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Bosonization in Arbitrary Genus

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We prove the equivalence between fermionic and scalar field theories on Riemann surfaces of arbitrary topology. The effects of global topology include a modification of the bosonic action.

Fermi-Bose equivalence has recently played an important role in several aspects of string theory. For example, bosonization figures prominently in the light-cone gauge proof of the equivalence of the Neveu-Schwarz-Ramond and Green-Schwarz formulations of the superstring[1]. Bosonization also plays a key role in understanding the spacetime gauge symmetries of the heterotic string [2]. Finally, bosonization is an important tool in the discussion of the spacetime supersymmetry of the superstring, *via* the fermion vertex operators [3] [4].

Most analyses of bosonization have concerned themselves with properties local on the world sheet of a two-dimensional field theory. While for many applications this suffices, one

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would also like to know how the global topology of a compact Euclidean surface modifies the procedure. On an arbitrary Riemann surface Σ, however, one faces the complication that not all background field configurations on the world sheet are gauge-equivalent, unlike the situation on the sphere or the plane. The space of inequivalent metrics on Σ is called the moduli space $\mathcal{M}$ of Σ.

Recently it has become clear that physicists can use the rich and beautiful structure of $\mathcal{M}$ to gain insight into problems of 2d field theory such as bosonization. Most importantly, $\mathcal{M}$ is a complex space, so that methods from algebraic geometry become applicable. The exact link between field theory and complex geometry comes from a remarkable theorem of D. Quillen[5][6], which we use. Quillen's theorem describes the determinant of a family of wave operators in complex-analytic terms. His result was later generalized by Belavin and Knizhnik to include families parametrized by $\mathcal{M}$, the case of interest in string theory [7][8][9] [10] [11]. On the other hand, it has been known for some time that the work of Quillen is closely related to Faltings' work on Arakelov geometry[12]. Faltings describes the bundles in which the determinants live in an inductive way, building them up from simpler ones. It has been suggested that some combination of Quillen's and Faltings' approaches would be of use in string theory[8].

In this letter we will use just such a combination to prove bosonization formulæ on Riemann surfaces of arbitrary genus. Our results are more general than the simple spin-1/2 answer described in[13]. Similar results have been obtained by E. Martinec[14]. One can see that such an approach could well lead to bosonization by examining the recent formulation of the string integrand given by [15]. Manin's formula is strongly reminiscent of bosonization. For example it contains the exponential of the Green function which we would expect from correlation functions of the form $\langle e^{\varphi}\, e^{\varphi}\rangle$; these look like the insertions of fields needed to soak up zero modes in a fermionic system with an index.

In the following we will describe what bosonization says about the partition function of a generalized ghost system. (The method extends to give similar formulæ for the correlation functions.) In particular, the answers we will work out necessarily involve some new global terms in the bosonic action. We will then sketch a mathematical proof that this action is in fact correct, deferring the details to a later publication. The insertion of fields mentioned above correspond precisely to the insertion of points used by Faltings to build up arbitrary-spin determinant bundles. We think this is a very pleasant interaction of mathematics and physics, one which is likely to yield further results about two-dimensional field theory in the future.

We will now outline the bosonization procedure for a first order Fermi system of weight λ [3][4]. Thus we consider two anticommuting fields b, c on Σ, where b is a λ-form, and c is a $(1-\lambda)$-form with action $S_{bc} = \int b\overline{\partial}c$, where $\overline{\partial}$ is the Dolbeault operator coupled to $L^{2-2\lambda}$, a power of a spin bundle L. The ghost number current has an anomaly given by $k = (2\lambda - 1)(g-1)$ so that we must insert the appropriate number of b and c fields into the partition function to obtain a nonvanishing path integral. On a higher genus Riemann surface we require no insertions for $\lambda = \frac{1}{2}$; g insertions of b and one insertion of c for $\lambda = 1$; and k insertions of b for $\lambda > 1$. Thus, denoting by ω_i and by ψ_i a basis of holomorphic 1-forms and λ-forms respectively we obtain for the partition functions with fields inserted at points P_i:

