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Covariant Insertion of General Vertex Operators

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We provide a very simple prescription for inserting an arbitrary state into a string amplitude. The corresponding string measure is defined without any additional information whenever the state is BRST-invariant, generalizing the usual physical conditions. In particular no world-sheet metric is needed. We recover and interpret in a simple way the "$\hat{b}$" prescription of Polchinski and explain geometrically how it secures the decoupling of BRST spurious states.

11/88

Vertex operators were originally understood as operators depending on formal Koba-Nielsen variables, useful in computing tree-level S-matrix elements in dual theory. The external lines had to be on-shell in order for the amplitudes so computed to be dual. As string theory developed, the formal variables took on meaning as positions on a surface, which could have any topology. In the Polyakov representation of string amplitudes vertex operators became functions of the world-sheet fields, metrics, and their derivatives; the on-shell restriction then became the requirement that the conformal factor of the metric drop out of amplitudes (see *e.g.* refs. [1]–[5]). Each of these papers found appropriate vertex operators for the emission of *physical* states, which among other things are on-shell and transversely polarized.

Today vertex operators appear in more general contexts. For example a string background obeying the tree-level equations of motion can lead to tadpole infinities, forcing us to modify the background to one *off-shell* from the point of view of the tree-level equations [6]. Thus we sometimes need to insert slightly off-shell states. Moreover the factorization of string amplitudes involves the insertions, on each side of the pinch, of vertex operators at zero momentum, which for most states is far off-shell. Finally, even on-shell one would like to be able to insert arbitrary longitudinal states in order to obtain gauge-covariant effective field equations for the backgrounds. Moreover, there are even some gauge-invariant, BRST-invariant states which do not meet the usual physical conditions, for example the dilaton state discussed in [7]; we need a prescription for coupling these, too.

The problem of general insertions has been studied in the BRST formalism by Polchinski in ref. [7]. (For earlier approaches see also [8][9].) In this letter we will simplify and generalize his prescription. Specifically we will explain geometrically his modification to the ghost insertion needed to convert the inserted state from the fixed to the integrated picture. The derivation makes no use of a world-sheet metric and so preserves the holomorphic structure of conformal field theory. The virtue of this minimalist viewpoint is that we will see quite clearly the sense in which the modified ghost insertion "covariantizes" the resulting string measure prescription.

We will for illustration consider mainly the closed bosonic string and the problem of inserting BRST invariant states. The motivation for the study, however, comes from the fermionic string, where the holomorphic structure plays a more central role. Using these methods one can show that the Fischler-Susskind mechanism suffices to remove the "ambiguity" of string perturbation theory [10]–[12], regardless of whether supermoduli space is canonically split. Details will appear elsewhere [13]. As in the bosonic case [7], the key is to find uniquely defined corrections to the background fields which cancel boundary obstructions to BRST decoupling. For now we will simply assume that such boundary terms have been dealt with. For example, we will not discuss the anomaly seen in [9].

We begin by recalling some ideas and notation from refs. [14]–[16]. Let $\mathcal{M}_0$ be the moduli space of surfaces of some genus g. Since we want to insert a state $|\psi\rangle$ into the amplitude associated to a surface Σ, we also let $\mathcal{M}$ be the moduli space of Riemann surfaces of genus g with one marked point $P \in \Sigma$. (It is easy to extend the treatment to handle several insertions.) Following [15] we will also let $\mathcal{P}$ denote the space of Riemann surfaces with marked point and a local complex coordinate centered at the point: (Σ, P, z) where $z(P) = 0$. Thus $\mathcal{P}$ is a bundle over $\mathcal{M}$. A section $\sigma : \mathcal{M} \to \mathcal{P}$ of this bundle is just a choice of local coordinate z for every (Σ, P). We will also refer to σ as a *family* of local coordinates.

