ESSAYS ON MATCHING WITH ENDOGENOUS EFFORT

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ABSTRACT

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This thesis extends the two-sided matching literature by including an endogenous effort choice after the matching stage. We examine how different matching rules affect incentives to exert effort and the costs paid to induce various effort levels in three settings: a legal system, an education system, and a labor market. In an indigent defense program, the government provides counsel for indigent defendants. After the assignment of an attorney to a defendant, the attorney exerts a costly effort; however, the government only observes a noisy signal of the effort. We model the problem as a one-to-one matching problem with moral hazard. We show that holding the total expenditure for counsel fixed and changing the matching procedure to accommodate defendants' and attorneys' preferences, i.e., switch from random matching to stable matching, defendants become worse off because a stable matching exacerbates the moral hazard problem on the part of counsel. In in the second case we consider a teacher who chooses a costly effort after observing the distribution of students assigned to his/her class. We model the problem as a many-to-one matching with a costly non-contractible effort choice. We show that the effect of policies that affect the student assignment to classes, such as tracking, implementing school choice, and voucher programs, depends on the curvature of teachers' marginal utility of effort. We find conditions under which the argmax of a maximization problem is strictly supermodular or strictly submodular. Subsequently, we characterize conditions under which sorting students based on their academic performances increases (decreases) the total effort of teachers and the average performance of students. In the third setting, we consider a labor market in which each worker chooses an effort after assignment to a firm. The effort choice, labor in this setting, is observable and contractible. We show that the profit maximizing labor maximizes

the total surplus of the match. Moreover, the unique matching in any equilibrium maximizes the total surplus; however, this matching may have a lower total output compared with any other matching. Stated differently, eliminating labor market frictions increases the efficiency; however, it may increase or decrease the total output.

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CHAPTER 1 : Overview

This thesis extends the two-sided matching literature by including an endogenous effort choice after the matching stage. We study the application of matching with an endogenous effort choice in three settings: a legal system, an education system, and a labor market. In the first setting, the matching is one-to-one and there is heterogeneity in both sides of the market. However, the government pays the cost of each contract and there is no transfer of utility between the two sides of the market. In this setting, only a noisy signal of effort is observable and contractible. In the second setting, the matching is many-to-one. We consider two cases: a case where there is heterogeneity in both sides of the market and a case where there is heterogeneity only in one side of the market. In this setting, effort is not contractible. In the third setting, the matching is one-to-one and there is heterogeneity in both sides of the market. In this setting, utility is transferable between two sides and effort is observable and contractible. In the first setting, the uniform random matching is more desirable than the stable matching. In the second setting, if the marginal utility of a teacher is supermodular and convex in effort, then the stable matching is the most desirable matching; however, if the marginal utility of a teacher is submodular and concave in effort, then the stable matching is the least desirable matching. In the third setting, the stable matching maximizes the total surplus; however, the total output under the stable matching can be lower than any other matching.

In the second chapter, we model the problem of assigning counsel to indigent defendants as a matching problem. A novel aspect of this matching problem is the moral hazard component on the part of counsel. Within the model, we show that holding the total expenditure for counsel fixed and changing the matching procedure to accommodate defendants' and attorneys' preferences, i.e., switch from random matching to stable matching, defendants become worse off because a stable matching exacerbates the moral hazard problem on the part of counsel. In addition, we show that under suitable conditions random matching is the efficient way to allocate defendants to counsel.

In the third chapter, we study the problem of sorting students based on their academic performances and its effect on teachers' effort choice. The effect of sorting students based on their academic performances depends not only on direct peer effects but also on indirect peer effects through teachers' efforts. We show that standard assumptions in the literature are insufficient to determine the effect of sorting on the performances of students and so are silent on the effect of policies such as tracking, implementing school choice, and voucher programs. In order to analyze the effect of sorting on the total performance of students, we need to understand the effect of sorting on the total effort of teachers, which depends on supermodularity and submodularity of the argmax of teachers' utility maximization problem. We find conditions under which the argmax of a maximization problem is strictly supermodular or strictly submodular. We show that the effect of policies such as tracking, implementing school choice, and voucher programs depends on the curvature of teachers' marginal utility of effort. We characterize conditions under which sorting increases (decreases) the total effort of teachers and the average performance of students.

In the fourth chapter, we model a labor market in which workers and firms are heterogeneous. A classic question in economics is the welfare consequences of different allocations of heterogeneous workers to heterogeneous firms. Welfare is unobservable because the cost of providing labor to the workers is unobservable. Many researchers use GDP instead of the total surplus as a proxy for welfare. Within our model, the total output measures the welfare only if the effective labor supplied by workers does not depend on the firm they are matched with. The assumption of fixed hours of labor supply may seem reasonable in a market for unskilled labor; however, in a market for skilled labor it seems less plausible. We show that if there is complementarities between a worker's ability and a firm's technology, positive assortative matching (PAM) is the only matching in any equilibrium and it's the unique efficient matching. However, the total output can be lower under PAM compared with other matchings. We investigate the effect of eliminating all friction, implementing a centralized clearing house, in a labor market. We characterize two sets of sufficient conditions on the production function and cost function under which the total output and welfare move in the same direction. Under the first set of conditions, the increase in total output overestimates the efficiency gain. In contrast, under the second set of conditions, the increase in the total output underestimates the efficiency gain. We identify a third set of conditions under which the total output and welfare move in opposite directions.

CHAPTER 2 : Matching with Moral Hazard: Assigning Attorneys to Indigent Defendants

2.1. Introduction

Each year, more than a hundred thousand individuals in the U.S. who are too poor to pay for counsel are subject to criminal prosecution.[1] The Sixth Amendment to the U.S. Constitution guarantees defendants the right to counsel in federal criminal prosecutions but does not specify how this right is to be exercised. The U.S. Supreme Court expanded these rights in a series of cases decided in the 1960s and 1970s. The most celebrated of these being *Gideon v. Wainwright* (1963), in which the court held that a defendant charged with a felony, including state crimes, had the right to government-provided counsel.

At present, the government provides counsel for indigent defendants using three different defender systems. The first is the practice of hiring defense attorneys from public defender organizations, in which salaried staff attorneys render criminal indigent defense services through a public or private nonprofit organization or as direct government employees (Cohen (2012)). The second is contract defense programs. In this system, contracts to represent indigent defendants are awarded through an "auction." The dollar value of the contract and its duration are specified before the auction. Private attorneys, bar associations, or law firms indicate their willingness to accept the specified contract. Then the government awards the contract to a subset of participants based on their quality. The duration of each contract is one year, and the dollar value is set in terms of a flat fee per criminal case or hourly rate with a cap, which turns into a flat fee per case if the attorney's work report exceeds the cap. The third system is to use assigned counsel programs, in which a judge assigns an attorney to the case, and the attorney accepts out of professional courtesy.

The common feature of all three systems is that the indigent defendant is not permitted to choose his/her attorney. Schulhofer and Friedman (1993) summarize this state of affairs as

[1] Langton and Farole (2010) report, "In 2007, 957 public defender offices across the nation received more than 5.5 million indigent defense cases."

follows:

> Most citizens would consider it shockingly unethical for an attorney representing one side in a lawsuit to be selected or paid, even indirectly, by the opposing party. Yet such principles are violated routinely in this country on a massive scale. In criminal cases, the great majority of defense attorneys are paid directly or indirectly by the prosecuting party, the state.

Drumgo v. Superior Court (1973) is an extreme example of the denial of choice. Fleeta Drumgo and five others were each charged with five counts of murder, one count of conspiracy, and one count of assault while serving a state prison sentence. Four features made Drumgo's case special: A private attorney had to be appointed because the public defender's office was unable to serve. Richard Hodge, the attorney requested by Drumgo was *qualified* and *willing* to represent Drumgo. Drumgo's request for representation by Hodge *preceded* the appointment of a different private attorney by the trial judge. The trial judge denied Drumgo's request to be represented by Hodge. Subsequently, the court of appeals ordered the trial judge to replace Drumgo's court chosen counsel with Hodge. This decision was overturned by the California Supreme Court on the grounds that the trial judge had the discretionary power to appoint counsel for an indigent defendant (Tague (1974)).

In this paper, we take up the question of how counsel should be matched to indigent defendants and analyze the effect of allowing indigent defendants a choice. In our model, the government moves first by announcing a contract. This is followed by an entry decision by attorneys, and then using the announced selection process, the government selects which attorneys to hire. Then there is a matching stage in which defendants are matched to hired attorneys. Subsequently, each attorney decides whether to exert effort for his/her assigned client or shirk the responsibility. Because the government has to provide funding for this system, the government is responsible for designing the contract using a selection process, a matching process, and a wage contract. To put it differently, a contract specifies a selection process, a matching process, and a wage contract.

What distinguishes this problem from other matching problems considered in the literature is the moral hazard component. The government that is charged with matching defendants to attorneys must ensure that sufficient incentives exist for each attorney to exert effort on behalf of his/her assigned defendant.

There is much evidence of a moral hazard problem in the representation of indigent defendants, especially under private contractor systems. Furthermore, shirking can be grounds for appeal (see *Strickland v. Washington* (1984)). One vivid instance of moral hazard comes from McDuffie County, Georgia. In an effort to cut costs on indigent defense, a contract was awarded to Bill Wheeler, who offered to perform *all* the county's indigent defense work for $25,000, almost $20,000 lower than the other two bids and $21,000 lower than the previous year's cost. As part of his contract, Wheeler continued to maintain a private practice as well. As Lemos (2000) reports, "most of Wheeler's indigent clients met him for the first time in court. After a brief, whispered conversation, Wheeler would recommend a guilty plea." Between 1993 and 1998, Wheeler filed only seven motions and tried only 14 cases in court, of which only two were jury trials.

The first part of this paper justifies the denial of choice in the indigent defense system. We compare the indigent defense system under three different matching rules. The first rule assigns indigent defendants uniformly at random to counsel. We view this as representative of how defendants are currently matched with counsel (Schulhofer and Friedman (1993) and Cohen (2012)). We then consider a setting in which defendants are permitted to choose a counsel from the same group of attorneys as before. Indeed, Tague (1974) and Schulhofer and Friedman (1993) have all argued for giving defendants a greater say in the choice of counsel. Schulhofer and Friedman (1993), in particular, suggest the use of vouchers. We model the outcome of such a voucher system as a stable matching; however, under the voucher system, the group of attorneys who are assigned to indigent defendants may be different. We show that holding the government's budget fixed, changing the matching from random to stable, i.e., accommodating defendants' and attorneys' preferences, makes defendants worse off. Moreover, we show that using a voucher system, i.e., using a stable matching *and* changing the set of hired attorneys, makes the indigent defendants worse off. There are two main reasons why permitting defendants a choice makes them worse off. First, institutional restrictions require that wage contracts be nondiscriminatory, i.e., the government cannot give different wages to different attorneys for different cases.[2] Under this restriction, if the government changes the matching rule from uniform random to stable, then there will be an attorney who knows that he/she will get the worst case after signing the contract. Hence, this attorney's participation constraint is violated under the previous wage contract. To satisfy this attorney's participation constraint, the government raises every attorney's wage contract. As a result, given a fixed budget, the government can't hire enough attorneys and incentivize them to exert effort.

The second reason is risk aversion on the part of the attorneys. The government has to compensate for the disutility of exerting effort for each attorney by providing a wage contract. Under the uniform random matching, the government has to compensate all attorneys for the expected disutility of exerting effort. However, under stable matching, each attorney is assigned to a specific indigent defendant, and the government has to compensate attorneys for different costs of exerting high effort. Consider a case in which hired attorneys have the same reservation wage and attorneys are risk averse, i.e., their utility function for money is concave. The cheapest way for the government to compensate all of them is to give them a uniform lottery over all indigent defendants and the same wage contract because their utility function for money is concave. In section 2.5, we formally show that after relaxing the institutional restriction to only nondiscriminatory wage contracts, if a condition on reservation wages of a subset of attorneys and a condition on cost function are satisfied, the government will optimally choose a nondiscriminatory contract and the uniform random matching.

The second part of this paper explores an optimal allocation and an optimal contract.

[2] We discuss reasons for restriction to nondiscriminatory wage contracts in section 2.5.

We characterize the optimal allocation of attorneys to indigent defendants, in which an allocation is a lottery over different matchings. We show that if the reservation wage of all hired attorneys is the same, the uniform random matching is optimal. In addition, if the cost function is separable, then the status quo indigent defense system is using *the optimal contract*, even if the government is allowed to use any discriminatory wage contract and any allocation of attorneys to indigent defendants.

2.2. Model

There is a finite set of indigent defendants J, and $|J| = N$. The difficulty of each indigent defendant $j \in J$'s case is exogenously given and denoted by $d_j \in D$.[3] Index indigent defendants according to their case difficulty, i.e., $d_j \leq d_{j+1} \forall j \in J$. There is a finite set of available attorneys I^a, and the number of available attorneys exceeds the number of indigent defendants, i.e., $|I^a| > |J|$. Each attorney $i \in I^a$ has an exogenous quality $q_i \in Q$.[4] Index attorneys in I^a according to their quality, $q_i \leq q_{i+1} \quad \forall i \in I^a$. Each indigent defendant is in need of an attorney. By law, the government has to provide each indigent defendant with one attorney. Furthermore, the government has to ensure that each indigent defendant receives representation that satisfies the "effective assistance of counsel" criterion, which we discuss later.

The game begins with the government announcing a contract. A contract specifies a selection rule, an allocation rule, and a wage contract, all of which we define later. Attorneys decide to participate given the announced contract. Denote the set of attorneys who participate by $I^p \subseteq I^a$.

The government hires a subset of participating attorneys based on the announced selection rule. The set of hired attorneys is denoted by $I \subseteq I^p$. The government has to hire N attorneys to ensure that each indigent defendant has an attorney. If $|I| < N$, then the

[3] One can interpret the difficulty of a case by the type of indigent defendant or the type of case assigned to the indigent defendant.

[4] One can determine the quality of an attorney by the amount of experience he/she has and the law school from which he/she graduated (Iyengar (2007)).

constitutional right of at least one indigent defendant is violated; hence, we require that $|I| = N$ for an indigent defense system. Based on the announced allocation rule, attorneys are assigned to a defendant. Subsequently, each attorney decides to exert high effort or low effort, denoted by $e \in \{e_l, e_h\}$, on behalf of his/her assigned client.

A wage contract (w_1, w_2) specifies payments to hired attorneys based on a noisy signal of the attorney's effort level. The effort is not observable by the government; therefore, it's not contactable. Based on the announced wage contract, each attorney is paid a contingent wage, i.e., w_1 if the signal is s_1 and w_2 if the signal is s_2. The signal $s \in \{s_1, s_2\}$ follows the distribution $p_l = Pr(s_2|e_l)$, $p_h = Pr(s_2|e_h)$. s_2 is more likely if the attorney exerts a high effort rather than a low effort, i.e., $p_h > p_l$. To put it differently, s_2 is good news about the attorney's effort being high, and s_1 is a bad news about the attorney's effort being high. The signal can be the number of visits before the trial that the attorney had with his client, the number of motions that the attorney filed, and other indicators of the attorney's effort. The government cannot rely on an indigent defendant's report about the attorney's effort, because every convicted indigent defendant will use his/her own report as grounds for appealing the court's decision.

An attorney with quality q_i has an outside option $r(q_i)$. If attorney i gets hired, his/her payoff is $u(w) - c(e, d)$, which is determined by wage, effort, and difficulty of the assigned case.

The government wants to minimize the sum of the expected payments to hired attorneys, i.e., $\sum_{i \in I} p_h w_2 + (1-p_h) w_1$, subject to providing every indigent defendant with one attorney who satisfies the minimum effort condition. Each attorney must choose high effort e_h in compliance with the effective assistance standard specified in the *Strickland v. Washington* (1984) ruling. In *Strickland v. Washington* (1984), the court announced the standard for evaluating postconviction claims of ineffective assistance.

Assumption 1 *(i) $u(w)$ is strictly increasing, continuous, and strictly concave in w.*

(ii) High effort costs more than low effort, i.e., $c(e_h, d_j) > c(e_l, d_j)$ *for any* $d_j \in D$.

(iii) $r(q_i)$ *is nondecreasing in* q_i.

(iv) $c(e_h, d)$ *is nondecreasing in* d *and* $c(e_h, d_1) \neq c(e_h, d_N)$.

(v) $c(e_h, d) - c(e_l, d)$ *is nondecreasing in* d.

