap that is strong enough to counteract the lower drag force pulling the particle along the channel. If the electrodes intersect the channel at an angle of $\alpha_{\rm el}$, then $F_{\rm DEP} \cos \alpha_{\rm el}$ is the component of the DEP force acting against the direction of the flow, while $F_{\rm DEP} \sin \alpha_{\rm el}$ is the component that deflects the particles across the channel. Under the combined effects of the drag force and the DEP force, captured particles move along the electrodes and are deflected until they intersect with the outlet channels.

A sample experiment is depicted in Figure 6. Here, 1.0, 2.5, and 4.8 μ m polystyrene particles are introduced in the particle stream of DI water, combine with the buffer stream, and flow through the pinched segment. This set of particle sizes is too small to be sorted by the PFF portion alone, but can be handled by the DEP portion, which uses a root mean square ($V_{\rm rms} = V/\sqrt{2}$) voltage of $V_0 = 24.6$ V. The pink circles over the electrodes indicate the locations of several particles, which appear as white against the dark background of the device. The 1.0 μ m particles experience little overall deflection from the DEP force and mostly exit through Outlet 1; the 2.5 μ m are deflected moderately and mostly exit through Outlet 3; the 4.8 μ m particles are deflected immediately by the first electrode gap and are carried along the first electrode to Outlet 5. By simply counting the number of particles that pass into each outlet channel, we can measure the effectiveness of the sorting (Figure 7). Notice that the largest and smallest particles have very little variation in their final locations. For the 4.8 μ m particles the DEP force is strong enough to deflect them right away along the





Figure 6: Top: Video frame from experiment showing combined PFF and DEP sorting of 1.0, 2.5, and 4.8 μ m polystyrene particles in DI water. Bottom: The same video frame including annotations. The microfluidic channels (blue) cause the particle and buffer streams to combine and flow through the pinched segment before they pass over an interdigitated electrode array (gold) and then separate into five outlet channels (numbered 1–5). The resulting particle traces are shown in pink.



Figure 7: Bar plot showing the percentage of polystyrene particles collected in each outlet channel from the experiment shown in Figure 6. The RMS voltage difference is $V_0 = 24.6$ V.

first electrode into Outlet 5. The DEP force is weak for the 1.0 μ m particles, and the drag force is able carry them across all the electrode gaps. The middle-sized 2.5 μ m particles have quite a large variation, as they exit through Outlets 1–4, although a large majority end up in Outlet 3. This is likely due to the initial vertical positions of the particles. The particles that are already near the top of the channel can be captured by the DEP force more easily than those particles that are initially near the bottom of the channel.

The voltage parameter V_0 must be adjusted to optimize the sorting of each set of particle sizes. Increasing the voltage parameter V_0 increases the particle migration due to DEP, deflecting them further toward Outlet 5. Also, in general, the smaller the particle sizes are, the larger the required voltage will be. The required V_0 does depend slightly on the flow rate as well, which is mainly determined by the buffer stream flow rate.

2.2. Numerical Comparison

We also solve the flow field and the electric potential within the device using the commercial flow solver COMSOL Multiphysics to provide a numerical comparison to the experimental results of the previous section. This numerical solution uses one-way coupling between the particle and the fluid. In other words, the fluid flow is solved ignoring the presence of any particles, and the particle motion is calculated afterward using post-processing. Since the largest particle size is 4.8 μ m, the particles are small when compared to the channel height of 50 μ m or the electrode width of $65 \ \mu m$. Furthermore, the particles are in a dilute sample. For small particles in a dilute sample, the particles do not greatly influence the overall fluid flow and there is little particle-particle interaction. Using one-way coupling permits us to solve the flow field once and use this flow field to calculate all the particle trajectories afterward. Two-way coupling necessitates updating the fluid motion around the particle for each timestep of the particle trace, greatly increasing the computational time. Recall also that the classical DEP force expression (1.3) uses the background electric field without the presence of the particle, relying on a similar assumption of a small particle with respect to the length scale of the electric field.

Once the flow field and the electric potential field are solved with the same parameters from the experiment, we seed particles in the sample stream inlet and calculate their trajectories in the following manner. The particles are convected along with the same velocity as the fluid due to the drag force, until the DEP force acts on the particles to move them relative to the fluid. Since this is within the Stokes' flow regime, we have

$$\mathbf{F}_{\text{DEP}} = 6\pi\mu a f_w(x, y, z) (\mathbf{v}_p - \mathbf{v}_m), \qquad (2.1)$$

where μ is the viscosity of the fluid, \mathbf{v}_p and \mathbf{v}_m are the velocities of the particle and fluid medium, respectively, and $f_w(x, y, z)$ is a wall correction factor to the Stokes' drag. The wall correction factor [20] for a sphere moving parallel to a plane wall is

$$f_w = \left[1 - \frac{9}{16}(a/h) + \frac{1}{8}(a/h)^3 - \frac{45}{256}(a/h)^4 - \frac{1}{16}(a/h)^5\right]^{-1}$$
(2.2)

and for a sphere moving normal to a wall is

$$f_w = \left[1 - \frac{9}{8}(a/h) + \frac{1}{2}(a/h)^3\right]^{-1}.$$
(2.3)

In (2.2) and (2.3) a is the particle radius and h is the distance between the center of the particle and the wall (i.e., h/a > 1). From (2.1) we directly see that

$$\Delta \mathbf{x}_p = \mathbf{v}_p \Delta t = \left[\mathbf{v}_m + \frac{\mathbf{F}_{\text{DEP}}}{6\pi\mu a f_w(x, y, z)} \right] \Delta t, \qquad (2.4)$$

where $\Delta \mathbf{x}_p$ is the change in the particle's position over a timestep of Δt .

Equation (2.4) allows us to calculate the trajectory for each individually seeded particle from its initial position within the sample stream inlet, and this calculation is done using COMSOL Multiphysics post-processing. Figure 8 shows a representative particle trace for each particle size with the flow field and electric potential solved numerically in COMSOL. The green, red, and blue trajectories correspond to a 1.0, 2.5, and 4.8 μ m particle, respectively. As expected, PFF alone cannot sort this set of particles. Although the particles are aligned with the wall of the pinched segment so that they are arranged by size as they reach the expanded channel section, the distance between the particles' trajectories as they exit the pinched segment is still not large enough to provide adequate sorting. Due to the PFF portion forcing the particles toward



Figure 8: Representative particle traces for each particle size with the flow field and electric potential solved numerically in COMSOL using the same parameters as in the experiment. The red, green, and blue trajectories correspond to the 1.0, 2.5, and 4.8 μ m particles, respectively. Note: Particles enlarged to show their sizes relative to one another.

the top wall, the initial x- and y-positions of the particles have little effect on the final outlet channel. However, the initial z-position (perpendicular to the page in Figure 8) of a particle can still have some effect, since the pinched segment does not focus the particles in the vertical direction. The z-position of a particle determines how far it must be deflected vertically by the negative DEP force in order to reach the slower fluid flow near the top wall of the channel that allows it to be captured along an electrode. Figure 9 displays the particle trajectories for 100 particles of each size initially evenly distributed within the inlet channel. The spread in the trajectories among the same-sized particles is due mainly to the different initial z-positions of the particles.

After the pinched segment, the particles follow along the fluid streamlines, but as they reach the electrode array, the effects of the DEP force become apparent. A 4.8 μ m particle is deflected from the horizontal direction of the fluid flow almost immediately as it encounters the first electrode and then is directed along it into Outlet 5. A 2.5 μ m particle takes longer to reach the top wall of the channel, and thus is not captured



Figure 9: Particle traces for each particle size with the flow field and electric potential solved numerically in COMSOL. Each particle size is seeded at 100 evenly distributed initial positions within the inlet channel. The red, green, and blue trajectories correspond to the 1.0, 2.5, and 4.8 μ m particles, respectively. The spread in the outlet positions of the particles is mainly due to the different initial z-positions of the particles in the inlet channel. Note: Particles enlarged to show their sizes relative to one another.

by the DEP force until it passes over the second or third electrode, depending on its initial z-position. A 1.0 μ m particle does not experience a DEP force strong enough to capture it completely, and thus is only deflected slightly even after passing over all four electrodes. These numerical results confirm what is seen in the experiments and illustrate how PFF and DEP combined can be used to sort particles by size.

2.3. Analytical Solution

We also examine an approximate analytical solution for the electric potential that can be used to calculate the DEP force and the particle trajectories. In elliptic coordinates [41] the electric potential solution for the case of two semi-infinite electrodes with potentials $\pm V/2$ and separated by a gap of 2*d* can be easily written. When the system is centered at the origin with the electrodes along the *x*-axis, the electric potential is given by [52]

$$\phi_1(x,z) = \frac{V}{2} \left(1 - \frac{2}{\pi} \theta_1 \right), \tag{2.5}$$

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Figure 10: Two electrodes with potentials $\pm V/2$ and separated by a gap of 2d are located on the bottom of a channel of height H.

where

$$\theta_1 = \cos^{-1} \left(\frac{\sqrt{(x+d)^2 + z^2} - \sqrt{(x-d)^2 + z^2}}{2d} \right).$$
(2.6)

The geometry of our electrode array does not exactly match this analytical case, however. The electrodes are not in a semi-infinite domain, but are within a channel of height H (Figure 10), with electrically insulating boundary conditions on the channel walls. Thus, we use the method of images and introduce a mirrored pair of electrodes located at z = 2H. This mirrored pair has a solution given by $\phi_2(x, z) = \phi_1(x, 2H - z)$, or, more explicitly, replacing θ_1 in (2.5) with

$$\theta_2 = \cos^{-1} \left(\frac{\sqrt{(x+d)^2 + (2H-z)^2} - \sqrt{(x-d)^2 + (2H-z)^2}}{2d} \right).$$
(2.7)

Adding the mirrored solution to the original allows us to match the boundary condition on the wall at z = H, but at the same time causes a mismatch with the boundary condition at z = 0. We can introduce more mirrored pairs of electrodes with electric potential solutions: $\phi_3(x, z) = -\phi_1(x, z - 2H)$ to cancel the effect of ϕ_2 on the boundary condition at z = 0; $\phi_4(x, z) = -\phi_1(x, 4H - z)$ to account for ϕ_3 and once again fix the boundary condition at z = H; and so on. Note that the sign in front of ϕ_n alternates every two terms. Each successive correction is smaller as the new mirrored electrodes are further away from the actual channel itself. The exact electric potential for the pair of electrodes in a channel is given by the sum of the infinite series from the mirrored electrodes:

$$\phi_{\text{exact}}(x,z) = \sum_{n=1}^{\infty} \phi_n(x,z) = \frac{V}{2} \sum_{n=1}^{\infty} (-1)^{\lfloor (n-1)/2 \rfloor} \left(1 - \frac{2}{\pi} \theta_n\right),$$
(2.8)

where the floor function $\lfloor \cdots \rfloor$ is defined as the greatest integer less than or equal to the enclosed quantity, and θ_n is given by

$$\theta_n = \cos^{-1} \left(\frac{\sqrt{(x+d)^2 + (z - \lfloor n/2 \rfloor 2H)^2} - \sqrt{(x-d)^2 + (z - \lfloor n/2 \rfloor 2H)^2}}{2d} \right).$$
(2.9)

(Note that $\theta_2 = \theta_3$, $\theta_4 = \theta_5$, etc.) Although this mirrored electrode process can go on to infinity, we must truncate it at some finite n in order to compute the solution. Since we are using negative DEP, the particles are deflected toward the wall at z = H. Thus, it is more critical to match the boundary condition at this wall, and we choose to truncate the series at n = 4 to satisfy this condition.

We also introduce a scaling factor β to match the nominal electric potential. Since the last mirrored electrodes affect the potential on the original pair of electrodes, changing them from their nominal voltages of $\pm V/2$, the boundary conditions on the channel walls cannot both be satisfied simultaneously. The scaling factor β is necessary to correct for the "extra" potential introduced by truncating the infinite series of mirrored electrodes. Our analytical solution for the original electric pair is now of the form

$$\phi_{\text{pair}}(x,z) = \phi_1(x,z) + \phi_2(x,z) + \beta\phi_3(x,z) + \beta\phi_4(x,z).$$
(2.10)

The superposition of $\phi_1 + \phi_2$ matches the boundary condition at z = H. Likewise, the superposition of $\phi_3 + \phi_4$ matches this boundary condition. Thus, we must scale both
ϕ_3 and ϕ_4 together by the factor β in order to keep the top wall boundary condition satisfied. Due to symmetry with respect to the *x*-axis, the potential $\phi_3 = -\phi_2$. Thus, we can rewrite (2.10) as

$$\phi_{\text{pair}}(x,z) = \phi_1(x,z) + (1-\beta)\phi_2(x,z) + \beta\phi_4(x,z).$$
(2.11)

The scaling factor β is defined such that the electric potential at the center of the right electrode, $(x, z) = (d + w_{\rm el}/2, 0)$ for electrodes of width $w_{\rm el}$, exactly matches the nominal potential +V/2. In other words,

$$V/2 = \phi_1(d + w_{\rm el}/2, 0) + (1 - \beta)\phi_2(d + w_{\rm el}/2, 0) + \beta\phi_4(d + w_{\rm el}/2, 0).$$
(2.12)

Since $\phi_1(d + w_{\rm el}/2, 0) = V/2$, we have

$$\beta = \frac{\phi_2(d + w_{\rm el}/2, 0)}{\phi_2(d + w_{\rm el}/2, 0) - \phi_4(d + w_{\rm el}/2, 0)}.$$
(2.13)

Because the potential is anti-symmetric about the z-axis, the scaled potential will exactly match the nominal potential at the center of both of the original electrodes. The factor β is a function of the gap distance 2d, the width of the electrodes $w_{\rm el}$, and the height of the channel H, but not the potential V, and thus it is determined solely by the geometry of the electrodes and channel. For a 50 μ m high channel with electrodes of 65 μ m width and separated by a gap distance of 25 μ m, we find $\beta = 0.6555$. We find that for $H \to \infty$, $\beta = 2/3$, and for $d \to \infty$ or $w_{\rm el} \to \infty$, $\beta \to 0.5$. Thus, the actual value for β for this channel geometry is only 1.67% different from the semi-infinite case.

The solution for a single electrode pair (2.10) can now be extended by superposition



Figure 11: Top: Contour plot of the analytical solution for the electric potential in the channel over a four-electrode array located at z = 0. Side view looking along the length of the electrodes. Bottom: Line plot of the potential at z = 0. The x-positions of the electrodes are indicated by the gray shaded regions. The nominal voltages of the electrodes are 25, 0, 20, and 0 V.

to multiple pairs of electrodes, with each pair of electrodes having their own mirrored solutions. For example, the electric potential near the second electrode gap is given by $\phi_{\text{pair}_2}(x, z) = \phi_{\text{pair}}(x - 2d - w_{\text{el}}, z)$, where w_{el} is the width of the electrodes. Furthermore, the voltage difference V between two successive electrodes can be adjusted independently for each pair, if needed, since the electrode solution is directly proportional to this voltage. Appendix A contains the various MATLAB files used in computing the analytical electrical potential, DEP force, and particle traces.

Figure 11 contains a plot of the total analytical solution for the electric potential in a 50 μ m high channel for the case of four electrodes, each of 65 μ m width and separated by gaps of 25 μ m, with potentials of 25, 0, 20, and 0 V, respectively. Notice how the boundary condition at the top channel wall is satisfied, as the isopotential contour lines are perpendicular to the wall. The bottom line plot of the potential at z = 0



Figure 12: Comparison of the analytical and numerical solutions for the DEP force in the x-direction (across the electrodes) on a 4.8 μ m diameter particle located at $z = 45 \mu$ m.

illustrates how the potential on the electrodes matches the nominal voltage only at the centers of the electrodes, but is not quite uniform across the whole electrode. For example, the third electrode has a potential difference between its left and right sides of 0.531 V, or 2.65%.

Nevertheless, this small variation in potential on the surface of the electrode does not greatly influence the DEP force. To confirm this, we compare the DEP force calculated from this analytical solution to our numerical solution in COMSOL (Figure 12). The analytical F_x DEP force (across the electrodes) calculated from (1.3) closely matches the COMSOL-derived values at $z = 45 \ \mu m$ (near the top channel wall). The F_z (vertical) DEP force (not shown) also is in close agreement to the COMSOL solution. The noise in the numerical DEP force is due to the fact that computing $\nabla E = \nabla(-\nabla\phi)$ in the DEP force expression (1.3) requires the calculation of the second gradient of the electric potential, while the analytical solution allows the gradients to be calculated in a much cleaner manner.



Figure 13: Revised geometry using five angled electrodes. Each electrode is initially at an angle $\alpha_{\rm el}$ with the channel and then bends to become parallel to it. Captured particles can follow the electrodes directly to the corresponding outlet channels.

In Figure 12 notice also that the DEP force in the x-direction crosses zero over each electrode and each electrode gap. At these stationary points no net DEP force is exerted on the particles to move them across the electrodes. The stationary points over the electrode gaps are unstable, while the ones over the electrodes are stable. Thus, with DEP alone the particles will tend to collect over the electrodes at these stationary points. In the full device we must also account for the angle of the electrodes by multiplying the DEP force by $\cos \alpha_{el}$ and also the flow of the fluid, which provides a uniform offset to the net force at any given z-value. Even when accounting for these two additional factors, the overall shape of the force plot still remains the same. However, the magnitude of the DEP force will still have a stationary point located over the electrodes in order to capture and direct the particles. This also confirms what was shown by the experiment — that particles captured by the DEP force tend to remain over the electrodes and travel along them to the outlet channels.

With the knowledge of the analytical solution and the paths that the particles take as they are captured by the DEP force, we can suggest a better, more optimal geometry for the electrode array (Figure 13). Already notice in Figure 8 that the electrode gaps do not go completely across the whole channel width. The reason for this is to prevent a strong DEP force from trapping the largest particles where the first electrode meets the lower wall. Shifting the electrode array slightly upward lowers the DEP force next to the lower wall and allows these large particles to exit through Outlet 5. Since we know that the captured particles travel along the electrodes, we can extend the same electrode pattern to the other electrodes as well. Each electrode initially intersects the channel at an angle α_{el} and then bends to become parallel to it near the outlet channels. Also, we use five electrodes, each one corresponding to an individual outlet channel. This permits us to determine directly for each particle its outlet channel depending on which electrode captures it.

Since in the experiments the electrode array is on a glass slide separate from the PDMS channels, it is difficult to position it in exactly the same place from run to run. The revised electrode geometry sketched in Figure 13 provides more repeatability in the experiments compared to the original design, since the outlet positions of the particles do not depend on where the electrodes intersect the outlet channels. Thus, with this angled electrode array design, the particles will still exit through the same outlets, regardless of small shifts in the electrode array placement along the flow (left-right) direction. Furthermore, since each electrode aligns with an outlet channel, it is also easier to place the electrode array more repeatably in the cross-channel (up-down) direction.

The angle α_{el} can also be adjusted to optimize the electrode design. A shallower angle permits the capture of smaller particles, all else being equal, since then a larger portion of the DEP force is directed against the fluid flow and the particles thus spend more time in the region near the electrode gap. Figure 14 shows the analytically calculated



Figure 14: Analytically calculated particle deflection across the channel versus particle size for $\alpha_{\rm el}$ of 15° and 30° over a single pair of electrodes. A shallower angle permits the capture of smaller particles, all else being equal.

particle deflection versus particle size, as the particles cross a single pair of 1000 μ m long electrodes. The shallower angle of $\alpha_{\rm el} = 15^{\circ}$ can capture particles around 4 or 5 μ m and larger, but when $\alpha_{\rm el}$ is increased to 30°, particles about 7 μ m and larger can be captured by the DEP force. However, a very shallow angle requires a much longer DEP channel section, and since we have five electrodes, we simply choose the angle to be 45°.

Finally, we can compare the sorting calculated from the analytical electric potential solution to the results from the experiment in Section 2.1. Once again, we seed particles at various vertical heights in the channel and calculate their trajectories to determine the outlet channels. Figure 15 shows the percentage of particles that exit through each outlet channel compared with the experiment's results that were shown in Figure 7 on page 15. The RMS voltage difference in both is $V_0 = 24.6$ V. Qualitatively, both methods predict similar particle sorting, although there are some differences in the percentages of particles in specific outlet channels. While for only three particle sizes,



Figure 15: Bar plot showing the percentage of polystyrene particles collected in each outlet channel from the analytical electric potential solution compared to the data from the experiment. The RMS voltage difference between successive electrodes is $V_0 = 24.6$ V for both cases.

this accuracy is fine, to take full advantage of all five outlet channels with five different particle sizes, we must be able to predict more accurately exactly which outlet channel corresponds to which sizes of particles.

There are several possible sources of the error between the experiment and our theoretical analysis. Our theory neglects the polarization of the electric double layer around the particle, but it may actually cause a dipolar force. There may also be induced electro-osmosis due to the interaction of the electric double layer next to the electrode with the induced electric field. The Dukhin number [26]

$$Du \equiv \frac{\kappa_p^{\sigma}}{\sigma_m a} = \frac{1}{2} \frac{\sigma_p}{\sigma_m}$$
(2.14)

determines the relative importance of the surface conductivity. Here, κ^{σ} is the surface conductivity of the particle and σ_m is the bulk conductivity of the fluid medium. For a 2.5 μ m polystyrene particle in water, we have Du = 0.137 < 1. Since the Dukhin number is small, the double layer effects should also be small. Our analysis also assumes an isothermal fluid. However, there may also be secondary flows due to electrothermal effects that may play a role in particle sorting, such as electrothermal flow and thermal DEP. Now, let us discuss the effects of a nonuniform temperature field.

2.4. Electrothermal Effects

All the analysis in the previous sections has implicitly assumed that the entire system is isothermal. However, this assumption is not necessarily true. Since the application of an electric field often involves transmission of current through the suspending solution, Joule heating may cause temperature gradients [4, 50]. For a current density **J** and an electric field **E**, the power per unit volume generated through Joule heating is equal to $\mathbf{J} \cdot \mathbf{E}$, or $\sigma_m E^2$. For a fluid medium with high electrical conductivity or with a strong applied electric field, the Joule heating within the fluid medium can be considerable. As it turns out, most biological fluids do have high electrical conductivity. For example, blood plasma has a conductivity $\sigma_m = 1.3$ S/m [22]. Common buffer solutions used for biological systems try to match the properties of blood and thus have similar conductivity values [31, 42]. Also, the conductivity of seawater is around $\sigma_m = 5$ S/m [36]. Thus, for biological and marine systems the thermal effects due to Joule heating may be significant.

Ramos et al. [50] introduced the simple model electrode arrangement shown in Figure 16 to estimate the temperature rise due to Joule heating from analytical solutions for the electric and temperature fields. The electric field is given by $\mathbf{E}(r) = (V/\pi r)\hat{\mathbf{n}}_{\theta}$. Solving the temperature balance equation with Joule heating, $(k_m/r^2)(d^2T/d\theta^2) + \sigma_m E^2 = 0$, yields the thermal field $T(\theta) = (\sigma_m V^2 \theta / 2\pi k_m)(1 - \theta/\pi)$. Here, k_m is the thermal conductivity of the fluid medium. The maximum temperature rise is found to be



Figure 16: Electrode arrangement for solving the electric field and the thermal field due to Joule heating [50]. The electric field is given by $\mathbf{E}(r) = (V/\pi r)\hat{\mathbf{n}}_{\theta}$. Solving the temperature balance equation with Joule heating $(k_m/r^2)(d^2T/d\theta^2) + \sigma_m E^2 = 0$ with boundary conditions $T(0) = T(\pi) = 0$ yields the thermal field $T(\theta) = (\sigma_m V^2 \theta/2\pi k_m)(1 - \theta/\pi)$. The maximum temperature rise is then $T_{\text{max}} = \sigma_m V^2/8k_m$ at $\theta = \pi/2$.

 $T_{\text{max}} = \sigma_m V^2/8k_m$ at $\theta = \pi/2$. An analysis of Joule heating in a parallel capacitor model found the same maximum temperature rise [50]. The thermal conductivity of blood plasma $k_m = 0.571 \text{ W/m/K}$ [48], which is similar to that of water, and a voltage difference of only 10 V could cause a Joule heating temperature rise of as much as 28°C in the device!

Temperature gradients may also result from external heating on the boundary [4]. Often, heating sources are deliberately introduced in order to control chemical reactions and biological interactions, but the external heating may also arise from unintended artifacts, such as microscope illumination [16, 17]. Even in low conductivity fluids a temperature rise of 15°C has been measured in experiments [17]. Thus, external heating sources may also be a significant source of temperature changes.

Since temperature affects the electrical properties of materials, a nonuniform temperature field causes permittivity and conductivity gradients that induce, respectively, bound and free charges. These induced charges, in turn, lead to the electrical body force [50]

$$\mathbf{f}_{\rm ET} = -\frac{1}{2} \Big[\frac{\left[\left(\frac{\nabla \sigma}{\sigma} - \frac{\nabla \epsilon}{\epsilon} \right) \cdot \mathbf{E} \right] \epsilon \mathbf{E}}{1 + (\omega \epsilon / \sigma)^2} + \frac{1}{2} |\mathbf{E}|^2 \nabla \epsilon \Big].$$
(2.15)



Figure 17: The electrical body force (2.15) can cause electrothermal flow in a fluid with nonuniform temperature. This flow is generally characterized by the presence of counter-rotating vortices. The orientation of the vortices depends on the material properties and the frequency of the electric field. In this case the fluid experiences a downward electrothermal force over the electrode gap, causing a clockwise eddy over the left electrode and a counterclockwise eddy over the right electrode. Image from Ramos et al. [50].

This body force can cause electrothermal flow in a fluid with nonuniform temperature (Figure 17). We have just seen that the Joule heating temperature rise scales with V^2 , which means that the changes in electrical permittivity and conductivity do likewise. The force expression (2.15) includes an additional multiplication by E^2 , so that the electrothermal force \mathbf{f}_{ET} scales as V^4 . Therefore, at higher voltages electrothermal flow can dominate over the DEP force, which scales with V^2 . In our particle sorting experiments electrothermal flow becomes dominant at voltages in the neighborhood of 35 or 40 V, but it may need to be considered at even lower voltages as well.

In addition to applications related to the sorting of biological particles and cells, another potential application area that includes both electric and thermal fields is the enhancement of nucleate boiling heat transfer through electrohydrodynamics (EHD). Boiling heat transfer is an important process in a wide array of applications, from everyday items such as household refrigerators or personal electronic devices, to largescale heat exchangers in nuclear reactor power plants, and even to space applications under microgravity conditions. Often, these applications take advantage of the higher heat transfer offered by the phase change in nucleate boiling. Two major engineering problems in the nucleate boiling regime are how to enhance the rate of heat transfer and how to increase the critical heat flux so that more efficient and compact designs of thermal transport systems are feasible.