To every $\widetilde{\Sigma} \equiv (\Sigma, P, z)$ in $\mathcal{P}$ we can associate a state $|\widetilde{\Sigma}\rangle$ in the Fock space of first-quantized string theory [17][14][16]. This state satisfies among other things a conserved charge condition and variational condition. The former says that

$$b(v)|\widetilde{\Sigma}\rangle = 0 \tag{1}$$

where $b(v) \equiv \oint b_{zz} v^z dz$, b is the antighost field associated to the Virasoro algebra, and v is any vector field on the circle $\{z = 1\}$ which extends analytically to all of $\widetilde{\Sigma}$, except possibly P. (We will call such v "Borel vectors".) The latter condition says that

$$\delta_v |\widetilde{\Sigma}\rangle = T(v)|\widetilde{\Sigma}\rangle \tag{2}$$

for any Virasoro generator v. Here δ_v is a derivative on $\mathcal{P}$ in the direction given by v and T is the stress tensor. See ref. [16] for the notation.

From $|\widetilde{\Sigma}\rangle$ one can readily construct a differential $(6g - 6)$-form $\widetilde{\mu}$ on $\mathcal{P}$ [16]. Simply let

$$\widetilde{\mu}(V_1, \cdots V_{3g-3}, \bar{V}_1, \cdots, \bar{V}_{3g-3}) = \langle 0|b(v_1) \cdots \bar{b}(\bar{v}_{3g-3})|\widetilde{\Sigma}\rangle.$$

Here V_i are tangents to $\mathcal{P}$ at $\widetilde{\Sigma}$ and v_i are corresponding elements of Virasoro. $\langle 0|$ is the $SL(2, \mathbf{C})$ - invariant vacuum. The v_i are in fact ambiguous by the addition of Borel vectors, but the condition (1) says that such changes do not matter. In the sequel we will suppress the antiholomorphic objects $\bar{V}_i, \bar{b}(\bar{v}_i)$, *etc.* from the notation.

Similarly if we are given a state ψ with ghost number appropriate for a vertex operator we can get a $(6g-4)$-form on $\mathcal{P}$ by defining

$$\widetilde{\mu}_{\psi}(V_1,\cdots,V_{3g-2}) = \langle\psi|b(v_1)\cdots b(v_{3g-2})|\widetilde{\Sigma}\rangle, \tag{3}$$

which again is well-defined by (1).

Unfortunately we don't want forms on $\mathcal{P}$. For the Polyakov measure we want a form μ on $\mathcal{M}_0$; for the insertion we want a form on $\mathcal{M}$, since we expect to integrate over all positions of the insertion. In fact in the former case one shows that $\widetilde{\mu} = \pi_0^*\mu$ where $\pi_0 : \mathcal{P} \to \mathcal{M}_0$ and μ is a top differential form on $\mathcal{M}_0$ [16]. A similar result obtains for the insertion:

$$\widetilde{\mu}_{\psi} = \pi^*\mu_{\psi}, \tag{4}$$

where $\pi : \mathcal{P} \to \mathcal{M}$ and μ_{ψ} is a top form on $\mathcal{M}$, but only if ψ obeys some physical state conditions [16]. These say that

$$L_n\psi = 0, \qquad b_n\psi = 0, \qquad n \geq 0; \tag{5}$$

they are the analogs in the BRST formalism of the conditions found in [1]–[5] and elsewhere. The first condition says that $\widetilde{\mu}$ is unchanged if we change (Σ, P, z) to (Σ, P, z'); the second says that $\widetilde{\mu}$ annihilates vertical tangent vectors, the $v \in \mathrm{Vir}_+$, which change z infinitesimally while leaving (Σ, P) unchanged. When these are satisfied one can define μ_{ψ} at (Σ, P) by choosing *any* z and evaluating $\widetilde{\mu}_{\psi}$; this is what the notation $\widetilde{\mu}_{\psi} = \pi^*\mu_{\psi}$ means.