Assumptions 1-i and 1-ii on the utility function and the cost function are standard in moral hazard literature; strict concavity of $u(w)$ follows from risk aversion of attorneys. The main result of the paper holds for risk-neutral attorneys as well. Assumption 1-iii about the reservation wage is plausible because the quality of an attorney is a qualitative measure, which represents demand for an attorney. More demand corresponds to a higher reservation wage. The first part of Assumption 1-iv is without loss of generality because we determined the order of the indigent defendants based on the difficulty of their case; however, the order of these difficulties are not specified. To put it differently, we can define a new difficulty measure for each case such that $c(e_h, d)$ is nondecreasing in d. The second part of Assumption 1-iv is satisfied if at least two indigent defendants have cases with different costs of exerting high effort. Assumption 1-v (increasing differences assumption) states that if the cost of exerting high effort for case j is higher than that for case j', then the difference between the cost of exerting high effort and low effort for case j should be higher than the difference between the cost of exerting high effort and low effort for case j'. We assume that all attorneys in I^p have a weak incentive to participate, and all other attorneys have a strict incentive not to participate in the indigent defense system.

The government selects a subset of attorneys to represent indigent defendants from the set of participating attorneys. A *selection rule* is a mapping from the power set of I^a into the power set of I^a, such that $\Gamma(\tilde{I}) \subseteq \tilde{I} \quad \forall \tilde{I} \subseteq I^a$. To put it differently, there is a set of available attorneys I^a, a subset of this set will participate I^p, and then the government uses the specified selection rule Γ to hire a subset of this set $I = \Gamma(I^p)$.

Given a set of hired attorneys I, a *matching* is a one-to-one mapping $\mu : J \to I$. Denote the set of all matchings by $\mathcal{M}(I)$. An *allocation* $\lambda(I) = (\lambda_\mu(I))_{\mu \in \mathcal{M}(I)}$ is a probability distribution over the set of all matchings. An allocation determines how to (randomly) assign a given set of attorneys to a set of indigent defendants. For each matching $\mu \in \mathcal{M}(I)$, $0 \leq \lambda_\mu(I) \leq 1$ and $\sum_{\mu \in \mathcal{M}(I)} \lambda_\mu(I) = 1$. Denote the set of all $\lambda(I)$ by $\Delta\mathcal{M}(I)$. Given an allocation $\lambda(I)$, define $\phi_\lambda(I)(i,j)$ as the probability that attorney $i \in I$ matches with indigent $j \in J$. Define Φ_λ as $N \times N$ matrix, where element (i,j) is $\phi_\lambda(i,j)$. Note that Φ_λ is a doubly stochastic matrix. A *random allocation*, $\lambda(I)$, is an allocation such that $\exists i \in I, \exists j \in J : 0 < \phi_\lambda(i,j) < 1$. Under a random allocation, at least one attorney does not know his assigned case when he/she signs the contract.

The government needs to announce an allocation for each set of hired attorneys. The government can announce the same rule for all sets of hired attorneys or the government can use different allocations for different sets of hired attorneys. In sections 2.3 and 2.4, the government uses the same allocation no matter which attorneys are defending the indigent defendants. However, in section 2.5, we specify an optimal allocation for each set of hired attorneys.

Define $\Omega(I^a)$ to be the set of all subsets of I^a of size N, i.e., $\Omega(I^a) = \{\tilde{I} | \tilde{I} \subseteq I^a, |\tilde{I}| = N\}$. An *allocation rule* Λ is a mapping from $\Omega(I^a)$ to $\Delta\mathcal{M}(I)$, $\Lambda : \Omega(I^a) \to \Delta\mathcal{M}(I)$, i.e., Λ specifies a probability distribution λ over matchings for any subset of size N of I^a.

2.3. Alternative Indigent Defense Systems

We compare the current indigent defense system (status quo) with two alternatives: an indigent defense system that uses a stable matching (stable matching system) and an indigent defense system that uses vouchers (a voucher system). First, we solve for an optimal wage contract for a given allocation λ and a selection rule Γ. Using this wage contract, we analyze the status quo indigent defense system, an indigent defense system under a stable matching, and an indigent defense system that uses vouchers.

An indigent defense system under a stable matching gives indigent defendants and attorneys the right to choose each other. However, the attorneys are selected by the government before the matching stage. Under a stable matching, an indigent defendant chooses an attorney from the set of attorneys selected by the government. If the attorney accepts, then he/she is matched to the defendant.

A voucher indigent defense system allows indigent defendants to select any attorney from the set of available attorneys. Under a voucher indigent defense system, the government does not select attorneys before the matching stage. In other words, the set of hired attorneys under a voucher indigent defense system is different from the set of hired attorneys under the status quo indigent defense system. The outcome of a voucher system is a stable matching.

2.3.1. Optimal wage contract for a given allocation and selection rule

We find the optimal wage contract from the government's point of view subject to two sets of constraints. First, the government wants to hire attorneys in the set $I = \{k', \ldots, k+N-1\}$. Second, the government wants every attorney who is hired to exert high effort no matter which case the attorney is assigned to. The objective of the government is to minimize the expected cost of providing counsel to indigent defendants.

Define $u_1 = u(w_1)$, $u_2 = u(w_2)$. Hence a wage contract (w_1, w_2) in the utility measure is (u_1, u_2). Define $h(.) = u^{-1}(.)$. Under Assumption 1, $h(.)$ is strictly convex. [5]

The following optimization problem identifies the optimal wage contract (u_1^*, u_2^*) for a given

[5] Such notations make the problem of finding the optimal way of implementing an action a convex programming problem (see Grossman and Hart (1983)).

λ, such that any attorney $i \in I$ is willing to participate and exert high effort:

$$\min_{u_1,u_2} \sum_{i \in I} p_h h(u_2) + (1-p_h)h(u_1)$$

$s.t.$

$$(p_h - p_l)(u_2 - u_1) \geq c(e_h, d) - c(e_l, d) \quad \forall d \in D \quad \text{(IC)},$$
$$u_1 + p_h(u_2 - u_1) \geq r(q_i) + E^i_\lambda(c(e_h, d)) \quad \forall i \in I \quad \text{(IR)},$$

where:

$$E^i_\lambda(c(e_h, d)) = \sum_{j \in J} \phi_\lambda(i, j) c(e_h, d_j),$$

is the expected cost of exerting high effort under the allocation λ for attorney i. Observe that because of restriction to nondiscriminatory wage contracts, the objective function can be simplified to:

$$\sum_{i \in I} p_h h(u_2) + (1-p_h)h(u_1) = N(p_h h(u_2) + (1-p_h)h(u_1)).$$

Note that the only place that allocation enters the government's problem for determining an optimal wage contract is in the right-hand side of individual rationality (IR) constraints.

Incentive compatibility (IC) constraints hold for $\forall d \in D$ and $\forall i \in I$ because the government wants all attorneys to exert high effort for every case. Observe that (IC) is the same for $\forall i \in I$. Therefore, we need to consider (IC) $\forall d \in D$. (IR) is satisfied only in expectation because attorneys are committed to accept any case assigned to them under λ.

We solve the government's problem for selecting an optimal wage contract, an optimal allocation rule, and an optimal selection rule in two steps. First, for any given allocation rule and selection rule, find the optimal wage contract u^*_1, u^*_2. Then using the optimal wage contract, in the second step, find an optimal allocation rule and an optimal selection rule. Lemma 1 specifies the optimal wage contract w^*_1, w^*_2 or equivalently u^*_1, u^*_2 for a given allocation rule and selection rule.

Lemma 1 *Given Assumption 1, for a fixed allocation λ and a fixed set of hired attorneys I, the optimal wage contract is:*

$$u_1^* = R_\lambda - p_h\big(\tfrac{c(e_h,d_N)-c(e_l,d_N)}{p_h-p_l}\big), \tag{2.1}$$

$$u_2^* = R_\lambda + (1-p_h)\big(\tfrac{c(e_h,d_N)-c(e_l,d_N)}{p_h-p_l}\big), \tag{2.2}$$

where R_λ is:

$$R_\lambda = max\{r(q_i) + E_\lambda^i(c(e_h,d))\}_{i\in I}.$$

Proof: See the appendix.

2.3.2. Status quo

The *uniform random allocation*, $\lambda_u(I)$, selects each element of $\mathcal{M}(I)$ with equal probability, i.e., it selects a matching from $\mathcal{M}(I)$ uniformly at random. Observe that under the uniform random allocation, any element of I has the same uniform probability of matching to any element of J, i.e., for any $i \in I$ and any $j \in J$, $\phi_{\lambda_u}(i,j) = \frac{1}{N}$. The *uniform random allocation rule* Λ_u, specifies the uniform random allocation for any set $I \in \Omega(I^a)$.

The status quo system for assigning counsel to defendants does not rely on the preferences of defendants and attorneys. The status quo allocation rule that the government uses is the uniform random allocation rule (Cohen (2012) and Schulhofer and Friedman (1993)). Furthermore, under the status quo system, the government selects the N highest quality attorneys from set I^p (Schulhofer and Friedman (1993)). We call this selection rule the *merit-based selection rule.*

In our model, under the uniform random allocation, the government is indifferent between selecting which N attorneys from set I^p to hire because fixing a nondiscriminatory wage contract the cost of hiring any attorney from the set of participating attorneys is the same. Hence, the government is behaving optimally when it uses the merit-based selection rule. We denote the index of the highest element of this set by $k+N-1$, i.e., $q_{k+N-1} \geq q_i \forall i \in I^p$

and $k+N-1 \in I^p$. Hence the set of hired attorneys under the merit-based selection rule is $\{k, k+1, ..., k+N-1\}$. To put it differently, given λ_u and a (w_1, w_2), the government is indifferent between different selection rules Γ as long as $|\Gamma(I^p)| = N$. One such selection rule is the merit-based selection rule. To model the status quo indigent defense system, we use the merit-based selection rule that the government currently uses (Schulhofer and Friedman (1993)). Under the uniform random allocation, R_λ depends on $r(q_{K+N-1})$ only, i.e., given the set I^p, which depends on u_1^*, u_2^*, the government's cost does not depend on other elements of I. Hence, the government hires the highest-quality attorneys from set I^p, i.e., the government uses the merit-based selection rule. Note that $r(q_i)$ is a nondecreasing function, not a strictly increasing function; therefore, it may be the case that $|I^p| > N$. In this case, because the cost of a contract with each attorney in I^p is the same for the government, the government is indifferent, so the government selects the highest-quality attorneys.

The optimal wage contract under uniform random allocation is:

$$u_1^* = r(q_{k+N-1}) + E_d(c(e_h, d)) - p_h\left(\tfrac{c(e_h,d_N)-c(e_l,d_N)}{p_h-p_l}\right), \tag{2.3}$$

$$u_2^* = r(q_{k+N-1}) + E_d(c(e_h, d)) + (1-p_h)\left(\tfrac{c(e_h,d_N)-c(e_l,d_N)}{p_h-p_l}\right), \tag{2.4}$$

where $E_d(c(e_h, d)) = \frac{1}{N}\sum_{j=1}^{N} c(e_h, d_j)$.

The cost of this indigent defense system for the government is:

$$\mathcal{C}_u = N\Bigg((1-p_h)\, h \left(r\left(q_{k+N-1}\right) + E_d\left(c\left(e_h, d\right)\right) - p_h \left(\frac{c\left(e_h, d_N\right) - c\left(e_l, d_N\right)}{p_h - p_l} \right) \right) + p_h h \left(r\left(q_{k+N-1}\right) + E_d\left(c\left(e_h, d\right)\right) + (1-p_h) \left(\frac{c\left(e_h, d_N\right) - c\left(e_l, d_N\right)}{p_h - p_l} \right) \right) \Bigg).$$

We view the *status quo contract* as the merit-based selection rule, the uniform random allocation rule, and the wage contract specified in equations (2.3) and (2.4). We view $\mathcal{C}_u$ as the budget of the status quo indigent defense system.

The following lemma characterizes the set of attorneys who would participate under the specified optimal wage contract, the uniform random allocation rule, and the merit selection rule. Recall that the index of the highest-quality attorney who participates is $k+N-1$ and any attorney with an index above $k+N-1$ does not participate.

Although the government is using the merit-based selection rule, hired attorneys are not the highest-quality attorneys available. The government uses the wage contract to incentivize attorneys to participate in the indigent defense system. If the wage contract is designed for the N lowest quality attorneys, then attorneys with reservation wages higher than $r(q_N)$ will not participate. Lemma 2 shows that, even under merit-based selection rule, any hired attorney has a reservation wage of at most $r(q_N)$. If we consider the reservation wage as a signal for an attorney's quality, we can conclude that the government is hiring attorneys with quality close to N lowest-quality attorneys.

Lemma 2 *Under the status quo contract the reservation wage of highest-quality attorney who participates is strictly lower than the reservation wages of attorneys who abstain, i.e.,* $r(q_{k+N-1}) < r(q_{k+N})$. *Furthermore, if* $k > 1$*, then reservation wages* $r(q_i)$ *for all* $i = N, N+1, \dots, k+N-1$ *are the same.*

Proof: The expected utility of attorney $k+N$ from participating in the indigent defense system is strictly less than his/her outside option $r(q_{k+N})$, otherwise he/she would participate in the indigent defense system. Moreover, the expected utility of attorney $k+N-1$ from participating in the indigent defense system is at least $r(q_{k+N-1})$. Under the uniform random allocation rule, the expected utility of attorney $k+N$ from participating in the indigent defense system is equal to the expected utility of attorney $k+N-1$ from participating in the indigent defense system. Hence, their outside options can not be equal, i.e., $r(q_{k+N-1}) \neq r(q_{k+N})$.

Suppose there exists attorney i' such that $N \leq i' < N+k-1$ and $r(q_{N+k-1}) > r(q_{i'})$. By hiring attorneys $i'-N+1, ..., i'$, the government satisfies all the equilibrium constraints, and the expected cost is lower than hiring attorneys $k, ..., N+k-1$ because $\mathcal{C}_u$ is a strictly increasing function of $r(q_{k+N-1})$, a contradiction. ∎

2.3.3. An indigent's right to an attorney of his/her choice

In this section, we study two indigent defense systems that permit defendants and attorneys to choose each other. A stable matching characterizes the outcome of incorporating the preferences of indigent defendants and attorneys. There are many arguments in favor of defendants' and attorneys' right to choose.

Tague (1974) and Schulhofer and Friedman (1993) have argued for giving defendants a greater say in their choice of counsel. There is a natural conflict of interest between the indigent's attorney and the prosecution side, hence giving the power of selecting and funding of both sides to one office will result in a conflict of interest. Moreover, other government funded systems, such as health care and the education system, incorporate the preferences of two sides of the market; one such instance is vouchers in education systems. The outcome of a system that gives indigent defendants and attorneys a choice is a stable matching. Gale and Shapley (1962) define a stable matching: A matching is stable if no matched agent prefers to be single and no pair of agents prefers each other to their assigned partner in the matching.

Schulhofer and Friedman (1993) suggest a voucher system for an indigent defense system. The outcome of a voucher system is a stable matching. However, under a voucher system, the set of attorneys who get hired is different than the set of attorneys who get hired under the status quo indigent defense system of assigning attorneys to indigent defendants. We consider a different set of hired attorneys under the voucher system when we compare the outcome of status quo indigent defense system with a voucher system.

Stable matching

First, we define a stable matching and a positive assortative matching. We show that all stable matchings are positive assortative matchings. Hence, we can restrict our attention to positive assortative matchings. At the end, we discuss the optimal wage contract under a stable matching.

To define a stable matching, we specify preferences for indigent defendants and attorneys. For a fixed wage contract, each indigent defendant j has a *preference* over attorneys denoted by $\prec_j$. If two attorneys exert the same amount of effort, all indigent defendants prefer the attorney with the higher quality to the attorney with the lower quality, i.e., if $q_i < q_{i'}$ then $i \prec_j i', \forall j \in J$. Moreover, if $q_i = q_{i'}$, then every indigent defendant is indifferent between attorney i and attorney i'. Given a wage contract, each attorney $i \in I^a$ has a preference over cases denoted by $\prec_i$. Each attorney prefers a case that gives him/her a higher utility to a case that gives him a lower utility under the specified wage contract. Furthermore, if an attorney is not committed to accepting at least one case and a case that gives him/her a strictly lower utility than his/her outside option, then he/she prefers his/her outside option.

Under a nondiscriminatory wage contract, all attorneys prefer cases with lower disutility of high effort, i.e., if $c(e_h, d_j) < c(e_h, d_{j'})$ then $j' \prec_i j, \forall i \in I^a$. Moreover, if $c(e_h, d_j) = c(e_h, d_{j'})$, then every attorney is indifferent between case j and case j'. Using the preferences of both sides of this market, we can define stable matching and a positive assortative matching for a fixed wage contract and find their relationship.