Heat transfer rates can be modified through active techniques involving external forces, such as from an electric field, or by passive methods, such as designing specialized surface or fluid properties [34]. The EHD field contains much research on using an externally applied electric field to enhance nucleate boiling heat transfer [1, 27, 34]. In particular the DEP force due to a nonuniform applied electric field is commonly used in nucleate boiling to modify the behavior of the nucleated vapor bubbles [56–59].

In nucleate boiling applications, the fluid may actually experience large temperature gradients due to the externally applied heat sources. An accurate understanding of the forces acting on vapor bubbles and/or solid particles in such media is essential to optimize the heat transfer processes in these systems. The enhancing effect of an electric field on heat transfer rates has been known for almost 100 years now, and the potential for using EHD to modify boiling heat transfer has been explored by numerous research groups since the 1960s, as shown by the review papers [1, 27, 34]. Increasingly, it has been realized since the work of Ogata and Yabe [43] that the augmented heat transfer rate in nucleate boiling is due to the electrostatic force that pushes vapor bubbles onto the heated surface and keeps the bubbles sliding on the surface, disrupting the thermal boundary layer.

The source of the force on the vapor bubbles was simply described as the DEP force [1, 43]. Most of the theoretical studies for bubble growth and motion conducted so far [5, 6, 10, 32, 33, 44] do not fully account for the couplings between the electric field and the temperature field. The classical DEP force has also often been used

to compute the force acting on solid particles even in the presence of temperature gradients [4, 21, 46, 50]. Thus, the electrostatic force on the particle is generally assumed to be given simply by the classical DEP force, even under the combined effects of electric and thermal fields, although it has been recognized that the electrical body force (2.15) can induce electrothermal flow in a fluid. The direct contribution of temperature gradients to the DEP force on a particle suspended in the fluid has not been determined. The classical DEP force expression (1.3) assumes that the system is isothermal. Since we know that a nonuniform temperature field modifies the electrical force on the fluid, it stands to reason that it may also modify the electrical force on particles within the fluid, and the classical DEP force expression may not fully account for the entire force in this case. This project addresses a fundamental question: what is the electrostatic force and moment acting on a particle immersed in a fluid medium under the combined effects of electric and thermal gradient fields? Our objective is to determine the additional electrostatic force, which we term thermal DEP (T-DEP) force, that acts on a spherical particle (either solid particle or vapor bubble) in the presence of temperature gradients and to demonstrate that, in fact, the classical DEP needs to be modified in the presence of temperature gradients. We derive a formula analogous to the classical DEP expression (1.3) that can be easily used to calculate the thermal DEP force on a particle in a non-isothermal fluid.

CHAPTER 3 : Classical DEP Force

In this chapter let us examine more closely the origins of the classical DEP force expression (1.3) to gain insight into a method for deriving the thermal DEP force.

3.1. Dipole Moment Approximation

The expression (1.3) is known as the dipole moment approximation for the DEP force. It is derived by placing the particle in a uniform background electric field and solving for the dipole moment $\bar{\mathbf{p}}$ induced by this particle. However, the DEP force is caused by the interaction of this dipole with a *nonuniform* electric field and thus requires a background electric field gradient to be nonzero. To take care of this apparent contradiction, one can calculate the dielectric force more rigorously by integrating the electromagnetic Maxwell stress tensor (MST)

$$\boldsymbol{\sigma}_{\rm MST} = \frac{\epsilon}{4} \left(\widetilde{\mathbf{E}} \widetilde{\mathbf{E}}^* + \widetilde{\mathbf{E}}^* \widetilde{\mathbf{E}} - (\widetilde{\mathbf{E}} \cdot \widetilde{\mathbf{E}}^*) \mathbf{I} \right)$$
(3.1)

over the particle's surface [54, 62]. In the MST expression $\tilde{\mathbf{E}}$ represents the actual electric field in the particle's presence, and \mathbf{I} is the identity tensor. Computing the force from the Maxwell stress tensor does not require the assumption of a uniform background electric field. While the MST approach can be used more generally, solving the actual electric field $\tilde{\mathbf{E}}$ and integrating the MST to obtain the force is impractical in many cases due to the abrupt change in dielectric properties at the particle surface causing numerical imprecision as well as the formidable job of continually recalculating the local electric field as the particle moves. As we shall see, however, the dipole moment approximation of the force exactly matches the leading order term of the force derived from the Maxwell stress tensor approach [62]. When higher order terms



Figure 18: A spherical particle of radius a, with permittivity and conductivity ϵ_p and σ_p , respectively, is placed in a fluid medium with permittivity and conductivity ϵ_m and σ_m , respectively. At the location of the particle, the background electric field is $\widetilde{\mathbf{E}}_c^{o}$ along the z-axis.

are negligible, the dipole moment approximation can be used to compute the DEP force at a much lower computational cost without much sacrifice in accuracy and is preferred over the Maxwell stress tensor approach.

3.2. Problem Setup

For this analysis of the classical DEP force, consider the case of a spherical particle of radius a, with permittivity and conductivity ϵ_p and σ_p , respectively, placed in an infinite fluid medium with permittivity and conductivity ϵ_m and σ_m , respectively (Figure 18). In the global coordinate system the particle is located at $\mathbf{X} = \mathbf{X}_c$. In the neighborhood of the particle, we use a spherical coordinate system (r, θ, φ) centered at the location of the particle:

$$x = r \sin \theta \cos \varphi$$

$$y = r \sin \theta \sin \varphi$$

$$z = r \cos \theta$$

(3.2)

Here, θ is the polar angle measured from the z-axis ($0 \le \theta \le \pi$) and ϕ is the azimuthal angle ($0 \le \phi < 2\pi$). The unit vectors in this spherical coordinate system are given by

$$\hat{\mathbf{e}}_{r} = \sin\theta\cos\varphi\hat{\mathbf{e}}_{x} + \sin\theta\sin\varphi\hat{\mathbf{e}}_{y} + \cos\theta\hat{\mathbf{e}}_{z}$$

$$\hat{\mathbf{e}}_{\theta} = \cos\theta\cos\varphi\hat{\mathbf{e}}_{x} + \cos\theta\sin\varphi\hat{\mathbf{e}}_{y} - \sin\theta\hat{\mathbf{e}}_{z}$$

$$\hat{\mathbf{e}}_{\varphi} = -\sin\varphi\hat{\mathbf{e}}_{x} + \cos\varphi\hat{\mathbf{e}}_{y}$$
(3.3)

Without loss of generality, we can assign the z-axis to be the direction of the background electric field at the location of the particle, $\tilde{\mathbf{E}}_{c}^{o} = \tilde{\mathbf{E}}^{o}(\mathbf{X}_{c}, t) = \mathbf{E}^{o}(\mathbf{X}_{c}) \exp(j\omega t)$. Throughout this analysis the $\exp(j\omega t)$ time-dependent portion is dropped, and only the position-dependent portion, i.e., the time-independent or time-averaged (RMS) electric field, is used. Thus, the time-averaged electric field $\mathbf{E}_{c}^{o} = \mathbf{E}^{o}(\mathbf{X}_{c}) = E_{c}^{o}\hat{\mathbf{e}}_{z}$, where E_{c}^{o} is the magnitude of the background electric field evaluated at the location of the particle. Furthermore, we divide the space around the particle into an outer (far) $r \sim O(L)$ region, where L is a length scale associated with variations in the background electric field, in which the presence of the particle has negligible effect, and an inner (local) $r \sim O(a)$ region in the neighborhood of the particle. In the outer region we scale length with L and use $X \sim O(1)$ to denote the outer region's coordinate system. In the inner region we introduce the stretched coordinate $\xi = r/a$. In the global coordinate system the gradient of a scalar function ϕ is given by

$$\nabla_{X}\phi = \frac{\partial\phi}{\partial r}\hat{\mathbf{e}}_{r} + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{\mathbf{e}}_{\theta} + \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\varphi}\hat{\mathbf{e}}_{\varphi}, \qquad (3.4)$$

and the Laplacian of ϕ by

$$\nabla_X^2 \phi = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \phi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \phi}{\partial \varphi^2}, \tag{3.5}$$

where the subscript X denotes the global (outer) coordinate system. Correspondingly, in the neighborhood of the particle we have $\nabla_{\xi}\phi = a\nabla_{X}\phi$ and $\nabla_{\xi}^{2}\phi = a^{2}\nabla_{X}^{2}\phi$, while $\hat{\mathbf{e}}_{\xi} = \hat{\mathbf{e}}_{r}$.

For an AC electric field of frequency ω we have $\widetilde{\mathbf{E}}(\mathbf{X}, t) = \overline{\mathbf{E}}(\mathbf{X}) \exp(j\omega t)$. The electric field is related to the free charge $\widetilde{\rho}_f$ through Gauss' law:

$$\nabla \cdot (\epsilon \widetilde{\mathbf{E}}) = \widetilde{\rho}_f, \tag{3.6}$$

while the change in free charge with time gives rise to a current density:

$$\nabla \cdot \widetilde{\mathbf{J}} + \frac{\partial \widetilde{\rho}_f}{\partial t} = 0, \qquad (3.7)$$

where the current density $\widetilde{\mathbf{J}}$ is defined as

$$\widetilde{\mathbf{J}} = \sigma \widetilde{\mathbf{E}}.\tag{3.8}$$

Since the time-dependence for the AC field is given by the expression $\exp(j\omega t)$, the time derivative $\partial/\partial t$ can be expressed as multiplication by $j\omega$. We see from (3.7)

and (3.8) that

$$\nabla \cdot (\sigma \bar{\mathbf{E}}) = -j\omega\rho_f. \tag{3.9}$$

Adding (3.6) with (3.9), we have

$$\nabla \cdot \left[\left(\epsilon + \frac{\sigma}{j\omega} \right) \bar{\mathbf{E}} \right] = 0. \tag{3.10}$$

Thus, for a lossy dielectric material we can replace the real permittivity ϵ with an equivalent complex expression,

$$\bar{\epsilon} = \epsilon + \frac{\sigma}{j\omega} = \epsilon - j\frac{\sigma}{\omega}.$$
(3.11)

The complex permittivity is a mathematical way to account for the lossy effects of a dielectric material under an applied AC electric field. Rather than solving the two separate equations (3.6) and (3.7), with the complex permittivity we can combine them into a single equation describing the AC electric field.

The governing equation for the time-independent electric field in the outer region is then given by

$$\nabla_X \cdot \left(\bar{\epsilon}_m \mathbf{E}^o \right) = 0. \tag{3.12}$$

This outer electric field can be computed by solving (3.12) using the usual numerical means for the particular device of interest, disregarding the presence of the particle, and its solution can then be expanded into a Taylor series in terms of $\mathbf{x} = \mathbf{X} - \mathbf{X}_c$ in the neighborhood of the particle:

$$\mathbf{E}^{o}(\mathbf{x}) = \mathbf{E}^{o}_{c} + \mathbf{x} \cdot \nabla_{X} \mathbf{E}^{o}_{c}.$$
(3.13)

The gradient of the background field evaluated at the location of the particle is given by

the term $\nabla_X \mathbf{E}_c^o$. This electric field gradient arises from the geometry of the electrodes designed to produce a nonuniform field. We assume that the particle radius a is small compared to the characteristic dimension L of the outer electric field, that is, $\chi = a/L \ll 1$. Thus, we keep only up to the linear term in (3.13) and drop any higher order terms.

In the inner region the governing equation is given by

$$\nabla_{\xi} \cdot \left(\bar{\epsilon} \bar{\mathbf{E}}^i\right) = 0. \tag{3.14}$$

To complete the solution process of the inner problem, we must match the inner dependent variables to the outer ones in the limit $\xi \to \infty$ and $\mathbf{X} \to \mathbf{X}_c$, i.e.,

$$\lim_{\xi \to \infty} \bar{\mathbf{E}}^i = \lim_{\mathbf{X} \to \mathbf{X}_c} \bar{\mathbf{E}}^o.$$
(3.15)

In response to the outer electric field (3.13) the corresponding inner electric field can be expressed as

$$\bar{\mathbf{E}}^i = \bar{\mathbf{E}}^i_0 + \delta_E \bar{\mathbf{E}}^i_1. \tag{3.16}$$

The terms on the right hand side of (3.16) are the electric field resulting from the applied far field electric field and the nonuniform component of the applied electric field due to the electrode geometry in the device, respectively. The small parameter δ_E arises directly from the dimensionless form of (3.13) and is similar to the ratio $\xi = a/l$. It is defined as

$$\delta_E \equiv \frac{a |\nabla_X \mathbf{E}_c^o|}{|\mathbf{E}_c^o|} \ll 1. \tag{3.17}$$

For concreteness, we consider the case where the applied electric field only varies in the z-direction, $\mathbf{x} \cdot \nabla_{\!X} \bar{\mathbf{E}}_c^o = \delta_E \frac{z}{a} E_c^o \hat{\mathbf{e}}_z$. Variations of the electric field in other space directions can be similarly treated, but due to symmetry do not yield any net DEP force.

3.3. Electric Potential Solution

For this isothermal case the complex permittivity $\bar{\epsilon}$ in the governing equation (3.14) is a constant and can be taken out of the expression. Since the electric field $\bar{\mathbf{E}}^i = -\nabla_{\xi} \bar{\phi}^i$, rewriting (3.14) in terms of the electric potential $\bar{\phi}^i$ yields Laplace's equation for the inner electric potential:

$$\nabla_{\xi}^2 \bar{\phi}^i = 0, \qquad (3.18)$$

Since there is no φ -dependence in the problem, we take advantage of axisymmetry to simplify the equation. Thus, (3.18) can be written as

$$\frac{1}{\xi^2} \frac{\partial}{\partial \xi} \left(\xi^2 \frac{\partial \bar{\phi}^i}{\partial \xi} \right) + \frac{1}{\xi^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \bar{\phi}^i}{\partial \theta} \right) = 0, \tag{3.19}$$

The general solution to (3.19) is of the form

$$\bar{\phi}^i(\xi,\theta) = \sum_{l=0}^{\infty} \left(\bar{A}_l \xi^l + \bar{B}_l \xi^{-(l+1)} \right) P_l(\cos\theta), \qquad (3.20)$$

where \bar{A}_l and \bar{B}_l are constants to be determined, and $P_l(\cos \theta)$ are the Legendre polynomials:

$$P_0(\cos \theta) = 1$$

$$P_1(\cos \theta) = \cos \theta$$

$$P_2(\cos \theta) = \frac{1}{2}(3\cos^2 \theta - 1)$$

$$\vdots$$
(3.21)

To solve the potential both inside the particle $\bar{\phi}_p^i$ and in the surrounding medium $\bar{\phi}_m^i$, we need a total of four boundary conditions. For the far-field boundary condition, we have the matching condition with the outer electric potential (3.13), which requires a quadratic form for the potential:

$$\bar{\phi}_m^i(\xi \to \infty) = -E_c^o a\xi \cos\theta - \frac{1}{2}\delta_E E_c^o a\xi^2 P_2(\cos\theta).$$
(3.22)

This far-field potential satisfies the general solution of Laplace's equation and also provides the required matching electric field $\mathbf{E}^o = E_c^o (1 + \delta_E z/a) \hat{\mathbf{e}}_z$.

The second and third boundary conditions are at the interface between the particle and the medium $\xi = 1$, where we have continuous potential:

$$\bar{\phi}^i_m|_{\xi=1} = \bar{\phi}^i_p|_{\xi=1}; \tag{3.23}$$

and continuous displacement normal to the sphere:

$$\left[\bar{\epsilon}_m \frac{\partial \bar{\phi}_m^i}{\partial \xi}\right]_{\xi=1} = \left[\bar{\epsilon}_p \frac{\partial \bar{\phi}_p^i}{\partial \xi}\right]_{\xi=1}.$$
(3.24)

Finally, the potential $\bar{\phi}^i_p$ must be finite as $\xi \to 0$:

$$\left. \bar{\phi}_p^i \right|_{\xi=0} = \text{finite.} \tag{3.25}$$

From the general solution (3.20) and the boundary condition (3.22), the potential in the medium is of the form

$$\bar{\phi}_{m}^{i} = -E_{c}^{o}a\xi\cos\theta - \frac{1}{2}\delta_{E}E_{c}^{o}a\xi^{2}P_{2}(\cos\theta) + \sum_{l=0}^{\infty}\bar{B}_{l}\xi^{-(l+1)}P_{l}(\cos\theta).$$
(3.26)

Similarly, (3.20) with the boundary condition (3.25) requires that

$$\bar{\phi}_p^i = \sum_{l=0}^{\infty} \bar{A}_l \xi^l P_l(\cos\theta).$$
(3.27)

Now, we solve for the coefficients \bar{A}_l and \bar{B}_l from the boundary conditions (3.23) and (3.24):

$$-E_{c}^{o}a\cos\theta - \frac{1}{2}\delta_{E}E_{c}^{o}aP_{2}(\cos\theta) + \bar{B}_{0} + \bar{B}_{1}\cos\theta + \bar{B}_{2}P_{2}(\cos\theta) + \cdots$$
$$= \bar{A}_{0} + \bar{A}_{1}\cos\theta + \bar{A}_{2}P_{2}(\cos\theta) + \cdots$$
(3.28)

and

$$-\bar{\epsilon}_m E_c^o a \cos\theta - \delta_E \bar{\epsilon}_m E_c^o a P_2(\cos\theta) - \bar{\epsilon}_m \bar{B}_0 - 2\bar{\epsilon}_m \bar{B}_1 \cos\theta - 3\bar{\epsilon}_m \bar{B}_2 P_2(\cos\theta) + \cdots$$
$$= \bar{\epsilon}_p \bar{A}_1 \cos\theta + 2\bar{\epsilon}_p \bar{A}_2 P_2(\cos\theta) + \cdots, \quad (3.29)$$

respectively. The orthogonality property of the Legendre polynomials allows us to match coefficients of $P_l(\cos\theta)$ and solve for the coefficients \bar{A}_l and \bar{B}_l . From the constant $P_0(\cos\theta)$ terms,

$$\bar{A}_0 = \bar{B}_0 = 0;$$
 (3.30)

from the $P_1(\cos\theta)$ terms,

$$\bar{A}_1 = -E_c^o a \frac{3\bar{\epsilon}_m}{\bar{\epsilon}_p + 2\bar{\epsilon}_m} = -E_c^o a (1 - \bar{K}_E),$$

$$\bar{B}_1 = E_c^o a \frac{\bar{\epsilon}_p - \bar{\epsilon}_m}{\bar{\epsilon}_p + 2\bar{\epsilon}_m} = E_c^o a \bar{K}_E;$$
(3.31)

and from the $P_2(\cos\theta)$ terms,

$$\bar{A}_{2} = -\frac{1}{2} \delta_{E} E_{c}^{o} a \; \frac{5\bar{\epsilon}_{m}}{2\bar{\epsilon}_{p} + 3\bar{\epsilon}_{m}} = -\frac{1}{2} \delta_{E} E_{c}^{o} a (1 - \bar{K}_{E,2}),$$

$$\bar{B}_{2} = -\frac{1}{2} \delta_{E} E_{c}^{o} a \frac{2(\bar{\epsilon}_{p} - \bar{\epsilon}_{m})}{2\bar{\epsilon}_{p} + 3\bar{\epsilon}_{m}} = -\frac{1}{2} \delta_{E} E_{c}^{o} a \bar{K}_{E,2}.$$
(3.32)

For $l \geq 3$, the coefficients $\bar{A}_l = \bar{B}_l = 0$. In the above expressions \bar{K}_E is the (first order) Clausius–Mossotti factor stated previously (1.4) and $\bar{K}_{E,2}$ is the second-order Clausius–Mossotti factor, where in general the *l*th-order factor is given by [62]

$$\bar{K}_{E,l} = \frac{l(\bar{\epsilon}_p - \bar{\epsilon}_m)}{l\bar{\epsilon}_p + (l+1)\bar{\epsilon}_m}.$$
(3.33)

In summary, the inner electric potential in the medium and in the particle can be written, respectively, as

$$\bar{\phi}_{m}^{i} = -E_{c}^{o}a\xi(1-\bar{K}_{E}\xi^{-3})\cos\theta - \frac{1}{2}\delta_{E}E_{c}^{o}a\xi^{2}(1-\bar{K}_{E,2}\xi^{-5})P_{2}(\cos\theta) = \bar{\phi}_{0m}^{i} + \delta_{E}\bar{\phi}_{1m}^{i}$$
(3.34)

and

$$\bar{\phi}_{p}^{i} = -E_{c}^{o}a\xi(1-\bar{K}_{E})\cos\theta - \frac{1}{2}\delta_{E}E_{c}^{o}a\xi^{2}(1-\bar{K}_{E,2})P_{2}(\cos\theta) = \bar{\phi}_{0p}^{i} + \delta_{E}\bar{\phi}_{1p}^{i}.$$
 (3.35)

Notice that the electric potentials in the inner region can be expressed in terms of a zeroth-order term and a first-order term proportional to δ_E , which corresponds to the perturbation form of the inner electric field stated in (3.16).

3.4. Maxwell Stress Tensor Integration

To compute the classical DEP force using the Maxwell stress tensor (3.1), we integrate the MST over the surface of a spherical control volume located within the medium, just outside the surface of the particle ($\xi = 1^+$). This choice of a control volume will include both the stress from the bulk charge inside the particle and the surface charge at the interface between particle and medium. If we were to use the electric field in the particle instead and evaluate the MST on the inside of the particle surface, we would also need to include the jump condition at the interface due to the effects of the accumulated surface charge on the surface of the particle in order to determine the total force on the particle. Thus, the electric field in the MST expression must be evaluated using the electric field in the medium.

The electric field in the medium follows directly from (3.34):

$$\bar{\mathbf{E}}_m^i = -\nabla_{\!X} \bar{\phi}_m^i = \bar{E}_r \hat{\mathbf{e}}_r + \bar{E}_\theta \hat{\mathbf{e}}_\theta, \qquad (3.36)$$

where the components \bar{E}_r and \bar{E}_{θ} are given by

$$\bar{E}_{r} = E_{c}^{o}(1 + 2\bar{K}_{E}\xi^{-3})\cos\theta + \frac{1}{2}\delta_{E}E_{c}^{o}\xi(2 + 3\bar{K}_{E,2}\xi^{-5})P_{2}(\cos\theta),$$

$$\bar{E}_{\theta} = -E_{c}^{o}(1 - \bar{K}_{E}\xi^{-3})\sin\theta - \frac{3}{2}\delta_{E}E_{c}^{o}\xi(1 - \bar{K}_{E,2}\xi^{-5})\cos\theta\sin\theta.$$
(3.37)

Again, notice that each electric field component has the perturbation form (3.16), with a zeroth-order term arising from the applied outer electric field and a first-order term proportional to δ_E due to the nonuniformity of this outer electric field. These electric field components \bar{E}_r and \bar{E}_{θ} are evaluated at $\xi = 1$ in order to compute the force.