$$
\begin{aligned}
Z^{bc}_{\lambda=\frac{1}{2}} &= det\overline{\partial}^{\dagger}_{L}\overline{\partial}_{L} \\
Z^{bc}_{\lambda=1} &\equiv \left| \int [db][dc] \prod_{i=1}^{g} b(P_i) c(Q) e^{-S_{bc}} \right|^2 = \frac{det\omega_i(P_j) \wedge det\overline{\omega}_i(P_j)}{det\langle \omega_i | \omega_j \rangle} \left(\frac{det' - \nabla^2}{\int \sqrt{g}} \right) \\
Z^{bc}_{\lambda} &\equiv \left| \int [db][dc] \prod_{i=1}^{k} b(P_i) e^{-S_{bc}} \right|^2 = \frac{det\psi_i(P_j) \wedge det\overline{\psi}_i(P_j)}{det\langle \psi_i | \psi_j \rangle} det\overline{\partial}^{\dagger}_{L^{2-2\lambda}}\overline{\partial}_{L^{2-2\lambda}}
\end{aligned}
\tag{1}
$$

Since we have inserted fields b which are differential forms, the partition functions should be regarded as (k,k)-forms on Σ^k.

Bosonization is the statement that the above first order field theories can be replaced by equivalent scalar field theories. It was shown in [3][4] that the local properties of the weight λ system are reproduced by a scalar action which is a sum of the usual action $S_1 = 4\pi i \int \partial\phi\overline{\partial}\phi$ and an anomaly term S_2

$$
S[\phi] = S_1 + S_2 \equiv S_1 + 4\pi i \int R\phi \tag{2}
$$

The second term accounts for the local anomaly in the ghost number current. To account for global properties we will need to consider "instanton" configurations which wind n_i times around the a_i-cycle and m_j-times around the b_j-cycle (fig.1). The solution to the equations of motion in the $(\vec{n}\,\vec{m})$-sector can be expressed in terms of holomorphic differentials as: $d\phi_{nm} = (m - \overline{\Omega}n)(\Omega - \overline{\Omega})^{-1}\omega + c.c.$ where Ω is the period matrix. To evaluate the $R\phi$ term we must define the multiple-valued field ϕ by choosing a system of curves intersecting in a single point R (fig.1). Such a choice lets us cut open the surface to obtain a polygon Σ_c. We must also choose a basepoint P_0; then $\phi = \int_{P_0}^{P} d\phi$ is well-defined on Σ_c.

Let us now investigate the dependence of the action on the various choices we have made. First, a change of basepoint shifts ϕ, and therefore S, by a constant. This is simply a reflection of the (integrated) $U(1)$ anomaly and is compensated by the the bosonized insertions that soak up zero modes. Next, let us consider the dependence on the curves a_i, b_i chosen to represent the homology basis. If we view ϕ_{nm} as a discontinuous function on Σ, then deforming a cycle through the discontinuity produces a change in the action. For example consider the two choices of representatives for the a_l cycle in fig. 2. If ϕ has a winding number around b_l then there is a discrepancy in the actions because

$$\int_{\widetilde{\Sigma}_c} R\phi - \int_{\Sigma_c} R\phi = \int_{b_l} d\phi \int_D R \tag{3}$$

where D is the region bounded by $\widetilde{a}_l$ and a_l. Thus, in the instanton sectors the anomaly term is not well-defined. We may compensate for this by adding a term to the action so that

$$S_2 = 4\pi i \int_{\Sigma_c} R\phi - \int_{b_k} d\phi f[a_k] + \int_{a_k} d\phi f[b_k] \tag{4}$$

where f is any functional of the curves such that if $\widetilde{a}_k$ is homologous to a_k then $f[\widetilde{a}_k] - f[a_k] = 4\pi i \int_D R$ where D is the region enclosed by the two curves. By a similar argument the action S_2 is independent of the choice of the intersection point R.