For a fixed wage contract, a matching μ is *stable* if:

1. Every attorney prefers his/her match to his/her outside option, i.e., $\gamma_i \prec_i \mu^{-1}(i), \forall i \in I$, where γ_i is the attorney's outside option. Note that any indigent defendant prefers any attorney to his/her outside option.

2. There is no blocking pair. A blocking pair is (i, j) such that $\mu(j) \neq i$, $\mu^{-1}(i) \prec_i j$ and

$\mu(j) \prec_j i$.

For a fixed wage contract, a *positive assortative matching* is a matching such that:

1. For any $i, i' \in I$ if $q_i < q_{i'}$ then $\mu^{-1}(i) \prec_{i'} \mu^{-1}(i')$.

2. For any $j, j' \in J$ if $c(e_h, d_j) > c(e_h, d_{j'})$ then $\mu(j) \prec_{j'} \mu(j')$.

Lemma 3 specifies the relationship between stable matching and a positive assortative matching.

Lemma 3 *Under any nondiscriminatory wage contract, if all attorneys have an incentive to exert high effort, any stable matching is a positive assortative matching.*

Proof: Suppose there exists a stable matching μ that is not a positive assortative matching, i.e., there exist i, i' and j, j' such that $\mu(j) = i$, $\mu(j') = i'$, $q_i < q_{i'}$ and $c(e_h, d_j) < c(e_h, d_{j'})$. Then (i', j) is a blocking pair, because $j' \prec_{i'} j$ and $i \prec_j i'$. Therefore, the matching μ is not stable, a contradiction. ■

Consider an indigent defense system that uses a positive assortative matching. Each attorney at the ex-ante stage of the game, i.e., deciding to participate in the system or abstain, knows exactly which case difficulty he will face. Under a positive assortative matching, if there exist an attorney i and case j such that $0 < \phi(i, j) < 1$, then $c(e_h, d_j) = c(e_h, d_{j'})$ for any other case $j' \in J$ with $0 < \phi(i, j') < 1$.

The following optimization solves for the optimal wage contract for hiring attorneys from the set $I = \{k, ..., k + N - 1\}$, under a positive assortative matching λ_s, when they have incentive to exert high effort:

$$\min_{u_1,u_2} \sum_{i=k}^{k+N-1} p_h h(u_2) + (1-p_h)h(u_1)$$

$s.t.$

$$(p_h - p_l)(u_2 - u_1) \geq c(e_h, d) - c(e_l, d) \forall d \in D,$$

$$u_1 + p_h(u_2 - u_1) \geq r(q_i) + (c(e_h, d_{k+N-i})) \forall i \in I.$$

We can show that the incentive constraint for the highest d binds.[6] The matching is a positive assortative, so we need to find $q_{\tilde{i}}$ such that $r(q_{\tilde{i}}) + (c(e_h, d_{k+N-i}))$ is maximized. We can relax this problem and only consider the (IR) for q_k. Attorney k will get case N. We can find a lower bound on an optimal wage contract, denote this by (u_1^{**}, u_2^{**}), by relaxing the problem and only considering (IR) for the attorney with the lowest quality, i.e., attorney k:[7]

$$u_1^{**} \geq r(q_k) + c(e_h, d_N) - p_h\left(\tfrac{c(e_h,d_N)-c(e_l,d_N)}{p_h-p_l}\right), \tag{2.5}$$

$$u_2^{**} \geq r(q_k) + c(e_h, d_N) + (1-p_h)\left(\tfrac{c(e_h,d_N)-c(e_l,d_N)}{p_h-p_l}\right). \tag{2.6}$$

Voucher system

Critics of the status quo indigent defense system have proposed the use of a voucher indigent defense system because this system gives indigent defendants the right to choose their attorneys. In a voucher indigent defense system, the government gives each indigent defendant a voucher that specifies a wage contract for the attorney who accepts the defendant's case.

Given that every indigent defendant has the same voucher, all indigent defendants will go

[6] See the proof of Lemma 1.

[7] Note that this is a lower bound on an optimal wage contract for any indigent defense system that uses a matching with the following property: Attorney k', where $k' \geq k$, is assigned to case N with probability 1, i.e., $\exists k' \geq k$ such that $\phi(k', N) = 1$. Negative assortative matching is one example. All the results in section 2.4 hold for this type of indigent defense system as well.

to their most preferred attorney in the first round, i.e., they will go to $i^* \in I^a$, where $i \prec_j i^* \forall i \in I^a, \forall j \in J$. Then i^* will either accept his/her most preferred case, i.e., j^* where $j \prec_{i^*} j^*, \forall j \in J$, in this case (i^*, j^*) are matched, or i^* will reject all indigent defendants' proposals. The same process happens for the remaining indigent defendants and attorneys until every indigent defendant is matched with one attorney or there are no remaining attorneys. At the end, if every indigent defendant has an attorney, the allocation is a positive assortative matching.

2.4. Comparison of Alternative Indigent Defense Systems

In this section, we compare the status quo with two alternative indigent defense systems: a stable matching system and a voucher system. First, we define the measure for these comparisons. Second, we specify the assumption that we need for these comparisons.

If the cost of providing the same N attorneys who exert high effort is lower under the one contract compared with another contract, then we say that the first contract is *superior* to the second contract. Consider two indigent defense systems with different contracts. Suppose the first contract is superior to the second contract. Moreover, the budget of the second indigent defense system is set equal to the budget of the first system. Then an indigent defense system that uses the second contract will result in one of the following: Either effective representation requirement for at least one indigent defendant is violated, i.e., at least one attorney is choosing e_l. Or the quality of the lowest-quality attorney under this contract is strictly lower than the quality of lowest-quality attorney under the superior contract.

Assumption 2 $r(q_N) - r(q_1) < c(e_h, d_N) - E_d(c(e_h, d))$.

Intuitively, the match-specific part of the utility function varies more than the reservation wage of the N lowest-quality attorneys. Assumption 2 states that the difference between the reservation wages of the N lowest-quality attorneys in the set I^a is less than the difference between the cost of exerting high effort when matched with the highest cost case and the expected cost of exerting high effort when matched uniformly at random. The N lowest-quality available attorneys have a very similar outside option. Therefore, their reservation wages are close to each other.

Theorem 1 *Under Assumptions 1 and 2, the status quo contract is superior to any contract that uses a positive assortative matching.*

Proof: We show that the cost of the government under a positive assortative matching for hiring $I = \{k, ..., k+N-1\}$ is strictly greater than the cost of the government under random allocation for hiring all attorneys in I. Recall that under the uniform random allocation rule $r(q_N) = r(q_{k+N-1})$, so:

$$r(q_{k+N-1}) + E_d(c(e_h, d)) = r(q_N) + E_d(c(e_h, d)) < r(q_1) + c(e_h, d_N) \leq r(q_k) + c(e_h, d_N),$$

which implies that:

$$u_1^* < u_1^{**}, u_2^* < u_2^{**}.$$

Because $h(.)$ is a strictly increasing function, we have:

$$h(u_1^*) < h(u_1^{**}), h(u_2^*) < h(u_2^{**})$$

$$\Rightarrow \sum_{i=k}^{k+N-1} p_h h(u_2^*) + (1-p_h)h(u_1^*) < \sum_{i=k}^{k+N-1} p_h h(u_2^{**}) + (1-p_h)h(u_1^{**}).$$

■

Theorem 1 states that under the status quo system budget, using a positive assortative matching instead of the uniform random allocation, and using the same set of attorneys, the government cannot induce every attorney to exert high effort. Theorem 1 shows that permitting indigent defendants and attorneys to choose each other, given the same budget that the status quo indigent defense system has, will result in a worse indigent defense system from the indigent defendants' point of view.

Lemma 3 states that any stable matching is a positive assortative matching. Theorem

1 implies the following statement about the comparison between status quo and a stable matching system.

Corollary 1 *Under Assumptions 1 and 2, the status quo contract that uses the uniform random allocation is superior to any contract that uses any stable matching.*

There are two reasons that Theorem 1 and corollary 1 hold: restriction of using only nondiscriminatory wage contracts and the risk aversion of attorneys.

The first reason is the restriction of using only nondiscriminatory wage contracts. Under any positive assortative matching, the lowest-quality attorney in set I is matched to the most difficult case. Consider the participation constraint of this attorney under the status quo contract and under any contract that uses a positive assortative matching. The reservation wage of this attorney is the same under these two contracts. However, the expected disutility of effort is strictly larger under a positive assortative matching. Therefore, the government has to increase the wage contract for this attorney to satisfy his participation constraint. The wage contract is nondiscriminatory; therefore, increasing a wage contract for one attorney implies that wage contracts are increased for every hired attorney. The cost of the government is strictly increasing in u_1 and u_2; hence, an increase in the wage contract of every attorney increases the cost of the indigent defense system for the government. Note that this argument doesn't depend on the risk aversion of attorneys.

The second reason is the risk aversion on the part of attorneys. Consider an example with two cases and two attorneys with the same reservation wages. In this example, suppose the government can pay the attorneys different wages. The government has to compensate the attorneys' expected disutility of effort and their forgone reservation wage. Under a positive assortative matching, the low-quality attorney is matched to the difficult case and the high-quality attorney is matched to the easy case. Recall that $u(w)$ is strictly concave. Therefore, it is cheaper for the government to pay equal wages to both attorneys and use the uniform random allocation, instead of paying a very high wage to the low-quality attorney and a low wage to the high-quality attorney. The role of risk aversion is discussed in more detail in section 2.5.

The following lemma shows that risk aversion is not essential for this result and that without risk aversion the same conclusion is true.

Lemma 4 *If the attorneys are risk neutral, i.e., $u(w) = w$, Theorem 1 holds, i.e., the status quo contract is superior to any contract that uses a positive assortative matching.*

Proof: Given the set I and λ, the minimum cost for the government to hire all attorneys in I and induce every $i \in I$ to exert high effort is $\mathcal{C}$:

$$\begin{aligned} \mathcal{C} = \quad & \min_{u_1,u_2} \textstyle\sum_{i=k}^{k+N-1} p_h w_2 + (1-p_h) w_1 \\ s.t. \quad & \\ & (p_h - p_l)(w_2 - w_1) \geq c(e_h, d) - c(e_l, d), \\ & \forall d \in D, \quad \forall i \in I, \\ & w_1 + p_h(w_2 - w_1) \geq r(q_i) + E^i_\lambda(c(e_h, d)), \\ & \forall i \in I. \end{aligned}$$

Note that the left-hand side of (IR) is equal to $\frac{1}{N}$ of $\mathcal{C}$. In order to find $\mathcal{C}$, we need to find which (IR) binds. Hence, the cost of the government given the set I is:

$$\mathcal{C} = N \times R_\lambda.$$

Under the uniform random allocation, the cost of the government is:

$$\mathcal{C}_u = N \times (r(q_{k+N-1}) + E_d c(e_h, d)).$$

Under a positive assortative matching, the cost of the government is:

$$\mathcal{C}_s = N \times R_\lambda \geq N \times (r(q_k) + c(e_h, d_N)).$$

The inequality follows from the definition of R_λ. Hence, under Assumption 2, $\mathcal{C}_u < \mathcal{C}_s$, i.e., given the same budget that the status quo system has, the outcome of the indigent defense system under any stable matching is worse than the outcome of the indigent defense system under the uniform random allocation. ∎

Next, we compare the outcome of the status quo indigent defense system with the outcome of a voucher system. Under a voucher system, indigent defendants and attorneys have the right to choose each other; hence, the allocation λ is not a choice of the government. Moreover, the indigent defendants can choose the set of hired attorneys, i.e., selection rule Γ is not under the government's control. These two features make a voucher indigent defense system an interesting alternative system at first glance. However, the following theorem shows that the status quo contract is superior to any contract that uses a voucher system.

In order to do this comparison, we find a lower bound on the cost of the optimal wage contract from the government's point of view, under a positive assortative matching as the allocation rule. Recall that the allocation under a voucher system is a positive assortative matching. Under a voucher system, we consider any set of hired attorneys with size N, where attorneys' qualities are at least q_k. One possible set of hired attorneys under a voucher system is $\{k, ..., k+N-1\}$. We require the wage optimal contract to give the hired attorneys incentive to exert high effort under a voucher system.

Theorem 2 *Given Assumptions 1 and 2, under the status quo system's budget, switching from the status quo contract to any contract that uses a voucher system results in one of the following:*

1. *At least one indigent defendant doesn't have an attorney.*

2. *At least one hired attorney doesn't have sufficient incentive to exert high effort.*

3. *The quality and the reservation wage of the lowest-quality attorney among hired attorneys is strictly lower than the quality and the reservation wage of the lowest-quality attorney among hired attorneys under the status quo contract.*

Proof: From equations 2.5 and 2.6, we know that the cost of the government under any stable matching, such that the quality of each attorney is at least q_k, is at least $N(p_h h(u_2^{**}) + (1-p_h)h(u_1^{**}))$, because $h(.)$ is strictly increasing in u_1^{**} and u_2^{**}, $N(p_h h(u_2^{**}) + (1-p_h)h(u_1^{**}))$ is a lower bound on the cost of the government under a positive assortative matching, too. If we show that the cost of the government under a voucher system for hiring the set I is greater than the cost of the government under the uniform random allocation rule for hiring I, then we can conclude that the cost of the government for hiring N attorneys using a voucher system such that the quality of each attorney is at least q_k is greater than the cost of the government under the uniform random allocation for hiring I. We proved this in Theorem 1, so no matter which set of attorneys from set I^a are recruited under the voucher system, as long as their quality is above q_k given the same budget, the outcome of the system under the uniform random allocation is superior to the outcome under the voucher system. We do not consider the situation in which a voucher system results in some unmatched indigent defendants, or some indigent defendants are matched with attorneys with quality lower than q_k, because in these situations it is clear that the status quo contract is superior to the contract that uses a voucher system. ■

2.5. Optimal Allocation

The status quo allocation rule that the government uses in the indigent defense system is the uniform random allocation rule, i.e., $\Lambda_u = \lambda_u(I) \quad \forall I$.

In this section, first, for any set of hired attorneys I, we characterize the optimal allocation. Second, for a fixed set of hired attorneys I, we specify the conditions on reservation wages of hired attorneys such that the uniform random allocation is optimal. At the end, we specify a set of conditions on reservation wages of attorneys $1, ..., N$ and the cost function such that the status quo contract is the optimal contract.

The government wants to minimize the cost, subject to hiring N attorneys and incentivizes them to exert high effort. The government can choose a selection rule, an allocation rule, and a wage contract. We showed that the merit-based selection rule is optimal under the uniform random allocation rule, and we specified the optimal wage contract in Lemma 1. In this section, we define and characterize the optimal allocation.

Given a set I, an allocation λ^* is *optimal* if the cost of the government under the contract that uses λ^* and the optimal wage contract given λ^*, which is specified in Lemma 1, is lower than the cost of the government under any other contract.

Theorem 3 *The following linear program identifies an optimal allocation for a given set I:*

$$\begin{aligned} &\textit{(LP*)} \qquad \min_{y,\{\phi(i,j)\}_{i\in I, j\in J}} \quad y \\ &s.t. \\ &r(q_i) + \textstyle\sum_{j\in J}\phi(i,j)c(e_h, d_j) \le y \quad \forall i \in I, \\ &\textstyle\sum_{j\in J}\phi(i,j) = 1 \quad \forall i \in I, \\ &\textstyle\sum_{i\in I}\phi(i,j) = 1 \quad \forall j \in J, \\ &\phi(i,j) \ge 0 \quad \forall i \in I, j \in J. \end{aligned}$$

Proof: See the appendix. ∎

The solution of linear program (LP*) specifies a doubly stochastic matrix $[\phi^*(i,j)]$ and y^*, using the Birkhoff-von Neumann decomposition algorithm, we can find the optimal allocation λ^*.

If the reservation wages of all hired attorneys are equal, the optimal allocation can be characterized easily from the solution of the linear program (LP*). Lemma 5 shows that the uniform random allocation is optimal under this condition. Therefore, the status quo allocation that the government uses is indeed optimal if reservation wages of all hired attorneys are equal. On the other hand, if under the status quo contract reservation wages of all hired attorneys are not equal, then we can improve the status quo indigent defense system by using a different allocation rule.