The time-averaged electric force on the particle is given by the integration of the

Maxwell stress tensor over the particle's surface:

$$\mathbf{F} = \int_{S} \boldsymbol{\sigma}_{\mathrm{MST}} \cdot \hat{\mathbf{n}} \, dS. \tag{3.38}$$

Note that the expression (3.1) is already the time-independent portion of the stress tensor [62]. For the spherical surface the outward unit normal vector $\hat{\mathbf{n}} = \hat{\mathbf{e}}_r$. Then,

$$\mathbf{F} = \int_{S} \frac{\epsilon_{m}}{4} \left[\bar{\mathbf{E}}_{m}^{i} (\bar{\mathbf{E}}_{m}^{i*} \cdot \hat{\mathbf{e}}_{r}) + \bar{\mathbf{E}}_{m}^{i*} (\bar{\mathbf{E}}_{m}^{i} \cdot \hat{\mathbf{e}}_{r}) - (\bar{\mathbf{E}}_{m}^{i} \cdot \bar{\mathbf{E}}_{m}^{i*}) \hat{\mathbf{e}}_{r} \right] dS$$

$$= \int_{S} \frac{\epsilon_{m}}{4} \left[(\bar{E}_{r} \bar{E}_{r}^{*} - \bar{E}_{\theta} \bar{E}_{\theta}^{*}) \hat{\mathbf{e}}_{r} + (\bar{E}_{r} \bar{E}_{\theta}^{*} + \bar{E}_{r}^{*} \bar{E}_{\theta}) \hat{\mathbf{e}}_{\theta} \right] dS.$$
(3.39)

Note that the permittivity in the MST expression is the real (actual) permittivity ϵ , not the complex permittivity $\bar{\epsilon}$, and it does not include the lossy effects of the conductivity. Due to symmetry, the MST integration over the spherical surface gives no net force in the $\hat{\mathbf{e}}_x$ and $\hat{\mathbf{e}}_y$ directions, and the only nonzero force component is in the $\hat{\mathbf{e}}_z$ direction, which is also the direction of the background electric field. This force component is given by

$$F_{z} = \mathbf{F} \cdot \hat{\mathbf{e}}_{z}$$

$$= \int_{S} \frac{\epsilon_{m}}{4} \left[\left(\bar{E}_{r} \bar{E}_{r}^{*} - \bar{E}_{\theta} \bar{E}_{\theta}^{*} \right) \cos \theta - \left(\bar{E}_{r} \bar{E}_{\theta}^{*} + \bar{E}_{r}^{*} \bar{E}_{\theta} \right) \sin \theta \right] dS$$

$$= \int_{0}^{2\pi} \int_{0}^{\pi} \frac{\epsilon_{m}}{4} \left[\left(\bar{E}_{r} \bar{E}_{r}^{*} - \bar{E}_{\theta} \bar{E}_{\theta}^{*} \right) \cos \theta - \left(\bar{E}_{r} \bar{E}_{\theta}^{*} + \bar{E}_{r}^{*} \bar{E}_{\theta} \right) \sin \theta \right] a^{2} \sin \theta \, d\theta \, d\varphi$$

$$= \frac{\pi}{2} \epsilon_{m} a^{2} \int_{0}^{\pi} \left[\left(\bar{E}_{r} \bar{E}_{r}^{*} - \bar{E}_{\theta} \bar{E}_{\theta}^{*} \right) \cos \theta - \left(\bar{E}_{r} \bar{E}_{\theta}^{*} + \bar{E}_{r}^{*} \bar{E}_{\theta} \right) \sin \theta \right] \sin \theta \, d\theta$$

$$(3.40)$$

with the electric field components \mathbf{E}_r and \mathbf{E}_{θ} from (3.37) evaluated at $\xi = 1$. To

simplify the integrations, we use the following integration formulas:

$$\int_{0}^{\pi} \cos^{m} \theta \sin \theta \, d\theta = -\frac{(-1)^{m+1} - 1}{m+1} = \begin{cases} 0, & m \text{ odd} \\ \frac{2}{m+1}, & m \text{ even} \end{cases}$$
(3.41)

$$\int_0^{\pi} \cos^m \theta \sin^3 \theta \, d\theta = \frac{(-1)^{m+3} - 1}{m+3} - \frac{(-1)^{m+1} - 1}{m+1} = \begin{cases} 0, & m \text{ odd} \\ \frac{4}{(m+1)(m+3)}, & m \text{ even} \end{cases}$$
(3.42)

In evaluating the integrals we only retain the linear terms in the small parameter δ_E , since any higher order (quadratic) terms in δ_E will be negligible. Evaluating the integral (3.40) yields

$$F_{z} = \frac{\pi}{2} \epsilon_{m} a^{2} \left[(E_{c}^{o})^{2} \cdot 2\delta_{E} (\bar{K}_{E} + \bar{K}_{E}^{*}) \right].$$
(3.43)

For a complex number $\bar{z} = x + jy$, we have the identity

$$\bar{z} + \bar{z}^* = (x + jy) + (x - jy) = 2x = 2 \operatorname{Re}(\bar{z}),$$
(3.44)

Also, δ_E is defined in (3.17) and is given by $a|\nabla_X \mathbf{E}_c^o|/E_c^o$. Thus, we rewrite the force expression (3.43) as

$$F_z = 2\pi\epsilon_m a^3 \operatorname{Re}(\bar{K}_E) E_c^o |\nabla_X \mathbf{E}_c^o|$$
(3.45)

or

$$F_z = \frac{3}{2} \epsilon_m V_p \operatorname{Re}(\bar{K}_E) E_c^o |\nabla_X \mathbf{E}_c^o|, \qquad (3.46)$$

where V_p is the volume of the spherical particle. Since both the electric field and its gradient are in the $\hat{\mathbf{e}}_z$ direction, the only possible vector combination that can match

the F_z force component (3.46) is given by $\mathbf{E}_c^o \cdot \nabla_X \mathbf{E}_c^o$. This dot product also gives the direction of the classical DEP force for any arbitrary directions of the electric field and electric field gradient. Thus, the classical DEP force evaluated from the Maxwell stress tensor is given by

$$\mathbf{F}_{\text{DEP}} = \frac{3}{2} \epsilon_m V_p \operatorname{Re}(\bar{K}_E) (\mathbf{E}_c^o \cdot \nabla_X \mathbf{E}_c^o).$$
(3.47)

Notice this expression exactly matches the classical DEP expression (1.3) that is derived from the dipole moment approximation, as expected and as shown in the literature [28, 29, 62]. We see that the dipole moment approximation merely uses the assumption of a small particle to calculate the leading order term for the DEP force. We have used this same assumption in the Maxwell stress tensor approach by truncating the Taylor series expansion of the outer electric field at the linear term (3.13). Thus, the resulting force from both methods is equal. The dipole moment approximation is thus valid only when the length scale of the electric field variation is large compared to the size of the particle. If this assumption is not accurate, then the dipole moment approximation must be modified to include the quadrupole or other higher multipole moments to accurately predict the DEP force on the particle.

The Maxwell stress tensor approach also yields some additional insight into the origins of the classical DEP force. The force expressed in (3.43) is proportional to δ_E , and this is the leading order term of the force. Due to symmetry the interaction of the zeroth order terms $\bar{\mathbf{E}}_0^i \bar{\mathbf{E}}_0^i$ in the MST do not cause any net force on the particle. Likewise the interaction of the first order terms $(\delta_E \bar{\mathbf{E}}_1^i)(\delta_E \bar{\mathbf{E}}_1^i)$ also cancel each other out over the particle (in addition to being $O(\delta_E^2)$ and thus already negligible) and yield no net force on the particle. The DEP force is thus due to the interaction between the zeroth order and first order terms, $\bar{\mathbf{E}}_0^i(\delta_E \bar{\mathbf{E}}_1^i)$, which directly shows why the leading order of the force is of $O(\delta_E)$. In other words, the DEP force is due to the interaction of the polarization induced in the particle by the zeroth-order electric field with the nonuniformity of the first-order electric field. Wang et al. [62] show the derivation of the classical DEP force to any power of δ_E , which is possible simply through keeping additional higher-order terms in the Taylor series expansion when computing the inner electric field in the Maxwell stress tensor approach.

CHAPTER 4 : Thermal DEP Force — Parallel Case

4.1. Temperature Effects

Now, let's see how the Maxwell stress tensor approach applies to the case of coupled electric and thermal fields, where the particle and medium are acted upon by both an applied electric field and a nonuniform temperature field. The temperature dependence of the electric properties is described by using Taylor series expansions in terms of the temperature deviation from T_c , the temperature at the location of the particle. Accordingly, we define the following parameters:

$$\zeta_{\epsilon} \equiv (1/\epsilon) (d\epsilon/dT),$$

$$\zeta_{\sigma} \equiv (1/\sigma) (d\sigma/dT),$$

$$\bar{\zeta} \equiv (1/\bar{\epsilon}) (d\bar{\epsilon}/dT).$$
(4.1)

These ζ parameters describe the relative change in the material properties with temperature and are modeled as constants.

In this analysis of the thermal dielectrophoretic (thermal DEP) force, we use the same problem setup as in Section 3.2, with a local spherical coordinate system centered at the location of the particle, $\mathbf{X} = \mathbf{X}_c$, except now we additionally include a nonuniform background temperature field by means of an applied temperature gradient (Figure 19). We have an AC electric field $\tilde{\mathbf{E}}^o(\mathbf{X}, t) = \bar{\mathbf{E}}^o(\mathbf{X}) \exp(j\omega t)$, but in this analysis we again use the time-independent portion $\bar{\mathbf{E}}^o(\mathbf{X})$ in order to calculate a time-averaged force [62]. We define the z-axis to be the direction of the background electric field evaluated at the location of the particle, $\bar{\mathbf{E}}^o_c$. At this location the background temperature gradient field $\nabla_X T^o_c$ intersects $\bar{\mathbf{E}}^o_c$ at some arbitrary angle α between 0 and π . Without loss



Figure 19: A spherical particle of radius a, with permittivity and conductivity ϵ_p and σ_p , respectively, is placed in a fluid medium with permittivity and conductivity ϵ_m and σ_m , respectively. At the location of the particle, the background electric field is $\mathbf{\tilde{E}}^o(\mathbf{X}_c, t) = \mathbf{\bar{E}}^o(\mathbf{X}_c) \exp(j\omega t)$ along the z-axis and the background temperature gradient field is $\nabla_X T_c^o$. Both $\mathbf{\bar{E}}_c^o$ and $\nabla_X T_c^o$ are perpendicular to the x-axis, and the angle between them is α .

of generality, we can specify that the temperature gradient is in the yz-plane, and both $\bar{\mathbf{E}}_c^o$ and $\nabla_X T_c^o$ are perpendicular to the *x*-axis, i.e., the unit vector $\hat{\mathbf{e}}_x$ is in the direction of the cross product $\nabla_X T_c^o \times \bar{\mathbf{E}}_c^o$. At the location of the particle, the electric field can only have an $\hat{\mathbf{e}}_z$ component, while the temperature gradient can have at most $\hat{\mathbf{e}}_y$ and $\hat{\mathbf{e}}_z$ components.

4.2. Decomposition

4.2.1. Outer Region

In the outer region the governing equation for the electric field with temperaturedependent permittivity is given by

$$\nabla_{X} \cdot \left(\bar{\epsilon}_{m}(T^{o})\bar{\mathbf{E}}^{o}\right) = 0.$$
(4.2)

For the temperature field we use the steady-state heat equation:

$$\nabla_X \cdot (k_m \nabla_X T^o) + \sigma_m (\bar{\mathbf{E}}^o \cdot \bar{\mathbf{E}}^{o*}) = 0, \qquad (4.3)$$

where k_m is the thermal conductivity of the medium. The solutions to these outer fields can be expanded into Taylor series in terms of $\mathbf{x} = \mathbf{X} - \mathbf{X}_c$ around the particle:

$$\bar{\mathbf{E}}^{o}(\mathbf{x}) = \bar{\mathbf{E}}_{c}^{o} + \mathbf{x} \cdot \nabla_{\!X} \bar{\mathbf{E}}_{E,c}^{o} + \mathbf{x} \cdot \nabla_{\!X} \bar{\mathbf{E}}_{T,c}^{o}$$
(4.4)

and

$$T^{o}(\mathbf{x}) = T^{o}_{c} + \mathbf{x} \cdot \nabla_{X} T^{o}_{c}.$$

$$(4.5)$$

Likewise, the temperature-dependent permittivity is expressed as

$$\bar{\epsilon}_m(T^o) = \bar{\epsilon}_{m,c} \left(1 + \bar{\zeta}_m(\mathbf{x} \cdot \nabla_X T_c^o) \right). \tag{4.6}$$

In (4.4) there are two separate and independent perturbations, and the effects of these two perturbations can be computed independently and then superimposed. The $\nabla_X \bar{\mathbf{E}}_E^o$ term, which is the gradient of the electric field due to the geometry of the electrodes, is the same electric field gradient term from (3.13) that we have examined in the previous chapter on the classical DEP force. The subscript E is added to denote that this electric field variation is due to the electrode geometry, and the subscript T denotes the electric field variation due to the nonuniform temperature. The $\nabla_X \bar{\mathbf{E}}_T^o$ term, which is the electric field gradient due to the temperature-varying complex permittivity, is what leads to the thermal DEP force. Again, we truncate the Taylor series expansions at the linear order, taking advantage of the assumption that the particle is small compared to the length scales of change in both the electric and

temperature fields

We substitute the temperature-dependent permittivity (4.6) into the governing equation (4.2) and expand the expression, keeping only the linear terms.

$$\bar{\epsilon}_{m,c}\nabla_X \cdot \left[\bar{\mathbf{E}}_c^o + \mathbf{x} \cdot \nabla_X \bar{\mathbf{E}}_{E,c}^o + \mathbf{x} \cdot \nabla_X \bar{\mathbf{E}}_{T,c}^o + \bar{\zeta}_m (\mathbf{x} \cdot \nabla_X T_c^o) \bar{\mathbf{E}}_c^o\right] = 0$$
(4.7)

or

$$\bar{\epsilon}_{m,c} \Big[\nabla_X \cdot \big[\bar{\mathbf{E}}_c^o + \mathbf{x} \cdot \nabla_X \bar{\mathbf{E}}_{E,c}^o \big] + \nabla_X \cdot \big[\mathbf{x} \cdot \nabla_X \bar{\mathbf{E}}_{T,c}^o + \bar{\zeta}_m (\mathbf{x} \cdot \nabla_X T_c^o) \bar{\mathbf{E}}_c^o \big] \Big] = 0.$$
(4.8)

The complex permittivity evaluated at the particle's location $\bar{\epsilon}_{m,c}$ is a constant and can be divided through the above equation. Furthermore, in the previous chapter on classical DEP we have seen that both $\bar{\mathbf{E}}^o$ and $\nabla_X \bar{\mathbf{E}}^o_E$ satisfy Laplace's equation,

$$\nabla_{X} \cdot \left[\bar{\mathbf{E}}^{o} + \mathbf{x} \cdot \nabla_{X} \bar{\mathbf{E}}^{o}_{E} \right] = 0.$$
(4.9)

Subtracting (4.9) from (4.8) yields

$$\nabla_X \cdot \left[\mathbf{x} \cdot \nabla_X \bar{\mathbf{E}}^o_{T,c} + \bar{\zeta}_m (\mathbf{x} \cdot \nabla_X T^o_c) \bar{\mathbf{E}}^o_c \right] = 0.$$
(4.10)

We see that

$$\nabla_{\!X}\bar{\mathbf{E}}^o_T = -\frac{1}{2}\bar{\zeta}_m \big(\nabla_{\!X}T^o\otimes\bar{\mathbf{E}}^o + \bar{\mathbf{E}}^o\otimes\nabla_{\!X}T^o\big) \tag{4.11}$$

can describe the effect of the nonuniform temperature field and the temperaturedependent complex permittivity on the electric field gradient. This choice of the form for $\nabla_X \bar{\mathbf{E}}_T^o$ is not unique, since the divergence operator in equation (4.10) only dictates for the three terms on the main diagonal of the tensor $\nabla_X \bar{\mathbf{E}}_T^o$, but does not give any information about the off-diagonal terms. The expression (4.11) satisfies the requirements of equation (4.10) for the main diagonal terms. Since we have already subtracted out the background field as well as the perturbation due to the geometry, what remains in $\nabla_X \bar{\mathbf{E}}_T^o$ is only due to the effects of the nonuniform temperature. Finally, the symmetry in (4.11) ensures that the corresponding electric field is irrotational and thus conservative. The expression (4.11) is the simplest vector form that satisfies all these requirements.

4.2.2. Inner Region

In the inner region the field equations (4.2) and (4.3) are written in terms of the stretched coordinate ξ . The Poisson equation for the temperature-dependent inner electric field is

$$\nabla_{\xi} \cdot \bar{\mathbf{E}}^{i} + \bar{\delta}_{T} \bar{\mathbf{E}}^{i} \cdot \nabla_{\xi} \Theta^{i} = 0, \qquad (4.12)$$

where

$$\Theta^i = (T^i - T_c) / \Delta T \tag{4.13}$$

is the dimensionless temperature, $\bar{\delta}_T = \bar{\zeta}_m \Delta T$, and $\Delta T = |\nabla_X T_c^o| a$ is the characteristic temperature difference across the particle. This parameter $\bar{\delta}_T$

$$\bar{\delta}_T \equiv \bar{\zeta}_m |\nabla_X T_c^o| a \tag{4.14}$$

arises directly from the governing equation and typically is small, $|\bar{\delta}_T| \ll 1$.

For the temperature's governing equation the small particle assumption allows us to neglect any additional Joule heating in the vicinity of the particle, since in the inner region it is of order $(a/L)^2$. The heat equation in the inner region is thus dominated by conduction with negligible Joule heating:

$$\nabla_{\xi} \cdot (\nabla_{\xi} \Theta^i) = 0. \tag{4.15}$$

To complete the solution process of the inner problem, we must match the inner dependent variables with the outer ones in the limit $\xi \to \infty$ and $\mathbf{X} \to \mathbf{X}_c$. In response to the outer electric field (4.4) the corresponding inner electric field can be expressed as

$$\bar{\mathbf{E}}^{i} = \bar{\mathbf{E}}_{0}^{i} + \delta_{E} \bar{\mathbf{E}}_{1E} + \bar{\delta}_{T} \bar{\mathbf{E}}_{1T}.$$
(4.16)

The three terms on the right hand side of (4.16) are the electric field resulting from the applied far field electric field, the nonuniform component of the applied electric field due to the electrode geometry in the device, and the modification to the electric field resulting from thermal effects. The parameters δ_E and $\bar{\delta}_T$ arise directly from the dimensionless form of (4.4).

Substituting the decomposition (4.16) back into the governing equation (4.12), we have the zeroth-order equation

$$\nabla_{\boldsymbol{\xi}} \cdot \left(\bar{\boldsymbol{\epsilon}}_c \bar{\mathbf{E}}_0^i\right) = 0 \tag{4.17}$$

and the $O(\bar{\delta}_T)$ first-order equation

$$\nabla_{\xi} \cdot \bar{\mathbf{E}}_{1T}^{i} + \bar{\mathbf{E}}_{0}^{i} \cdot \nabla_{\xi} \Theta^{i} = 0.$$
(4.18)

4.2.3. Temperature Gradient Decomposition

Since the perturbed electric field (4.16) due the thermal effect is linear in temperature, the source term of the Poisson equation (4.18) is linear with the temperature gradient. We can express the general thermal gradient field as

$$\nabla_{X}T^{o} = |\nabla_{X}T^{o}|\sin\alpha\hat{\mathbf{e}}_{y} + |\nabla_{X}T^{o}|\cos\alpha\hat{\mathbf{e}}_{z} = \nabla_{X}T^{o}_{\perp} + \nabla_{X}T^{o}_{\parallel}, \qquad (4.19)$$

where $\nabla_X T^o_{\perp}$ and $\nabla_X T^o_{\parallel}$ are the thermal gradient components perpendicular and parallel to the electric field, respectively. The expression for $\nabla_X \bar{\mathbf{E}}^o_T$ from (4.11) can likewise be straightforwardly decomposed into parallel and perpendicular components:

$$\nabla_{X}\bar{\mathbf{E}}_{T}^{o} = -\frac{\bar{\zeta}_{m}}{2} \left(\nabla_{X}T_{\parallel}^{o} \otimes \bar{\mathbf{E}}^{o} + \bar{\mathbf{E}}^{o} \otimes \nabla_{X}T_{\parallel}^{o} \right) - \frac{\bar{\zeta}_{m}}{2} \left(\nabla_{X}T_{\perp}^{o} \otimes \bar{\mathbf{E}}^{o} + \bar{\mathbf{E}}^{o} \otimes \nabla_{X}T_{\perp}^{o} \right)$$

$$= (\nabla_{X}\bar{\mathbf{E}}_{T}^{o})_{\parallel} + (\nabla_{X}\bar{\mathbf{E}}_{T}^{o})_{\perp}.$$

$$(4.20)$$

Likewise, the temperature perturbation in the Taylor series expansion of the outer electric field (4.4) can be split into parallel and perpendicular components,

$$\bar{\mathbf{E}}^{o}(\mathbf{x}) = \bar{\mathbf{E}}^{o}_{c} + \mathbf{x} \cdot \nabla_{X} \bar{\mathbf{E}}^{o}_{E,c} + \mathbf{x} \cdot (\nabla_{X} \bar{\mathbf{E}}^{o}_{T,c})_{\parallel} + \mathbf{x} \cdot (\nabla_{X} \bar{\mathbf{E}}^{o}_{T,c})_{\perp}, \qquad (4.21)$$

with corresponding inner electric field

$$\bar{\mathbf{E}}^{i} = \bar{\mathbf{E}}_{0}^{i} + \delta_{E} \bar{\mathbf{E}}_{1E}^{i} + \bar{\delta}_{T} (\bar{\mathbf{E}}_{1T}^{i})_{\parallel} + \bar{\delta}_{T} (\bar{\mathbf{E}}_{1T}^{i})_{\perp}.$$

$$(4.22)$$

Recall from the classical DEP problem that we saw the force is due to the interaction between the background zeroth-order electric field with the nonuniformity of the first-order electric field. Likewise, the same holds true in this thermal DEP problem.



Figure 20: The general problem can be decomposed into (a) the parallel case ($\alpha = 0^{\circ}$), and (b) the perpendicular case ($\alpha = 90^{\circ}$). Superposition of these two solutions provides the general solution for an abitrary temperature gradient (abitrary α).

With our linearized equations, it is easy to show from the Maxwell stress tensor that the total electrical force can be expressed as the superposition of the contributions from the three separate small perturbation terms in (4.22). Any cross-interaction between the perturbations to the electric field will be at most second-order terms in the δ_E and $\bar{\delta}_T$ parameters and will be negligible for these small parameters. The interaction of the zeroth-order electric field with the δ_E term results in the classical DEP force, and its interaction with the two $\bar{\delta}_T$ terms result in the thermal DEP force. The superposition of the classical DEP and the thermal DEP forces give the total electrical force on the particle, while the thermal DEP force itself can be expressed as the superposition of the thermal DEP forces from the separate parallel and perpendicular cases.

Since we can use superposition to determine the total force, in the following analysis we will temporarily drop the δ_E geometry perturbation and concentrate solely on the thermal DEP force, considering cases where $|\bar{\delta}_T| \ll 1$. Again, we compute everything using the time-averaged values. Also, we analyze the parallel and the perpendicular thermal gradient fields separately (Figure 20). Although α can take any value between 0° and 180°, both $\alpha = 0°$ and $\alpha = 180°$ can be described by the parallel case with just a sign difference, so that the parallel ($\alpha = 0^{\circ}$) and perpendicular ($\alpha = 90^{\circ}$) cases together describe the full possible range of α .

4.3. Electric Potential Solution

4.3.1. Zeroth Order Solution

Let's examine first the parallel case. In this case the outer electric field $\mathbf{E}_c^o = E_c^o \hat{\mathbf{e}}_z$ and the temperature gradient $\nabla_X T_c^o = |\nabla_X T_c^o| \hat{\mathbf{e}}_z$ are both in the z-direction. (The subscript \parallel is dropped for convenience.) The corresponding electric potential ϕ_c^o $(\mathbf{E}_c^o = -\nabla_X \phi_c^o)$ and dimensionless temperature Θ_c^o are given respectively by

$$\phi_c^o = -E_c^o z = -E_c^o a \xi \cos\theta \tag{4.23}$$

and

$$\Theta_c^o = z/a = \xi \cos \theta. \tag{4.24}$$

Notice that the equations (4.15) and (4.17) in the inner region for the dimensionless temperature Θ^i and the zeroth-order electric potential $\bar{\phi}_0^i$, respectively, are both Laplace's equation and thus mathematically identical. This is a classical textbook problem, with the boundary conditions at the particle-medium interface ($\xi = 1$) of continuous electric potential (temperature) and continuous electric displacement (heat flux) normal to the surface. The zeroth-order solution for the electric field has no temperature dependence and therefore is exactly identical to the zeroth-order solution from the classical DEP case discussed in the previous chapter. Inn the particle and in
the medium it is given by (3.34) and (3.35), respectively, with $\delta_E = 0$.

$$\bar{\phi}^{i}_{0m} = -E^{o}_{c}a\xi\cos\theta(1-\bar{K}_{E}\xi^{-3}), \qquad (4.25)$$

$$\bar{\phi}^i_{0p} = -E^o_c a\xi \cos\theta (1 - \bar{K}_E); \qquad (4.26)$$

The solutions for the dimensionless temperature directly follow:

$$\Theta_m^i = \xi \cos\theta (1 - G\xi^{-3}), \qquad (4.27)$$

$$\Theta_p^i = \xi \cos \theta (1 - G). \tag{4.28}$$

In the above temperature solutions

$$G = \frac{k_p - k_m}{k_p + 2k_m} \tag{4.29}$$

is a factor analogous to the Clausius–Mossotti factor, where the thermal conductivity kreplaces the complex electrical permittivity $\bar{\epsilon}_m$.

4.3.2. First Order Solution

The first-order equation (4.18) can be expressed in terms of the electric potential as

$$\nabla_{\xi}^2 \bar{\phi}_{1T}^i + \nabla_{\xi} \bar{\phi}_0^i \cdot \nabla_{\xi} \Theta^i = 0.$$
(4.30)

The source term can be calculated from (4.25)-(4.28). In the medium we have

$$\nabla_{\xi} \bar{\phi}_{0m}^{i} \cdot \nabla_{\xi} \Theta_{m}^{i} = -E_{c}^{o} a \left[\cos \theta (1 + 2\bar{K}_{E}\xi^{-3}) \hat{\mathbf{e}}_{r} - \sin \theta (1 - \bar{K}_{E}\xi^{-3}) \hat{\mathbf{e}}_{\theta} \right] \\ \left[\cos \theta (1 + 2G\xi^{-3}) \hat{\mathbf{e}}_{r} - \sin \theta (1 - G\xi^{-3}) \hat{\mathbf{e}}_{\theta} \right] \\ = -E_{c}^{o} a \cos^{2} \theta (1 + 2\bar{K}_{E}\xi^{-3}) (1 + 2G\xi^{-3}) \\ - E_{c}^{o} a \sin^{2} \theta (1 - \bar{K}_{E}\xi^{-3}) (1 - G\xi^{-3}) \\ = -E_{c}^{o} a (1 - \bar{K}_{E}\xi^{-3}) (1 - G\xi^{-3}) \\ - 3E_{c}^{o} a \cos^{2} \theta \left((\bar{K}_{E} + G)\xi^{-3} + \bar{K}_{E}G\xi^{-6} \right), \end{cases}$$
(4.31)

while the source term in the particle is given by

$$\nabla_{\xi} \bar{\phi}^{i}_{0p} \cdot \nabla_{\xi} \Theta^{i}_{p} = -E^{o}_{c} a \left[\cos \theta (1 - \bar{K}_{E}) \hat{\mathbf{e}}_{r} - \sin \theta (1 - \bar{K}_{E}) \hat{\mathbf{e}}_{\theta} \right] \cdot \left[\cos \theta (1 - G) \hat{\mathbf{e}}_{r} - \sin \theta (1 - G) \hat{\mathbf{e}}_{\theta} \right]$$

$$= -E^{o}_{c} a (1 - \bar{K}_{E}) (1 - G).$$

$$(4.32)$$

In the parallel thermal DEP case there is no φ -dependence, and we can again take advantage of the axisymmetry to simplify the differential equation. The homogeneous solution of (4.30) is simply the general solution to Laplace's equation with no φ dependence and is given by

$$\bar{\phi}_{1T}^{i,H} = \sum_{n=0}^{\infty} \left(\bar{A}_n \xi^{-(n+1)} + \bar{B}_n \xi^n \right) P_n(\cos \theta), \tag{4.33}$$

where \bar{A}_n and \bar{B}_n are constants determined by the boundary conditions and $P_n(\cos\theta)$ are the Legendre polynomials of $\cos\theta$ given in (3.21). The particular solution for the source term in the medium is given by

$$\bar{\phi}_{1T,m}^{i,P} = \frac{E_c^o a}{6} (\xi^2 + \bar{K}_E G \xi^{-4}) + \frac{E_c^o a}{3} \left(-(\bar{K}_E + G) \xi^{-1} + \bar{K}_E G \xi^{-4} \right) P_2(\cos \theta) \quad (4.34)$$

and in the particle by

$$\bar{\phi}_{1T,p}^{i,P} = \frac{E_c^o a}{6} (1 - \bar{K}_E)(1 - G)\xi^2.$$
(4.35)

These can be verified by substituting them back into equation (4.30) with the source terms given by (4.31) and (4.32), respectively.