The problem is that (5) excludes some interesting states. For example, the dilaton state $|D\rangle = (c_1c_{-1} - \bar{c}_1\bar{c}_{-1})|k\rangle$, where $k^2 = 0$, is a perfectly good, BRST-invariant state which is not annihilated by b_1. One would also like to insert the longitudinal graviton state $n_{(\mu}k_{\nu)}a^{\mu}_{-1}\bar{a}^{\nu}_{-1}c_1\bar{c}_1|k\rangle$. To avoid complications with the dilaton take $k\cdot n = 0$; then this state describes a coordinate transformation of the spacetime metric $G_{\mu\nu} \to G_{\mu\nu} + \partial_{(\mu}n_{\nu)}e^{ikx}$ for any k whatsoever. For $k^2 \neq 0$, however, this state fails to satisfy (5). Callan and Gan found that a related vertex could be inserted, once a suitable auxiliary field was introduced [8]. In BRST form this is [7]

$$|L\rangle = \left[c_1\bar{c}_1k_{(\mu}n_{\nu)}a^{\mu}_{-1}\bar{a}^{\nu}_{-1} + \tfrac{1}{4}(c_0+\bar{c}_0)k^2n_{\mu}(c_1a^{\mu}_{-1} - \bar{c}_1\bar{a}^{\mu}_{-1})\right]|k\rangle\ . \tag{6}$$

This state still fails (5), but like $|D\rangle$ it is BRST-invariant. In fact it is BRST-exact.

To deal with these cases momentarily suppose that a global family of local coordinates, $\sigma : \mathcal{M} \to \mathcal{P}$, exists and has been chosen. Then we can let

$$\mu_{\psi,\sigma} \equiv \sigma^* \widetilde{\mu}_\psi , \tag{7}$$

the *pullback* of $\widetilde{\mu}$ by σ. This certainly defines a form on $\mathcal{M}$, and if ψ satisfies the physical state conditions we get precisely that $\mu_{\psi,\sigma} \equiv \mu_\psi$, independent of σ. This follows from (4) since $\pi \circ \sigma$ is the identity for any section σ. More generally, though, $\mu_{\psi,\sigma}$ depends on the family of local coordinates chosen.

For insertions implementing the Fischler-Susskind mechanism this unwanted dependence on σ cancels certain boundary terms. For the above states no such cancellation occurs. But consider not the density μ but the full amplitude:

$$\langle\!\langle \mathcal{V}_\psi \rangle\!\rangle \; \equiv \; \int_{\mathcal{M}} \mu_{\psi,\sigma} = \int_{\sigma(\mathcal{M})} \widetilde{\mu}_\psi . \tag{8}$$

If $\widetilde{\mu}_\psi$ is a *closed* differential form on $\mathcal{P}$ then this quantity will not change as we deform σ. Moreover if $\widetilde{\mu}_\psi$ is *exact* then $\langle\!\langle \mathcal{V}_\psi \rangle\!\rangle = 0$ and ψ decouples. A simple adaptation of an argument in [16] shows, however, that $\widetilde{\mu}_\psi$ is closed (exact) precisely when the state ψ is itself closed (exact) under the BRST generator Q. Thus we can use (7)–(8) to insert the states $|D\rangle, |L\rangle$ and other BRST cohomology classes, provided that a global slice σ can be found, and any other σ' is homotopy equivalent to σ. In particular (6) decouples as desired.

In fact no such choice of σ exists for $g \neq 1$. We can nevertheless proceed using the observation [7] that if $z' = e^{i\alpha} z$ for some real phase α independent of z, then $\widetilde{\mu}_\psi$ changes by the action of $L_0 - \bar{L}_0$; thus for ψ annihilated by $L_0 - \bar{L}_0$ we can find a slice which is "global enough" to define $\langle\!\langle \mathcal{V}_\psi \rangle\!\rangle$. Also any two such slices are homotopic modulo constant phases.

Thus we have a global prescription for inserting any state invariant under Q and $b_0 - \bar{b}_0$. Note that no covariant derivative on the bundle $\mathcal{P} \to \mathcal{M}$ has entered; indeed no natural choice of such a connection appears to exist. The prescription is automatically covariant in the sense that $\sigma^* \widetilde{\mu}$ is closed (exact) whenever $\widetilde{\mu}$ is, a basic property of the pullback [18], and this is what ensures BRST decoupling. (Recall that in this letter we are not concerned with subtleties at the boundary of $\mathcal{M}$.)