Lemma 5 *For a fixed set of hired attorneys I, the uniform random allocation is the optimal allocation if and only if $r(q_i)$ is the same for all $i \in I$.*

Proof: See the appendix. ■

Corollary 2 *If the reservation wages of at least two hired attorneys under the status quo contract are not equal, then there exists a superior contract that hires the same set of hired attorneys.*

Define a separable cost function as $c(e,d) = c(e) - g(d)$. This special cost function represents the following cost structure: The utility of an attorney is the utility from wage minus disutility of effort plus nonpecuniary utility that depends on the type of his/her match, i.e., $u(w) + g(d) - c(e)$. The separable cost function represents a situation in which there is no complementarity between case difficulty and effort level.

For the rest of the paper, we relax the restriction to the nondiscriminatory wage contracts. A discriminatory wage contract specifies a contingent wage for each possible assignment of attorneys and indigents, i.e., $\{u_1(i,j), u_2(i,j)\}_{i \in I, j \in J}$. An *optimal contract* is a contract that minimizes the cost of the government. Note that the government can choose any selection rule, any allocation rule, and any discriminatory wage contract to minimize the cost. Recall that the merit-based selection rule is optimal under the uniform random allocation rule, and based on Lemma 5, the uniform random allocation rule is optimal if $r(q_i) = r \quad \forall i \in I$. Currently, the government is using nondiscriminatory wage contracts.

There are several reasons that the government should in fact use nondiscriminatory wage contracts. First, the government is prosecuting the defendant. At the same time, the prosecution and attorney are involved in the plea bargaining process. Using discriminatory contracts signals the government's perception of the likelihood of winning or losing the case. This signal affects the plea bargaining process. Second, the right to counsel is a constitutional right. Paying different wages for different cases based on any criteria other than the case type may seem as discrimination among the indigent defendants. Third, it may be the case that the government does not have the same information as the counsel about the difficulty of each case. In addition to those reasons, we identify a condition such that nondiscriminatory wage contracts are optimal even when discriminatory wage contracts are available. The following theorem identifies conditions on reservation wages of a subset of available attorneys and cost function such that the status quo contract is optimal among a very broad class of contracts, such as discriminatory wage contracts; stable, deterministic allocation rules; random allocation rules; and any selection rule that doesn't violate the constitutional rights of indigent defendants.

Theorem 4 *Under Assumptions 1 and 2, separable cost function, and* $r(q_i) = r$ *for all* $i \in \{1, \ldots, N\}$*, the status quo contract is* **the optimal contract***.*

Proof: See the appendix.

One can use Theorem 4 for comparison and show that Theorem 1 holds because of two different forces; first, institutional restrictions to nondiscriminatory wage contracts, and second, the risk aversion of attorneys.

COROLLARY 3 *Under Assumptions 1 and 2, a separable cost function, and* $r(q_i) = r$ *for all* $i \in \{1, \ldots, N\}$*, even if the government can announce discriminatory wage contracts, the status quo contract is superior to any contract that uses a positive assortative matching and any contract that uses a stable matching.*

Corollary 3 shows that without restriction to nondiscriminatory wage contracts, the status quo contract is superior to any contract that uses a stable matching. This result is due to the fact that attorneys are strictly risk averse. Note that under discriminatory wage contracts, there may exist a stable matching that is not a positive assortative matching.

2.6. Conclusion

We model the assignment of indigent defendants to attorneys as a matching with a moral hazard component. Using this model, we show that the matching process is a part of the contract and that changing the matching process will affect the incentives of attorneys. Specifically, accommodating defendants' and attorneys' preferences encourages some attorneys who are hired under status quo to either exit the indigent defense system or to put in less effort making defendants worse off. Furthermore, using a voucher system makes defendants worse off.

We characterize an optimal allocation. Using this characterization, we show that the uniform random matching is optimal if and only if the reservation wage of all hired attorneys is the same. Hence, if under the status quo contract at least two hired attorneys have different reservation wages, then there exists a superior contract. The superior contract is the merit-based selection rule, an optimal allocation, and the optimal wage contract, all of which we characterize in this article. Under this superior contract, the government can hire the same set of attorneys and give them sufficient incentive to exert high effort with a strictly lower expenditure. Finally, we show that if the cost function is separable and the reservation wage of all hired attorneys is the same, then the status quo indigent defense system is using the optimal contract.

CHAPTER 3 : Sorting and Peer Effects

3.1. Introduction

We introduce a model to analyze the effect of student sorting on the total effort of teachers and the average or total performance of students. Our model allows for both direct and indirect peer effects. First, consider the case of homogeneous teachers who choose effort after observing their classes' composition. Each teacher chooses an effort based on the distribution of students' abilities in his/her class, i.e., the teacher's choice may depend on the whole distribution, not just the mean of students' abilities. The effect of sorting on the teachers' total effort choice is ambiguous. Because the teachers' total effort may increase or decrease as a result of student sorting, the effect of sorting on the average or total performance of students is ambiguous, too. We characterize conditions on the utility function of teachers under which the total effort of teachers strictly increases or strictly decreases by sorting. If the teachers' marginal utility of effort is supermodular (submodular) and convex (concave) in effort, then the total effort of teachers increases (decreases) as a result of sorting of students. Subsequently, in the absence of direct peer effects, if performance is convex (concave) in a teacher's effort, the total performance of an education system increases (decreases) as a result of sorting.[1] We show that this result persists even when we allow for heterogeneous teachers.

Sorting affects students' outcomes because of peer effects. There are two types of peer effects: direct and indirect. Direct peer effects are the result of student-to-student spillovers (see Sacerdote (2000), Sacerdote (2011), and Epple and Romano (2011) for a review of the literature). Indirect peer effects happen through a teacher's effort choice (see Duflo et al. (2011) and Todd and Wolpin (2012)). Duflo et al. (2011) report that both direct and indirect peer effects exist in the data and that the data cannot be explained using only one kind of peer effects.

[1] Because most of the results for one set of conditions are parallel in wording to results under the other set of conditions, instead of stating results under each set of conditions separately, we state both results in one statement using parentheses.

Two types of sorting are present in an education system: within-school sorting and between-school sorting. Within-school sorting, or tracking, is an explicit policy that sorts students into different classes based on their abilities. Sorting between schools happens in different ways, such as: (i) Sorting between public and private schools.[2] (ii) Sorting as a result of voucher programs.[3] Chakrabarti (2009) states that "There is strong and robust evidence in favor of stratification by ability" as a result of Milwaukee Voucher Program. Hsieh and Urquiola (2006) report that they "find evidence that the voucher program led to increased sorting, as the 'best' public school students left for the private sector." (iii) Standardized admissions tests. MacLeod and Urquiola (2012) state that "the introduction of standardized admissions tests will lead to stratification by ability." (iv) Public information regarding schools' qualities. Hastings and Weinstein (2007) find that "providing parents with direct information on school test scores resulted in significantly more parents choosing higher-scoring schools for their children.". (v) Different school choice policies.[4] Levin (1998) reports that "evidence is consistent that educational choice leads to greater socioeconomic (SES) and racial segregation of students." We incorporate both types of sorting in our model.

Our paper also contributes to the literature on comparative statics. That literature is focused on the monotonicity of the argmax of a maximization problem;[5] however, to analyze the effect of sorting on the total performance of students, we need to understand the effect of sorting on the total effort of teachers, which depends on supermodularity and submodularity of the argmax of teachers' utility maximization problem. We find conditions under which the argmax of a maximization problem is strictly supermodular or strictly submodular. More concretely, if the marginal utility of a teacher is supermodular (submodular) and convex (concave) in effort, then the argmax of the teacher's maximization problem — the optimal effort of the teacher — is supermodular (submodular); therefore, the total perfor-

[2]see Epple and Romano (1998) and Epple et al. (2002) for more details.

[3]See Barrow and Rouse (2008).

[4]See Avery and Pathak (2015) for the effect of implementing school choice instead of neighborhood assignment rule on student sorting.

[5]See Topkis (1998), Milgrom and Shannon (1994), and Edlin and Shannon (1998).

mance of all teachers increases (decreases) as a result of sorting students. In the absence of direct peer effects, the only channel through which sorting changes the total/average performance of students is through indirect peer effects. Hence, if teachers are putting in more effort altogether and performance is a convex function in the teacher's effort, then the total/average students' performance increases. On the other hand, if teachers are putting in less effort altogether and performance is a concave function in the teacher's effort, then the total/average students' performance decreases. We state the results for an education system; one can use the same tools to analyze any one-to-many matching with endogenous effort choice and evaluate the effect of different matchings. Moreover, conditions for strict supermodularity (submodularity) of the argmax are derived for a general maximization problem; hence, these results can be used in any maximization problem that has the same structure.

Sorting increases inequality in students' performances; however, if the marginal utility of teachers is supermodular and convex in effort, sorting increases the total/average performance of students. In this situation, sorting is desirable under the Utilitarian welfare function. On the other hand, if the marginal utility of teachers is submodular and concave in effort, sorting decreases the total/average performance of students and increases inequality in students' performances. In this situation, sorting reduces both the Utilitarian welfare function and the Rawlsian welfare function. Affirmative action policies have the opposite effect of sorting, i.e., these policies reduce sorting; therefore, in this situation, affirmative action policies increase both the Utilitarian welfare function and the Rawlsian welfare function.

The effect of sorting on the total/average performance through the channel of indirect peer effects is robust, whether direct peer effects exist, even when teachers are heterogeneous in quality and utility function. Furthermore, if teachers are heterogeneous and classes are sorted, the standard results regarding the benefit of positive assortative matching (PAM) versus negative assortative matching (NAM) may not hold.[6] We show that even if the performance function of classes and the utility of teachers are supermodular, the total performance of students can be higher under negative assortative matching compared with positive assortative matching. Moreover, inequality is lower under negative assortative matching compared to positive assortative matching. To put it differently, the value of the Utilitarian welfare function and the Rawlsian welfare function are higher under NAM compared with PAM.

The effectiveness of monetary incentives on teachers' effort choices is debated in the literature. There is some evidence of a positive effect of monetary incentives in developing countries (Lavy (2002)). However, in developed countries such as the U.S., the evidence suggests that monetary incentives have an insignificant effect on teachers' effort choices (Fryer (2013)). We consider the U.S. as the main application, i.e., monetary incentives don't affect teachers' effort choices. Under a pay-per-performance system in which a teacher's wage depends on the performance of his/her students, sorting has an impact on the budget of the education system and results in inequality in teachers' salaries. Sorting increases inequality in teachers' salaries when teachers are homogeneous and wage increases are based on students' performances. Sorting increases (decreases) the total payment to teachers if the total students' performance increases (decreases) and payment to teachers is an increasing and a convex (concave) function of class performance. Our model can incorporate monetary incentives, too. We consider a general utility function that can incorporate monetary incentives; hence, we can analyze the effect of sorting under different monetary incentive systems.

In section 3.2, we set up the model and develop the required mathematical tool to handle the student sorting problem. In section 3.3, we analyze the effect of student sorting on the total effort of teachers by finding conditions on the utility function of teachers such that the argmax of their utility maximization problem is supermodular (submodular). Subse-

[6] See Tincani (2014).

quently in section 3.4, we show how sorting affects students' outcomes under three settings: (1) homogeneous teacher with indirect peer effects, (2) homogeneous teacher with direct and indirect peer effects, and (3) heterogeneous teacher with direct and indirect peer effects.

3.2. Model

Let T be a finite set of homogeneous teachers and I a finite set of students, where $|I| = n|T|$. A student $i \in I$ has a type $\theta_i \in \mathcal{R}_+$. The type can represent a student's ability, the prior year's test score, parents' education/income, or any other characteristic that affects the students' performance. We interpret type as ability.

A matching is an assignment of students to teachers, denoted by $\mu : T \to I$, such that $|\mu(t)| = n$, where n is the size of the class. Each student is assigned to only one teacher, i.e., $\mu^{-1}(i)$ is a function. We denote a class by the profile of types $\theta \in \mathcal{R}_+^n$ assigned to it. We denote the class assigned to teacher t by $\theta^t = (\theta_i)_{i \in \mu(t)}$.

There is a measure of performance for each student $i \in \mu(t)$, denoted by $p(e_t, \theta_i, \theta^t)$. We interpret a student's performance as his/her end-of-year test score. There is an aggregate measure of performance for each class t, denoted by $p(e_t, \theta^t)$. We consider the aggregate measure of performance for each class as the average performances of students in that class.

Teacher $t \in T$ chooses an effort $e_t \in [0, 1]$. Each teacher gets a payment — wage plus bonus — based on the aggregate performance of the class, denoted by $w(p(e_t, \theta^t))$.[7] Each teacher t has a utility function $f(e_t, \theta^t)$. The utility function is the same for all teachers.[8] $f(e_t, \theta^t)$ represents the induced utility of a teacher; the utility of a teacher may depend on the performance of his/her class, his/her wage, and the amount of effort he/she exerts.[9] We assume that the wage structure is fixed; hence, if the utility of a teacher depends on the

[7] Wage can be a constant function.

[8] We relax this assumption in section 3.4.3.

[9] For example, the utility function of a teacher can be the non pecuniary utility that he/she gets from his/her class's performance minus the cost of effort, i.e., $f(e, \theta) = u(p(e, \theta)) - c(e)$. The cost of effort may depend on effort and the class composition, i.e., $f(e, \theta) = u(p(e, \theta)) - c(e, \theta)$. The utility function of a teacher may include the utility he/she gets from wages or bonuses plus the non pecuniary utility that he/she gets from his/her class's performance minus the cost of effort, i.e., $f(e, \theta) = u(p(e, \theta)) + v(w(p(e, \theta))) - c(e)$.

wage, then $f(e, \theta)$ is the induced utility function for a fixed wage structure. The marginal utility of effort at $e = 1$ is strictly negative for any class θ. The marginal utility of effort at $e = 0$ is strictly positive for any class θ.

We assume that a teacher's utility function and performance of his/her class are symmetric functions in students' type, i.e., any permutation of a class θ generates the same performance and utility for a teacher: if θ' is a permutation of θ, then:

$$f(e, \theta) = f(e, \theta'), p(e, \theta_i, \theta) = p(e, \theta_i, \theta'), \text{ and } p(e, \theta) = p(e, \theta').$$

We assume a teacher's utility function and the performance of his/her class are three times continuously differentiable, i.e., $f(e, \theta), p(e, \theta) \in \mathcal{C}^3$.

First, we define sorting of two classes, and then we show the mathematical relationship between sorting and the coordinate-wise maximum and minimum of two classes. We order all the students in the two classes by their types, and then we put the top half of students in one class and bottom half in the other class. This process is called *sorting*.[10]

One-step sorting of two classes is defined as the coordinate-wise maximum and minimum of two classes:

$$\forall \theta, \theta^\dagger : \theta' = \theta \vee \theta^\dagger, \theta'' = \theta \wedge \theta^\dagger,$$

where for any two vectors $\theta = (\theta_1, ..., \theta_n), \theta^\dagger = (\theta_1^\dagger, ..., \theta_n^\dagger)$:

$$\theta \vee \theta^\dagger = (max(\theta_1, \theta_1^\dagger), ..., max(\theta_n, \theta_n^\dagger)), \theta \wedge \theta^\dagger = (min(\theta_1, \theta_1^\dagger), ..., min(\theta_n, \theta_n^\dagger)).$$

Lemma 6 *There exists a reordering of two classes such that sorting is achieved by one-step sorting.*

[10] This process is called sorting only if the two new classes have different student compositions than they did before sorting.

Proof: In the appendix.

For any class θ, define $\tilde{\theta}$ as the reordering of the vector θ in descending order, i.e., the permutation of the class in which the first element is the greatest type in the class, the second element is the second greatest type in the class, and so on: $\tilde{\theta} = (\tilde{\theta}_1, \ldots, \tilde{\theta}_n)$ such that $\tilde{\theta}_i \geq \tilde{\theta}_{i+1} \quad \forall i = 1, \ldots, n-1$. Given this reordering of the two classes θ^1, θ^2, define a partial ordering of two classes $\succeq^*$ as the vector ordering in $\mathcal{R}^n$:

$$\theta^1 \succeq^* \theta^2 \Leftrightarrow \tilde{\theta}^1_i \geq \tilde{\theta}^2_i \quad \forall i = 1, \ldots, n, \tag{3.1}$$

we call θ^1 *a better class* than θ^2. Note that after sorting of two classes $\theta, \theta^\dagger$, the sorted classes θ', θ'', have the following property:

$$\theta \succeq^* \theta', \theta^\dagger \succeq^* \theta', \theta'' \succeq^* \theta, \theta'' \succeq^* \theta^\dagger.$$

We call θ' *the lower track* and θ'' *the higher track*. Observe that the higher track is a better class than the lower track.