Finally, the solutions to $\bar{\phi}_{1T}^i$ are given by the sum of the homogeneous and particular solutions:

$$\bar{\phi}_{1T,m}^{i} = \sum_{n=0}^{\infty} \left(\bar{A}_{n} \xi^{-(n+1)} + \bar{B}_{n} \xi^{n} \right) P_{n}(\cos \theta) + \frac{E_{c}^{o} a}{6} (\xi^{2} + \bar{K}_{E} G \xi^{-4}) + \frac{E_{c}^{o} a}{3} \left(-(\bar{K}_{E} + G) \xi^{-1} + \bar{K}_{E} G \xi^{-4} \right) P_{2}(\cos \theta),$$
(4.36)

$$\bar{\phi}_{1T,p}^{i} = \sum_{n=0}^{\infty} \left(\bar{A}_{n}' \xi^{-(n+1)} + \bar{B}_{n}' \xi^{n}\right) P_{n}(\cos\theta) + \frac{E_{c}^{o} a}{6} \xi^{2} (1 - \bar{K}_{E})(1 - G).$$
(4.37)

4.3.3. Boundary Conditions

The four boundary conditions allow us to solve for the constants \bar{A}_n , \bar{B}_n , \bar{A}'_n , and \bar{B}'_n . First, as $\xi \to \infty$, the inner solution must match the outer electric potential that gives rise to the electric field gradient (4.11). Thus, we have the far-field boundary condition,

$$\lim_{\xi \to \infty} \bar{\phi}^i_{1T,m} = \frac{1}{2} E^o_c a \xi^2 \cos^2 \theta = \frac{E^o_c}{2a} z^2.$$
(4.38)

The $\cos^2 \theta$ term can be rewritten in terms of the Legendre polynomials $P_n(\cos \theta)$, i.e., $\cos^2 \theta = (1 + 2P_2(\cos \theta))/3$. Taking advantage of orthogonality, we compare like coefficients of $P_n(\cos \theta)$ between equations (4.36) and (4.38) to discover that

$$\bar{B}_2 = \frac{E_c^o a}{3} \tag{4.39}$$

and all the remaining \bar{B}_n are zero. The second boundary condition at the center of the particle ($\xi = 0$) is that the electric potential must be finite, which requires all \bar{A}'_n to be zero. We can now simplify equations (4.36) and (4.37) somewhat:

$$\bar{\phi}^{i}_{1T,m} = \sum_{n=0}^{\infty} \bar{A}_{n} \xi^{-(n+1)} P_{n}(\cos \theta) + \frac{E_{c}^{o} a}{6} (\xi^{2} + \bar{K}_{E} G \xi^{-4}) + \frac{E_{c}^{o} a}{3} \xi^{2} (1 - \bar{K}_{E} \xi^{-3}) (1 - G \xi^{-3}) P_{2}(\cos \theta), \qquad (4.40)$$

$$\bar{\phi}_{1T,p}^{i} = \sum_{n=0}^{\infty} \bar{B}_{n}' \xi^{n} P_{n}(\cos\theta) + \frac{E_{c}^{o}a}{6} \xi^{2} (1 - \bar{K}_{E})(1 - G).$$
(4.41)

The remaining two boundary conditions are at the interface between the particle and the medium ($\xi = 1$). The potential must be continuous:

$$\bar{\phi}^{i}_{1T,m}|_{\xi=1} = \bar{\beta}\bar{\phi}^{i}_{1T,p}|_{\xi=1},\tag{4.42}$$

or,

$$\sum_{n=0}^{\infty} \bar{A}_n P_n(\cos\theta) + \frac{E_c^o a}{6} (1 + \bar{K}_E G) + \frac{E_c^o a}{3} (1 - \bar{K}_E) (1 - G) P_2(\cos\theta)$$
$$= \bar{\beta} \sum_{n=0}^{\infty} \bar{B}'_n P_n(\cos\theta) + \frac{E_c^o a}{6} \bar{\beta} (1 - \bar{K}_E) (1 - G); \quad (4.43)$$

and the displacement normal to the surface must also be continuous:

$$\bar{\epsilon}_m (1 + \bar{\delta}_T \Theta_m^i) \left(\frac{\partial \bar{\phi}_{0m}^i}{\partial \xi} + \bar{\delta}_T \frac{\partial \bar{\phi}_{1T,m}^i}{\partial \xi} \right) \Big|_{\xi=1} = \bar{\epsilon}_p (1 + \bar{\beta} \bar{\delta}_T \Theta_p^i) \left(\frac{\partial \bar{\phi}_{0p}^i}{\partial \xi} + \bar{\beta} \bar{\delta}_T \frac{\partial \bar{\phi}_{1T,p}^i}{\partial \xi} \right) \Big|_{\xi=1}.$$
(4.44)

(Note that because $\bar{\delta}_T$ is defined using the material properties of the medium, we require the additional factor $\bar{\beta} = \bar{\zeta}_p/\bar{\zeta}_m$ in the temperature dependence of $\bar{\epsilon}_p$. By a similar token, the first-order perturbation to the potential in the particle is given by $\bar{\beta}\bar{\delta}_T\bar{\phi}^i_{1T,p}$ while the perturbation in the medium is simply $\bar{\delta}_T\bar{\phi}^i_{1T,m}$.) Collecting all the $O(\bar{\delta}_T)$ terms in (4.44) and using the following derivatives,

$$\frac{\partial \bar{\phi}_{0m}^i}{\partial \xi}\Big|_{\xi=1} = -E_c^o a \cos \theta (1+2\bar{K}_E),$$
$$\frac{\partial \bar{\phi}_{0p}^i}{\partial \xi}\Big|_{\xi=1} = -E_c^o a \cos \theta (1-\bar{K}_E),$$

$$\frac{\partial \bar{\phi}_{1T,m}^{i}}{\partial \xi}\Big|_{\xi=1} = \sum_{n=0}^{\infty} -(n+1)\bar{A}_{n}P_{n}(\cos\theta) + \frac{E_{c}^{o}a}{3}(1-2\bar{K}_{E}G) + \frac{E_{c}^{o}a}{3}(2+\bar{K}_{E}+G-4\bar{K}_{E}G)P_{2}(\cos\theta) + \frac{\partial \bar{\phi}_{1T,p}^{i}}{\partial \xi}\Big|_{\xi=1} = \sum_{n=0}^{\infty} n\bar{B}_{n}'P_{n}(\cos\theta) + \frac{E_{c}^{o}a}{3}(1-\bar{K}_{E})(1-G),$$

we can express this last boundary condition as

$$-\bar{\epsilon}_{m}\sum_{n=0}^{\infty}(n+1)\bar{A}_{n}P_{n}(\cos\theta) + \frac{\bar{\epsilon}_{m}E_{c}^{o}a}{3}(-2\bar{K}_{E}+G) + \bar{\epsilon}_{m}E_{c}^{o}a(-\bar{K}_{E}+G)P_{2}(\cos\theta)$$
$$=\bar{\epsilon}_{p}\bar{\beta}\sum_{n=0}^{\infty}n\bar{B}_{n}'P_{n}(\cos\theta) - \frac{2\bar{\epsilon}_{p}E_{c}^{o}a}{3}\bar{\beta}(1-\bar{K}_{E})(1-G)P_{2}(\cos\theta). \quad (4.45)$$

Once again, we compare like coefficients of $P_n(\cos\theta)$ in (4.43) and (4.45), which yield

2n equations to solve for the 2n unknown constants \bar{A}_n and \bar{B}'_n . For n = 0,

$$\bar{A}_0 + \frac{E_c^o a}{6} (1 + \bar{K}_E G) = \bar{\beta} \bar{B}'_0 + \frac{E_c^o a}{6} \bar{\beta} (1 - \bar{K}_E) (1 - G),$$

$$-\bar{\epsilon}_m \bar{A}_0 + \frac{\bar{\epsilon}_m E_c^o a}{3} (-2\bar{K}_E + G) = 0.$$
 (4.46)

Thus,

$$\bar{A}_{0} = \frac{E_{c}^{o}a}{3} (-2\bar{K}_{E} + G),$$

$$\bar{B}_{0}' = \frac{E_{c}^{o}a}{6\bar{\beta}} \Big[3(-\bar{K}_{E} + G) + (1 - \bar{\beta})(1 - \bar{K}_{E})(1 - G) \Big].$$
(4.47)

For n = 2,

$$\bar{A}_{2} + \frac{E_{c}^{o}a}{3}(1 - \bar{K}_{E})(1 - G) = \bar{\beta}\bar{B}_{2}',$$

$$-3\bar{\epsilon}_{m}\bar{A}_{2} + \bar{\epsilon}_{m}E_{c}^{o}a(-\bar{K}_{E} + G) = 2\bar{\epsilon}_{p}\bar{\beta}\bar{B}_{2}' - \frac{2}{3}\bar{\epsilon}_{p}\bar{\beta}E_{c}^{o}a(1 - \bar{K}_{E})(1 - G),$$

$$(4.48)$$

yielding

$$\bar{A}_{2} = \frac{E_{c}^{o}a}{3} \Big[\frac{3\bar{\epsilon}_{m}(-\bar{K}_{E}+G)}{2\bar{\epsilon}_{p}+3\bar{\epsilon}_{m}} - \frac{2\bar{\epsilon}_{p}(1-\bar{\beta})(1-\bar{K}_{E})(1-G)}{2\bar{\epsilon}_{p}+3\bar{\epsilon}_{m}} \Big],$$

$$\bar{B}_{2}' = \frac{E_{c}^{o}a}{3\bar{\beta}} \Big[\frac{3\bar{\epsilon}_{m}(1-2\bar{K}_{E}+\bar{K}_{E}G)}{2\bar{\epsilon}_{p}+3\bar{\epsilon}_{m}} + \frac{2\bar{\epsilon}_{p}\bar{\beta}(1-\bar{K}_{E})(1-G)}{2\bar{\epsilon}_{p}+3\bar{\epsilon}_{m}} \Big].$$
(4.49)

For n = 1 and $n \ge 3$, we find that all these coefficients \bar{A}_n and \bar{B}'_n are zero. In the force evaluation in the following sections we shall see that even \bar{A}_2 has no net contribution to the force, and only the constant \bar{A}_0 is needed to derive the analytical force expression. The potentials are given by

$$\bar{\phi}^{i}_{1T,m} = \bar{A}_{0}\xi^{-1} + \bar{A}_{2}\xi^{-3}P_{2}(\cos\theta) + \frac{E^{o}_{c}a}{6}(\xi^{2} + \bar{K}_{E}G\xi^{-4}) + \frac{E^{o}_{c}a}{3}\xi^{2}(1 - \bar{K}_{E}\xi^{-3})(1 - G\xi^{-3})P_{2}(\cos\theta), \bar{\phi}^{i}_{1T,p} = \bar{B}'_{0} + \bar{B}'_{2}\xi^{2}P_{2}(\cos\theta) + \frac{E^{o}_{c}a}{6}\xi^{2}(1 - \bar{K}_{E})(1 - G),$$

or

$$\begin{split} \bar{\phi}_{1T,m}^{i} &= \frac{E_{c}^{o}a}{3} (-2\bar{K}_{E} + G)\xi^{-1} + \frac{E_{c}^{o}a}{6} (\xi^{2} + \bar{K}_{E}G\xi^{-4}) \\ &+ \frac{E_{c}^{o}a}{3} \Big[\frac{3\bar{\epsilon}_{m}(-\bar{K}_{E} + G)}{2\bar{\epsilon}_{p} + 3\bar{\epsilon}_{m}} - \frac{2\bar{\epsilon}_{p}(1 - \bar{\beta})(1 - \bar{K}_{E})(1 - G)}{2\bar{\epsilon}_{p} + 3\bar{\epsilon}_{m}} \Big] \xi^{-3}P_{2}(\cos\theta) \\ &+ \frac{E_{c}^{o}a}{3}\xi^{2}(1 - \bar{K}_{E}\xi^{-3})(1 - G\xi^{-3})P_{2}(\cos\theta), \end{split}$$
(4.50)
$$\bar{\phi}_{1T,p}^{i} &= \frac{E_{c}^{o}a}{6\bar{\beta}} \Big[3(-\bar{K}_{E} + G) + (1 - \bar{\beta})(1 - \bar{K}_{E})(1 - G) \Big] + \frac{E_{c}^{o}a}{6}\xi^{2}(1 - \bar{K}_{E})(1 - G) \\ &+ \frac{E_{c}^{o}a}{3\bar{\beta}} \Big[\frac{3\bar{\epsilon}_{m}(1 - 2\bar{K}_{E} + \bar{K}_{E}G)}{2\bar{\epsilon}_{p} + 3\bar{\epsilon}_{m}} + \frac{2\bar{\epsilon}_{p}\bar{\beta}(1 - \bar{K}_{E})(1 - G)}{2\bar{\epsilon}_{p} + 3\bar{\epsilon}_{m}} \Big] \xi^{2}P_{2}(\cos\theta).$$
(4.51)

4.4. Calculation of the Thermal DEP Force

4.4.1. Electric Field

The total electric potential is given by

$$\bar{\phi}^{i}_{m} = \bar{\phi}^{i}_{0m} + \bar{\delta}_{T} \bar{\phi}^{i}_{1T,m},$$

$$\bar{\phi}^{i}_{p} = \bar{\phi}^{i}_{0p} + \bar{\beta} \bar{\delta}_{T} \bar{\phi}^{i}_{1T,p}.$$
(4.52)

To calculate the force on the particle using the Maxwell stress tensor (MST), we again need to evaluate the electric field $\bar{\mathbf{E}}_m^i = -\nabla_X \bar{\phi}_m^i$ on the outside surface of the particle $(\xi=1^+).$ This electric field can be expressed as

$$\bar{\mathbf{E}}_m^i = \bar{E}_r \hat{\mathbf{e}}_r + \bar{E}_\theta \hat{\mathbf{e}}_\theta, \qquad (4.53)$$

where the components \bar{E}_r and \bar{E}_{θ} are derived from (4.50) using the gradient operator (3.4):

$$\bar{E}_r = E_c^o(1 + 2\bar{K}_E)\cos\theta + \bar{\delta}_T E_c^o F_1(\bar{K}_E, G) + \bar{\delta}_T E_c^o F_2(\bar{K}_E, G, \bar{\beta}) P_2(\cos\theta), \quad (4.54)$$

$$\bar{E}_{\theta} = E_c^o(-1 + \bar{K}_E)\sin\theta + \bar{\delta}_T E_c^o F_3(\bar{K}_E, G, \bar{\beta})\cos\theta\sin\theta; \qquad (4.55)$$

and the functions F_1 , F_2 , and F_3 are defined as:

$$F_1 = -\frac{1}{3}(1+2\bar{K}_E)(1-G), \qquad (4.56)$$

$$F_{2} = -\frac{2\bar{\epsilon}_{m}(1+2\bar{K}_{E})(1-G)}{2\bar{\epsilon}_{p}+3\bar{\epsilon}_{m}} - \frac{2}{3}\frac{\bar{\epsilon}_{p}(5-2\bar{K}_{E}-2G-\bar{K}_{E}G)}{2\bar{\epsilon}_{p}+3\bar{\epsilon}_{m}} + \frac{2\bar{\beta}\bar{\epsilon}_{p}(1-\bar{K}_{E})(1-G)}{2\bar{\epsilon}_{p}+3\bar{\epsilon}_{m}},$$
(4.57)

$$-\frac{3\bar{\epsilon}_m(1-2\bar{K}_E+\bar{K}_EG)}{4.58} + \frac{2\bar{\beta}\bar{\epsilon}_p(1-\bar{K}_E)(1-G)}{4.58}$$

$$F_3 = \frac{3\epsilon_m(1-2R_E+R_EG)}{2\bar{\epsilon}_p + 3\bar{\epsilon}_m} + \frac{2\rho\epsilon_p(1-R_E)(1-G)}{2\bar{\epsilon}_p + 3\bar{\epsilon}_m}.$$
(4.58)

4.4.2. MST Integration

Once again, the time-averaged Maxwell stress tensor is given by

$$\boldsymbol{\sigma}_{\rm MST} = \frac{\epsilon}{4} \big(\widetilde{\mathbf{E}} \widetilde{\mathbf{E}}^* + \widetilde{\mathbf{E}}^* \widetilde{\mathbf{E}} - (\widetilde{\mathbf{E}} \cdot \widetilde{\mathbf{E}}^*) \mathbf{I} \big).$$
(4.59)

Note that the MST expression contains the real permittivity ϵ and not the complex permittivity $\bar{\epsilon}$ [62]. Also, the permittivity is a function of the temperature:

$$\epsilon_m(T)|_{\xi=1} = \epsilon_m \left(1 + \delta_T \cos\theta(1 - G)\right),\tag{4.60}$$

where $\delta_T = \zeta_{\epsilon m} \Delta T$. Then, the force acting on the particle can be computed by integrating the stress tensor over the surface of the particle:

$$\mathbf{F} = \int_{S} \boldsymbol{\sigma}_{\text{MST}} \cdot d\mathbf{S}, \qquad (4.61)$$

where for the spherical particle of radius *a* the surface element $d\mathbf{S} = a^2 \sin \theta \, d\theta \, d\varphi \, \hat{\mathbf{e}}_r$. We can rewrite (4.61) as

$$\mathbf{F} = \frac{\epsilon_m a^2}{4} \int_0^{2\pi} \int_0^{\pi} \left[1 + \delta_T \cos \theta (1 - G) \right] \cdot \\ \left[(\bar{E}_r \bar{E}_r^* - \bar{E}_\theta \bar{E}_\theta^*) \hat{\mathbf{e}}_r + (\bar{E}_r \bar{E}_\theta^* + \bar{E}_r^* \bar{E}_\theta) \hat{\mathbf{e}}_\theta \right] \sin \theta \, d\theta \, d\varphi \\ = \frac{\epsilon_m \pi a^2}{2} \int_0^{\pi} \left[1 + \delta_T \cos \theta (1 - G) \right] \cdot \\ \left[(\bar{E}_r \bar{E}_r^* - \bar{E}_\theta \bar{E}_\theta^*) \hat{\mathbf{e}}_r + (\bar{E}_r \bar{E}_\theta^* + \bar{E}_r^* \bar{E}_\theta) \hat{\mathbf{e}}_\theta \right] \sin \theta \, d\theta.$$

$$(4.62)$$

The electric field components \bar{E}_r and \bar{E}_{θ} are given by (4.54) and (4.55), respectively. We expect the F_x and F_y components of the force to be zero due to axisymmetry, and performing the integration confirms these results. We use the unit vectors (3.3) to compute $F_z = \mathbf{F} \cdot \hat{\mathbf{e}}_z$, which is the only nonzero force component. The integrand in (4.62) has components of zeroth-order, first-order, and second-order. The zerothorder components are symmetric about the sphere and provide no net force. The first-order components proportional to $\bar{\delta}_T$, $\bar{\delta}_T^*$, and δ_T are the ones that yield the thermal DEP force. The second-order terms are neglected because in our perturbation analysis $|\bar{\delta}_T| \ll 1$.

In evaluating F_z we use the following integral formulas:

$$\int_{0}^{\pi} \sin^{3} \theta \, d\theta = \frac{4}{3} \qquad \text{and} \qquad \int_{0}^{\pi} \cos^{2} \theta \sin^{3} \theta \, d\theta = \frac{4}{15}.$$
 (4.63)

Then, we find

$$F_{z} = \frac{\pi}{15} a^{2} \epsilon_{m} (E_{c}^{o})^{2} \Big[\bar{\delta}_{T} \big(15F_{1} + 6\bar{K}_{E}^{*}(F_{2} - F_{3}) \big) \\ + \bar{\delta}_{T}^{*} \big(15F_{1}^{*} + 6\bar{K}_{E}(F_{2}^{*} - F_{3}^{*}) \big) \\ + \delta_{T} \big(5 + 10\bar{K}_{E} + 10\bar{K}_{E}^{*} + 2\bar{K}_{E}\bar{K}_{E}^{*} \big) \big(1 - G \big) \Big],$$

$$(4.64)$$

or

$$F_z = \frac{1}{2} \epsilon_m V_p \operatorname{Re}\left(\bar{\zeta}_m \bar{K}_T^{\parallel}\right) (E_c^o)^2 |\nabla_X T_c^o|.$$
(4.65)

In (4.65), V_p is the volume of the particle, $\operatorname{Re}(\cdot)$ denotes the real portion of the enclosed quantity, and

$$\bar{K}_{T}^{\parallel} = \frac{1}{10} \Big[10(2\bar{K}_{E} + 1)(G - 1) \\ + 4\bar{K}_{E}^{*}(\bar{K}_{E}G + 2\bar{K}_{E} + 2G - 5) \\ - (2\bar{K}_{E}\bar{K}_{E}^{*} + 20\bar{K}_{E} + 5)(G - 1)\zeta_{\epsilon m}/\bar{\zeta}_{m} \Big].$$

$$(4.66)$$

Note that because the force from (4.64) includes the quantity $F_2 - F_3$, the net effect of the constant \bar{A}_2 turns out to be zero. Furthermore, the parameter $\bar{\beta} = \bar{\zeta}_p/\bar{\zeta}_m$ also does not appear in the force expression. We will discuss the vector form of the force (4.65) in Chapter 6 on the general thermal DEP force.

4.4.3. Electric Buoyancy Correction

The coefficient \bar{K}_T^{\parallel} (4.66) requires a correction term to be useful in determining the net force on the particle relative to the fluid medium. If we consider the case of a single fluid medium, we do not expect any net force on a hypothetical spherical "particle" of fluid carved out of this fluid medium, since every element of fluid will have the same force acting on it. In other words, when the electric and thermal properties of the particle and medium are identical, we expect to find zero net force. However, setting $\bar{K}_E = 0$ and G = 0 still yields a nonzero coefficient \bar{K}_T^{\parallel} from (4.66). To resolve this apparent paradox, we must account for the force from the hydrostatic pressure in the fluid due to the electrostatic body force — the "electric buoyancy" [4, 17, 50]. Thus, we subtract the force that would have acted on a hypothetical particle of fluid of the same size and position as the actual particle.

This electric buoyancy correction is analogous to the process of calculating the apparent weight of a body immersed in a fluid. Since the immersed body displaces the fluid, the weight of the displaced fluid must be subtracted from the weight of the body itself. Likewise, for the thermal DEP force, subtracting the force from the MST integration in the single medium case will yield only the net force acting on the particle relative to the fluid medium.

To examine this issue more concretely, let us consider the parallel plate geometry shown in Figure 21, with a single fluid medium and no particle located between the plates. The plates are maintained at voltages $\pm V$ and temperature T_c and T_h , providing a uniform electric field and temperature gradient within this fluid. For this case the temperature-dependent electric field can be solved exactly.

We assume that the thermal conductivity k_m of the medium is constant. Therefore,



Figure 21: A fluid medium between two parallel plates, separated by a distance 2H, where the plates are maintained at two different temperatures, $T = T_1$ at the top plate and $T = T_2$ at the bottom plate. The voltages of the top and bottom plates are V and -V, respectively.

the temperature field in the medium is given by

$$T = T_c + \Theta z, \tag{4.67}$$

where $T_c = (T_1 + T_2)/2$ is the reference temperature and $\Theta = (T_2 - T_1)/2H$ is the temperature gradient. The electric potential is determined from

$$\nabla \cdot (\epsilon \nabla \bar{\phi}) = 0, \tag{4.68}$$

and the linearized permittivity is given by

$$\bar{\epsilon}_m(T) = \bar{\epsilon}_{m,c}(1 + \bar{\zeta}\Theta z), \tag{4.69}$$

where the temperature profile (4.67) is used, and $\epsilon_{m,c} = \epsilon_m(T_c)$. Let us introduce the

small parameters

$$\bar{\delta}_z = \bar{\zeta} \Theta z \tag{4.70}$$

and

$$\bar{\delta}_H = \bar{\zeta} \Theta H. \tag{4.71}$$

In the configuration shown in Figure 21, we expect the electric field to be a function of z only. Thus, the equation (4.68) reduces to

$$\frac{d}{dz} \Big[\bar{\epsilon}_{m,c} (1 + \bar{\delta}_z) \frac{d\bar{\phi}}{dz} \Big] = 0.$$
(4.72)

The solution to (4.72) with the boundary conditions $\phi = V$ at z = H and $\phi = -V$ at z = -H is

$$\bar{\phi} = V - \frac{2V}{\ln\frac{1+\bar{\delta}_H}{1-\bar{\delta}_H}} \ln\frac{1+\delta_H}{1+\bar{\delta}_z}.$$
(4.73)

We may expand (4.73) in terms of this small parameter,

$$\bar{\phi} = -Ez - \frac{1}{2}\bar{\delta}_H E\left(1 - \frac{z^2}{H^2}\right) + O\left((\bar{\delta}_H)^2\right) \equiv \phi_0 + \bar{\delta}_H \phi_1, \qquad (4.74)$$

where

$$E = -\frac{V}{H} \tag{4.75}$$

is the electric field intensity.