One natural choice of f may be described as follows. The metric on a holomorphic line bundle $\mathcal{L}$ is specified (up to a constant) by the curvature. We may then choose the unique holomorphic connection compatible with that curvature and compute the holonomy about a curve γ which we denote by $h[\gamma; \mathcal{L}]$. Thus, when bosonizing a weight λ system in a background with curvature R we have $f[a_l] = h[a_l; L^{2-2\lambda}] + v_l$ and $f[b_l] = h[b_l; L^{2-2\lambda}] + w_l$ where v and w are constants which will be determined momentarily.

The action should also be independent of the choice of homology basis, *i.e.*, it should be modular invariant. It is easy to check that if the parameters v, w in the action are independent of marking then we can have modular invariance only if $v = w = 0$. The change of $S_2[\phi_{nm}]$ under a change of marking is not obvious and requires a computation. We will establish modular invariance for one particular metric, then since S_2 is conformally invariant the action will then be basis-independent for all metrics. A natural metric is the Arakelov metric [16] [12] which is defined up to a constant by specifying the curvature of the holomorphic line bundle $L^{2-2\lambda}$ by $R = k\frac{i}{2g}\omega^i (Im\Omega)^{-1}_{ij}\overline{\omega}^j \equiv k\mu$. Note that $\int \mu = 1$. Every metric is gauge-equivalent to a single Arakelov metric. We will see that the choice of the Arakelov metric is is particularly convenient for both physics and mathematics.

To write the answer we must parametrize line bundles of a given degree. Since the difference of two bundles of the same degree is a flat bundle we may choose a fiducial spin structure L and parametrize bundles of degree $(2-2\lambda)(g-1)$ by $L^{2-2\lambda} \otimes F_{\theta_1,\theta_2}$ where F_{θ_1,θ_2} is the holomorphic flat bundle with holonomy θ_1, θ_2 around the a, b cycles [17] [9]. We will choose L to be the spin bundle corresponding to the (marking-dependent) vector of Riemann constants Δ. [1] This parametrization is particularly natural when considering functional determinants of $\overline{\partial}$ operators because of the Riemann vanishing theorem [18] [9]. For the bundle $L^{2-2\lambda} \otimes F_{\theta_1,\theta_2}$ one can show that

$$S_2[\phi_{nm}] = 4\pi i \left[(m - \overline{\Omega} n)(\Omega - \overline{\Omega})^{-1}((2\lambda - 1)\Delta + \theta_1 + \Omega\theta_2) + c.c. \right] \tag{5}$$

Using the transformation law of Δ under a change of marking one can show that $exp - (S_1 + S_2)$ is invariant up to a change of sign, reflecting a global anomaly. This sign may be cancelled by adding a third term to the action given by the product of the winding numbers $S_3 = 4\pi i \int_{a_k} d\phi \int_{b_k} d\phi = 4\pi i n \cdot m$. Thus if we choose the action $S = S_1 + S_2 + S_3$ then e^{-S} is independent of *all* choices and is the correct action corresponding to the weight λ system.

To complete the Fermi-Bose correspondence we must express the Fermi fields b, c in terms of the Bose field ϕ. In the Lagrangian formulation this is accomplished by

$$\begin{aligned} b &= (dz)^{\lambda} N_z(e^{4\pi i \phi_+}) \\ c &= (dz)^{1-\lambda} N_z(e^{-4\pi i \phi_+}) \end{aligned} \tag{6}$$