These two classes can be in one school or in two different schools. The former represents within-school sorting; The latter represents between-school sorting. In between-school sorting, every class in one school is a better class than any class in the other school (based on the partial order $\succeq^*$ defined in (3.1)). However, we may be unable to order two classes in the same school (based on the partial order $\succeq^*$) after between-school sorting. All the following results hold for both between-school sorting and within-school sorting.

A function $h : \mathcal{R}^n \to \mathcal{R}$ is *supermodular* if it is pairwise supermodular in any of its two arguments, i.e., the cross-partial derivatives in any of its two arguments are positive.[11] For example, $p(e, \theta)$ is supermodular if $p_{e\theta_i}(e, \theta) \geq 0, p_{\theta_i\theta_j}(e, \theta) \geq 0 \quad \forall i, j \in I, \forall e, \forall \theta$. If the performance function is supermodular then two types of complementarities exist: complementarity between effort of the teacher and a student's ability and complementarity

[11] See Topkis (1998).

between students' abilities. If cross-partials are strictly positive, then the function is strictly supermodular. A function is *modular* if the cross-partial derivatives in any of its two arguments are zero. For example, $p(e, \theta)$ is modular if $p_{e\theta_i}(e, \theta) = 0, p_{\theta_i\theta_j}(e, \theta) = 0, \quad \forall e \in [0, 1] \quad \forall i, j, \theta_i \in \mathcal{R}_+, \theta_j \in \mathcal{R}_+$. A function is *submodular* if it is pairwise submodular in any of its two arguments, i.e., the cross-partial derivatives in any of its two arguments are negative. For example, $p(e, \theta)$ is submodular if $p_{e\theta_i}(e, \theta) \leq 0, p_{\theta_i\theta_j}(e, \theta) \leq 0 \quad \forall i, j \in I, \forall e, \forall \theta$. If cross-partials are strictly negative, then the function is strictly submodular.

3.3. Characterizing Teachers' Optimal Efforts

In this section, we characterize the conditions under which the argmax of a maximization problem is strictly supermodular or strictly submodular.[12] Furthermore, these conditions determine the effect of sorting on the total effort of teachers. Each teacher maximizes a utility function; the argmax of a teacher's maximization problem is his/her optimal effort, which is unique under the following assumption (Assumption 1). We show that if the marginal utility of effort is supermodular and convex in effort, then the argmax is supermodular. Subsequently, we show that sorting increases the total effort of teachers in this case. Similarly, we show that if the marginal utility of effort is submodular and concave in effort, then the argmax is submodular. We conclude that sorting decreases the total effort of teachers in this case.

Assumption 1

i) A teacher's utility function is pairwise supermodular in effort and each student's type, i.e., $f_{e\theta_i}(e, \theta) \geq 0 \quad \forall i, e \in [0, 1], \theta \in \mathcal{R}_+^n$.

ii) Performance of a class is strictly increasing in the effort of teacher and each student's type, i.e., $p_e(e, \theta) > 0$, $p_{\theta_i}(e, \theta) > 0$. *Performance of student* i *is strictly increasing in the effort of teacher and increasing in other student's type, i.e.,* $p_e(e, \theta_i, \theta) > 0$, $p_{\theta_j}(e, \theta_i, \theta) \geq 0 \quad \forall j : \mu^{-1}(i) = \mu^{-1}(j)$.

[12]We consider only maximization problems that have a unique argmax.

iii) $f_{ee}(e, \theta) < 0$*, teachers' utility function is strictly concave in effort.*

Assumption 1-i captures a complementarity between a teacher's effort and a student's type in teacher's utility function. To put it differently, the marginal utility of effort is increasing in a student's type. Assumption 1-ii states that the performance of a class increases as the teacher puts in more effort or as the ability of a student increases. Assumption 1-iii ensures that a teacher's maximization problem has a unique interior solution given by the first-order condition, i.e., a teacher chooses an effort such that the marginal utility of effort is zero.

Given Assumption 1, increasing a student's type — having a better class — results in a higher effort by the teacher. The following lemma shows this result formally.

Lemma 7 *Given Assumption 1, the optimal effort of a teacher (*$e^*(\theta) = \text{argmax}_{e\in[0,1]} f(e, \theta)$*) is increasing in any student's type.*

Proof: Increasing any student's type in a class results in a better class: $\theta_i \geq \theta'_i \forall i \Rightarrow \theta \succeq^* \theta'$. Using the Topkis theorem, we have $\theta \succeq^* \theta' \Rightarrow e^*(\theta) \geq e^*(\theta')$. ■

Consider two classes with two teachers. After sorting, the two teachers are assigned to two new sorted classes, the higher track and the lower track. The teacher assigned to the higher track puts in more effort after sorting because the higher track is a better class compared with both initial classes, based on the partial order defined in (3.1). Sorting increases the effort of the teacher assigned to the higher track and decreases the effort of the teacher assigned to the lower track. Hence, every student in the higher track has a higher performance after sorting, and every student in lower track has a lower performance after sorting. We call this an *increase in inequality* of students' performances.

Proposition 1 *Given Assumption 1, sorting increases inequality in students' performances, i.e., every student in the higher track has a higher performance after sorting, and every student in the lower track has a lower performance after sorting.*

Proof: By Lemma 7, a teacher's effort increases in the higher track. Because performance is increasing in teacher's effort, every student in the higher track has a higher performance after sorting. Similarly for students in the lower track, performance decreases after sorting. ■

COROLLARY 4 *Under the Rawlsian welfare function, i.e., Max-Min of all students' performances, sorting decreases welfare.*

To understand the effect of sorting on the average/total performance of students, first we need to analyze another problem: What is the effect of sorting on the total effort of teachers? In the following theorem, we show that if the marginal utility of effort is supermodular and convex in effort, then the argmax of a teacher's utility maximization problem — the optimal effort of a teacher — is strictly supermodular. Under this condition, sorting increases the total effort of teachers. Similarly, if the marginal utility of effort is submodular and concave in effort, then the argmax of the teacher's utility maximization problem — the optimal effort of a teacher — is strictly submodular.

Condition 1 *The marginal utility of effort is supermodular and convex in effort:*

$$f_{e\theta_i\theta_j}(e,\theta) \geq 0, f_{ee\theta_i}(e,\theta) \geq 0, f_{eee}(e,\theta) \geq 0, \forall e \in [0,1], i, j, \theta_i \in \mathcal{R}_+, \theta_j \in \mathcal{R}_+,$$

with at least one strict inequality.

Condition 2 *The marginal utility of effort is submodular and concave in effort:*

$$f_{e\theta_i\theta_j}(e,\theta) \leq 0, f_{ee\theta_i}(e,\theta) \leq 0, f_{eee}(e,\theta) \leq 0, \forall e \in [0,1], i, j, \theta_i \in \mathcal{R}_+, \theta_j \in \mathcal{R}_+,$$

with at least one strict inequality.

Theorem 5

1. *If Condition 1 is satisfied, then the optimal effort is strictly supermodular.*

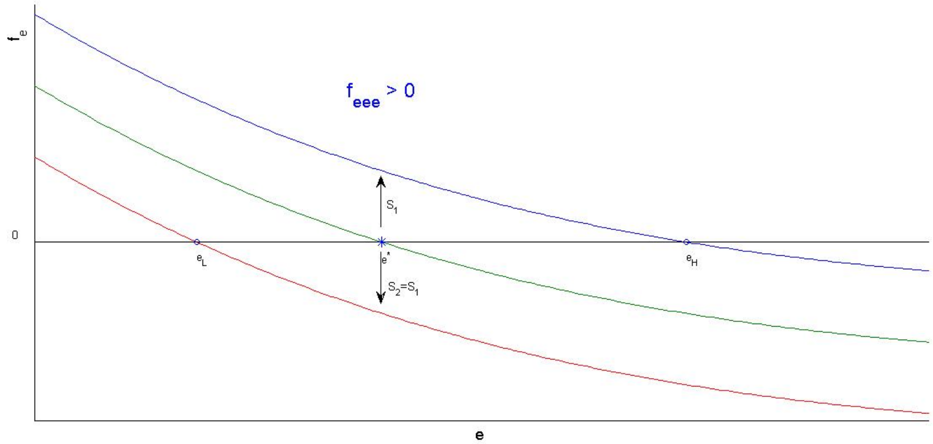

Figure 1: The teacher's marginal utility of effort is convex.

2. If Condition 2 is satisfied, then the optimal effort is strictly submodular.

Proof: In the appendix.

Theorem 5 holds under weaker conditions, which we characterize in the appendix.

The following thought experiment shows the effect of each inequality in Condition 1 on the teacher's total effort. Consider a teacher assigned to two identical classes. The teacher chooses an effort level such that the marginal utility of effort is zero, e^* in Figure 1, for both classes. Hence, the total effort is $2e^*$. After sorting, the teacher's marginal utility changes in both classes. More concretely, the teacher's marginal utility in the higher track shifts upward and in the lower track shifts downward at e^*. Suppose these two shifts are equal, i.e., $s_1 = s_2$ in Figure 1. Because the marginal utility of effort is convex, recall that $f_{eee}(e, \theta) \geq 0$, the increase in the teacher's optimal effort in the higher track is more than the decrease in the teacher's optimal effort in the lower track, i.e., $e_h - e^* \geq e^* - e_l$ in Figure 1, with strict inequality if the marginal utility of effort is strictly convex.

On top of that, $f_{e\theta_i\theta_j}(e, \theta) \geq 0$ ensures that the marginal utility of effort is supermodular in students' type. Therefore, the upward shift in the marginal utility function in the higher track is greater than or equal to the downward shift in the teacher's marginal utility of

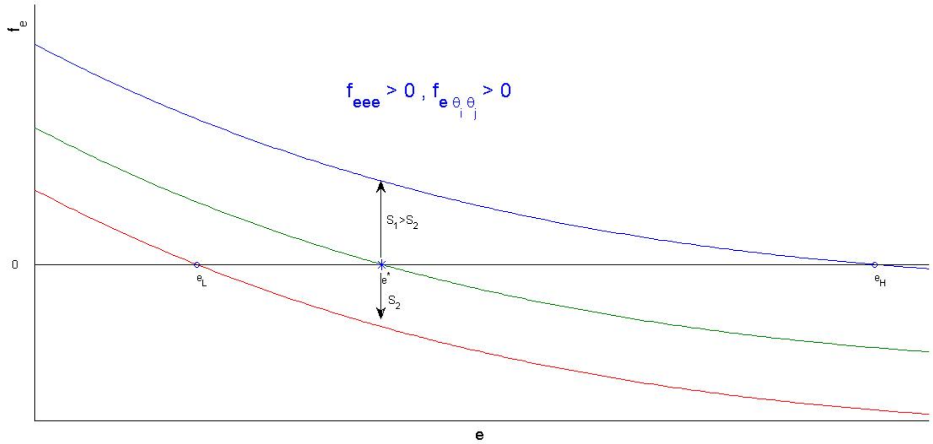

Figure 2: The teacher's marginal utility of effort is pairwise supermodular in students' type.

effort in the lower track, $s_1 \geq s_2$ in Figure 2. This implies that $e_h - e^* \geq e^* - e_l$ in Figure 2, with strict inequality if the marginal utility of effort is strictly supermodular.

$f_{ee\theta_i}(e, \theta) \geq 0$ ensures that the slope of the marginal utility of effort for the teacher in the higher track is greater than or equal to the slope of the marginal utility of effort for the teacher in the lower track. To put it differently, in Figure 3, the marginal utility of effort for the higher track is flatter than the marginal utility of effort before sorting, which is flatter than the marginal utility of effort for the lower track. Hence, the teacher's optimal effort in the lower track, e_l^*, is to the right of e_l, and the teacher's optimal effort in the higher track, e_h^*, is to the right of e_h. Therefore, $e_h^* - e^* \geq e^* - e_l^*$ with strict inequality if $f_{ee\theta_i}(e, \theta) \geq 0$ holds with strict inequality. To conclude, each of the three inequalities in Condition 1 ensure that the total effort of the teacher increases after sorting. We state the result for any initial class composition in the following proposition.

Proposition 2

1. *Given Condition 1, sorting strictly increases the total effort of teachers.*

2. *Given Condition 2, sorting strictly decreases the total effort of teachers.*

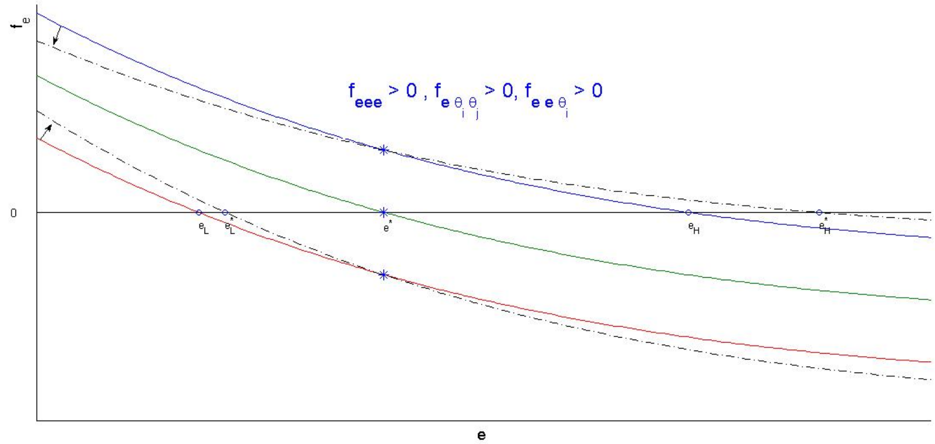

Figure 3: The teacher's marginal utility of effort is pairwise supermodular in the teacher's effort and student's type.

Proof:

1. By Lemma 6, sorting is achieved by one-step sorting. By Theorem 5, the total effort strictly increases if a class composition changes. Hence, sorting strictly increases the total effort of teachers.

2. By Lemma 6, sorting is achieved by one-step sorting. By Theorem 5, the total effort strictly decreases if a class composition changes. Hence, sorting strictly decreases the total effort of teachers.

■

The effect of sorting on the total performance of students, under standard assumptions used in the literature, is ambiguous.

Example 1 *(Value-added system)* *This example is inspired by the value-added measure* *used in Koedel et al. (2015). Suppose* $p(e, \theta_i, \theta) = e\theta_i$ *and*

$$P(e, \theta) = \frac{1}{n}\left(\sum_{i \in \mu(t)} p(e, \theta_i, \theta) - p'(\theta_i)\right),$$

where $p'(\theta_i)$ *is student* i*'s last year's test score. This performance function measures the average of a teacher's contribution to the increase in the students' scores from their scores from last year. Consider the following utility function:*

$$f(e, \theta) = u(P(e, \theta)) - c(e).$$

Suppose $\theta_i > 1$ *and* $e \in [\epsilon, 1]$ *for some* $0 < \epsilon < 1$, $u'(.) > 0$, $u''(.) < 0$, $c'(.) > 0$, $c''(.) \geq 0$, *and* $c'''(.) \geq 0$.

If $u'''(.) \leq 0$, *then sorting decreases the total performance of students; however, if* $u'''(.) \geq max\{-u''(.)\left(\frac{2n}{e}\right), c'''(.)\}$, *then sorting increases the total performance of students. (The proof is in the appendix.)*

3.4. Impact of Sorting on Students' Performances

In this section, we characterize the effect of sorting on the average of students' performances under three different settings: (1) only indirect peer effects exist;[13] (2) both direct and indirect peer effects exist; and (3) teachers are heterogeneous in quality and utility function, and both direct and indirect peer effects exist.

First, we consider a setting with homogeneous teachers and without direct peer effects. The only impact of sorting on students' performances is through indirect peer effects. Using the characterization of the total effort of teachers after sorting, we can characterize the effect of sorting on the average of students' performances. If teachers are putting in more (less) effort in total and the performance function is convex (concave) in effort, then the total/average

[13] We assume students benefit from high achieving peers in all sections; however, we consider direct peer effects as the direct peer effects that have different effects on students with different abilities. Stated differently, if the performance is a modular function in students' type, then there is no direct peer effects.

students' performance increases (decreases). This setting allows us to isolate the effect of teachers' effort and the role of the curvature of the teachers' marginal utility function in answering our main question: Does sorting increase or decrease the average performances of students? Is sorting a desirable outcome based on the utilitarian welfare criterion?

Second, we consider a setting with both direct and indirect peer effects with homogeneous teachers. Duflo et al. (2011) report that both direct and indirect peer effects exist in the data and that excluding either is inconsistent with their data. Sorting has two effects on the average of students' performances. If both effects go in the same direction then we can determine whether sorting increases or decreases the average of students' performances. However, if these two effects go in opposite directions, then the effect of sorting depends on the magnitude of each effect; we provide a general method to evaluate the effect of sorting on average students' performances in this case.