Next, consider an imaginary sphere of radius *a* carved out at the center of the medium, as shown in Figure 21. We can rewrite the exact potential (4.73) into spherical coordinates centered on the sphere, where $z = r \cos \theta$:

$$\bar{\phi} = V - \frac{2V}{\ln\frac{1+\bar{\delta}_H}{1-\bar{\delta}_H}} \ln\frac{1+\bar{\delta}_H}{1+\bar{\delta}_H r\cos\theta/H},\tag{4.76}$$

or, in the perturbation form,

$$\bar{\phi} = -Er\cos\theta - \frac{1}{2}\bar{\delta}_H E\left(1 - \frac{(r\cos\theta)^2}{H^2}\right). \tag{4.77}$$

The corresponding electric fields $\bar{\mathbf{E}} = -\nabla \bar{\phi}$ on the surface of the sphere (r = a) are given as

$$\bar{\mathbf{E}} \equiv \bar{E}_r \hat{\mathbf{e}}_r + \bar{E}_\theta \hat{\mathbf{e}}_\theta = \hat{\mathbf{e}}_r \Big[\frac{\bar{C}}{1 + \bar{\delta}_T \cos \theta} \cos \theta \Big] + \hat{\mathbf{e}}_\theta \Big[\frac{-\bar{C}}{1 + \bar{\delta}_T \cos \theta} \sin \theta \Big], \tag{4.78}$$

$$\bar{\mathbf{E}} = \hat{\mathbf{e}}_r [E\cos\theta - E\bar{\delta}_T\cos^2\theta] + \hat{\mathbf{e}}_\theta [-E\sin\theta + E\bar{\delta}_T\cos\theta\sin\theta], \qquad (4.79)$$

respectively, where the constant

$$\bar{C} = \frac{2\bar{\delta}_H V/H}{\ln\frac{1+\bar{\delta}_H}{1-\bar{\delta}_H}}.$$
(4.80)

With the electric field from (4.78), we find the integration of the Maxwell stress tensor over the surface of the sphere yields a net force in the z-direction:

$$F_z = \int_S \left(\sigma_{\rm MST} \cdot \hat{\mathbf{e}}_r\right) \cdot \hat{\mathbf{e}}_z \, dS = \pi a^3 \epsilon_{m,c} \bar{C}^2 \frac{1}{\bar{\delta}_T} \left(2 - \frac{1}{\bar{\delta}_T} \ln \frac{1 + \bar{\delta}_T}{1 - \bar{\delta}_T}\right). \tag{4.81}$$

Obviously, this force is not zero. We may also use the perturbation form of the electric field (4.79), given by

$$\bar{E}_r = E\cos\theta(1 - \bar{\delta}_T\cos\theta), \qquad (4.82)$$

$$\bar{E}_{\theta} = -E\sin\theta(1-\bar{\delta}_T\cos\theta), \qquad (4.83)$$

and similarly evaluate this force as the real portion of

$$F_{z} = \pi a^{2} \epsilon_{m,c} \int_{0}^{\pi} (1 + \delta_{T} \cos \theta) \Big[(\bar{E}_{r} \bar{E}_{r}^{*} - \bar{E}_{\theta} \bar{E}_{\theta}^{*}) \cos \theta - (\bar{E}_{r} \bar{E}_{\theta}^{*} + \bar{E}_{r}^{*} \bar{E}_{\theta}) \sin \theta \Big] \sin \theta \, d\theta$$

$$= \pi a^{2} \epsilon_{m,c} \int_{0}^{\pi} (1 + \delta_{T} \cos \theta) \Big[E^{2} \operatorname{Re}(1 - \bar{\delta}_{T} \cos \theta)^{2} \cos \theta \Big] \sin \theta \, d\theta$$

$$= \pi a^{2} \epsilon_{m,c} E^{2} \Big[\int_{0}^{\pi} \operatorname{Re}(1 - \bar{\delta}_{T} \cos \theta)^{2} \cos \theta \sin \theta \, d\theta$$

$$+ \int_{0}^{\pi} \delta_{T} \cos \theta \operatorname{Re}(1 - \bar{\delta}_{T} \cos \theta)^{2} \cos \theta \sin \theta \, d\theta \Big]$$

$$= \pi a^{2} \epsilon_{m,c} E^{2} \operatorname{Re}\Big[-\frac{4}{3} \bar{\delta}_{T} + \frac{2}{3} \delta_{T} \Big] + O(\bar{\delta}_{T})^{2}$$

$$= -\frac{2}{3} \pi a^{2} \epsilon_{m,c} E^{2} \operatorname{Re}(2\bar{\delta}_{T} - \delta_{T}), \qquad (4.84)$$

or

$$F_z = -\frac{2}{3}\pi a^3 \epsilon_{m,c} \operatorname{Re}(2\bar{\zeta} - \zeta_{\epsilon m}) E^2 |\nabla T|$$
(4.85)

to the leading order. The force expression (4.85) can also be recovered from (4.81) by taking the Taylor series expansion of small $\bar{\delta}_T$, and noticing that $\bar{C} = E$ for small δ_H . This force can also be found by setting \bar{K}_E and G to be zero in the \bar{K}_T^{\parallel} expression (4.66) and inserting this coefficient into the MST-integrated force (4.65).

The force (4.85) can be absorbed into a modified hydrodynamic pressure and does not cause any net motion of the fluid for this single medium case with only an imaginary spherical particle. Furthermore, note that this force is simply the electrical body force (2.15) multiplied by the volume of the particle, which in a more general case gives rise to the electrothermal flow. Either way, however, we must still include the electric buoyancy correction to compute the net force on the particle relative to the fluid medium, independent of whatever electrothermal flow exists within the fluid itself. The "thermal" Clausius–Mossotti factor for the parallel case is thus given by

$$\bar{K}_{T}^{\parallel} = \frac{1}{10} \Big[10(2\bar{K}_{E}+1)(G-1) + 10 \\ + 4\bar{K}_{E}^{*}(\bar{K}_{E}G+2\bar{K}_{E}+2G-5) \\ - \big(5 + (2\bar{K}_{E}\bar{K}_{E}^{*}+20\bar{K}_{E}+5)(G-1)\big)\zeta_{\epsilon m}/\bar{\zeta}_{m} \Big].$$

$$(4.86)$$

The modified expression for \bar{K}_T^{\parallel} (4.86) does indeed yield $\bar{K}_T^{\parallel} = 0$ when the particle and the suspending fluid have identical properties.

4.5. Importance of the Thermal DEP Force

As discussed in Section 4.2.3, the total electrostatic force on the particle is given by the superposition of the classical DEP force and the thermal DEP force. How important is the thermal DEP force (4.65)? Let's consider some example cases where the temperature gradient is aligned with the electric field, so that the thermal DEP force only has the parallel component. For these cases, we can compare the magnitude of the thermal DEP force to the classical DEP force.

4.5.1. Nucleated Steam Bubble

For the first case, let us consider two concentric spherical electrodes of radius R_1 and R_2 (> R_1) with $L = R_2 - R_1 \gg a$, with a nucleated bubble of water vapor located on the wall at R_1 , in the thermal boundary layer above the heated electrode (Figure 22). A voltage difference of ΔV is maintained between the two electrodes. There is a temperature difference of ΔT_L across the thermal boundary layer located next to the heated surface.

For nucleate boiling heat transfer in water at 1 atm pressure, a temperature excess of 5–10°C should provide isolated bubbles which form on the surface, grow, and then



Figure 22: Parallel thermal DEP example. Two concentric spherical electrodes of radius R_1 and R_2 (> R_1) with $L = R_2 - R_1 \gg a$, with a nucleated bubble of water vapor located at R_1 in the thermal boundary layer above the heated electrode. A voltage difference of ΔV and temperature difference of ΔT_L are maintained between the two electrodes.

detach [25]. The corresponding surface heat flux q''_s ranges from about 6×10^3 to 1×10^5 W/m². Fourier's law of conduction [25],

$$q_s'' = -k_m \nabla T \approx -k_m \Delta T_L / L_{\text{TBL}}, \qquad (4.87)$$

allows us to estimate the width of the thermal boundary layer L_{TBL} given a temperature difference of ΔT_L across it. At $T_c = 100^{\circ}$ C, the thermal conductivity of water is $k_m = 0.6729 \text{ W/(m \cdot K)}$ [36]. Thus, for nucleate boiling in water, the thermal boundary layer thickness can range from about 67 μ m to as large as 0.561 mm. We choose a bubble radius $a = 250 \ \mu$ m so that the bubble will be located inside the thermal boundary layer.

This case readily provides a simple 1D analytical solution for the outer electric field. Due to the curvature of the electrodes, there will be a nonzero classical DEP force. Spherical electrodes are chosen to allow us to take advantage of axisymmetry in the φ -direction in evaluating the force numerically. Additionally, due to the very different densities ρ between the gas bubble and the fluid medium, we must consider

	Material Property	Units	Case 1	Case 2
Fluid Medium	ϵ_m		$55.6\epsilon_0$ [36]	$78.4\epsilon_0$ [36]
	σ_m	S/m	7.65×10^{-5} [36]	0.015 [50]
	$\zeta_{\epsilon m}$	K^{-1}	-0.004 [36]	-0.004 [36]
	$\zeta_{\sigma m}$	K^{-1}	0.02 [36]	0.02 [36]
	k_m	$W/(m \cdot K)$	0.6729 [36]	0.606 [36]
	$ ho_m$	$\rm kg/m^3$	958.4 [36]	997.0 [36]
Bubble/Particle	ϵ_p		$1.00589\epsilon_0$ [36]	$2.54\epsilon_0$ [51]
	σ_p	S/m	2×10^{-4} [13]	$5.12 \times 10^{-3}/a$ [23]
	k_p	$W/(m \cdot K)$	0.0240 [36]	0.10 [55]
	$ ho_p$	$\rm kg/m^3$	0.598 [36]	1060 [36]
	a	$\mu { m m}$	250	2.5
	T_c	°C	100	25
	$\Delta V_{ m rms}$	V	1000 [38]	20 [50]
	ΔT_L	$^{\circ}\mathrm{C}$	10 [25]	15 [17]
	R_1	$\mu \mathrm{m}$	5000	300
	R_2	$\mu \mathrm{m}$	10,000	350

Table 1: Relevant material properties for the calculation of the classical DEP and thermal DEP forces. Case 1 — nucleated water vapor bubble in DI water. Case 2 — polystyrene particle in water.



Figure 23: The analytically-solved classical DEP, thermal DEP, and buoyancy force on a nucleated bubble of water vapor in the thermal boundary layer as functions of AC frequency of the electric field.

the buoyancy force

$$F_b = (\rho_m - \rho_p)gV_p, \tag{4.88}$$

where $g = 9.81 \text{ m/s}^2$ is the acceleration of gravity. The relevant material properties at 100°C for water as well as for the water vapor bubble are given in Table 1. $\epsilon_0 = 8.854 \times 10^{-12} \text{ F/m}$ is the permittivity of free space.

Figure 23 depicts the various electrostatic forces associated with the DEP and thermal DEP forces and the buoyancy force as functions of the AC frequency for a nucleated water vapor bubble located within the thermal boundary layer. (See Appendix B for the MATLAB code used to calculate the forces.) Since the Clausius–Mossotti factor \bar{K}_E is a function of the AC frequency, the DEP and thermal DEP forces are as well. Notice that there are two distinct regimes: a low frequency region where the electrical conductivity is dominant, and a high frequency region where the electrical permittivity is dominant. The crossover frequency is approximately given by $\omega = \sigma_m/\epsilon_m$, which for this case corresponds to a frequency $f = \omega/2\pi = 24.7$ kHz.

This agrees nicely with what we see in Figure 23.

Furthermore, notice that at frequencies below the crossover frequency, the magnitude of the thermal DEP is much larger, while above the crossover frequency, the thermal DEP force is almost negligible. This is due to the fact that the conductivity has a much larger relative change with temperature than the permittivity does. (Compare $\zeta_{\sigma m}$ and $\zeta_{\epsilon m}$ in Table 1.) In the lower frequency region, we have $\bar{K}_E = 0.3499$ and G = -0.4737. The classical DEP force has been shown to be of similar order of magnitude to the buoyancy force [57]. At low frequencies the thermal DEP force can also be the same order of magnitude. The magnitude of the thermal DEP force in this region is approximately 50% of the classical DEP force magnitude, and both of these forces are directed toward the surface, opposite the direction of the buoyancy force. Thus, the electrostatic force here causes the bubble to be held on the surface for a longer time, thus allowing the bubble to grow larger before it detaches and to carry a larger amount of heat away from the surface once it does detach. Although the classical DEP and thermal DEP forces are proportional to the volume, just as the buoyancy force is, as the bubble grows, its center will be further away from the wall and experience a lower strength electric field, so that the bubble will still eventually detach. Alternatively, a lower applied voltage difference will allow the bubble to detach faster.

4.5.2. Polystyrene Particle in Water

Now, let's examine a second case where we have a polystyrene particle in water and a similar geometry of concentric spherical electrodes. The particle radius $a = 2.5 \ \mu m$, and the distance between the two electrodes is 50 μm . The RMS voltage difference is 20 V, and the applied temperature difference is $\Delta T_L = 15^{\circ}$ C [16, 17]. These values along with the relevant material properties are stated in Case 2 of Table 1. The



Figure 24: The analytically-solved classical DEP force and thermal DEP force on a polystyrene particle in water as functions of AC frequency of the electric field.

results are plotted in Figure 24. Since the density of polystyrene is similar to that of water, the buoyancy force in this case is negligible (approximately 4×10^{-14} N). Once again, we see identifiable low-frequency and high-frequency regimes, with a crossover frequency of about 3.44 MHz. In the low-frequency regime the ratio of the magnitudes of the thermal DEP and classical DEP forces is 1.53, while in the high-frequency regime the thermal DEP force is -15% of the classical DEP force strength. In the low frequency limit, $\bar{K}_E = -0.4041$ and G = -0.3857.

In Figures 23 and 24 the classical DEP (1.3) is computed using only the geometryinduced gradient $\nabla_X \bar{\mathbf{E}}_E^o$ (4.21), the thermal DEP is calculated using (4.65) with the factor \bar{K}_T^{\parallel} given by (4.86), and the total electrostatic force is given by their sum:

$$\mathbf{F}_{\rm es} = \mathbf{F}_{\rm DEP} + \mathbf{F}_{\rm T-DEP}.\tag{4.89}$$

Below the crossover frequency is the range where the electrical conductivity is dominant. Since $\zeta_{\sigma m}$ is much larger than $\zeta_{\epsilon m}$, the thermal DEP force is significantly larger compared to frequencies above this crossover frequency. Both the thermal DEP force and the classical DEP force are proportional to the particle volume V_p and to the square of the background electric field intensity $(E_c^o)^2$. However, even though their forms are similar, they arise from two very different physical mechanisms, as discussed in Section 4.2. Using only the classical DEP force formula will not properly capture the thermal effects contained in the coefficient \bar{K}_T^{\parallel} .

4.5.3. Effects of Different Material Properties

Besides the two specific cases we have just examined, we can also estimate the effects of material properties on the relative importance of the thermal DEP force. We choose $\zeta_{\sigma m} = 0.02$, which is a typical value for electrolytes [3], and we select a frequency below the crossover frequency so that the thermal DEP force is significant.

To compare the forces, we assume that the relative change of the magnitude of the electric field across the particle is comparable to the relative change in the temperature,

$$\frac{\Delta E_c^o}{E_c^o} = \frac{\Delta T}{\Delta T_L}.\tag{4.90}$$

A particle in this system will experience a thermal DEP force of magnitude

$$F_{\text{T-DEP}}^{\parallel} = \frac{1}{2} \epsilon_m V_p (E_c^o)^2 \frac{\Delta T}{a} \operatorname{Re} \left(\bar{\zeta}_m \bar{K}_T^{\parallel} \right)$$
(4.91)

and a classical DEP force of magnitude

$$F_{\rm DEP} = \frac{3}{2} \epsilon_m V_p E_c^o \frac{\Delta E_c^o}{a} \operatorname{Re}(\bar{K}_E).$$
(4.92)



Figure 25: Nondimensional forces $\hat{F}_{\text{DEP}}^{\parallel}$ and $\hat{F}_{\text{T-DEP}}$ for G = -0.5 (thermally insulating particle), G = 0 (same thermal properties), and G = 1 (thermally conducting particle) as functions of $\text{Re}(\bar{K}_E)$. $\Delta T_L = 15$ K.

Removing the common factor $\epsilon_m V_p E_c^o(\Delta E_c^o/a)$, we have the nondimensional forces

$$\hat{F}_{\text{T-DEP}}^{\parallel} = \operatorname{Re}(\bar{\zeta}_m \bar{K}_T) \,\Delta T_L \tag{4.93}$$

and

$$\hat{F}_{\text{DEP}} = 3 \operatorname{Re}(\bar{K}_E). \tag{4.94}$$

Figure 25 depicts \hat{F}_{DEP} and $\hat{F}_{\text{T-DEP}}$ as functions of $\text{Re}(\bar{K}_E)$ at a frequency below the crossover frequency across the possible spectrum of material properties, -0.5 < $\text{Re}(\bar{K}_E) < 1$ and -0.5 < G < 1. The thermal DEP force is more pronounced for a thermally insulating particle (G = -0.5). We also see that the direction of the thermal DEP force may change depending on the material properties. This is a phenomenon similar to positive and negative classical DEP, where particles move, respectively, toward or away from regions of high electric field intensity. Thus, we denote a thermal DEP force moving particles toward regions of higher temperature as positive and toward regions of lower temperatures as negative. The direction of the thermal DEP force is determined by the material properties and the direction of the temperature gradient. A positive thermal DEP force will push particles in the direction of the temperature gradient, while a negative thermal DEP force will push particles in the opposite direction.

4.6. Range of Validity of Linear Model

Recall that our model is based on using only linear perturbations to the electric field in the Taylor series expansion, which requires that the parameters δ_E and $\bar{\delta}_T$ are small. The dipole moment approximation (linear model) for the classical DEP is well-understood, and this approximation has been compared to the exact force calculated from the Maxwell stress tensor to determine a range of validity of the dipole moment approximation [37].

For our thermal DEP analysis, we likewise determine a range of validity of the small perturbation $\bar{\delta}_T = \bar{\zeta}_m a |\nabla_X T_c^o|$ by comparing the linear analytical model with a fully nonlinear model computed in COMSOL using an axisymmetric 2D computational domain. For the concentric spherical electrodes (Figure 26) described in Case 2 above, we have $\delta_E = 0.0154$ and $|\bar{\delta}_T| = 0.0149$. We find only a 0.31% difference in the thermal DEP force between the linear model (4.65) and the COMSOL-evaluated force from the Maxwell stress tensor integration of the nonlinear model results.

The magnitude of $\bar{\delta}_T$ can be easily changed in COMSOL by choosing a different



Figure 26: Left: The full computational mesh domain used in COMSOL to evaluate the thermal DEP force on a particle located between two concentric spherical electrodes. The particle is located in the region with the very fine mesh at the upper left of the domain, midway between the electrodes. Right: A zoomed view near the location of the particle. The particle surface is denoted by the smallest hemisphere, while the other two hemispheres are the surfaces are used for interpolation in evaluating the integral of the Maxwell stress tensor. The colors correspond to the local electric potential ϕ^i .

value for $\bar{\zeta}_m$, while keeping everything else the same. Recalculating the force with $|\bar{\delta}_T| = 0.1491$, a factor of ten times larger than before, we find the error in the force calculation only increases to 3.19%, which is still small. For practical cases, it is nearly impossible to find any materials that will provide a $\bar{\delta}_T$ value this large. As stated previously, the value of ζ_{σ} for electrolytes is generally around 0.02–0.05 K⁻¹. For biological applications, the temperature difference $a|\nabla_X T_c^o|$ across the small particles or cells cannot be very large, or else the temperature difference across the overall system will be large enough to kill the cells. For nucleate boiling applications, increasing the temperature difference will cause a departure from the isolated bubble regime into jets and columns of vapor, where this analysis will not directly apply. Thus, for practical intents and purposes, this linear model should be able to predict the thermal DEP force within a few percent error.

CHAPTER 5 : Thermal DEP Force — Perpendicular Case

Let's return to the decomposition of the temperature gradient into parallel and perpendicular components. Now, we will examine the case where the thermal gradient field is along the direction perpendicular to the electric field (Figure 20). The perturbed electric potential due to the temperature variation will no longer be axisymmetric, and the thermal DEP force will no longer necessarily be in the same direction as the electric field. However, the derivation of this perpendicular case follows a similar process to the parallel case of the previous chapter.

5.1. Electric Potential Solution

5.1.1. Zeroth Order Solution

The outer electric field is still in the z-direction and is given by $\mathbf{E}_c^o = E_c^o \hat{\mathbf{e}}_z$, but now the temperature gradient $\nabla_X T_c^o = |\nabla_X T_c^o| \hat{\mathbf{e}}_y$ is in the y-direction. The corresponding electric potential ϕ_c^o and dimensionless temperature Θ_c^o are given respectively by

$$\phi_c^o = -E_c^o z = -E_c^o a \xi \cos\theta \tag{5.1}$$

and

$$\Theta_c^o = y/a = \xi \sin \theta \sin \varphi. \tag{5.2}$$

The Laplace equations (4.15) and (4.17) in the inner region for the dimensionless temperature Θ^i and the zeroth-order electric potential $\bar{\phi}_0^i$, respectively, can each be solved using axisymmetry. The electric potential solution is symmetric about the z-axis and is identical to the parallel case (4.25)-(4.26):

$$\bar{\phi}^{i}_{0m} = -E^{o}_{c}a\xi\cos\theta(1-\bar{K}_{E}\xi^{-3}), \qquad (5.3)$$

$$\bar{\phi}^i_{0p} = -E^o_c a\xi \cos\theta (1 - \bar{K}_E). \tag{5.4}$$

The temperature solution is symmetric about the y-axis, and the solution is given simply by replacing z with y, i.e., replacing $\cos \theta$ in (4.27) and (4.28) with $\sin \theta \sin \varphi$:

$$\Theta_m^i = \xi \sin \theta \sin \varphi (1 - G\xi^{-3}), \qquad (5.5)$$

$$\Theta_p^i = \xi \sin \theta \sin \varphi (1 - G).$$
(5.6)

5.1.2. First Order Solution

The source term of the first-order equation (4.30) can be calculated from (5.3)–(5.6). In the medium we have

$$\nabla_{\xi} \bar{\phi}_{0m}^{i} \cdot \nabla_{\xi} \Theta_{m}^{i} = -E_{c}^{o} a \left[\cos \theta (1 + 2\bar{K}_{E}\xi^{-3}) \hat{\mathbf{e}}_{r} - \sin \theta (1 - \bar{K}_{E}\xi^{-3}) \hat{\mathbf{e}}_{\theta} \right] \cdot \left[\sin \theta \sin \varphi (1 + 2G\xi^{-3}) \hat{\mathbf{e}}_{r} + \cos \theta \sin \varphi (1 - G\xi^{-3}) \hat{\mathbf{e}}_{\theta} + \cos \varphi (1 - G\xi^{-3}) \hat{\mathbf{e}}_{\varphi} \right]$$

$$= -E_{c}^{o} a \cos \theta \sin \theta \sin \varphi \left[(1 + 2\bar{K}_{E}\xi^{-3})(1 + 2G\xi^{-3}) - (1 - \bar{K}_{E}\xi^{-3})(1 - G\xi^{-3}) \right]$$

$$= -3E_{c}^{o} a\xi^{-3} \cos \theta \sin \theta \sin \varphi (\bar{K}_{E} + G + \bar{K}_{E}G\xi^{-3}).$$

$$(5.7)$$

The source term in the particle is simply

$$\nabla_{\xi} \bar{\phi}^{i}_{0p} \cdot \nabla_{\xi} \Theta^{i}_{p} = -E^{o}_{c} a(1 - \bar{K}_{E}) \left[\cos \theta \hat{\mathbf{e}}_{r} - \sin \theta \hat{\mathbf{e}}_{\theta} \right] \cdot (1 - G) \left[\sin \theta \sin \varphi \hat{\mathbf{e}}_{r} + \cos \theta \sin \varphi \hat{\mathbf{e}}_{\theta} + \cos \varphi \hat{\mathbf{e}}_{\varphi} \right]$$
(5.8)
= 0.

Thus, in the particle we have a Poisson's equation with source term given by (5.7):

$$\frac{1}{\xi^2} \frac{\partial}{\partial \xi} \left(\xi^2 \frac{\partial \bar{\phi}^i_{1T,m}}{\partial \xi} \right) + \frac{1}{\xi^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \bar{\phi}^i_{1T,m}}{\partial \theta} \right) + \frac{1}{\xi^2 \sin^2 \theta} \frac{\partial^2 \bar{\phi}^i_{1T,m}}{\partial \varphi^2} - 3E_c^o a \xi^{-3} \cos \theta \sin \theta \sin \varphi (\bar{K}_E + G + \bar{K}_E G \xi^{-3}) = 0.$$
(5.9)

In the particle the source term is zero, so that we simply have Laplace's equation.

The homogeneous solution is generally given by spherical harmonics and can be found through separation of variables,

$$\bar{\phi}_{1T}^{i,H}(\xi,\theta,\varphi) = K(\xi)H(\theta)Y(\varphi).$$
(5.10)

Substituting (5.10) into the homogeneous portion of the differential equation (5.9), we find

$$\frac{\xi^2 \sin^2 \theta}{K} \frac{d^2 K}{d\xi^2} + \frac{2\xi \sin^2 \theta}{K} \frac{dK}{d\xi} + \frac{\sin^2 \theta}{H} \frac{d^2 H}{d\theta^2} + \frac{\cos \theta \sin \theta}{H} \frac{dH}{d\theta} = -\frac{1}{Y} \frac{d^2 Y}{d\varphi^2}.$$
 (5.11)

Since the left hand side of the above equation is a function only of ξ and θ while the right hand side is a function of φ , both sides must equal a constant, which we choose to be of the form m^2 . The general solutions to $Y(\varphi)$ are $\sin(m\varphi)$ and $\cos(m\varphi)$.

The left hand side of (5.11), which is also equal to m^2 , can be further split into

$$\frac{\xi^2}{K}\frac{d^2K}{d\xi^2} + \frac{2\xi}{K}\frac{dK}{d\xi} = -\left[\frac{1}{H}\frac{d^2H}{d\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{1}{H}\frac{dH}{d\theta} - \frac{m^2}{\sin^2\theta}\right].$$
(5.12)

Again, both sides must be equal to a constant, since the left hand side is a function of ξ and the right hand side is a function of θ . We choose this constant to be of the form l(l+1). The general solutions to $K(\xi)$ are ξ^l and $\xi^-(l+1)$. The equation for $H(\theta)$ is the associated Legendre differential equation, with general solutions given by the associated Legendre polynomials, $P_l^m(\cos \theta)$, where $-m \leq l \leq m$. The first several associated Legendre polynomials are as follows:

$$P_0^0(\cos \theta) = 1$$

$$P_1^0(\cos \theta) = \cos \theta$$

$$P_1^1(\cos \theta) = -\sin \theta$$

$$P_2^0(\cos \theta) = \frac{1}{2}(3\cos^2 \theta - 1)$$

$$P_2^1(\cos \theta) = -3\cos \theta \sin \theta$$

$$P_2^2(\cos \theta) = 3\sin^2 \theta$$

$$\vdots$$

Since $P_l^{-m}(\cos\theta)$ is simply a constant multiple of $P_l^m(\cos\theta)$, including only the nonnegative $m \ge 0$ will suffice. The general homogeneous solution is thus given by

$$\bar{\phi}_{1T}^{i,H} = \sum_{l=0}^{\infty} \sum_{m=0}^{l} \left(\bar{A}_l \xi^{-(l+1)} + \bar{B}_l \xi^l \right) P_l^m(\cos\theta) \left(\bar{C}_m \sin(m\varphi) + \bar{D}_m \cos(m\varphi) \right).$$
(5.14)

The constants \bar{A}_l , \bar{B}_l , \bar{C}_m , and \bar{D}_m are determined by the boundary conditions. The homogeneous solution can describe the electric potential both in the medium and in

the particle, so that we again use a ' to distinguish the constants in the particle.