We now need to show that the above simple prescription reproduces that of Polchinski when a certain special choice of coordinate family σ has been chosen. We will see that his modification of the ghost insertions for the moduli (changing b to $\hat{b}$) are nothing but the Jacobian factors needed to make $\widetilde{\mu}$ transform as a differential form under pullback by σ^*.

Let $\{V_i\}$ be tangents to $\mathcal{M}$ at (Σ, P), and let σ take (Σ, P) to $(\Sigma, P, z_{(\Sigma,P)})$. Then we want to compute

$$(\sigma^* \widetilde{\mu})(V_1, \cdots)|_{(\Sigma,P)} = \widetilde{\mu}(\sigma_* V_1, \cdots)|_{(\Sigma,P,z)}.$$

We will for illustration let $V + \bar{V}$ be the tangent which moves P leaving Σ fixed. So we will drop Σ from the notation, writing z_P for $z_{(\Sigma,P)}$. Let us expand $z_{P'}(\cdot)$ for P' near P as

$$\begin{aligned} z_{P'}(\cdot) = z_P(\cdot) - z_P(P') + \overline{z_p(P')} \sum_{n=1}^{\infty} \beta_n(P) z_P(\cdot)^{n+1} \\ + z_P(P') \sum_{n=1}^{\infty} \gamma_n(P) z_P(\cdot)^{n+1} + \mathcal{O}\left(z_P(P')^2\right) \ . \end{aligned} \tag{9}$$

Here $z_P(\cdot)$ is a different function on Σ for each P, while $z_P(P')$ is a small number. $z_{P'}(\cdot)$ must have this form, as it is for fixed P' an analytic coordinate centered at P' and reducing to $z_P(\cdot)$ when $P' = P$.

Since V just moves P, its image under σ_* consists of a piece corresponding to L_{-1}, plus a piece describing how the slice varies as P' changes in (9). That is, $\sigma_* V = \delta_v$, where $v = \ell_{-1} - \sum_{n=1}^{\infty} \bar{\beta}_n \bar{\ell}_n - \sum_{n=1}^{\infty} \gamma_n \ell_n$ and ℓ_n are abstract generators of Virasoro.

One way to get a family σ of local coordinates z_P is to introduce a metric on Σ and require that for each P, z_P makes the chosen metric "as flat as possible" at P [7]. With this choice one has $\gamma_n \equiv 0$ and $\beta_n(P) = -\frac{1}{4} \frac{1}{(n+1)!} \nabla_{z_P}^{n-1} R(P)$, where R is the Ricci scalar. Thus our prescription applied with this family instructs us to evaluate $\widetilde{\mu}$ with $v = \ell_{-1} + \frac{1}{8} R \bar{\ell}_1 + \cdots$, or in other words to replace $b_{-1}\mathcal{V}_\psi$ in the path integral by $\hat{b}_{-1}\mathcal{V}_\psi$ where $\hat{b}_{-1} = b_{-1} + \frac{1}{8} R b_1 + \cdots$. The resulting function is to be multiplied by the 2-form dual to $V \wedge \bar{V}$, but this is just $\sqrt{g_{ab}} \mathrm{d}\sigma^1 \wedge \mathrm{d}\sigma^2$, where g_{ab} is the given metric expressed in terms of any convenient fixed coordinates σ^a. Thus we recover Polchinski's prescription as a special case. In particular $\langle\langle \mathcal{V}_\psi \rangle\rangle$ is Weyl invariant as a special case of its general slice independence shown above.

The other moduli insertions work similarly. In each case we can, if we please, decompose $\sigma_* V_i$ into a bit in $\mathrm{Vir}_{<0}$, corresponding to a naive insertion of $b(z)$ integrated with a Beltrami differential, plus a bit in $\mathrm{Vir}_{\geq 0}$, corresponding to the new terms in [7]. The point is that such a decomposition is not necessary. Also the slice-independence of $\langle\langle \mathcal{V}_\psi \rangle\rangle$ and BRST decoupling are quite general and not limited to the family in [7].