Third, we consider a general environment in which teachers have different qualities and utility functions, and both direct and indirect peer effects exist. We show that — by extending Assumption 1, Condition 1, and Condition 2 to include teachers' type — the previous results are robust. Furthermore, if classes are ordered by the partial ordering $\succ^*$ defined in (3.1), we can analyze the welfare implications of positive assortative matching (PAM) of teachers and classes compared with negative assortative matching (NAM) of teachers and classes. The curvature of the marginal utility of effort of teachers has an important impact on this welfare comparison. There are simple examples in which the usual results about the benefits of PAM compared with NAM don't hold. More precisely, in these examples, switching from PAM to NAM decreases the inequality and increases the average performances of students, i.e., increases both the utilitarian welfare function and the Rawlsian welfare function.

3.4.1. Indirect Peer Effects

Consider an environment in which all teachers have the same quality and utility function. Suppose there is no direct peer effect.[14] What is the effect of sorting on the total students' performance in this environment? Can we increase the average performance of students by changing the composition of the classes? In other words, which matching of students to classes maximizes the total performance of students? In this subsection,we show that the answers to these questions depend on the curvature of the marginal utility of effort of teachers. More concretely, if Condition 1 is satisfied and performance is convex in effort, sorting increases the total performance of students, i.e., sorting is a desirable outcome based on the utilitarian criterion. On the other hand, if Condition 2 is satisfied and performance is concave in effort, sorting is the least desirable matching of students to classes under both the utilitarian and the Rawlsian criteria.

Suppose there is no direct peer effect, i.e., $p_{e\theta_i}(e, \theta) = 0, p_{\theta_i\theta_j}(e, \theta) = 0 \ \forall e \in [0, 1], i, j, \theta_i \in \mathcal{R}_+, \theta_j \in \mathcal{R}_+$. Then, the only effect of sorting on students' performances is through indirect peer effects. The direction of indirect peer effects depends on whether Condition 1 is satisfied and performance is convex in effort or Condition 2 is satisfied and performance is concave in effort. Under the former, teachers put in more effort in total, and the composition of classes for a fixed level effort doesn't affect the total performance of these classes; therefore, sorting increases the total performance. Under the latter, teachers put in less effort in total; hence, sorting decreases total performance.

Theorem 6

1. *If Condition 1 holds and performance is modular and convex in effort, sorting strictly increases the total performance of students.*

2. *If Condition 2 holds and performance is modular and concave in effort, sorting strictly decreases the total performance of students.*

[14]See Foster (2006).

Proof: In the appendix.

The proof specifies a general method for analyzing other situations as well; for example, when Condition 1 is satisfied but performance is concave in effort. Simply put, if equation (A.10) defined in the appendix is positive (negative), then sorting increases (decreases) the total performance.

Note that even when there is no direct peer effect and the total effort of teachers increases, the total performance may increase or decrease by sorting. For example, consider two teachers who exert the same level of effort for two identical classes; one teacher increases his/her effort by an amount equal to the amount by which the other teacher decreases his/her effort. The total performance of these two classes (strictly) increases if the performance function is (strictly) convex and (strictly) decreases if the performance function is (strictly) concave.

If Condition 1 holds and performance is modular and convex in effort, the effect of sorting depends on the welfare function that we use, i.e., based on different welfare objectives sorting maybe desirable or undesirable. If Condition 2 holds and performance is modular and convex in effort, sorting results in the worst classes' compositions among all other classes' compositions. Therefore, a policymaker needs to consider the curvature of a teacher's marginal utility of effort in order to make a decision that increases or decreases the sorting of students.

Corollary 5

1. *If condition 1 holds and performance is modular and convex in effort, sorting improves the utilitarian welfare function but decreases the Rawlsian welfare function.*

2. *If condition 2 holds and performance is modular and concave in effort, sorting decreases both the utilitarian welfare function and the Rawlsian welfare function.*

Because sorting changes the performance of classes, under a pay-per-performance system, sorting has an effect on the total payment to teachers — budget of an education system — and on inequality in payments to teachers. For example, under the value-added system in the U.S. education system, tracking increases teachers' income inequality; however, it may increase or decrease the average payment to teachers. The following corollary states these effects formally.

Corollary 6

1. *If Condition 1 holds, performance is modular and convex in effort, and payment to teachers is an increasing and convex function of class performance, then sorting strictly increases the total payment to teachers.*

2. *If Condition 2 holds, performance is modular and concave in effort, and payment to teachers is an increasing and concave function of class performance, then sorting strictly decreases the total payment to teachers.*

3. *If payment to teachers is an increasing function of class performance, then sorting strictly increases inequality in teachers' payments.*

3.4.2. Direct and Indirect Peer Effects

Suppose both direct and indirect peer effects are present. Direct and indirect peer effects may affect the total performances of students after sorting in the same direction or in opposite directions. If both direct and indirect peer effects increases (decrease) the total performances of students, then the effect of sorting is clear, which we state in the following theorem. On the other hand, if they have the opposite effect on the total performances of students, then the effect of sorting on the total performance depends on the magnitude of direct versus indirect peer effects. We provide a general method to evaluate the effect of sorting on the total performances of students when the direct and indirect peer effects have the opposite effect on the total performances of students.

Theorem 7

1. *If Condition 1 holds and performance is increasing, supermodular, and convex in effort, sorting strictly increases the total performance of students.*

2. *If Condition 2 holds and performance is increasing, submodular, and concave in effort, sorting strictly decreases the total performance of students.*

Proof: In the appendix.

Note that Proposition 1 and Corollary 4 hold in this subsection. Hence, sorting increases inequality in students' performance.

Corollary 7

1. *If Condition 1 holds and performance is increasing, supermodular, and convex in effort, sorting improves the utilitarian welfare but decreases the Rawlsian welfare.*

2. *If Condition 2 holds and performance is increasing, submodular, and concave in effort, sorting decreases both the utilitarian welfare and the Rawlsian welfare.*

Consider a situation in which sorting increases the total performances of students because of direct peer effects but decreases the total performances of students because of indirect peer effects. The effect of sorting on the total performance depends on the magnitude of these two forces. A *general method* for finding the effect of the tradeoff between direct and indirect peer effects for any performance function $p(e, \theta)$ and any utility function $f(e, \theta)$ is:
Step 1: Use equation (A.11) in the appendix to find the sign of the function $\Psi(e, \theta)$, which is defined in equation (A.11) in the appendix.
Step 2: If $\Psi(e, \theta)$ is positive everywhere, then sorting increases the total performance; if $\Psi(e, \theta)$ is negative everywhere, then sorting decreases the total performance (proof in the appendix).
Step 3: If the sign of $\Psi(e, \theta)$ is unclear for all effort levels and class compositions, we can find the sign of $\Psi(e, \theta)$ for a given level of effort and a given class composition. If $\Psi(e, \theta)$ is positive, then sorting increases the total performance at the given effort level and class

composition; if $\Psi(e, \theta)$ is negative, then sorting decreases the total performance at the given effort level and class composition.

In Theorems 6 and 7, we used Assumption 1; specifically, we used the assumption that $p_{\theta_i} \geq 0$ and $f_{e\theta_i} > 0$. Note that we can define the order on θ_i such that $p_{\theta_i} \geq 0$ holds. To put it differently, assumption of $p_{\theta_i} \geq 0$ is without loss of generality; however, the assumption that both p_{θ_i} and $f_{e\theta_i}$ are strictly positive is not without loss of generality. One can relax this assumption: suppose teachers prefer low-ability students, i.e., suppose $f_{e\theta_i} < 0$. Then the optimal effort of a teacher is decreasing in a student's type. Moreover, we can use the general method, i.e., finding the sign of $\Psi(e, \theta)$, to determine the effect of sorting on the total performance of students in this situation.

Using the results from Theorem 7, we can specify the effect of sorting on the teachers' payment when both direct and indirect peer effects exist.

Corollary 8

1. *If Condition 1 holds, performance is increasing and supermodular, and payment to teachers is an increasing and convex function of class performance, then sorting strictly increases the total payment to teachers.*

2. *If Condition 2 holds, performance is increasing and submodular, and payment to teachers is an increasing and concave function of class performance, then sorting strictly decreases the total payment to teachers.*

3. *If payment to teachers is an increasing function of class performance, then sorting strictly increases inequality in teachers' payments.*

3.4.3. Heterogeneous Teachers and Direct and Indirect Peer Effects

Suppose each teacher has a quality — teacher's type — $q_t \in \mathcal{R}_+$ that enters the performance function and teachers' utility function, i.e., teachers' utility function is $f(e, q, \theta)$ and the performance of a class is $p(e, q, \theta)$. We extend Assumption 1 and Conditions 1 and 2 to include the teachers' type as well.

Assumption 2 *A teacher's utility function is pairwise supermodular in effort and the teacher's type, i.e.,* $f_{eq}(e, q, \theta) \geq 0 \quad \forall e \in [0,1], q \in \mathcal{R}_+, \theta \in \mathcal{R}_+^n$.

Condition 3 *The marginal utility of effort is supermodular and convex in effort, i.e.,*

$$f_{eq\theta_i} \geq 0, f_{eeq} \geq 0, f_{e\theta_i\theta_j} \geq 0, f_{ee\theta_i} \geq 0, f_{eee} \geq 0, \forall e \in [0,1], \forall \theta \in \mathcal{R}_+^n, \forall q \in \mathcal{R}_+,$$

with at least one strict inequality.

Condition 4 *The marginal utility of effort is submodular and concave in effort, i.e.,*

$$f_{eq\theta_i} \leq 0, f_{eeq} \leq 0, f_{e\theta_i\theta_j} \leq 0, f_{ee\theta_i} \leq 0, f_{eee} \leq 0, \forall e \in [0,1], \forall \theta \in \mathcal{R}_+^n, \forall q \in \mathcal{R}_+,$$

with at least one strict inequality.

We maintain the assumption of anonymity for students of a class, i.e., teachers' utility functions and the performance of a class are symmetric functions in students' types. The one-step sorting is defined as before with the assignment of the higher-quality teacher to the upper track. Sorting is defined as: Order all the students in the two classes by their types, then put the top half of the students in one class with the teacher who has the higher quality and put the bottom half in the other class with the teacher who has the lower type. If the two new classes have different student and teacher compositions as did the two classes before sorting, then this process is called *sorting*. After sorting of two classes, we can order the students of these classes by the binary relation $\succ^*$ defined in (3.1). Define *student sorting with positive assortative matching*(PAM) as the sorting of students of two classes and assigning the higher-quality teacher to the higher-track class and the lower-type teacher to the lower-track class. Sorting is equivalent to student sorting with PAM. Similarly, define *student sorting with negative assortative matching*(NAM) as the sorting of students of two classes and assigning the lower-quality teacher to the higher-track class and the higher-quality teacher to the lower-track class.

The following theorem states the parallel result of Theorem 6 when teachers are heterogeneous. We drive the result parallel to Proposition 2 — the effect of sorting on the total teachers' effort — in the appendix.

Theorem 8 *Given Assumptions 1 and 2:*

1. *If Condition 3 holds and performance is supermodular and convex, sorting of two classes increases the total performance.*

2. *If Condition 4 holds and performance is submodular and concave, sorting of two classes decreases the total performance.*

Proof: In the appendix.

As we stated in Corollary 6, under a pay-per-performance system — such as the value-added system in the U.S. — sorting changes the inequality in teachers' payment and the total payment to teachers.

Corollary 9 *Suppose Assumptions 1 and 2 are satisfied.*

1. *If Condition 3 holds, performance is supermodular and convex, and payment to teachers is an increasing and convex function of class performance, then sorting strictly increases the total payment to teachers.*

2. *If Condition 4 holds, performance is submodular and concave, and payment to teachers is an increasing and concave function of class performance, then sorting strictly decreases the total payment to teachers.*

3. *If payment to teachers is an increasing function of class performance, then sorting strictly increases inequality in teachers' payments.*

Changing the matching from PAM to NAM when students are sorted — or classes are ordered by the binary relationship $\succ^*$ defined in (3.1) — has the opposite effect of sorting. The following theorem shows one implication of such a change in the matching of classes and teachers. Note that the standard assumption on supermodularity or submodularity of the performance function is insufficient for analyzing the advantage or disadvantage of PAM versus NAM.

Theorem 9 *Suppose Assumptions 1 and 2 and Condition 4 are satisfied and performance is submodular and concave:*

1. *The total performance under student sorting with NAM is higher than student sorting with PAM.*
2. *Inequality in students' performance under student sorting with NAM is lower than student sorting with PAM.*

Proof: In the appendix.

COROLLARY 10 *Suppose Assumptions 1 and 2 and Condition 4 are satisfied and performance is submodular and concave:*

1. *The total payment to teachers under student sorting with NAM is higher than student sorting with PAM.*
2. *Inequality in teachers' payment under student sorting with NAM is lower than student sorting with PAM.*

Remark 1 ***Affirmative Action****:*

Let I be a set of students and each student belongs to either the minority group or the majority group. These students are assigned to two classes θ^1, θ^2, such that $\theta^1 \succ^* \theta^2$. Suppose the percentage of minority students in class θ^1 in less than ϕ and the percentage of minority students in class θ^2 is more than ϕ.

Affirmative action policies such as implementing a quota, i.e., assigning at least ϕ percent of seats to minority students in each class, have the opposite effect of sorting. Denote the two classes after implementing quotas by θ_a^1,θ_a^2. The following relation between these classes hold: $\theta^1 \succ^* \theta_a^1$, $\theta^2 \succ^* \theta_a^1$, $\theta_a^2 \succ^* \theta^2$, and $\theta_a^2 \succ^* \theta^1$. Affirmative action policies have the opposite effect of sorting on the total/average effort of teachers and the total/average performance of students in Proposition 2 and Theorems 6, 7, and 8.

3.5. Conclusion

We model an education system in which teachers choose their effort level based on the whole distribution of students, not only the mean of students' abilities. Furthermore, in our model both direct and indirect peer effects exist. The model incorporates both between-school sorting and within-school sorting, i.e., tracking.

We show that the standard assumptions in the literature are insufficient to understand the effect of sorting on the total effort of teachers and the total performance of students. We show that the change in the total performance of students after sorting depends on teachers' utility function. Even in the absence of direct peer effects, when teachers are homogeneous, sorting has an effect on the total effort of teachers and the total performance of students that depends on the curvature of teachers' marginal utility of effort.

We characterize conditions on the utility function of a teacher under which the optimal effort of a teacher is strictly supermodular. Under these conditions, sorting increases the total effort of teachers and the total performance of students, even though sorting increases inequality in students' performances. Therefore, under these conditions, how one evaluates the effect of sorting on students' performances depends on the welfare criteria chosen, i.e., under the Utilitarian criterion, sorting increases welfare; however, under the Rawlsian criterion, sorting decreases welfare. Similarly, we characterize conditions on the utility function of a teacher under which the optimal effort of a teacher is strictly submodular. Under these conditions, under both welfare criteria, any assignment of students to teachers is strictly preferred to sorting.

CHAPTER 4 : Total Output as the Measure of Welfare in a Labor Market

4.1. Introduction

A classic question in economics is what are the welfare consequences of different allocations of heterogeneous workers to heterogeneous firms. The total surplus of all firms and all workers, welfare, is unobservable because the cost of providing effective labor to the workers is unobservable. Even though many economists have noted that the total output (GDP) is a flawed measure of economic welfare,[1] many researchers in the empirical literature, especially in the macro search literature, use GDP instead of the total surplus as a proxy for welfare. Hsieh et al. (2013) use aggregate output per worker to measure the macroeconomic consequences of the convergence in the occupational distribution between white men, women, and blacks, which captures the change in the allocation of talent to occupations.[2] Hagedorn et al. (2012) use output as the objective of the optimal assignment of workers to firms under different assumptions about the production function of a matched worker-firm pair. Lise et al. (2015a) develop an empirical search-matching model to estimate the potential gain from optimal regulation. They define optimal policy as the policy that maximizes total output and home production, i.e., GDP.

Within our model, the total output measures the welfare, the total surplus of workers and firms, only if the effective labor supplied by workers does not depend on the firm they are matched with. If labor is a complement to or a substitute for a firm's technology, then the marginal product of labor is different when a worker is matched with a more productive firm; therefore, a worker's labor choice is different when he/she matches with a more productive firm. The assumption of fixed hours of labor supply may seem reasonable in a market for unskilled labor; however, in a market for skilled labor, such as the market for CEOs,[3] it

[1] Jones and Klenow (2010) states "Leisure, inequality, mortality, morbidity, crime, and the natural environment are just some of the major factors affecting living standards within a country that are incorporated imperfectly, if at all, in GDP."