The particular solution in the medium is given by

$$\bar{\phi}_{1T,m}^{i,P} = \frac{1}{2} E_c^o a \cos \theta \sin \theta \sin \varphi \left(-(\bar{K}_E + G)\xi^{-1} + \bar{K}_E G\xi^{-4} \right).$$
(5.15)

This solution can be verified by substitution into equation (5.9). Finally, the solutions to $\bar{\phi}^i_{1T}$ are given by the sum of the homogeneous and particular solutions:

$$\bar{\phi}_{1T,m}^{i} = \sum_{l=0}^{\infty} \sum_{m=0}^{l} \left(\bar{A}_{l} \xi^{-(l+1)} + \bar{B}_{l} \xi^{l} \right) P_{l}^{m} (\cos \theta) \left(\bar{C}_{m} \sin(m\varphi) + \bar{D}_{m} \cos(m\varphi) \right) + \frac{1}{2} E_{c}^{o} a \cos \theta \sin \theta \sin \varphi \left(-(\bar{K}_{E} + G) \xi^{-1} + \bar{K}_{E} G \xi^{-4} \right),$$
(5.16)

$$\bar{\phi}_{1T,p}^{i} = \sum_{l=0}^{\infty} \sum_{m=0}^{l} \left(\bar{A}_{l}' \xi^{-(l+1)} + \bar{B}_{l}' \xi^{l} \right) P_{l}^{m} (\cos \theta) \left(\bar{C}_{m}' \sin(m\varphi) + \bar{D}_{m}' \cos(m\varphi) \right).$$
(5.17)

5.1.3. Boundary Conditions

We use the four boundary conditions to solve for the unknown constants. First, the far-field matching condition requires that

$$\lim_{\xi \to \infty} \bar{\phi}^i_{1T,m} = \frac{1}{2} E^o_c a \xi^2 \cos \theta \sin \theta \sin \varphi = \frac{E^o_c}{2a} yz.$$
(5.18)

The associated Legendre polynomials $P_l^m(\cos\theta)$ are orthogonal just like the Legendre polynomials $P_n(\cos\theta)$ from the parallel case. Using this orthogonality property, we can compare like coefficients of $P_l^m(\cos\theta)$ between equations (5.16) and (5.18), we find that

$$\bar{B}_2 \bar{C}_1 = -\frac{E_c^o a}{6} \tag{5.19}$$

and the products $\bar{B}_l \bar{C}_m$ and $\bar{B}_l \bar{D}_m$ are zero for all other l and m. The second boundary condition at the center of the particle ($\xi = 0$) is that the electric potential must be finite, which requires all \bar{A}'_l to be zero. We can now simplify equations (5.16) and (5.17) somewhat:

$$\bar{\phi}_{1T,m}^{i} = \sum_{l=0}^{\infty} \sum_{m=0}^{l} \xi^{-(l+1)} P_{l}^{m} (\cos \theta) \left(\bar{C}_{lm} \sin(m\varphi) + \bar{D}_{lm} \cos(m\varphi) \right) + \frac{1}{2} E_{c}^{o} a \cos \theta \sin \theta \sin \varphi \left(\xi^{2} - (\bar{K}_{E} + G)\xi^{-1} + \bar{K}_{E}G\xi^{-4} \right), \qquad (5.20)$$

$$\bar{\phi}_{1T,p}^{i} = \sum_{l=0}^{\infty} \sum_{m=0}^{l} \xi^{l} P_{l}^{m}(\cos\theta) \left(\bar{C}_{lm}^{\prime} \sin(m\varphi) + \bar{D}_{lm}^{\prime} \cos(m\varphi) \right).$$
(5.21)

In the above equations, we have simplified the constants by defining $\bar{C}_{lm} = \bar{A}_l \bar{C}_m$, $\bar{D}_{lm} = \bar{A}_l \bar{D}_m$, $\bar{C}'_{lm} = \bar{B}'_l \bar{C}'_m$, and $\bar{D}'_{lm} = \bar{B}'_l \bar{D}'_m$.

The remaining two boundary conditions are at the interface between the particle and the medium ($\xi = 1$). The potential must be continuous:

$$\bar{\phi}^{i}_{1T,m}|_{\xi=1} = \bar{\beta}\bar{\phi}^{i}_{1T,p}|_{\xi=1}, \qquad (5.22)$$

or,

$$\sum_{l=0}^{\infty} \sum_{m=0}^{l} P_l^m(\cos\theta) \left(\bar{C}_{lm} \sin(m\varphi) + \bar{D}_{lm} \cos(m\varphi) \right) - \frac{1}{6} E_c^o a P_2^1(\cos\theta) \sin\varphi (1 - \bar{K}_E) (1 - G) = \sum_{l=0}^{\infty} \sum_{m=0}^{l} P_l^m(\cos\theta) \left(\bar{C}'_{lm} \sin(m\varphi) + \bar{D}'_{lm} \cos(m\varphi) \right), \quad (5.23)$$

and the displacement normal to the surface must also be continuous:

$$\bar{\epsilon}_m (1 + \bar{\delta}_T \Theta_m^i) \left(\frac{\partial \bar{\phi}_{0m}^i}{\partial \xi} + \bar{\delta}_T \frac{\partial \bar{\phi}_{1T,m}^i}{\partial \xi} \right) \Big|_{\xi=1} = \bar{\epsilon}_p (1 + \bar{\beta} \bar{\delta}_T \Theta_p^i) \left(\frac{\partial \bar{\phi}_{0p}^i}{\partial \xi} + \bar{\beta} \bar{\delta}_T \frac{\partial \bar{\phi}_{1T,p}^i}{\partial \xi} \right) \Big|_{\xi=1}.$$
(5.24)

Collecting all the $O(\delta_T)$ terms in (5.24), where

$$\Theta_m^i|_{\xi=1} = \Theta_p^i|_{\xi=1} = \sin\theta\sin\varphi(1-G),$$

and using the following known derivatives,

$$\frac{\partial \bar{\phi}_{0m}^i}{\partial \xi}\Big|_{\xi=1} = -E_c^o a \cos \theta (1+2\bar{K}_E),$$
$$\frac{\partial \bar{\phi}_{0p}^i}{\partial \xi}\Big|_{\xi=1} = -E_c^o a \cos \theta (1-\bar{K}_E),$$

$$\bar{\phi}_{1T,m}^{i} = \sum_{l=0}^{\infty} \sum_{m=0}^{l} -(l+1)P_{l}^{m}(\cos\theta) \left(\bar{C}_{lm}\sin(m\varphi) + \bar{D}_{lm}\cos(m\varphi)\right) + \frac{1}{2}E_{c}^{o}a\cos\theta\sin\theta\sin\varphi\left(2 + \bar{K}_{E} + G - 4\bar{K}_{E}G\right), \bar{\phi}_{1T,p}^{i} = \sum_{l=0}^{\infty} \sum_{m=0}^{l} lP_{l}^{m}(\cos\theta) \left(\bar{C}_{lm}'\sin(m\varphi) + \bar{D}_{lm}'\cos(m\varphi)\right),$$

we can express this last boundary condition as

$$-\bar{\epsilon}_{m}\sum_{l=0}^{\infty}\sum_{m=0}^{l}(l+1)P_{l}^{m}(\cos\theta)\left(\bar{C}_{lm}\sin(m\varphi)+\bar{D}_{lm}\cos(m\varphi)\right)$$
$$+\frac{\bar{\epsilon}_{m}E_{c}^{o}a}{2}(\bar{K}_{E}-G)P_{2}^{1}(\cos\theta)\sin\varphi$$
$$=\bar{\epsilon}_{p}\bar{\beta}\sum_{l=0}^{\infty}\sum_{m=0}^{l}lP_{l}^{m}(\cos\theta)\left(\bar{C}_{lm}'\sin(m\varphi)+\bar{D}_{lm}'\cos(m\varphi)\right)$$
$$+\frac{\bar{\epsilon}_{p}E_{c}^{o}a}{3}\bar{\beta}(1-\bar{K}_{E})(1-G)P_{2}^{1}(\cos\theta)\sin\varphi.$$
(5.25)

Once again, we compare like coefficients of $P_l^m(\cos\theta)$ in (5.23) and (5.25) to solve for the unknown constants \bar{C}_{lm} , \bar{D}_{lm} , \bar{C}'_{lm} , and \bar{D}'_{lm} . Again, most of these constants are zero, and the only nonzero ones are \bar{C}_{21} and \bar{C}'_{21} . For l = 2 and m = 1 we have

$$\bar{C}_{21} - \frac{1}{6} E_c^o a (1 - \bar{K}_E) (1 - G) = \bar{C}'_{21},$$

$$-3\bar{\epsilon}_m \bar{C}_{21} + \frac{1}{2} \bar{\epsilon}_m E_c^o a (\bar{K}_E - G) = 2\bar{\epsilon}_p \bar{\beta} \bar{C}'_{21} + \frac{1}{3} \bar{\epsilon}_p \bar{\beta} E_c^o a (1 - \bar{K}_E) (1 - G),$$
(5.26)

yielding

$$\bar{C}_{21} = E_c^o a \Big[\frac{1}{2} \frac{\bar{\epsilon}_m (\bar{K}_E - G)}{2\bar{\epsilon}_p + 3\bar{\epsilon}_m} + \frac{1}{3} \frac{\bar{\epsilon}_p (1 - \bar{\beta})(1 - \bar{K}_E)(1 - G)}{2\bar{\epsilon}_p + 3\bar{\epsilon}_m} \Big],$$

$$\bar{C}_{21}' = -\frac{E_c^o a}{\bar{\beta}} \Big[\frac{1}{2} \frac{\bar{\epsilon}_m (1 - 2\bar{K}_E + \bar{K}_E G)}{2\bar{\epsilon}_p + 3\bar{\epsilon}_m} - \frac{1}{3} \frac{\bar{\epsilon}_p \bar{\beta}(1 - \bar{K}_E)(1 - G)}{2\bar{\epsilon}_p + 3\bar{\epsilon}_m} \Big].$$
(5.27)

This nonzero constant \bar{C}_{21} from the homogeneous solution does not provide any net force on the particle, as we shall see later.

The potentials are given by

$$\bar{\phi}^{i}_{1T,m} = \left[\bar{C}_{21}\xi^{-3} - \frac{1}{6}E^{o}_{c}a\left(\xi^{2} - (\bar{K}_{E} + G)\xi^{-1} + \bar{K}_{E}G\xi^{-4}\right)\right]P_{2}^{1}(\cos\theta)\sin\varphi,$$
$$\bar{\phi}^{i}_{1T,p} = \bar{C}'_{21}\xi^{2}P_{2}^{1}(\cos\theta)\sin\varphi,$$

$$\begin{split} \bar{\phi}_{1T,m}^{i} &= -\frac{1}{6} E_{c}^{o} a \left(\xi^{2} - (\bar{K}_{E} + G)\xi^{-1} + \bar{K}_{E}G\xi^{-4}\right) P_{2}^{1}(\cos\theta) \sin\varphi \\ &+ E_{c}^{o} a \left[\frac{1}{2} \frac{\bar{\epsilon}_{m}(\bar{K}_{E} - G)}{2\bar{\epsilon}_{p} + 3\bar{\epsilon}_{m}} + \frac{1}{3} \frac{\bar{\epsilon}_{p}(1 - \bar{\beta})(1 - \bar{K}_{E})(1 - G)}{2\bar{\epsilon}_{p} + 3\bar{\epsilon}_{m}}\right] \xi^{-3} P_{2}^{1}(\cos\theta) \sin\varphi, \end{split}$$
(5.28)
$$\bar{\phi}_{1T,p}^{i} &= -\frac{E_{c}^{o} a}{\bar{\beta}} \left[\frac{1}{2} \frac{\bar{\epsilon}_{m}(1 - 2\bar{K}_{E} + \bar{K}_{E}G)}{2\bar{\epsilon}_{p} + 3\bar{\epsilon}_{m}} - \frac{1}{3} \frac{\bar{\epsilon}_{p}\bar{\beta}(1 - \bar{K}_{E})(1 - G)}{2\bar{\epsilon}_{p} + 3\bar{\epsilon}_{m}}\right] \xi^{2} P_{2}^{1}(\cos\theta) \sin\varphi. \tag{5.29}$$

5.2. Calculation of the Thermal DEP Force

5.2.1. Electric Field

To calculate the force, we need to evaluate the total electric field in the medium on the outside surface of the particle, $\bar{\mathbf{E}}_m^i|_{\xi=1^+} = -\nabla_X \bar{\phi}_m^i|_{\xi=1^+}$, where the total electric potential is given by $\bar{\phi}_m^i = \bar{\phi}_{0m}^i + \bar{\delta}_T \bar{\phi}_{1T,m}^i$. This electric field can be expressed as

$$\bar{\mathbf{E}}_{m}^{i} = \bar{E}_{r}\hat{\mathbf{e}}_{r} + \bar{E}_{\theta}\hat{\mathbf{e}}_{\theta} + \bar{E}_{\varphi}\hat{\mathbf{e}}_{\varphi}, \qquad (5.30)$$

where the three components are found by taking the gradient of (5.28):

$$\bar{E}_r = E_c^o(1+2\bar{K}_E)\cos\theta + \bar{\delta}_T E_c^o F_1(\bar{K}_E, G, \bar{\beta})\cos\theta\sin\theta\sin\varphi, \qquad (5.31)$$

$$\bar{E}_{\theta} = E_c^o(-1 + \bar{K}_E)\sin\theta + \bar{\delta}_T E_c^o F_2(\bar{K}_E, G, \bar{\beta})(1 - 2\sin^2\theta)\sin\varphi, \qquad (5.32)$$

$$\bar{E}_{\varphi} = \bar{\delta}_T E_c^o F_2(\bar{K}_E, G, \bar{\beta}) \cos \theta \cos \varphi.$$
(5.33)

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The functions F_1 and F_2 are:

$$F_{1} = -\left[\frac{3\bar{\epsilon}_{m}(1+2\bar{K}_{E})(1-G)}{2\bar{\epsilon}_{p}+3\bar{\epsilon}_{m}} + \frac{\bar{\epsilon}_{p}(5-2\bar{K}_{E}-2G-\bar{K}_{E}G)}{2\bar{\epsilon}_{p}+3\bar{\epsilon}_{m}} - \frac{3\bar{\beta}\bar{\epsilon}_{p}(1-\bar{K}_{E})(1-G)}{2\bar{\epsilon}_{p}+3\bar{\epsilon}_{m}}\right],$$
(5.34)

$$F_{2} = -\left[\frac{3}{2}\frac{\bar{\epsilon}_{m}(1-2\bar{K}_{E}+\bar{K}_{E}G)}{2\bar{\epsilon}_{p}+3\bar{\epsilon}_{m}} + \frac{\bar{\beta}\bar{\epsilon}_{p}(1-\bar{K}_{E})(1-G)}{2\bar{\epsilon}_{p}+3\bar{\epsilon}_{m}}\right].$$
(5.35)

5.2.2. MST Integration

The Maxwell stress tensor (3.1) includes the real permittivity as a function of temperature:

$$\epsilon_m(T)|_{\xi=1} = \epsilon_m \left(1 + \delta_T \sin \theta \sin \varphi (1 - G) \right).$$
(5.36)

Then, the force acting on the particle is given by the integral of the MST over the particle surface (4.61), which, for this spherical particle, can be expressed as

$$\mathbf{F} = \frac{\epsilon_m a^2}{4} \int_0^{2\pi} \int_0^{\pi} \left[1 + \delta_T \sin \theta \sin \varphi (1 - G) \right] \cdot \left[(E_r E_r^* - E_\theta E_\theta^* - E_\varphi E_\varphi^*) \hat{\mathbf{e}}_r + (E_r E_\theta^* + E_r^* E_\theta) \hat{\mathbf{e}}_\theta + (E_r E_\varphi^* + E_r^* E_\varphi) \hat{\mathbf{e}}_\varphi \right] \sin \theta \, d\theta \, d\varphi.$$
(5.37)

Using the unit vectors (3.3) with (5.37), we compute the F_x , F_y , and F_z components of the force. Because of symmetry in integrating over the sphere, only those integrals containing even powers of $\cos \theta$ and containing even powers of both $\cos \varphi$ and $\sin \varphi$ are nonzero. Thus, we find that only $F_y = \mathbf{F} \cdot \hat{\mathbf{e}}_y$ is nonzero. The first-order components proportional to $\bar{\delta}_T$, $\bar{\delta}_T^*$, and δ_T provide the leading-order thermal DEP force for this perpendicular case. In evaluating F_y we use the following integral formulas:

$$\int_{0}^{\pi} \sin^{3} \theta \, d\theta = \frac{4}{3} \qquad \int_{0}^{\pi} \cos^{2} \theta \sin \theta \, d\theta = \frac{2}{3}$$
$$\int_{0}^{\pi} \sin^{5} \theta \, d\theta = \frac{16}{15} \qquad \int_{0}^{\pi} \cos^{2} \theta \sin^{3} \theta \, d\theta = \frac{4}{15}$$
$$\int_{0}^{2\pi} \cos^{2} \varphi \, d\varphi = \pi \qquad \int_{0}^{2\pi} \sin^{2} \varphi \, d\varphi = \pi$$

Then, we have

$$F_{y} = \frac{\pi}{15} a^{2} \epsilon_{m} (E_{c}^{o})^{2} \Big[\bar{\delta}_{T} \big(3\bar{K}_{E}^{*}(F_{1} + 3F_{2}) \big) \\ + \bar{\delta}_{T}^{*} \big(3\bar{K}_{E}(F_{1}^{*} + 3F_{2}^{*}) \big) \\ + \delta_{T} \big(-5 + 5\bar{K}_{E} + 5\bar{K}_{E}^{*} + 4\bar{K}_{E}\bar{K}_{E}^{*} \big) \big(1 - G \big) \Big],$$
(5.38)

or

$$F_y = \frac{1}{2} \epsilon_m V_p \operatorname{Re}(\bar{\zeta}_m \bar{K}_T^{\perp}) (E_c^o)^2 |\nabla_X T_c^o|, \qquad (5.39)$$

where

$$\bar{K}_{T}^{\perp} = \frac{1}{10} \Big[3\bar{K}_{E}^{*}(\bar{K}_{E}G + 2\bar{K}_{E} + 2G - 5) - (4\bar{K}_{E}\bar{K}_{E}^{*} + 10\bar{K}_{E} - 5)(G - 1)\zeta_{\epsilon m}/\bar{\zeta}_{m} \Big].$$
(5.40)

Note that because the force from (5.38) includes the quantity $F_1 + 3F_2$, the net effect of the constant \bar{C}_{21} turns out to be zero. Thus, the homogeneous portion of $\bar{\phi}^i_{1T,m}$ provides no net contribution to the force on the particle.

Again, we must correct the coefficient \bar{K}_T^{\perp} to account for the electric buoyancy force, following the argument of Section 4.4.3. Setting \bar{K}_E and G to be zero in (5.40), we directly determine this correction term. Thus, the proper coefficient to be used in


Figure 27: The thermal DEP coefficients $\operatorname{Re}(\bar{K}_T^{\parallel})$ (dashed lines) and $\operatorname{Re}(\bar{K}_T^{\perp})$ (solid lines) as functions of $\operatorname{Re}(\bar{K}_E)$, evaluated at G = -0.5 (thermally insulating particle), G = 0 (same thermal properties), and G = 1 (thermally conducting particle).

calculating the thermal DEP force for the perpendicular case is

$$\bar{K}_{T}^{\perp} = \frac{1}{10} \Big[3\bar{K}_{E}^{*}(\bar{K}_{E}G + 2\bar{K}_{E} + 2G - 5) \\ + \big(5 - (4\bar{K}_{E}\bar{K}_{E}^{*} + 10\bar{K}_{E} - 5)(G - 1) \big) \zeta_{\epsilon m} / \bar{\zeta}_{m} \Big].$$
(5.41)

Now, we can calculate the value of the coefficient \bar{K}_T^{\perp} (5.41) across the range of possible material properties (Figure 27). Again, we choose a frequency below the crossover frequency to illustrate a region where the thermal DEP force can be significant, and use the values of $\zeta_{\epsilon m}$ and $\bar{\zeta}_m$ for water. The coefficient for the perpendicular case is strongest for a thermally insulating particle (G = -0.5), just as we saw for the parallel case. Although the magnitude of \bar{K}_T^{\perp} is generally smaller compared to \bar{K}_T^{\parallel} , it is not always negligible. For example, when $\operatorname{Re}(\bar{K}_E) = G = -0.5$, we have $\operatorname{Re}(\bar{K}_T^{\parallel}) = 2.585$



Figure 28: The computational domain used in COMSOL to evaluate the perpendicular thermal DEP force. The particle of radius a is located in a box with sides of length 2H = 20a. The far-field boundary conditions for the electric potential and the temperature are applied on the outer surface of the box.

and $\operatorname{Re}(\bar{K}_T^{\perp}) = 1.1825$, or about 54% as large. Note that this example of a particle that is both electrically and thermally insulating is similar to the case of polystyrene particles in water discussed in Section 4.5.2. Thus, the perpendicular component of the thermal DEP force may be of similar order magnitude to the classical DEP in some instances. Appendix B contains the MATLAB code used for this force calculation.

5.3. Comparison with Numerical Solution

To validate the analytical solution of the thermal DEP force in the perpendicular case, we compare it with a numerical integration of the Maxwell stress tensor in COMSOL. We choose a simple geometry of a small spherical particle located at the center of a box (Figure 28), with an applied electric field in the z-direction and an applied temperature gradient in the y-direction. The boundary conditions on the domain match the far-field boundary condition (5.18). We use the material properties for polystyrene particles in water and geometry values from Case 2 of the previous chapter, such that $|\bar{\delta}_T| = 0.015$. We take advantage of the symmetry in the x-direction and solve the fields only for x > 0. The mesh contains 190,942 elements and varies in size from a resolution of a/80 near the surface of the particle to a maximum mesh size of a near the surface of the box.

This choice of a problem setup will not have any contribution from either classical DEP or the parallel component of the thermal DEP, since both $\nabla_X \bar{\mathbf{E}}^o_{E,c}$ and $(\nabla_X \bar{\mathbf{E}}^o_{T,c})_{\parallel}$ in (4.21) are zero. Thus, the electrostatic force from the integration of the Maxwell stress tensor should be solely due to the perpendicular thermal DEP. The COMSOL-evaluated integration of the Maxwell stress tensor yields an electrostatic force of 4.98×10^{-11} N on the particle, while our analytical formula yields a perpendicular thermal DEP force of 5.37×10^{-11} N, so that the error between the two values is about 7.4%.

Although this error is higher than in the parallel case, the integration is much more difficult to perform numerically in the perpendicular case. First of all, we do not have axisymmetry in the perpendicular case, so that a 3D model must be used instead of the 2D model used in the parallel case. Also, the jump in material properties at the particle surface makes it difficult to integrate directly on the surface. Rather, we integrate the Maxwell stress tensor on a virtual surface of integration in the fluid at $\xi = 1.10$ and $\xi = 1.05$, and then use linear interpolation to estimate the value at the particle surface $(\xi = 1)$. Finally, within the Maxwell stress tensor itself, there is a lot of cancellation between the large positive values on one half of the sphere with the large negative values on the other half. Numerically, this greatly reduces the accuracy in determining the net force over the entire sphere. Despite all these issues, the numerically calculated force matches reasonably well with the expected force from our analytical expression.

CHAPTER 6 : General Thermal DEP Force and Torque

6.1. General Thermal DEP Force

With the results from Chapters 4 and 5 of the parallel and perpendicular cases, where the temperature gradient has been decomposed according to (4.19), we can now examine the general form of the thermal DEP force. The parallel component of the thermal DEP force is given by (4.65) with \bar{K}_T^{\parallel} from (4.86), and the perpendicular component of the thermal DEP force is given by (5.39) with \bar{K}_T^{\perp} from (5.41).

The general thermal DEP force must be given by some combination of these two cases, but we must determine the vector form of the force. Notice that both components of the thermal DEP force are proportional to $(E_c^o)^2 |\nabla_X T_c^o|$. The only possible vector combinations that can satisfy this requirement are $(\mathbf{\bar{E}} \cdot \mathbf{\bar{E}}^*) \nabla T$, $(\mathbf{\bar{E}}^* \cdot \nabla T) \mathbf{\bar{E}}$, and $(\nabla T \cdot \mathbf{\bar{E}}^*) \mathbf{\bar{E}}$ and their complex conjugates. However, $\mathbf{\bar{E}}^* \cdot \nabla T = \nabla T \cdot \mathbf{\bar{E}}^*$, so that we really only have two unique combinations. The general thermal DEP force must then be proportional to a linear combination of these two:

$$\mathbf{F}_{\text{T-DEP}} \propto \text{Re}\Big[\bar{C}_T(\bar{\mathbf{E}}\cdot\bar{\mathbf{E}}^*)\nabla T + \bar{C}_E(\bar{\mathbf{E}}^*\cdot\nabla T)\bar{\mathbf{E}}\Big].$$
(6.1)

Now, let us determine the coefficients \bar{C}_T and \bar{C}_E . In the parallel case, both the electric field and the temperature gradient are in the z-direction, which means that $(\bar{\mathbf{E}} \cdot \bar{\mathbf{E}}^*) \nabla T$ and $(\bar{\mathbf{E}}^* \cdot \nabla T) \bar{\mathbf{E}}$ are both equal to $(E_c^o)^2 |\nabla_X T_c^o|$. The coefficient \bar{K}_T^{\parallel} is thus related to the term $\bar{C}_T + \bar{C}_E$. In the perpendicular case, the electric field is in the z-direction, while the temperature gradient is in the y-direction. Here, $(\bar{\mathbf{E}} \cdot \bar{\mathbf{E}}^*) \nabla T$ still equals $(E_c^o)^2 |\nabla_X T_c^o|$, but $(\bar{\mathbf{E}}^* \cdot \nabla T) \bar{\mathbf{E}}$ equals zero because the dot product $\bar{\mathbf{E}}^* \cdot \nabla T$ is zero. Since only a single vector combination remains, the coefficient \bar{K}_T^{\perp} in (5.41)

must correspond to the coefficient \bar{C}_T for $(\bar{\mathbf{E}} \cdot \bar{\mathbf{E}}^*) \nabla T$:

$$\bar{C}_T = \bar{K}_T^{\perp} = \frac{1}{10} \Big[3\bar{K}_E^* (\bar{K}_E G + 2\bar{K}_E + 2G - 5) \\ + \big(5 - (4\bar{K}_E \bar{K}_E^* + 10\bar{K}_E - 5)(G - 1) \big) \zeta_{\epsilon m} / \bar{\zeta}_m \Big].$$
(6.2)

Knowing \bar{C}_T , we return to the parallel case, where the two equal vector combinations implies that $\bar{C}_E + \bar{C}_T = \bar{K}_T^{\parallel}$. Thus, we find

$$\bar{C}_{E} = \bar{K}_{T}^{\parallel} - \bar{K}_{T}^{\perp} = \frac{1}{10} \Big[10(2\bar{K}_{E} + 1)(G - 1) + 10 \\ + \bar{K}_{E}^{*}(\bar{K}_{E}G + 2\bar{K}_{E} + 2G - 5) \\ + (-10 + (2\bar{K}_{E}\bar{K}_{E}^{*} - 10\bar{K}_{E} - 10)(G - 1))\zeta_{\epsilon m}/\bar{\zeta}_{m} \Big].$$
(6.3)

Finally, the general thermal DEP force is given by

$$\mathbf{F}_{\text{T-DEP}} = \frac{1}{2} \epsilon_m V_p \Big[\text{Re}(\bar{\zeta}_m \bar{C}_T) (\bar{\mathbf{E}}_c^o \cdot \bar{\mathbf{E}}_c^{o*}) \nabla_X T_c^o + \text{Re}(\bar{\zeta}_m \bar{C}_E (\bar{\mathbf{E}}_c^{o*} \cdot \nabla_X T_c^o) \bar{\mathbf{E}}_c^o) \Big].$$
(6.4)

In the above equation (6.4) the two terms on the right hand side point in the direction of $\nabla_X T_c^o$ and $\bar{\mathbf{E}}_c^o$, respectively. The coefficients \bar{C}_T and \bar{C}_E , then, correspond to the portion of the thermal DEP force acting along with the temperature gradient and the electric field, respectively. Notice that for the two special parallel and the perpendicular cases, the (positive or negative) thermal DEP force acts in the same direction as the temperature gradient. In the perpendicular case this is because the second term on the right hand side of (6.4) drops out, and only the term aligned with the temperature gradient remains. In the parallel case, since both the temperature gradient and electric field are aligned, the two terms on the right hand side are also aligned, i.e., act in the direction of the temperature gradient. However, for an arbitrary angle α between the electric field and the temperature gradient, this is not generally the case, and the direction of the thermal DEP force is given by a combination of a portion along the electric field and a portion along the temperature gradient. The force must still be in the plane determined by the vectors $\bar{\mathbf{E}}_c^o$ and $\nabla_X T_c^o$, however, and the force (6.4) can be used to determine the thermal DEP force for any arbitrary angle α .