where $\phi_+ = \int_{P_0}^{P} \partial\phi$ is the right-moving part of ϕ; the factor of $4\pi i$ is determined by demanding that the expressions on the right hand side have the correct conformal weight, and the normal ordering prescription N_z cancels the coordinate-dependence so that b, c are well-defined differential forms. We may now evaluate the Gaussian path integral with insertions by introducing Arakelov's Green function [12] which satisfies $\partial\overline{\partial} logG(P,Q) = i\pi\mu(P) - i\pi\delta(P,Q)$ and $\int \mu logG(\cdot, Q) = 0$. The normal-ordering of the Green functions at coincident points is fixed by the requirement that the expression be coordinate independent and finite:

$$: logG(P,P) := \lim_{Q \to P} \left(logG(Q,P) - 2\lambda log|z(P) - z(Q)| - (1 - 2\lambda) logd(P,Q) \right) \tag{7}$$

[1] In the following we consider Δ as a point in the Jacobian.

where $d(P,Q)$ is the invariant distance. The Arakelov metric has the important property that the normal ordering of the scalar ($\lambda=0$) field gives zero at the coincident points.

In addition to the Gaussian integrals we must sum over instanton sectors. By (6) we must allow ϕ to shift by an integer or half-integer about the cycles. Thus the Bose partition functions are:

$$
\begin{aligned}
Z^{Bose}_{\lambda=\frac{1}{2}} &= \left(\frac{det' -\nabla^2}{\int\sqrt{g}}\right)^{-\frac{1}{2}} Z_{inst} \\
Z^{Bose}_{\lambda=1} &= \prod_{i=1}^{g} \frac{\sqrt{-1}}{2} dz_i(P_i)\wedge d\overline{z}_i(P_i) e^{:logG(P_i,P_i):}\left(\frac{det' -\nabla^2}{\int\sqrt{g}}\right)^{-\frac{1}{2}} \frac{\prod_{i<j} G(P_i,P_j)^2}{\prod_i G(P_i,Q)^2} Z_{inst} \\
Z^{Bose}_{\lambda} &= \prod_{i=1}^{k} (\frac{\sqrt{-1}}{2})^{\lambda} (dz_i(P_i))^{\lambda}\wedge (d\overline{z}_i(P_i))^{\lambda} e^{:logG(P_i,P_i):}\left(\frac{det' -\nabla^2}{\int\sqrt{g}}\right)^{-\frac{1}{2}} \prod_{i<j} G(P_i,P_j)^2 Z_{inst}
\end{aligned}
\tag{8}
$$

where Z_{inst} denotes the instanton sum. This may be expressed as

$$
Z_{inst} = \sum_{n,m\in(\frac{1}{2}\mathbf{Z})^g} e^{-S_1[\phi_{nm}]+4\pi i[(m-\overline{\Omega}n)(\Omega-\overline{\Omega})^{-1}z+c.c.]+4\pi imn} \tag{9}
$$

where $z=\Delta+\theta_1+\Omega\theta_2$ for a twisted $\lambda=\frac{1}{2}$ system, while $z=I[\sum_{i=1}^{g}P_i-Q]+\Delta$ for $\lambda=1$ (here $I[\cdot]$ is the Jacobian map, and P_i,Q are the insertion points), and $z=I[\sum_{i=1}^{k}P_i]+\Delta$ for all other spins. Twists may be included in the latter two systems by adding $\theta_1+\Omega\theta_2$ to z [18][9]. After an application of the Poisson summation formula this sum may be expressed in terms of a function $\mathcal{N}[z]$:

$$
Z_{inst} = (detIm\Omega)^{\frac{1}{2}}\mathcal{N}[z] \equiv (detIm\Omega)^{\frac{1}{2}} e^{-2\pi(Imz)(Im\Omega)^{-1}(Imz)}|\vartheta(z|\Omega)|^2 \quad . \tag{10}
$$

where ϑ is the Riemann theta function. Bosonization states that the Fermi and Bose partition functions are equal. Equating these we obtain the following formulae for the determinants of the Laplacians for any spin [2]:

$$
det\overline{\partial}^{\dagger}_L\overline{\partial}_L = \left(\frac{det' -\nabla^2}{\int\sqrt{g}detIm\Omega}\right)^{-\frac{1}{2}} \mathcal{N}[\theta_1+\Omega\theta_2] \tag{11a}
$$

$$
\left(\frac{det' -\nabla^2}{\int\sqrt{g}detIm\Omega}\right)^{\frac{3}{2}} = \frac{\prod_{i<j} G(P_i,P_j)^2}{\parallel det\omega_i(P_j)\parallel^2} \frac{\mathcal{N}\big[I(\sum_1^g P_i - Q)-\Delta\big]}{\prod G(P_i,Q)^2} \tag{11b}
$$

$$
\frac{det\overline{\partial}^{\dagger}_{L^{2-2\lambda}}\overline{\partial}_{L^{2-2\lambda}}}{det\langle\psi_i|\psi_j\rangle} = \left(\frac{det' -\nabla^2}{\int\sqrt{g}detIm\Omega}\right)^{-\frac{1}{2}} \frac{\prod_{i<j} G(P_i,P_j)^2}{\parallel det\psi_i(P_j)\parallel^2}\mathcal{N}\big[I[\sum_i^k P_i]-(2\lambda-1)\Delta\big] \tag{11c}
$$

[2] These equalities hold up to a constant which depends only on g and λ.

where in the first formula we have considered a twisted spin bundle, and the notation $\| det\psi_i(P_j) \|^2$ is the ratio of differential forms

$$\frac{det\psi_i(P_j) \wedge det\overline{\psi}_i(P_j)}{\prod_{i=1}^{k}(dz_i(P_i))^{\lambda} \wedge (d\overline{z}_i(P_i))^{\lambda} e^{:logG(P_i,P_i):}}$$

Actually, bosonization asserts the equality of *all* the correlation functions of the two theories. These may be easily computed in using the above rules. We will now justify our bosonization procedure by describing how the identities (11)and the equalities of correlation functions can be proved rigorously à la Faltings.

The spin-$\frac{1}{2}$ partition function, eq. ($11a$), has already been derived in [9]. We will give a full proof of it elsewhere[13]. Thus what we would like is to find a mathematical operation corresponding to the insertion of fields at points P_i. We can then use such an operation to build up arbitrary spins $L^{2-2\lambda}$ from the known case L^1.

$U(1)$ line bundles on a 2d surface are familiar from the theory of magnetic monopoles, where Σ is the sphere. The total magnetic charge inside Σ can be found by counting the net number of string singularities, which are points P_i of Σ. Turning this around, we can specify the bundle by naming k points $P_1, \ldots, P_k$ and putting transition functions $\exp(i \arg z_{(i)})$ near each. $z_{(i)}$ is a local coordinate vanishing at P_i. Similarly in the analytic case we can choose $P_1, \ldots, P_k$ with transitions simply given by $z_{(i)}$. Let $\mathcal{O}(P_1 + \cdots + P_k)$ denote the resulting line bundle. Clearly its smooth sections can also be viewed as ordinary functions, possibly with simple poles at $\{P_i\}$. In particular the section $\sigma^{(P)}$, which equals one away from P, vanishes at P. We can then put a smooth metric on $\mathcal{O}(P)$ by setting [12][16]

$$\|\sigma^{(P)}\|(Q) = G(P,Q) \quad , \tag{12}$$

and similarly for $\mathcal{O}(P_1 + \cdots + P_k)$.

We can now ask what happens to the fermion determinant when we replace $L^{2-2\lambda}$ by $L^{2-2\lambda} \otimes \mathcal{O}(P)$. Once we know that, we can apply the operation k times to raise $L^{2-2\lambda}$ up to degree $g-1$, *i.e.* to a twisted spin bundle, then apply the known formula. For conciseness we will only give the answer for $\lambda \geq 3/2$, but the general formula needed to prove (11) is not much harder.