[2] Firms in our model are the same as occupations.

[3] See Scheuer and Werning (2015); in their model, a CEO works more at a larger firm because of the complementarity between the CEO's effective labor and the firm's size.

seems less plausible.

We show that when one considers the endogenous labor choice of workers, relaxing the assumption of fixed hours of labor supply,[4] the total output can be a misleading measure of welfare: change in the total output may underestimate or overestimate an increase in the total surplus; moreover, the total output may decrease when the total surplus increases. We characterize two sets of sufficient conditions on the production function and cost function for which the total output and welfare move in the same direction. Under the first set of conditions, if the total surplus increases, the total output increases more than the total surplus. Stated differently, the total output overestimates the efficiency gains. Under the second set of conditions, if the total surplus increases, the total output increases but the increase is less than the increase in the total surplus. In other words, the total output underestimates the efficiency gains. We identify a third set of conditions under which the total output decreases if the total surplus increases; i.e., if one uses the total output to measure the welfare effect of a policy, then the conclusion is wrong. To our knowledge, this is the first study to model this problem formally, we specify when the total output is a misleading measure of efficiency and in which direction. Put differently, we specify whether the bias is upward or downward and when the total output changes in the opposite direction of welfare.

We develop a model in which firms post personalized wage schedules before matching with workers, and after the matching stage, workers choose an amount of labor. An equilibrium outcome in this setting is a collection of wages, the induced matching, and the induced effort levels which we formally define in section 4.2.1. Our model is closely related to Bulow and Levin (2006) and Jungbauer (2016) with an endogenous effort choice after the matching. There is heterogeneity in both sides of the market. We assume that the marginal product of labor depends on both the firm's technology and the worker's ability. The worker's ability and the firm's technology are complements; however, labor can be a complement to or a

[4]Lise et al. (2015b) report that they "find mixed evidence in support of the assumption of fixed hours of labor supply."

substitute for the ability of the worker or technology of the firm. We assume that labor is observable and contractible. We show that, in both cases, positive assortative matching (PAM) is the most efficient matching and is the only matching in any contracting equilibrium. However, PAM may minimize or maximize the total output. Furthermore, PAM, compared with other matchings, may have higher or lower total labor and total leisure.[5] We conclude that the total output does not always measure the efficiency of the labor market when the amount of labor depends on the matching. More concretely, when efficiency increases, the total output may decrease. We characterize a set of sufficient conditions on the production function and cost function under which the total output increases if efficiency increases.

The model is consistent with some empirical finding that may seem in contradiction with the assumption that there is a complementarity between the worker's ability and the firm's technology. For example, our results are consistent with better workers working fewer hours compared with other workers. More concretely, if the production function is submodular in labor and worker's ability, submodular in labor and firm's technology, and supermodular in worker's ability and firm's technology, better workers match with better firms and they work less. In other words, in any equilibrium, the most able worker is matched with the firm with the highest technology and the most able worker works less than all other workers. In contrast, the least able worker works more than all other workers. Hagedorn et al. (2012)'s findings is another example. They use a large German matched employer-employee data set to estimate the production function. They solve for the optimal assignment of workers and firms to maximize output in their data set. Subsequently, they show that this output is higher than the output under PAM.[6] Hence, they state that the production function is not globally supermodular. We show that, even if the production function is globally supermodular in workers' type and firms' type, the output maximizing assignment can be

[5] Labor and leisure move in opposite directions. A policy that increases labor, decreases leisure.

[6] "Solving the optimal output maximizing assignment problem we find that optimally assigning individual workers to individual firms increases output only by 4.47%. In contrast, reassigning workers to the main diagonal, as would be optimal given the typical assumption of a globally supermodular production function would imply a 1.43% decline in output."

different from the efficient assignment of workers to firms.

In Section 4.2, we formally describe the model and define the appropriate notion of equilibrium for this model. In Section 4.3, we define a benchmark contract between a given worker and a firm in this setting and the induced effort choice under this contract; then, we show that the social planner chooses this effort level for any matched worker-firm pair. Moreover, the social planner chooses positive assortative matching as the efficient matching. In Section 4.4, we show that an equilibrium exists, and the equilibrium outcome is unique and efficient. In Section 4.5, we provide an example in which the production function is strictly supermodular in firm's technology and worker's ability. However, the total output is the same under any matching. In this example, the total surplus is strictly higher under PAM than under any other matching. The main results are presented in Section 4.5, where we show that the efficiency always increases when all frictions in a labor market are eliminated; however, the increase or decrease in the total output depends on the sign and magnitude of the third derivative of the production function.

4.2. Model

There is a finite set of workers I and a finite set of firms J, where $|I| = |J|$. Worker $i \in I$ has an ability, the worker's type; which with a slight abuse of notation, we denote worker i's ability by i. If $i < i'$, then worker i has a lower ability than worker i'. Firm $j \in J$ has a technology, the firm's type, which we denote by j. If $j < j'$, then firm j has a lower level of technology than firm j'. The types of all firms and workers are common knowledge. A matching is a one-to-one mapping $\mu : I \to J$ that assigns each worker to a firm. Worker i chooses an effort $e \in [0, 1]$ after assignment to a firm. We use the words effort and labor interchangeably. Effort is observable and contactable. Worker i's utility from being matched to firm j and exerting effort e is $u(i, j, e) = w(i, j, e) - c(e)$, where $w(i, j, e)$ is the wage worker i gets if he/she matches with firm j and exerts effort e and $c(e)$ is the cost of exerting effort e. The reservation wage is zero for all workers. Firm j's production (output) when matched with worker i who exerts effort e, is $v(i, j, e)$; hence, firm j's profit is $\pi(i,j,e) = v(i,j,e) - w(i,j,e)$.

The market unfolds in four stages:

(i) Firms announce personalized wage schedules: $\{w(i,j,e)\}_{e\in[0,1],i\in I,j\in J}$.

(ii) Firms and workers match in the matching stage.

(iii) Each worker chooses a utility maximizing effort level given the matching and wage schedule.

(iv) Each worker receives a wage based on the wage schedule and effort choice.

We model the matching stage as a cooperative game. For any worker-firm pair, given the wage schedule and the anticipated worker's effort choice, the worker knows the utility that he/she gets from the match. Moreover, given the wage schedule and the anticipated worker's effort choice, the firm knows its profit in the match. We assume that, if a worker is indifferent between two firms, he/she will rank firms based on the firm technology; i.e., ties are broken in favor of the most efficient match.

We derive our results under a set of standard assumptions, as we want to show that, even under these assumptions, a change in the total output does not measure the change in the total surplus.

Assumption 3

1. *$v(i,j,e)$ and $c(e)$ are three times continuously differentiable.*

2. *$c(e)$ is strictly increasing and convex.*

3. *$v(i,j,e)$ is strictly increasing, $v(i,j,e)$ is concave in effort, and $\frac{\partial^2(v(i,j,e)-c(e))}{\partial e^2} < 0$.*

4. *$\forall i,j : v(i,j,0) = c(0) = 0, \exists \epsilon > 0 : v(i,j,\epsilon) - c(\epsilon) > 0$, and $v(i,j,1) - c(1) < 0$.*

5. *$v(i,j,e)$ has strictly increasing differences in worker's type i and firm's type j; i.e.,*

$v_{ij}(i, j, e) > 0.$

The first four parts are mainly technical assumptions which ensure that the problem of maximizing the surplus of a worker-firm pair over different labor choices has an interior solution. The last part states that there is a complementarity between a firm's technology and a worker's ability in the production. All five parts of Assumption 3 are standard assumptions.

Assumption 4

1. *$v(i, j, e)$ has increasing differences in worker's type i and effort; i.e., $v_{ie}(i, j, e) \geq 0$.*

2. *$v(i, j, e)$ has increasing differences in firm's type j and effort; i.e., $v_{je}(i, j, e) \geq 0$.*

Assumption 5

1. *$v(i, j, e)$ has decreasing differences in worker's type i and effort; i.e., $v_{ie}(i, j, e) \leq 0$.*

2. *$v(i, j, e)$ has decreasing differences in firm's type j and effort; i.e., $v_{je}(i, j, e) \leq 0$.*

Under Assumption 4, there is a complementarity between labor and the firm's technology and between labor and the worker's ability. In contrast, under Assumption 5, labor is a substitute for the firm's technology and for the worker's ability. Most of our results hold under either of these assumptions.

4.2.1. Contracting Equilibrium

In this section, we define a contracting equilibrium and the outcome of an equilibrium. The definition of equilibrium combines two notions of equilibria, cooperative and noncooperative, with the requirement that they hold for all histories. This equilibrium definition is not restricted to a frictionless market.[7]

Following Cole et al. (2001), we define a **contracting equilibrium** as wage schedules for

[7]One can model a market with frictions as a set of restrictions on the set of feasible worker-firm pairs. Infeasible pairs cannot be part of an equilibrium or block an equilibrium.

all firms and workers (on and off the equilibrium path) $\{\hat{w}(i,j,e)\}_{e\in[0,1],i\in I,j\in J}$, workers' effort choice $\{\hat{e}(i,j)\}_{i\in I,j\in J}$, and the matching of firm and workers such that:

1. Firms maximize their profits given other firms' and workers' strategy.

2. Given the matching and wage schedule, each worker chooses an effort that maximizes his/her utility on and off the equilibrium path; i.e., $\hat{e}(i,j) \in \text{argmax}_{e\in[0,1]} u(i,j,\hat{w}(i,j,e))$.

3. The matching is stable. More concretely, (i) there is no worker, firm, wage schedule between them, and an effort choice such that both the worker and the firm strictly prefer it to their current match (no blocking pair). (ii) Each worker gets at least zero utility, and each firm makes a positive profit (individual rationality).

Note that the no blocking pair condition is an extension of the standard no blocking pair condition (see Roth and Sotomayor (1992)). This extension of no blocking pair condition embodies the no regret condition into the standard no blocking pair condition. We assume that, if a worker is indifferent between two or more effort levels, he/she will choose the highest effort level.

This equilibrium notion combines a noncooperative notion (Nash) and a cooperative notion (stability), with the requirement that the stability holds after all histories and workers choose the optimal effort level after all histories. Observe that each firm is best replying to the other firms' strategy and that the future consequences of any strategy is correctly foreseen.

The outcome of a contracting equilibrium consists of:

(1) a matching $\hat{\mu}$,

(2) an effort $\hat{e}(i) = \hat{e}(i, \hat{\mu}(i))$ for worker i,

(3) a wage $\hat{w}(i) = w(i, \hat{\mu}(i), \hat{e}(i))$ for worker i.

Therefore, given the outcome of a contracting equilibrium, we have:

(1) a utility $\hat{u}(i) = \hat{w}(i) - c(\hat{e}(i))$ for worker i,

(2) an output $\hat{v}(j) = v(\hat{\mu}^{-1}(j), j, \hat{e}(\hat{\mu}^{-1}(j)))$ for firm j,

(3) a profit $\hat{\pi}(j) = \hat{v}(j) - \hat{w}(\hat{\mu}^{-1}(j))$ for firm j,

(4) A surplus $\hat{s}(i, j) = \hat{v}(j) - c(\hat{e}(i)) = \hat{\pi}(j) + \hat{u}(i)$ for a matched pair $(j = \hat{\mu}(i))$.

4.3. Efficiency

In this section, we characterize the efficient matching and the efficient labor choice for any matched pair by solving the social planner's problem. We specify a benchmark contract for a given worker-firm pair; the benchmark contract is the optimal contract from firm's point of view when the firm makes a take-it-or-leave-it offer to a single worker. Subsequently, we show that the social planner selects this contract for any given worker-firm pair (Section 4.3.2). In Section 4.3.3, we characterize the efficient matching. Observe that, given the matching and effort level induced by the wage contract, we can find the total surplus in the economy.

4.3.1. Benchmark Contract

We consider a benchmark contract between a pair of worker-firm (i, j) that are matched together, where the worker has the outside option of $r(i, j)$ from leaving the match. This is a contract between one firm and one worker without any regard for the labor market. Firm j's problem is as follows:

$$\begin{aligned} \max_{e, w(i,j,e)} \quad & v(i, j, e) - w(i, j, e) \\ \text{s.t.} \quad & w(i, j, e) - c(e) \geq r(i, j), \end{aligned}$$

where $r(i, j)$ is an outside option of the worker that depends only on firm and worker type. Note that, in this contract, the incentive compatibility constraint is irrelevance, since the effort is observable. However, the individual rationality constraint is binding:

$$w(i, j, e) = c(e) + r(i, j)$$

Hence, firm j's problem becomes the following:

$$\max_e \pi(i, j, e) = v(i, j, e) - c(e) - r(i, j) \tag{4.1}$$

The solution to this problem is the induced effort choice in the benchmark contract. Note that, under Assumption 3, this effort is unique and maximizes the firm's profit.

4.3.2. Efficient Effort

In this section, we show that the induced effort choice in the benchmark contract is the social planner's effort choice for any given matched worker-firm pair. The social planner's problem is:

$$\max_e s(i, j, e) = v(i, j, e) - c(e) \tag{4.2}$$

Lemma 8 *The social planner's effort choice, for any matched pair, is the effort that the firm chooses in the benchmark contract.*

Proof: Given Assumption 3, the solution to the benchmark contract, firm's profit maximization problem, is unique. Moreover, the solution to the social planner's surplus maximization problem is unique, too. The surplus of a match is equal to the profit of the firm up to a constant in the benchmark contract. Because in the firm j's profit maximization problem, problem 4.1, we have: $s(i, j, e) = \pi(i, j, e) + r(i, j)$. Hence:

$$e^*(i, j) = \text{argmax}_e s(i, j, e) = \text{argmax}_e \pi(i, j, e) \forall i, j \tag{4.3}$$

The social planner's effort choice is the same as the firm's effort choice in the benchmark contract. ■

We call this effort **the efficient effort** level for (i, j) pair. We denote the surplus at the efficient effort by $s^*(i, j) = \max_e s(i, j, e) = \max_e \pi(i, j, e) + r(i, j)$.

4.3.3. Efficient Matching

Consider the problem of a social planner who wants to maximize the total surplus in the economy. The social planner can choose the matching and the effort level of each matched pair. Recall that the social planner selects the efficient effort level for any matched pair. The following lemma shows that the total surplus at the efficient effort level is strictly supermodular; therefore, positive assortative matching maximizes the total surplus and the social planner selects positive assortative matching.

Lemma 9 *Under Assumption 3:*

1. *Given Assumption 4 or 5, the total surplus at the efficient effort level is strictly supermodular.*

2. *Given Assumption 4, the efficient effort $e^*(i, j)$ is increasing in the worker's type and the firm's type.*

3. *Given Assumption 5, the efficient effort $e^*(i, j)$ is decreasing in the worker's type and the firm's type.*

Proof:

1. By the proof of Lemma 8, the efficient effort level maximizes the surplus of a match. Therefore, by first order condition, we have:

$$s_e(i, j, e^*(i, j)) = 0. \tag{4.4}$$

We want to show that, the total surplus at the efficient effort level is supermodular under either Assumption 4 or 5. By taking cross partial derivative of the surplus, at the efficient effort level, we have:

$$s^*_{ij} = s_{ij} + s_{ie}e^*_j + s_{je}e^*_i + s_{ee}e^*_i e^*_j + s_e e^*_{ij}.$$

By equation (4.4), s_e is zero at the efficient level of effort:

$$\begin{aligned} & s^*_{ij} = s_{ij} + s_{ie}e^*_j + s_{je}e^*_i + s_{ee}e^*_i e^*_j \\ \Rightarrow \quad & s^*_{ij} = s_{ij} + s_{ie}(-\tfrac{s_{ej}}{s_{ee}}) + s_{je}(-\tfrac{s_{ei}}{s_{ee}}) + s_{ee}(-\tfrac{s_{ei}}{s_{ee}})(-\tfrac{s_{ei}}{s_{ee}}) \\ \Rightarrow \quad & s^*_{ij} = s_{ij} - \tfrac{s_{ie}s_{ej}}{s_{ee}} - \tfrac{s_{ei}s_{je}}{s_{ee}} + \tfrac{s_{ei}s_{ej}}{s_{ee}} \end{aligned}$$

$$\Rightarrow \quad s^*_{ij} = s_{ij} - \tfrac{s_{ie}s_{ej}}{s_{ee}}. \tag{4.5}$$

Note that $s_{ij} = v_{ij}$; which is strictly positive by Assumption 3. Therefore, if $s_{ie}s_{ej} \geq 0$, then the total surplus at the efficient effort level is strictly supermodular. Under Assumption 4, we have that $s_{ie}s_{ej} \geq 0$, because $s_{ie} \geq 0$ and $s_{ej} \geq 0$. Under Assumption 5, we have that $s_{ie}s_{ej} \geq 0$, because $s_{ie} \leq 0$ and $s_{ej} \leq 0$. Hence, the surplus at the efficient effort level is strictly supermodular under either Assumption 4 or Assumption 5.