6.2. Rotating Electric Field

6.2.1. Thermal DEP Torque

Now, let us examine the torque due to thermal DEP. The torque acting on the particle can be evaluated by integrating the moment due to the Maxwell stress tensor over the surface of the particle:

$$\boldsymbol{\tau} = \int_{S} \mathbf{x} \times \left(\boldsymbol{\sigma}_{\text{MST}} \cdot \hat{\mathbf{n}}\right) dS, \tag{6.5}$$

where $\mathbf{x} = a\hat{\mathbf{n}}$ is the position vector on the surface of the particle. Since a stationary electric field will not generate a net torque, a rotating electric field is generally used for these cases [62]. This rotating electric field can be expressed as

$$\widetilde{\mathbf{E}}_{c}^{o}(\mathbf{X},t) = E_{c}^{o}(\widehat{\mathbf{e}}_{z}\cos\omega t - \widehat{\mathbf{e}}_{y}\sin\omega t)$$
(6.6)

for an electric field of magnitude E_c^o rotating counterclockwise in the yz-plane. Alternatively, we can use a complex electric field with the three components given explicitly as

$$E_x = 0,$$

$$E_y = E_c^o e^{j(\omega t + \frac{\pi}{2})},$$

$$E_z = E_c^o e^{j\omega t}.$$
(6.7)

The y-component of the electric field simply has a phase lag of $\pi/2$ compared to the z-component. Using the complex form, we can express the above equation (6.6) as [62]:

$$\widetilde{\mathbf{E}}_{c}^{o}(\mathbf{X},t) = E_{c}^{o}(\hat{\mathbf{e}}_{z} + j\hat{\mathbf{e}}_{y})e^{j\omega t}.$$
(6.8)

Notice that all the time-dependence is in the $\exp(j\omega t)$ term, so under time averaging, the electric field is given by

$$\bar{\mathbf{E}}_{c}^{o}(\mathbf{X}) = E_{c}^{o}(\hat{\mathbf{e}}_{z} + j\hat{\mathbf{e}}_{y}).$$
(6.9)

We specify the temperature gradient to be in the z-direction: $\nabla_X T_c^o = |\nabla_X T_c^o| \hat{\mathbf{e}}_z$. Then, the z- and y-components of the electric field (6.9) correspond to a parallel and a perpendicular case, respectively. Thus, $\bar{\mathbf{E}}_c^o(\mathbf{X}) = \bar{\mathbf{E}}_c^{o\parallel} + j\bar{\mathbf{E}}_c^{o\perp}$. The parallel portion is given by (4.54)–(4.55):

$$\bar{E}_{r}^{\parallel} = E_{c}^{o}(1+2\bar{K}_{E})\cos\theta + \bar{\delta}_{T}E_{c}^{o}F_{1}^{\parallel}(\bar{K}_{E},G) + \bar{\delta}_{T}E_{c}^{o}F_{2}^{\parallel}(\bar{K}_{E},G,\bar{\beta})P_{2}(\cos\theta), \quad (6.10)$$

$$\bar{E}^{\parallel}_{\theta} = E^o_c(-1 + \bar{K}_E)\sin\theta + \bar{\delta}_T E^o_c F^{\parallel}_3(\bar{K}_E, G, \bar{\beta})\cos\theta\sin\theta, \qquad (6.11)$$

$$\bar{E}_{\varphi}^{\parallel} = 0; \tag{6.12}$$

with the functions F_1^{\parallel} , F_2^{\parallel} , and F_3^{\parallel} given by F_1 , F_2 , and F_3 from (4.56)–(4.58).

The perpendicular portion can be determined straightforwardly from (5.31)–(5.33) with the E_0 terms in the y-direction:

$$\bar{E}_r^{\perp} = E_c^o(1+2\bar{K}_E)\sin\theta\sin\varphi + \bar{\delta}_T E_c^o F_1^{\perp}(\bar{K}_E, G, \bar{\beta})\cos\theta\sin\theta\sin\varphi, \qquad (6.13)$$

$$\bar{E}^{\perp}_{\theta} = E^{o}_{c}(1 - \bar{K}_{E})\cos\theta\sin\varphi + \bar{\delta}_{T}E^{o}_{c}F^{\perp}_{2}(\bar{K}_{E}, G, \bar{\beta})(1 - 2\sin^{2}\theta)\sin\varphi, \qquad (6.14)$$

$$\bar{E}_{\varphi}^{\perp} = E_c^o(1 - \bar{K}_E)\cos\varphi \qquad + \bar{\delta}_T E_c^o F_2^{\perp}(\bar{K}_E, G, \bar{\beta})\cos\theta\cos\varphi.$$
(6.15)

The functions F_1^{\perp} and F_2^{\perp} are F_1 and F_2 from (5.34)–(5.35). Finally, for the temperature gradient in the z-direction, the variation of ϵ_m with temperature is given by

$$\epsilon_m(T) = \epsilon_{m,c} \left(1 + \delta_T \cos \theta (1 - G) \right). \tag{6.16}$$

We now have everything necessary to calculate the torque from the integration (6.5):

$$\boldsymbol{\tau} = \frac{\epsilon_m a^2}{4} \int_0^{2\pi} \int_0^{\pi} \left[1 + \delta_T \cos \theta (1 - G) \right] \cdot \left[-(\bar{E}_r \bar{E}_{\varphi}^* + \bar{E}_r^* \bar{E}_{\varphi}) \hat{\mathbf{e}}_{\theta} + (\bar{E}_r \bar{E}_{\theta}^* + \bar{E}_r^* \bar{E}_{\theta}) \hat{\mathbf{e}}_{\varphi} \right] \sin \theta \, d\theta \, d\varphi,$$

$$(6.17)$$

where $\bar{E}_i = \bar{E}_i^{\parallel} + j\bar{E}_i^{\perp}$ for each of the r-, θ -, and φ -components.

Because of symmetry in integrating over the sphere, only those integrals containing even powers of $\cos \varphi$, $\sin \varphi$, and also $\cos \theta$ are nonzero. Thus, $\tau_x = \boldsymbol{\tau} \cdot \hat{\mathbf{e}}_x$ is the only nonzero component in this problem, and it is given by

$$\tau_x = \frac{\epsilon_m}{4} a^3 (E_c^o)^2 \Big[j(1+2\bar{K}_E)(1-\bar{K}_E^*) - j(1+2\bar{K}_E^*)(1-\bar{K}_E) \Big] \Big(\frac{2\pi}{3} + \frac{4\pi}{3} + \frac{2\pi}{3}\Big)$$
(6.18)

or

$$\tau_x = -3\epsilon_m V_p \operatorname{Im}(\bar{K}_E) (E_c^o)^2, \qquad (6.19)$$

where Im(...) denotes the imaginary portion of the enclosed complex expression.

Notice that the above torque expression does not have any dependence on the temperature or the parameter G. In fact, this expression (6.19) is exactly the same as the classical formula for the predicted electrorotational torque [62]. Although this result may seem surprising at first, it is actually what should be expected. Recall in our discussion about the thermal DEP force derivation from the Maxwell stress tensor that the electrostatic force is due to the interaction between the zeroth-order and first-order electric field. However, due to symmetry, the zeroth-order terms by themselves do not yield any net force on the particle. In calculating the force the second-order terms and any higher-order even terms are likewise symmetric. Thus, the leading order of the electrostatic force is the first-order term proportional to $\bar{\delta}_T$.

Now, compare the integral of the Maxwell stress tensor (4.61) used to derive the force with the expression (6.5) used to find the torque. Due to the additional multiplication by the moment arm \mathbf{x} in (6.5) when calculating the torque, all the odd-order terms are symmetric about the particle and produce no net torque, while the even-order terms now provide a nonzero result. Therefore, the leading order of the torque expression is proportional to $E_0 \cdot E_0$, while the first-order terms proportional to $\bar{\delta}_T$ have no net contribution to the torque. Since the temperature perturbation is a first-order effect, the zeroth-order electric field does not include this modification, and thus the torque still matches the classical expression. Although the second-order terms can provide a nonzero contribution, they are proportional to $(\bar{\delta}_T)^2$ and are negligible.

6.2.2. Thermal DEP Force

We can also find the thermal DEP force for the rotating electric field case. The electric field can be expressed as

$$\bar{\mathbf{E}}_{\text{ROT}} = \bar{\mathbf{E}}^{\parallel} + j\bar{\mathbf{E}}_{\perp} \tag{6.20}$$

where the components of $\bar{\mathbf{E}}^{\parallel}$ and $\bar{\mathbf{E}}_{\perp}$ are given in (6.10)–(6.15). Upon performing the MST integration (4.62), we find

$$F_{y} = \frac{\pi}{15}a^{2}\epsilon_{m}(E_{c}^{o})^{2} \left[j\bar{\delta}_{T} \left(-15F_{1}^{\parallel} + 3\bar{K}_{E}^{*} \left(F_{2}^{\parallel} - F_{3}^{\parallel} + F_{1}^{\perp} + 3F_{2}^{\perp} \right) \right) - j\bar{\delta}_{T}^{*} \left(-15F_{1}^{\parallel*} + 3\bar{K}_{E} \left(F_{2}^{\parallel*} - F_{3}^{\parallel*} + F_{1}^{\perp*} + 3F_{2}^{\perp*} \right) \right) - j\delta_{T} \cdot 15 \left(\bar{K}_{E} - \bar{K}_{E}^{*} \right) (1 - G) \right]$$

$$(6.21)$$

and

$$F_{z} = \frac{\pi}{15}a^{2}\epsilon_{m}(E_{c}^{o})^{2} \left[-\bar{\delta}_{T} \left(15F_{1}^{\parallel} + 3\bar{K}_{E}^{*} \left(2F_{2}^{\parallel} - 2F_{3}^{\parallel} + F_{1}^{\perp} + 3F_{2}^{\perp} \right) \right) + \bar{\delta}_{T}^{*} \left(15F_{1}^{\parallel*} + 3\bar{K}_{E} \left(2F_{2}^{\parallel*} - 2F_{3}^{\parallel*} + F_{1}^{\perp*} + 3F_{2}^{\perp*} \right) \right) + \delta_{T} \cdot 3 \left(5\bar{K}_{E} + 5\bar{K}_{E}^{*} + 2\bar{K}_{E}\bar{K}_{E}^{*} \right) (1 - G) \right].$$

$$(6.22)$$

The total thermal DEP force for the rotational case can thus be expressed as

$$\mathbf{F} = \frac{\pi}{15} a^{2} \epsilon_{m} (E_{c}^{o})^{2} \left\{ \begin{bmatrix} \operatorname{Re} \left[\bar{\delta}_{T} \left(10(2\bar{K}_{E}+1)(G-1) + 10 + 7\bar{K}_{E}^{*}(\bar{K}_{E}G+2\bar{K}_{E}+2G-5) \right) \right] \\ + 7\bar{K}_{E}^{*}(\bar{K}_{E}G+2\bar{K}_{E}+5\bar{K}_{E}+2G-5) \end{bmatrix} \right] \\ + \left[\operatorname{Im} \left[\bar{\delta}_{T} \left(-10(2\bar{K}_{E}+1)(G-1) - 10 + 5\bar{K}_{E}^{*}(\bar{K}_{E}G+2\bar{K}_{E}+2G-5) \right) \right] \\ + 30\delta_{T} \operatorname{Im}(\bar{K}_{E})(G-1) \right] \hat{\mathbf{e}}_{y} \right\}.$$
(6.23)

From inspection we see that F_z in (6.22) is simply the sum of the parallel case (4.64) and the perpendicular case (5.38). (Note that for the rotational case the force (5.38) is actually in the z-direction, not in the y-direction as in Chapter 5, because the temperature gradient here is in the z-direction.)

The force F_y (6.21) is due to the fact that the effects of the complex portion of the electric field do not completely cancel over a rotational period, since $\bar{\zeta}_m$ and \bar{K}_E are complex as well. However, at very low or at very high frequencies, both $\bar{\zeta}_m$ and \bar{K}_E have only a real component, and in these regions the force (6.21) equals zero. Thus, this term is only nonzero near the crossover AC frequency, and in most cases can be neglected. For example, generally a frequency below the crossover frequency is chosen to maximize the effects of the thermal DEP force. When the complex portion of the Clausius–Mossotti factor \bar{K}_E is small compared to the real portion, this force F_y is also small compared to the force F_z .

6.3. Practical Implementation of the Thermal DEP Force

Our analysis calculates the time-averaged classical DEP and thermal DEP forces due to the effects the electric and temperature fields. In general the particles are located within a moving fluid which satisfies Stokes' equation.

$$\mu \nabla^2 \mathbf{u} - \nabla p + \mathbf{f} = 0, \tag{6.24}$$

where μ is the viscosity of the fluid, **u** is the fluid velocity field, p is the pressure field, and **f** is an applied body force, e.g., the classical DEP and thermal DEP forces. The fluid is also incompressible: $\nabla \cdot \mathbf{u} = 0$. The velocity and pressure fields in the fluid can be solved generally using these two equations. Since the time constants for the electric field and the temperature field are very small, electrostatics provides a good approximation for the electric body force that acts on the fluid and the particle. The velocity of the particle can be determined using equation (2.4) from Section 2.2, where now the total electrostatic force (classical DEP plus thermal DEP) is included. Other body forces, such as Brownian motion, may be included as well. The RMS velocity of a particle due to Brownian motion is given by [35]

$$v_{\rm Brownian} = \sqrt{\frac{k_B T}{\rho_p V_p}},\tag{6.25}$$

where $k_B = 1.38065 \times 10^{-23}$ J/K is the Boltzmann constant [36], T is the temperature, and $\rho_p V_p$ is the particle's mass. Brownian motion becomes important for small particle sizes. We can compare the importance of Brownian motion (6.25) to the velocity from classical DEP, $v_{\text{DEP}} = F_{\text{DEP}}/6\pi\mu a$. For polystyrene particles in water and an electric field that matches our experimental case, we find that these two velocities are equal when the particle radius

$$a = \left(\frac{\sqrt{3}k_B T \mu}{4\sqrt{\pi}\rho_p \epsilon_m \operatorname{Re}(\bar{K}_E) V^2 / d_{gap}^3}\right)^{1/7} \approx 0.47 \ \mu \mathrm{m.}$$
(6.26)

Thus, a particle diameter of approximately 1 μ m is about the smallest that can theoretically be sorted using classical DEP.

CHAPTER 7 : Microfluidic Sorting Revisited

Now, let us return briefly to the microfluidic sorting device of Chapter 2 and examine what role the thermal DEP may play in this case with polystyrene particles in water. We use COMSOL to solve the fully-coupled electric potential and temperature fields and the flow field with the electrothermal body force (2.15) included in the Navier–Stokes equation. The inlet flow rates are 0.01 and 0.1 mL/hr for the particle stream and buffer stream, respectively, and a RMS voltage difference of 24.6 V is applied over each electrode gap.

In the experimental setup the bottom of the microfluidic channel is a glass slide containing the electrodes, while the top and side walls of the channel are made of PDMS. We have the material properties $k_{\text{glass}} = 1.4 \text{ W/(m\cdot K)}$ and $k_{\text{PDMS}} = 0.18 \text{ W/(m\cdot K)}$ [11]. For the thermal boundary conditions we specify the gold electrodes to be at ambient room temperature $T_a = 298 \text{ K}$ because of their high thermal conductivity. We use convective boundary conditions

$$q_s'' = h(T - T_a) \tag{7.1}$$

for the remaining walls. Here, q''_s is the surface heat flux and h is the heat transfer coefficient. The glass slide on the bottom has a thickness Δx of 1 mm, so the heat transfer coefficient is $h_{\text{glass}} = k_{\text{glass}}/\Delta x = 1400 \text{ W}/(\text{m}^2 \cdot \text{K})$. The PDMS has a thickness of approximately 3 mm, and we model its heat transfer coefficient as $h_{\text{PDMS}} = 60 \text{ W}/(\text{m}^2 \cdot \text{K})$.

Due to Joule heating, the temperature in this system is nonuniform. Since the conductivity of the fluid is only $\sigma_m = 0.015 \text{ S/}m$, the COMSOL simulation yields

a maximum temperature rise of only 1.75°C. However, this result agrees with the predictions of Ramos et al. [50], which we have discussed in Figure 16 of Section 2.4. The temperature rise predicted theoretically is about 1.87°C. For this low conductivity fluid medium, the Joule heating effects are not very large. Thus, we find that the thermal DEP force is small compared to the classical DEP force, with a thermal DEP to classical DEP ratio around 1%. Then, in this case the thermal DEP force is negligible and does not greatly influence the outlet positions of the particles shown in Figure 15.

While the Joule heating is small for this low-conductivity fluid, it is difficult to account for external heating sources, such as microscope illumination, which has been shown to be significant in certain cases [16, 17]. The 15°C temperature rise found by Green et al. [17], for example, would provide a much larger thermal DEP force. Since the actual temperature in the experiments was not determined, it is tough to say exactly what the thermal DEP force is for this case, and more experiments may be necessary to provide a more accurate picture.

CHAPTER 8 : Summary and Conclusions

8.1. Future Work

Although we have discussed the physical mechanism leading to the thermal DEP force and have done a theoretical analysis to derive the force expression, much more could be done to understand this force. Here are some possible future extensions to this work.

8.1.1. Direct Measurement of Thermal DEP Force

This theoretical analysis has been compared with numerical results from COMSOL, but a direct measurement of the thermal DEP force will provide another verification and corroboration of this analysis. Thus, we propose the following experimental setup to measure the thermal DEP force directly and verify our derived theoretical expression for the thermal DEP force $\mathbf{F}_{\text{T-DEP}}^{\parallel}$ when the electric field and the thermal gradient are aligned. As shown in Figure 29, the test section consists of a PMMA channel of 200 μm width and 500 μm depth that has vertical platinum electrodes of 20 mm length embedded on either side, supplying a voltage difference between 0 and 10 V across the channel. A microheating strip is embedded behind one electrode to provide a temperature difference up to 10°C across the channel. An array of temperature sensors is embedded to measure the actual temperature variation along the top and bottom surfaces of the channel. We also plan to use a temperature sensitive dye such as Rhodamine B to provide a direct measure of the temperature variation inside the channel [53]. This setup provides uniform electric and thermal gradient fields in the test section. There are three inlet streams to the channel, two buffer streams and the middle inlet stream containing polystyrene particles, as indicated in Figure 29.



Figure 29: Schematic of a microfluidic device for direct measurement of the thermal DEP force.

A particle introduced into this channel will initially travel along the center of the channel. As it passes through the test section, the thermal DEP force acting on the particle will cause it to migrate across the channel. At a precise location along the channel, the deflection of the particle from its initial position can be measured under a microscope. This deflection is directly correlated to the magnitude of the thermal DEP force experienced by the particle during its migration. The particle deflection can be controlled by changing the electric field strength, the applied temperature gradient, and the particle size.

For a particle of radius $a = 10 \ \mu \text{m}$ passing through the test section with voltage difference of 5 V and temperature difference of 5°C, according to the theoretical analysis, its migration velocity due to the parallel thermal DEP force is estimated to be 12 μ m/s. We can select a flow rate for the buffer streams such that the particle will be carried the 20 mm length of the test section in the time it takes for the particle to be deflected from the center streamline to the channel wall. In the test section, since the electric and thermal gradient fields are uniform, the electrothermal flow and the classical DEP force on the particle will be zero, although we do expect some edge effects from the electric field at the entrance into the test section that may cause a nonzero DEP force and also electrothermal flow. The velocity due to the DEP force at the entrance to the test section is estimated to be around 36 μ m/s, but in the direction against the flow (and perpendicular to the thermal DEP force). Thus it will not affect the thermal DEP force measurement as long as the particles are forced into the test section. For electrothermal flow, according to the estimate given in [3] the maximum velocity is around 35 μ m/s. The electrothermal flow causes the particle to move transverse in the channel. However, due to the strong flow, the time a particle spends in the entrance section is very short, and the actual deflection experienced by the particle due to electrothermal flow will be less than 7 μ m. We can also compensate for this initial deflection by starting our thermal DEP force measurement at a position slightly downstream of the entrance of the test section. Since within the channel the thermal DEP force should be uniform, we need only to measure the location of a particle at two positions within the channel to determine the amount it has been deflected due to thermal DEP.

Other possible forces are negligible in this experimental setup. Since the densities of polystyrene and water only differ by 5%, the sedimentation velocity of the particle is negligible (0.011 μ m/s). The natural convection induced by the temperature difference between vertical channel walls is also negligible (only $6.8 \times 10^{-6} \ \mu$ m/s) using the theory developed by Batchelor [2].

8.1.2. Particle Shapes and Properties

This theoretical analysis for a spherical particle could also be extended to other particle shapes, such as ellipsoidal particles, as has been done for the dipole moment approximation of the classical DEP force [28, 29]. This natural extension of this project will provide the ability to estimate the thermal DEP force in many more cases. In fact, in nucleate boiling, bubbles are known to deform into an elongated ellipsoidal shape due to the presence of an electric field [5, 33]. Thus, developing the theory for ellipsoidal particles is critical in a theoretical model for boiling applications.

Also, an analysis for multilayered particles or spherical shells will be very useful for biological applications. For example, a cell may have different electrical and thermal properties for the outer cell membrane, the inside of the cell, and its nucleus. A thermal DEP model that accounts for the different material properties of a multilayered particle will provide better modeling for biological systems. A similar analysis for multilayered particles has been done for classical DEP by Jones [29].

8.1.3. Electric Double Layer Effects

The effects of electric double layers have not been included in this thermal DEP analysis. An electric double layer forms when counterions are attracted to a charged body in a fluid. An applied electric field induces a net dipole moment in the particle (Figure 4), and the direction of this dipole alternates with the frequency of the AC electric field. At high frequencies the direction of the dipole switches fast enough to prevent the formation of a double layer. For very low-frequency AC electric fields, however, the dipole direction does not change very fast, so that ions may have time to dissociate from the fluid and form an electric double layer that screens some of the charge on the particle. By changing the effective net dipole moment of the particle, the electric double layer modifies the DEP force and also likely the thermal DEP force.

Our thermal DEP analysis results in only two specific frequency domains: a conductivitydominant region and a permittivity-dominant region, with a crossover frequency of about $\omega = \sigma_m/\epsilon_m$. An induced charge double layer has been shown to cause a modification to the classical DEP force at very low frequencies [60]. A similar effect is likely for the thermal DEP force.

8.2. Summary

It is well-known that a dielectric particle in a dielectric fluid medium experiences a dielectrophoretic force due to an applied electric field. It is also well-known that the additional presence of a nonuniform temperature field can induce electrothermal flow in a fluid due to the temperature-dependence of material properties including electrical permittivity and conductivity. However, even with a nonuniform temperature field, the force on a particle is usually still modeled using the classical DEP force. Our analysis finds that temperature gradients give rise to a thermal DEP force.

We used a small perturbation analysis to solve the governing equations for electric potential and temperature analytically, and then used the integration of the Maxwell stress tensor to derive an expression for the thermal force on the particle. The general case with an arbitrary angle between the electric field and temperature gradient may be decomposed into a parallel case and a perpendicular case. We computed the thermal DEP force for each of these cases from the Maxwell stress tensor. Just as the dipole moment approximation provides an easy way to calculate the classical DEP force to the leading order, we derive an analogous expression for the leading order of the thermal DEP force. This thermal DEP force is proportional to the temperature gradient, the square of the electric field strength, and the particle's volume. We use the theoretical analysis to examine the case of a rotating electric field and find the torque on the particle exactly matches the classical case.

Since the source of the thermal DEP force is due to a temperature-induced modification to the electric field, it cannot simply be lumped into the classical DEP force, which arises from a nonuniform electric field caused by the geometry of the electrodes. The isothermal electric potential, including the perturbation due to geometry, is still a solution to Laplace's equation. The temperature-dependence of the electric field gives rise to a nonzero source term, so that the temperature-perturbation solution comes from a more general Poisson's equation. For our small perturbation analysis, we see that superposition of the classical DEP from the geometric perturbation and the thermal DEP from the thermal perturbation gives the total electrostatic force on the particle. Subtracting the electric buoyancy force provides the corrected expression that gives the net electrostatic force on the particle relative to the fluid.