The family chosen in [7] is not holomorphic, as can be seen by the fact that $\beta_n \neq 0$. If we like, we can instead consider slices varying holomorphically with (Σ, P), a procedure useful when dealing with chiral theories At once, however, we confront the fact that global *holomorphic* families cannot in general be found, even modulo $U(1)$. Instead, we need to cover $\mathcal{M}$ with patches $\mathcal{U}_\alpha$ with a different σ_α on each. To get well-defined answers we must find compensating contributions $W_{\alpha\beta}$ to $\langle\!\langle \mathcal{V}_\psi \rangle\!\rangle$ on patch boundaries, as suggested in [5]. (Such compensators are reminiscent of the "Wu-Yang" terms appearing in [19].) We will see that they are easy to work out in the present approach.

Suppose we wish to couple the dilaton state $|D\rangle$ at zero momentum to a sphere, $\Sigma = S^2$. Naively the answer is zero: $c_1 c_{-1}$ can absorb two b operators but not $b\bar{b}$, and so $\widetilde{\mu}_D$ is a (2,0)-form on $\mathcal{P}$ (minus its conjugate). Since only $(1,1)$-forms can be integrated over Σ, $|D\rangle$ seems to decouple. If σ is not a holomorphic section, however, $\sigma^* \widetilde{\mu}_D$ can have a (1,1)-form bit which gives a non-zero answer when integrated over Σ. The idea is now to construct a σ which is holomorphic everywhere except in a very narrow strip about the equator E of S^2. Then $\langle\!\langle \mathcal{V}_D \rangle\!\rangle$ will get contributions only near E; we will see that it is just proportional to the Euler number of Σ, as befits a dilaton.

Let z be the usual coordinate on S^2 away from the north pole, and $w = z^{-1}$ a coordinate away from the south pole. Near a point Q let $z_Q(\cdot) = z(\cdot) - z(Q)$ be a centered coordinate; obviously z_Q depends holomorphically on Q. Similarly one has $w_Q(\cdot) = w(\cdot) - w(Q)$, related by

$$w_Q(\cdot) = -z(Q)^{-2} z_Q(\cdot) + z(Q)^{-3} z_Q(\cdot)^2 + \cdots . \tag{10}$$

Near E we have $z = e^a$ where $a = y + i\theta$ and $y \sim 0$. We would like a single coordinate family $u_Q(\cdot)$ equal to $z_Q(\cdot)$ south of E, to $w_Q(\cdot)$ north of E, and interpolating smoothly in a narrow strip $0 < y < \epsilon$, at least up to $U(1)$. A suitable choice is

$$u_Q(\cdot) = z_Q(\cdot) - \epsilon^{-1} y z(Q)^{-1} z_Q(\cdot)^2 \qquad , 0 < y < \epsilon.$$

We have multiplied (10) by the phase $-e^{-2i\theta}$ and retained only leading terms in ϵ^{-1}.

Differentiating $u_Q(\cdot)$ we find that $\beta_1 = -\epsilon^{-1} + \mathcal{O}(1)$. Thus

$$\begin{aligned} \langle\!\langle \mathcal{V}_D \rangle\!\rangle &\propto \int_{S^2} |\mathrm{d}z|^2 \left[\langle 0| c_1 c_0 c_{-1} \bar{c}_1 \bar{c}_0 \bar{c}_{-1} \cdot b_{-1} \cdot \beta_1 b_1 \cdot c_1 c_{-1} |0\rangle + \mathcal{O}(1) \right] \\ &\propto \int_0^{2\pi} \int_0^{\epsilon} 2 \mathrm{d}y \mathrm{d}\theta \left(-\frac{1}{\epsilon} \right) = -2. \end{aligned}$$

The extra insertions of c take care of the conformal Killing vectors on S^2. β is the coefficient defined in (9). More generally we find that $|D\rangle$ couples to the Euler number of any Riemann surface as required.

The point of this exercise was to demonstrate how one can use only holomorphic coordinate families and still get global, covariant insertions of general vertex operators. This was accomplished through the agency of correction terms first envisioned in [5]. For the dilaton case this term was the whole story; it was a topological invariant by itself. More generally it combines with integrals over the patches; the combination is then independent of the chosen family of slices as shown earlier.

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