If $\lambda \geq 3/2$, then $\bar{\partial}' \equiv \bar{\partial}_{L^{2-2\lambda} \otimes \mathcal{O}(P)}$ and $\bar{\partial} \equiv \bar{\partial}_{L^{2-2\lambda}}$ have no zero modes, while $\bar{\partial}'^{\dagger}$ has exactly one fewer zero mode than $\bar{\partial}^{\dagger}$. We will denote by $\chi_i \;\; i = 1, \ldots, k-1$ the zero modes of $\bar{\partial}'^{\dagger}$, by $\psi_i \;\; i = 1, \ldots, k-1$ the corresponding zero modes of $\bar{\partial}^{\dagger}$, and by ψ_k the extra mode. There is an Arakelov norm on $\{\psi_i\}$, and the same norm times (12) on $\{\chi_i\}$. With this notation, we get

$$\frac{\det\bar\partial^\dagger\bar\partial}{\det\langle\psi_i|\psi_j\rangle} = C_{g,k}\cdot\frac{\det\bar\partial'^\dagger\bar\partial'}{\det\langle\chi_i|\chi_j\rangle}\cdot\|\psi_k(P)\|^2 \quad . \tag{13}$$

The last factor is defined below (11). The proof of (13) follows lines similar to [13]. In particular the key step equates the curvatures of two Quillen norms. Since curvature only determines a norm up to a constant, we have an undetermined $C_{g,k}$ depending on the genus and the index. The form of (13) would have been more complicated had we not used the Arakelov metric slice.

Eq. (13) remains valid when we replace $L^{2-2\lambda}$ by a more general bundle. Applying it a second time we get the determinant for $L^{2-2\lambda}\otimes\mathcal{O}(P_1+P_2)$ expressed in terms of the norm $\|\psi_{k-1}\|^2$ in $L^{2-2\lambda}\otimes\mathcal{O}(P_1)$. We can rewrite this in terms of the usual norm times $G(P_1,P_2)$ using (12). Continuing in this way we arrive at formulæ $(11b,c)$ when proper care is taken with the last few steps, when $\lambda\le 1$. Thus we have put the bosonization procedure on a completely rigorous footing.

The bosonization formulae should prove useful in investigating properties of multiloop string amplitudes. For example, using either Faltings' approach or the present one it is possible to write various formulæ for the string integrand similar to those in [15]. One can also use (11)to investigate the behavior of the string integrand on the boundary of moduli space. Moreover $(11c)$ with $\lambda = 3/2$ and similar formulae for correlation functions should be useful for investigations of the modular invariance of multiloop superstring amplitudes. Finally, these formulae should help further our understanding of the ultraviolet structure of string perturbation theory and superstring finiteness.

In conclusion, we have shown that the algebraic geometry of determinant line bundles and the physics of bosonization are two aspects of the same thing. The mathematics allows us to prove bosonization formulae rigorously while the physics suggests both the existence of new identities for determinants, zero-modes, and Green functions, and an as yet unexplored connection with the representation theory of Kac-Moody algebras. We believe that bosonization will be a useful tool for further exploration of the deep connection between algebraic geometry and string theory.

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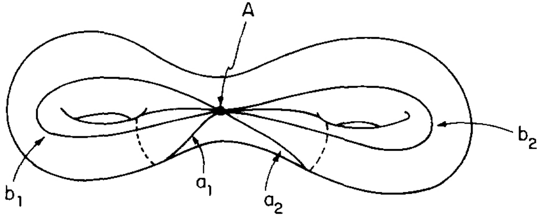

Fig. 1.

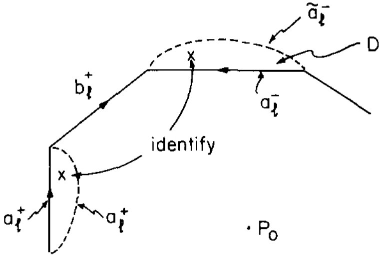

Fig. 2.