Since many microfluidic devices use an applied electric field to control and manipulate particles, the thermal DEP force can play an important role due to Joule heating of fluids with high electrical conductivity, including many common biological buffers. For low conductivity fluids externally applied heating elements can also induce large temperature gradients and a thermal DEP force. The temperature field may not only be an unintended artifact of the Joule heating, but it is possible to apply a temperature field specifically to take advantage of thermal DEP as well as classical DEP to control and manipulate cells or bacteria.

Large temperature gradients are also present in nucleate boiling applications. Thus, thermal DEP may play a role here as well. The heat transfer rates in nucleate boiling applications are highly dependent on bubble growth and departure. The thermal DEP force acting on a nucleated bubble can keep it on the surface longer, causing each bubble to carry more heat from the surface when it detaches, thereby modifying the heat transfer rates.

This analysis of the thermal DEP force can be extended to different particle shapes and properties, such as ellipsoidal particles or multilayered particles. Also, we propose an experiment to provide a direct measurement of the thermal DEP force. The thermal DEP force offers many new ways to take advantage of combined electric and thermal fields, and helps to provide a more accurate understanding of the electrostatic force acting on a dielectric particle.

APPENDIX A : Analytical Potential

In Section 2.3, the analytical electrical potential, DEP force, and particle traces are calculated using the following MATLAB files:

- phi_2_pairs.m calculates the electric potential and DEP force on a particle within the channel for a given geometry and material properties
- get_phi.m calculates the electric potential, including the mirrored electrodes and scaling factor β
- get_E.m calculates the electric field for a given potential
- get_Fx_DEP.m calculates the DEP force in the x-direction
- get_Fy_DEP.m calculates the DEP force in the y-direction
- particle_trace.m calculates particle traces in the channel for a given initial position, fluid velocity profile, and DEP force

phi 2 pairs.m

```
clear all
close all
directory = 'S:/models/analytical/';
d_gap = 12.5; % micron
height = 50; % micron
w_elect = 65; % micron
width = 2*(d_gap+w_elect);
V_el = [25 0 20 0]; % rms voltage
num_el_gap = length(V_el)-1;
dp = 4.8; % micron
```

```
rp = dp/2;
vol = 4/3*pi*(rp*1e-6)^3; % m^3
eps_0 = 8.854e-12; % A*s/V/m
eps_p = 2.5;
eps_m = 80;
sigma_p = 5e-9/(rp*1e-6); % A/V/m
sigma_m = 15e-3; % A/V/m
freq = 2.2e6; % Hz
omega = 2*pi*freq; % rad/s
A = eps_0*(eps_p-eps_m);
B = (sigma_m-sigma_p)/omega;
C = eps_0*(eps_p+2*eps_m);
D = (2*sigma_m+sigma_p)/omega;
fCM = (A*C-B*D)/(C^2+D^2);
% fCM = -0.4259;
eta_m = 8.89e-4; % viscosity Pa*s
x_coor = -width/2:1e-1:width/2+(num_el_gap-1)*(w_elect+2*d_gap); % micron
y_coor = 0:1e-1:height; %micron
[x_grid,y_grid] = meshgrid(x_coor,y_coor);
C = zeros(1, 2*length(V_el)-2);
C(1) = 1;
for ind = 3:2:length(C)
    C(ind) = get_phi(ind*(d_gap+w_elect/2),0,[0 1],d_gap,height,w_elect,1);
    C(ind+1) = 1 - C(ind);
end
C_{locs} = [length(C):-1:1 \ 1:length(C)];
for ind = 1:length(V_el)
    C_loc(ind,:) = C_locs(length(C)+1-2*(ind-1):2*length(C)-2*(ind-1));
end
coeff = C(C_{loc});
V_an = reshape(coeff(1:length(V_el),1:(length(V_el)*2-2))\V_el',...
               2, length(V_el) - 1)';
dx = 0.01; % micron
dy = 0.01; \% micron
phi = get_phi(x_coor,y_coor,V_an,d_gap,height,w_elect,num_el_gap);
%x_mid = w_elect + 2*d_gap;
E = get_E(x_coor,y_coor,V_an,d_gap,height,w_elect,dx,dy,num_el_gap);
% 2*E*gradE
```

```
Fx_DEP = 3/2*eps_m*eps_0*vol*fCM*Ex_gradient;
Fy_DEP = 3/2*eps_m*eps_0*vol*fCM*Ey_gradient;
```

get phi.m

```
function [phi_tot] = get_phi(x_coor,y_coor,V_an,...
                             d_gap,height,w_elect,num_el_gap)
[x_grid,y_grid] = meshgrid(x_coor,y_coor);
theta = zeros(length(y_coor),length(x_coor),4);
phi = zeros(length(y_coor),length(x_coor),4);
phi_sum = zeros(length(y_coor),length(x_coor),4);
phi_tot = zeros(length(y_coor),length(x_coor));
for gap = 1:num_el_gap
    x_offset = (gap-1)*(w_elect+2*d_gap);
    for i = 1:4
       theta(:,:,i) = acos((sqrt((x_grid-x_offset+d_gap).^2...
            +(y_grid-(-1)^i*2*round((i-1)/2)*height).^2)...
            -sqrt((x_grid-x_offset-d_gap).^2...
            +(y_grid-(-1)^i*2*round((i-1)/2)*height).^2))/(2*d_gap));
        phi(:,:,i) = (-1)^(round(i/2)+1)*((V_an(gap,2)-V_an(gap,1))/2...
            *(1-2/pi*theta(:,:,i)));
        phi_sum(:,:,i) = sum(phi(:,:,1:i),3)+(V_an(gap,2)+V_an(gap,1))/2;
    end
    % solve for scaling factor beta (geometry dependent only)
    x_beta = d_gap + w_elect/2 + x_offset;
    y_beta = 0;
    for i = 1:4
```

```
theta_beta(i) = acos((sqrt((x_beta-x_offset+d_gap).^2...
+(y_beta-(-1)^i*2*round((i-1)/2)*height).^2)...
-sqrt((x_beta-x_offset-d_gap).^2...
+(y_beta-(-1)^i*2*round((i-1)/2)*height).^2))/(2*d_gap));
phi_beta(i) = (-1)^(round(i/2)+1)*((V_an(gap,2)-V_an(gap,1))/2...
*(1-2/pi*theta_beta(i)));
phi_sum_beta(i) = sum(phi_beta(1:i))+(V_an(gap,2)+V_an(gap,1))/2;
end
phi2_temp = phi_sum_beta(2);
phi4_temp = phi_sum_beta(4);
beta = (V_an(gap,2)-phi2_temp)/(phi4_temp-phi2_temp);
%height -> Inf -- beta -> 2/3
```

```
%beta = 0.655546299051403;
phi_temp = phi_sum(:,:,2)+beta*(phi_sum(:,:,4)-phi_sum(:,:,2));
```

```
phi_tot = phi_tot + phi_temp;
```

%d_gap = 12.5, height = 50, w_elect = 65

```
end
```

get E.m

function [E] = get_E(x_coor,y_coor,V_num,... d_gap,height,w_elect,dx,dy,num_el_gap)

- Ex = (get_phi(x_coor+dx/2,y_coor,V_num,d_gap,height,w_elect,num_el_gap)... -get_phi(x_coor-dx/2,y_coor,V_num,d_gap,height,w_elect,num_el_gap))... ./dx*1e6;
- Ey = (get_phi(x_coor,y_coor+dy/2,V_num,d_gap,height,w_elect,num_el_gap)... -get_phi(x_coor,y_coor-dy/2,V_num,d_gap,height,w_elect,num_el_gap))... ./dy*1e6;

 $E = sqrt(Ex.^2 + Ey.^2);$

```
Fx_DEP = 3/2*eps_m*eps_0*vol*fCM*Ex_gradient;
```

Fy_DEP = 3/2*eps_m*eps_0*vol*fCM*Ey_gradient;

```
particle_trace.m
%%
x0 = -width/2+rp;
y0 = dp;
z0 = rp;
file_positions = fopen([directory 'positions.txt'],'w');
```

```
fclose('all');
for y0 = [round(rp+0.5) 0:5:height height-round(rp+0.5)]
    if y0 <= rp || y0 >= height-rp
        continue;
    end
%dt = 1e-4; % s
timesteps = 2000;
angle = 45; % degrees
U_max = 2.2917e-3; % m/s
x_trace = ones(1,timesteps)*NaN;
y_trace = ones(1,timesteps)*NaN;
z_trace = ones(1,timesteps)*NaN;
x_trace(1) = x0;
y_trace(1) = y0;
z_{trace}(1) = z0;
x_trace_fix = x_trace;
y_trace_fix = y_trace;
z_trace_fix = z_trace;
x_el = ones(1,timesteps)*NaN;
y_el = zeros(1,timesteps);
z_el = ones(1,timesteps)*NaN;
x_el(1) = -width/2;
y_{el}(1) = 0;
z_{el}(1) = z_{trace}(1);
x_el_fix = x_el;
y_el_fix = y_el;
z_el_fix = z_el;
for n = 1:timesteps
    x_coor = x_trace(n);
    y_coor = y_trace(n);
    phi = get_phi(x_trace(n),y_trace(n),V_an,...
                  d_gap,height,w_elect,num_el_gap);
    E = get_E(x_trace(n),y_trace(n),V_an,...
              d_gap,height,w_elect,dx,dy,num_el_gap);
    Fx_DEP_an = get_Fx_DEP(x_trace(n),y_trace(n),V_an,...
```

```
d_gap,height,w_elect,dx,dy,...
                       eps_m,eps_0,vol,fCM,num_el_gap);
Fy_DEP_an = get_Fy_DEP(x_trace(n),y_trace(n),V_an,...
                       d_gap,height,w_elect,dx,dy,...
                       eps_m,eps_0,vol,fCM,num_el_gap);
d_h = rp./min(y_trace(n),height-y_trace(n)); % distance from wall
% drag correction factors
faxen = max(0, (1 - 9/16 * d_h + 1/8 * d_h.^3 - 45/256 * d_h.^4...)
               - 1/16 * d_h.^5)); % parallel
normal = 1 + 1.10904083*(1/d_h-1)^-0.98196170; % normal
Vx_DEP_an = Fx_DEP_an.*faxen/(6*pi*eta_m*rp*1e-6); % m/s
Vy_DEP_an = Fy_DEP_an./(6*pi*eta_m*rp*1e-6*normal); % m/s
U = 4*U_max.*y_trace(n).*(height-y_trace(n))/height^2; % fluid velocity
% particle velocity
v_x = Vx_DEP_an + U*sin(angle(n)*pi/180);
v_y = Vy_{DEP_an};
v_z = U*cos(angle(n)*pi/180);
if x_trace(n) <= -width/2 + rp</pre>
    x_trace(n) = -width/2 + rp;
    v_x = max(0, v_x);
end
if y_trace(n) >= height - rp
    y_trace(n) = height - rp;
   v_y = \min(0, v_y);
end
if x_trace(n) >= width/2+(num_el_gap-1)*(w_elect+2*d_gap)
    break;
end
if z_{trace_fix(n)} \ge 400
    break;
end
dt = 1e-6/sqrt(v_x.^2 + v_y.^2 + v_z.^2);
x_trace(n+1) = x_trace(n) + v_x*dt*1e6;
y_trace(n+1) = y_trace(n) + v_y*dt*1e6;
z_trace(n+1) = z_trace(n) + v_z*dt*1e6;
angle(n+1) = angle(n);
x_el(n+1) = -w_elect - d_gap;
z_el(n+1) = z_trace(n+1);
```

```
x_trace_fix(n+1) = x_trace_fix(n) ...
+ sind(angle(n))*(x_trace(n+1)-x_trace(n)) ...
+ cosd(angle(n))*(z_trace(n+1)-z_trace(n));
y_trace_fix(n+1) = y_trace_fix(n) + y_trace(n+1) - y_trace(n);
z_trace_fix(n+1) = z_trace_fix(n) ...
- cosd(angle(n))*(x_trace(n+1)-x_trace(n)) ...
+ sind(angle(n))*(z_trace(n+1)-z_trace(n));
if z_trace_fix(n+1) > 350
break;
end
x_el_fix(n+1) = x_el_fix(n) + cosd(angle(n))*(z_el(n+1)-z_el(n));
z_el_fix(n+1) = z_el_fix(n) + sind(angle(n))*(z_el(n+1)-z_el(n));
```

```
end
```

```
file_positions = fopen([directory 'positions.txt'],'a+');
fprintf(file_positions,...
        '%8g\t%8g\t%10e\t%10e\t%10e\t%10e\t%10e\t%10e\t%10e\t%10e\n',...
        dp,y0,n,x_trace(n),y_trace(n),z_trace(n),...
        x_trace_fix(n),y_trace_fix(n),z_trace_fix(n));
fclose(file_positions);
%
z_max = 400/sind(angle(1));
x_{offset} = 577.5;
z_{offset} = 150;
figure(10)
subplot(2,1,1)
%find(x_trace>-d_gap & x_trace< d_gap)</pre>
plot3(x_trace,y_trace,z_trace,'r-','Linewidth',2)
%set(gca, 'OuterPosition', [0 0 0.5 .95])
hold on:
for ind = 1:num_el_gap+1
%
     plot3(x_trace(x_trace>-d_gap+(ind-1)*(w_elect+2*d_gap) ...
%
                 & x_trace< d_gap+(ind-1)*(w_elect+2*d_gap)),...
%
         y_trace(x_trace>-d_gap+(ind-1)*(w_elect+2*d_gap) ...
%
                & x_trace< d_gap+(ind-1)*(w_elect+2*d_gap)),...</pre>
%
         z_trace(x_trace>-d_gap+(ind-1)*(w_elect+2*d_gap) ...
%
                & x_trace< d_gap+(ind-1)*(w_elect+2*d_gap)),...</pre>
%
         'b-','Linewidth',2)
```

```
if y0 == round(rp+0.5)
        plot3([x_el(1:n) x_el(n:-1:1)+w_elect x_el(1)]...
                 +(ind-1)*(w_elect+2*d_gap),...
             [y_el(1:n) y_el(n:-1:1) y_el(1)],...
             [z_el(1:n) z_el(n:-1:1) z_el(1)], 'k-', 'Linewidth',3)
    end
end
xlabel('$x_{\textrm{el}}$', 'FontSize', 16, 'Interpreter', 'Latex')
ylabel('$y$','FontSize',16,'Interpreter','Latex')
zlabel('$z$','FontSize',16,'Interpreter','Latex')
set(gca,'ZDir','reverse')
ylim([0 height])
view([0 90])
%axis image
grid on;
subplot(2,1,2)
plot3(x_trace_fix+x_offset,y_trace_fix,z_offset-z_trace_fix,...
      'r-','Linewidth',2)
%set(gca, 'OuterPosition', [0.5 0 0.5 .95])
hold on;
for ind = 1:num_el_gap+1
%
     plot3(x_offset+x_trace_fix(x_trace>-d_gap+(ind-1)*(w_elect+2*d_gap) ...
%
                  & x_trace< d_gap+(ind-1)*(w_elect+2*d_gap)),...
%
         y_trace_fix(x_trace>-d_gap+(ind-1)*(w_elect+2*d_gap) ...
                   & x_trace< d_gap+(ind-1)*(w_elect+2*d_gap)),...
%
%
         z_offset-z_trace_fix(x_trace>-d_gap+(ind-1)*(w_elect+2*d_gap) ...
%
                             & x_trace< d_gap+(ind-1)*(w_elect+2*d_gap)),...</pre>
%
         'b-','Linewidth',2)
    if y0 == round(rp+0.5)
        plot3([450,850,850+w_elect*sqrt(2),450+w_elect*sqrt(2),450]...
               +(ind-1)*(w_elect+2*d_gap)./sind([45,45,45,45,45]),...
             [0,0,0,0,0],\ldots
             [200, -200, -200, 200, 200], ...
            'k-','Linewidth',2)
    end
    zlim([-202 202])
end
xlabel('$x$','FontSize',16,'Interpreter','Latex')
ylabel('$y$','FontSize',16,'Interpreter','Latex')
zlabel('$z$','FontSize',16,'Interpreter','Latex')
%set(gca,'ZDir','reverse')
ylim([0 height])
```

```
view([6 4])
view([0 0])
%axis image
grid on;
print(figure(10),'-depsc',...
    [directory 'trace_' num2str(round(20*V_0)) 'V_' ...
    num2str(round(10*dp)) 'dp_' num2str(angle(1)) 'theta.eps'])
```

APPENDIX B : Thermal DEP Calculations

- F_TDEP_Parallel.m calculates the thermal DEP force for the parallel case, the classical DEP force, and the buoyancy force over a range of AC frequencies.
- F_TDEP_Perpendicular.m calculates the perpendicular and parallel thermal DEP coefficients \bar{K}_T^{\perp} and \bar{K}_T^{\parallel} over the possible range of material properties.

F TDEP Parallel.m

```
clear all
close all
r_p = 250; % micron
Vol_p = 4/3*pi*(r_p*1e-6)^3; % m^3
rho_m = 958.4; % kg/m^3
rho_p = 0.598; % kg/m^3
g = 9.81; \ \mbox{m/s}^2
F_buoyancy = (rho_m - rho_p)*g*Vol_p; % N
eps_0 = 8.854e-12; % F/m
eps_m = 55.6*eps_0; % water at 100 C
eps_p = 1.00589*eps_0; % steam at 100 C
sgm_m = 7.87e-5; % DI water
sgm_p = 2e-4; % steam
eps_m_dT = -0.004 * eps_m;
sgm_m_dT = 0.02*sgm_m;
% frequency of electric field
for i = 1:106
    omega = 2*pi*10^((i-1)/15);
    omega_xplot(i) = omega;
    % complex permittivity
    eps_m_bar = eps_m - j * sgm_m/omega;
    eps_p_bar = eps_p - j * sgm_p/omega;
```

```
f_m = (eps_p - eps_m)/(eps_p + 2*eps_m);
% complex CM factor
f_m_bar = (eps_p_bar - eps_m_bar)/(eps_p_bar + 2*eps_m_bar);
\% relative permittivity change with temperature (1/eps * d eps / dT)
zeta_m = 1/eps_m * eps_m_dT;
zeta_m_bar = (eps_m_dT - j * sgm_m_dT/omega)/eps_m_bar;
k_m = 0.6729;
k_p = 0.0240;
% thermal CM factor
g_m = (k_p - k_m)/(k_p + 2*k_m);
a = r_p*1e-6; % 0.250e-3
H = 10*a;
            % 2.5e-3
              % 5e-3
L = 2 * H;
R1 = 5000e-6; \% 5e-3
R2 = R1 + L; \% 10e-3
R_c = R1 + a; % 5.25e-3
V_rms = 1e3; % 1 kV
l_V = L;
             % 5e-3
T_c = 373;
delta_T = 10; \% 10 K
l_T = 2*a; % 0.5e-3
E_{rms} = V_{rms}/1_V; % 4e5
VO = V_{rms*H/L};
del_T = delta_T*H/L;
Tinf = T_c-del_T/H*R1*R2/R_c+del_T/H*0.5*(R1+R2);
gradTinf = del_T/H*R1*R2/R_c^2;
Viso = V0/H*R1*R2/R_c - V0*(R2/H-1);
Vinf = V0+2*V0/log((1+zeta_m_bar*-del_T/H*(R1*R2/R1-0.5*(R1+R2)))...
        /(1+zeta_m_bar*-del_T/H*(R1*R2/R2-0.5*(R1+R2))))...
        *(log(1+zeta_m_bar*-del_T/H*(R1*R2/R_c-0.5*(R1+R2)))...
         -log(1+zeta_m_bar*-del_T/H*(R1*R2/R1-0.5*(R1+R2))));
Eiso = VO/H*R1*R2/R_c^2;
Einf = 2*V0/log((1+zeta_m_bar*-del_T/H*(R1*R2/R1-0.5*(R1+R2)))...
        /(1+zeta_m_bar*-del_T/H*(R1*R2/R2-0.5*(R1+R2))))...
        *zeta_m_bar*-del_T/H*R1*R2/R_c^2...
        /(1+zeta_m_bar*-del_T/H*(R1*R2/R_c-0.5*(R1+R2)));
gradEiso = -2*VO/H*R1*R2/R_c^3;
gradEinf = 2*V0/log((1+zeta_m_bar*-del_T/H*(R1*R2/R1-0.5*(R1+R2)))...
            /(1+zeta_m_bar*-del_T/H*(R1*R2/R2-0.5*(R1+R2))))...
            *zeta_m_bar*-del_T/H*R1*R2...
            *((1+zeta_m_bar*-del_T/H*(R1*R2/R_c-0.5*(R1+R2)))...
```

```
*-2/R_c^3+1/R_c^2*zeta_m_bar*-del_T/H*R1*R2/R_c^2)...
                /(1+zeta_m_bar*-del_T/H*(R1*R2/R_c-0.5*(R1+R2)))^2;
    K_T_bar_UC = 1/10*(10*(2*f_m_bar+1)*(g_m-1)+4*conj(f_m_bar)...
         *(f_m_bar*g_m+2*f_m_bar+2*g_m-5)...
          -(2*f_m_bar*conj(f_m_bar)+20*f_m_bar+5)...
           *(g_m-1)*zeta_m/zeta_m_bar);
    K_T_bar = 1/10*(10*(2*f_m_bar+1)*(g_m-1)+10+4*conj(f_m_bar)...
         *(f_m_bar*g_m+2*f_m_bar+2*g_m-5)...
          -(5+(2*f_m_bar*conj(f_m_bar)...
            +20*f_m_bar+5)*(g_m-1))*zeta_m/zeta_m_bar);
    K_T_bar_DEP = 1/10*(10*(2*f_m_bar+1)*(g_m-1)+10...)
         +4*conj(f_m_bar)*(f_m_bar*g_m+2*f_m_bar+2*g_m-5)...
         -(5+(2*f_m_bar*conj(f_m_bar)+20*f_m_bar+5)*(g_m-1))...
          *zeta_m/zeta_m_bar)+3*f_m_bar;
    %DEP_iso(i) = 3*eps_m*Vol_p*real(f_m_bar*Eiso*conj(gradEiso));
    DEP_iso(i) = 3*eps_m*Vol_p*real(f_m_bar*Einf*conj(gradEiso));
    DEP_inf(i) = 3*eps_m*Vol_p*real(f_m_bar*Einf*conj(gradEinf));
    TDEP_inf_UC(i) = eps_m*Vol_p*real(zeta_m_bar*K_T_bar_UC...
                          *Einf*gradTinf*conj(Einf));
    TDEP_inf(i) = eps_m*Vol_p*real(zeta_m_bar*K_T_bar...
                          *Einf*gradTinf*conj(Einf));
    TDEP_inf_DEP(i) = eps_m*Vol_p*real(zeta_m_bar*K_T_bar_DEP...
                          *Einf*gradTinf*conj(Einf));
    F_single_medium_coeff(i) = real(10*zeta_m_bar-5*zeta_m);
end
% F_DEP, F_TDEP vs f [Hz]
figure(102)
semilogx(omega_xplot/2/pi,DEP_iso,'k-','LineWidth',2)
hold on;
semilogx(omega_xplot/2/pi,TDEP_inf,'b-','LineWidth',2)
semilogx(omega_xplot/2/pi,TDEP_inf + DEP_iso,'r-.','LineWidth',2)
semilogx([min(omega_xplot) max(omega_xplot)]/2/pi,...
         [F_buoyancy F_buoyancy], 'm-', 'LineWidth',2)
hold off;
h_legend = legend('DEP', 'T-DEP', 'Electrostatic', 'Buoyancy');
set(h_legend,'Location','SouthEast');
xlabel('Frequency [Hz]', 'Interpreter', 'Latex', 'FontSize', 16)
ylabel('Force Coefficient','Interpreter','Latex','FontSize',16)
xlim([min(omega_xplot) max(omega_xplot)]/2/pi)
grid on;
```

%matlab2tikz('F_boiling.tikz');

F TDEP Perpendicular.m

```
clear all
close all
[K_E,G] = meshgrid(-0.5:0.01:1,-0.5:0.01:1);
zeta_eps_m = -0.004;
zeta_sgm_m = 0.02;
zeta_ratio = zeta_eps_m/zeta_sgm_m; % low frequency in water
%zeta_ratio = 1; % high frequency in water
Delta_T_L = 15;
K_T_{parallel} = real(0.1*(10*(2*K_E + 1).*(G-1) + 10 ...)
                    + 4*conj(K_E).*(K_E.*G + 2*K_E + 2*G - 5) ...
                    - 5*zeta_ratio - (2*K_E.*conj(K_E) + 20*K_E + 5)...
                         .*(G - 1)*zeta_ratio));
K_T_perp = real(0.1*(3*conj(K_E).*(-5+2*K_E+2*G+K_E.*G) ...
                + (5*G + 2*K_E.*(5+2*conj(K_E)).*(1-G))*zeta_ratio));
% nondimensional T-DEP force, low frequency
F_TDEP_parallel = K_T_parallel*zeta_sgm_m*Delta_T_L;
% nondimensional T-DEP force, low frequency
F_TDEP_perp = K_T_perp*zeta_sgm_m*Delta_T_L;
figure(1)
surf(K_E,G,K_T_perp)
xlabel('Re ($\bar{K}_E$)','Interpreter','Latex')
ylabel('$G$','Interpreter','Latex')
view([0 90])
shading flat
colorbar
figure(2)
hold on;
plot(K_E(G==-0.5),K_T_perp(G==-0.5),'Linewidth',2,'Color',[1 0 0])
plot(K_E(G==0),K_T_perp(G==0),'Linewidth',2,'Color',[0 0 0])
plot(K_E(G==1),K_T_perp(G==1),'Linewidth',2,'Color',[0 0 1])
```
```
plot(K_E(G==-0.5),K_T_parallel(G==-0.5),'--','Linewidth',2,'Color',[1 0 0])
plot(K_E(G==0),K_T_parallel(G==0),'--','Linewidth',2,'Color',[0 0 0])
plot(K_E(G==1),K_T_parallel(G==1),'--','Linewidth',2,'Color',[0 0 1])
hold off;
grid on;
xlabel('Re ($\bar{K}_E$)','Interpreter','Latex')
ylabel('$\bar{K}_T^{\perp}$','Interpreter','Latex')
h = legend('$G=-0.5$','$G=0$','$G=1$');
set(h,'Interpreter','Latex')
```

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