2. Follows from Topkis Theorem (Topkis (1998)).

3. Follows from Topkis Theorem (Topkis (1998)).

■

The main point in the proof of Lemma 9 is that the super-modularity or sub-modularity of the efficient effort does not affect the super-modularity and sub-modularity of the total surplus at the efficient effort level. Therefore, the curvature of the marginal product of effort or the marginal cost of effort does not affect the efficiency of PAM because the total surplus is evaluated at the surplus maximizing effort level. In section 4.5.1, we show that whether PAM has a higher or lower total output compared with other matchings depends on the curvature of the marginal product of effort and the marginal cost of effort. Similarly, whether PAM has a higher or lower total labor compared with other matchings depends on the curvature of the marginal product of effort and the marginal cost of effort.

Note that a worker may work less at a more productive firm. Moreover, a better worker may work less than a less able worker at a given firm. Nevertheless, the unique efficient matching is positive assortative matching. Stated differently, efficiency requires the matching of best worker to the best firm even though the best worker puts in the least amount of work compared with any other worker at this firm.

4.4. Equilibrium Existence and Uniqueness

In this section, we construct a contracting equilibrium. We characterize the set of all contracting equilibria. We show that the outcome of any contracting equilibrium involves the efficient effort level for any matched worker-firm pair and the efficient matching.

The following lemma shows that an equilibrium exists.[8]

Lemma 10 *Under Assumptions 3 and 4, the efficient effort for each worker-firm pair, positive assortative matching, and the subsequent wage schedule forms a contracting equilibrium:*

1. $\hat{w}(1,1,e^*(1,1)) = c(e^*(1,1))$,

2. $\hat{w}(j,j,e^*(j,j)) = b_j + c(e^*(j,j))$ *for* $j > 1$,[9]

3. $\hat{w}(j+1,j,e^*(j+1,j)) = v(j+1,j,e^*(j+1,j)) + b_j + c(e^*(j,j)) - v(j,j,e^*(j,j))$ *for* $j > 1$,

[8] We assume that $I = J = \{1, 2, 3, \ldots, n\}$ for clarity in this lemma.

[9] b_j is large enough to make worker j indifferent between working at firm $j-1$ and firm j.

4. $\hat{w}(i,j,e) = 0$ *for any other match or effort level.*

Proof: In the appendix.

In this equilibrium, firm j offers worker j a rent for working at firm j plus compensation for the cost of exerting the efficient amount of labor. Moreover, firm j offers worker $j+1$ a wage such that the firm's profit when hiring worker $j+1$ is equal to its profit when the firm hires worker j, given that both workers will exert the efficient effort. In other words, firm j competes with firm $j+1$ for worker $j+1$. Therefore, the rent that firm $j+1$ pays the worker $j+1$ is pinned down by firm j's wage offer to worker $j+1$.

The following lemma shows that, in any contracting equilibrium, the effort on the equilibrium path for any matched worker-firm pair is unique and efficient.

Lemma 11 *Under Assumptions 3 and 4 or Assumptions 3 and 5, in any contracting equilibrium, on the equilibrium path, workers choose the efficient effort level* $\hat{e}(i) = e^*(i, \hat{\mu}(i))$.

Proof: In the appendix.

Given that the effort choice on the equilibrium path is efficient, we show that, in any contracting equilibrium, the matching is unique and efficient.

Theorem 10 *Under Assumptions 3 and 4 or 3 and 5, in any contracting equilibrium, the matching is positive assortative.*

Proof: In the appendix.

This result holds under either Assumption 4 or Assumption 5. Recall that, under Assumption 5, the best worker works less than any other worker at the best firm. Moreover, given the efficient matching, the best worker works less than all other workers. Similarly, the least able worker has the highest working hours in the economy.

Corollary 11 *Under Assumptions 3 and 5, in any contracting equilibrium, the best worker is matched with the best firm and works less than any other worker. In contrast, the least* *able worker works more than any other worker.*

To summarize, a contracting equilibrium exists, and matching and the effort levels on the equilibrium path are unique, so the total output and the total surplus in any contracting equilibrium are the same.

4.5. Eliminating all Frictions

In Section 4.5.1, we analyze the effect of eliminating all frictions in a labor market on the total surplus and total output. If there is no friction in the marker, the unique matching in any equilibrium is PAM. We consider any departure from PAM as a friction in the labor market. Stated differently, if the observed matching in a labor market is not PAM, then there are some frictions in the labor market. We show that, under Assumption 3 and either Assumption 4 or Assumption 5, the frictionless market has a higher total surplus than a labor market with some frictions but it may have lower total output. First, we provide an example where the production function is strictly supermodular in firm's technology and worker's ability. At the efficient level of effort, all firms produce the same amount of output irrespective of their matched workers. Therefore, the total output is the same under any matching, but the total surplus is strictly higher under PAM than under any other matching. We characterize three sets of sufficient conditions and provide simple examples for each set of conditions. Under the first set of conditions, the total output increases when all frictions are eliminated; however, this increase is an upper bound on the increase in the total surplus. Under the second set of conditions, the total output increases, but the increase in the total output is a lower bound for the increase in the total surplus. However, under the third set of conditions, the total output decreases when all search frictions are eliminated. In Section 4.5.2, we show that, fixing an equilibrium, the total profit of all firms always increases when all search frictions are eliminated.

4.5.1. The Total Output

Suppose that the labor market is not efficient because of a friction in labor market. Consider a policy that decreased the total output. Can we conclude that efficiency has decreased as the result of this policy? In contrast, if a policy increased the total output, can we say that there was a efficiency gain? In this section, we answer these questions.

Given two pairs of matched worker-firm pairs, $(i,j),(i',j')$, we define meet of the two vectors as the coordinate-wise minimum of the two vectors $(i,j)\wedge(i',j')=(\min\{i,i'\},\min\{j,j'\})$ and joint of the two vectors as the coordinate-wise maximum of the two vectors $(i,j)\vee(i',j')=(\max\{i,i'\},\max\{j,j'\})$.

Lemma 12 *Given any matching μ, there exists a sequence of meet and joint operations with length n such that the resulting matching is PAM.*

Proof: In the appendix.

In the following example, there is a complementarity in the production function between worker's type and firm's technology. Hence, one may think that, by facilitating the match between the best worker and the best firm, production will increase (by allowing the matched pair to choose the efficient effort). However, considering the efficient labor choice, the production of a firm is constant. Moreover, the production of all firms is the same independent of which worker they are matched to. Stated differently, by changing the matching, the total surplus changes but the total output is constant. Therefore, if a researcher uses the output to measure the efficiency gain (or loss) from a policy regrading a change in labor market frictions, the conclusion will be wrong.

Example 2

$$v(i,j,e)=log(ij+e),c(e)=\frac{e}{k},$$

where $k\in[1.69,2]$ and $I,J\subset[1,1.3]$. Note that $s_{ij}>0,s_{ei}<0,s_{ej}<0$.

The efficient effort is $e^*(i,j) = k - ij$*; hence, the efficient effort is submodular. The total output at the efficient effort is* $v(i,j,e^*(i,j)) = log(k)$*. The total surplus at the efficient effort is* $s(i,j,e^*(i,j)) = log(k) - k + ij$*. The total output is constant for any matching. Therefore, PAM maximizes the total surplus and the total leisure. PAM minimizes the total labor. However, changing the matching does not affect the total output.*

It is not always the case that the total output is constant at the efficient effort level. Consider the following example:

Example 3

$$v(i,j,e) = ije, c(e) = ke^2, s(i,j,e) = ije - ke^2,$$

where $k > 0$ *is large enough to satisfy Assumption 3; i.e.,* $\forall i \in I, j \in J : k > ij$*.*

The efficient effort is $e^*(i,j) = \frac{ij}{2k}$*; hence, the efficient effort is supermodular. The total output at the efficient effort is* $v(i,j,e^*(i,j)) = \frac{(ij)^2}{2k}$*. The total surplus at the efficient effort is* $s(i,j,e^*(i,j)) = \frac{(ij)^2}{2k} - \frac{(ij)^2}{4k} = \frac{(ij)^2}{4k}$*. Therefore, PAM maximizes the total surplus, the total labor, and the total output, but it minimizes the total leisure.*

We characterize a set of sufficient conditions under which positive assortative matching maximizes the total surplus and total output. Under these conditions, if the total output increases as a result of a change in labor market, then we know that the total surplus has increased as well. However, the increase in the total output is an upper bound on the increase in the total surplus under Assumption 4. We use the result from Kamali-Shahdadi (2016) to analyze the argmax of the firm's profit maximization problem (the efficient labor choice). If the marginal profit of labor is supermodular and convex in labor, then the the efficient labor choice is supermodular,[10] and PAM maximizes the total labor, total surplus, and total output.

[10]See Kamali-Shahdadi (2016) for formal proof.

Condition 5 *The marginal surplus of effort is supermodular and convex in effort:*

$$s_{eij}(e,i,j) \geq 0, s_{eei}(e,i,j) \geq 0, s_{eej}(e,i,j) \geq 0, s_{eee}(e,i,j) \geq 0, \forall e \in [0,1], i, j,$$

with at least one strict inequality.

Theorem 11

Under Condition 5:

1. *Given Assumptions 3 and 4 or Assumptions 3 and 5, the frictionless labor market, compared with a labor market with some frictions, has a higher total surplus, higher total output, and higher total labor.*

2. *Under Assumptions 3 and 4, eliminating all frictions in the labor market results in a higher total output and efficiency gain; however, the increase in the total output is greater than the increase in the total surplus.*

Proof: In the appendix.

Under the second set of conditions that we characterize, if the total output increases as a result of a policy change related to labor market frictions, then we can conclude that the total surplus increased as a result of this policy; moreover, the increase in the total surplus is at least equal to the increase in the total output. Consider the following example:

Example 4

$$v(i,j,e) = e(i^{\frac{1}{\rho}} + j^{\frac{1}{\rho}})^{\rho}, c(e) = \frac{e^{k+1}}{k+1}, \rho > 1, k > 1, \frac{\rho}{k} < 1,$$

where $\forall i \in I, j \in J : 0 < (i^{\frac{1}{\rho}} + j^{\frac{1}{\rho}})^{\rho} < \frac{1}{k+1}$.

The efficient effort is $e(i,j) = (i^{\frac{1}{\rho}} + j^{\frac{1}{\rho}})^{\frac{\rho}{k}}$. $\frac{\rho}{k} < 1$*; hence, the efficient effort is submodular. The total output at the efficient effort is* $v(i,j,e^*(i,j)) = (i^{\frac{1}{\rho}} + j^{\frac{1}{\rho}})^{\frac{\rho(1+k)}{k}}$. *The total surplus* *at the efficient effort is* $s(i,j,e^*(i,j)) = (\frac{k}{k+1})(i^{\frac{1}{\rho}} + j^{\frac{1}{\rho}})^{\frac{\rho(1+k)}{k}}$. *Therefore, PAM maximizes the total surplus, total leisure, and total output, but it minimizes the total labor.*

Condition 6 *The marginal surplus of effort is submodular and concave in effort:*

$$s_{eij}(e,i,j) \leq 0, s_{eei}(e,i,j) \leq 0, s_{eej}(e,i,j) \leq 0, s_{eee}(e,i,j) \leq 0, \forall e \in [0,1], i, j,$$

with at least one strict inequality. Moreover,

$$(s_{ei}s_{ej})(v_e s_{eee} + 2s_{ee}^2 - v_{ee}s_{ee}) + (s_{ee}^2)(v_e s_{eij} - s_{ij}s_{ee}) - (v_e s_{ee})(s_{eei}s_{ej} + s_{eej}s_{ei}) \geq 0.$$

Theorem 12

Under Condition 6:

1. *Given Assumptions 3 and 4 or Assumptions 3 and 5, the frictionless market, compared with a market with some frictions, has a higher total surplus, higher total output, and lower total labor.*

2. *Under Assumptions 3 and 5, eliminating frictions in the labor market results in higher total output and efficiency gain; moreover, the increase in the total surplus is greater than the increase in the total output.*

Proof: In the appendix.

It is not always the case that an increase in the total output implies an increase in the total surplus. Under the third set of conditions that we characterize, if the total output decreases as a result of policy in the labor market, the total surplus increases as a result of this policy.

Example 5

$$v(i,j,e) = log(ij + e + 1) - ((\frac{1}{3})(log(ij + 3))), c(e) = \frac{e}{k},$$

where $k \in (2.69, 3)$ *and* $I, J \in [1, 1.3]$*. Note that* $s_{ij} = \frac{1+e}{(ij+e+1)^2} - \frac{1}{(ij+3)^2} > 0, s_{ei} < 0, s_{ej} < 0$.

The efficient effort is $e(i,j) = k - ij - 1$*; hence, the efficient effort is submodular. The total output at the efficient effort is* $v(i,j,e^*(i,j)) = log(k) - ((\frac{1}{3})(log(ij+3)))$*; hence, the total output at the efficient effort is submodular. The total surplus at the efficient effort is* $s(i,j,e^*(i,j)) = log(k) - ((\frac{1}{3})(log(ij+3))) + \frac{ij}{k} - 1 + \frac{1}{k}$*. Therefore, PAM maximizes the total surplus and total leisure. However, PAM minimizes the total output and total labor.*

Condition 7 *The marginal surplus of effort is submodular, concave in effort, and:*

$$s_{eij}(e,i,j) \leq \left(\frac{s_{ee}}{v_e}\right)(s_{ij}), s_{eei}(e,i,j) \leq 0, s_{eej}(e,i,j) \leq 0,$$
$$s_{eee}(e,i,j) \leq \left(\frac{s_{ee}}{v_e}\right)(c_{ee} - s_{ee}), \forall e \in [0,1], i, j,$$

with at least one strict inequality.

Theorem 13

Under Condition 7:

1. *Given Assumptions 3 and 4 or Assumptions 3 and 5, the frictionless market has the highest total surplus, lowest total output, and lowest total labor.*

2. *Given Assumptions 3 and 4 or Assumptions 3 and 5, eliminating frictions in the labor market results in a lower total output and higher total surplus.*

Proof: In the appendix.

4.5.2. The Total Profit

In this section, we show that, for a given equilibrium wage schedule, if a policy eliminates frictions in the labor market, then the total surplus and the total profit move in the same direction. [11]

[11] However, if firms react to changes in labor market frictions by changing the wage schedule that they offer to workers, the total profit may decrease while the total surplus increases.

Theorem 14 *Given a contracting equilibrium,*

1. *if the total surplus is supermodular at the optimum level of effort, then the total profit of firms is supermodular, and*

2. *if the total surplus is submodular at the optimum level of effort, then the total profit of firms is submodular.*

Proof: In the appendix.

4.6. Conclusion

If the effective labor supplied by workers does not depend on the firm they are matched with, then the total output measures the welfare. However, if labor is a complement to or a substitute for a firm's technology, then a worker's labor choice changes when he/she matches with a more productive firm. We show that PAM is efficient and incurs in any contracting equilibrium. However, PAM may minimize or maximize the total output. Therefore, if one considers the endogenous labor choice of workers, then the total output can be a misleading measure of welfare: change in the total output may underestimate or overestimate an increase in the total surplus; moreover, the total output may decrease when the total surplus increases. We characterize two sets of sufficient conditions on the production function and cost function for which the total output and welfare move in the same direction. Under the first set of conditions, if the total surplus increases, the total output increases more than the total surplus. Stated differently, the total output overestimates the efficiency gains. Under the second set of conditions, if the total surplus increases, the total output increases but the increase is less than the increase in the total surplus. In other words, the total output underestimates the efficiency gains. We identify a third set of conditions under which the total output decreases while the total surplus increases; i.e., the total output misrepresents the welfare effect of a policy.

APPENDIX

Proof of Lemma 1. We find an optimal wage contract by finding binding constraints. Consider an optimal solution to the minimization problem. The minimization problem is a standard convex problem. The existence of an optimal solution is guaranteed (see Grossman and Hart (1983)). Denote the optimal solution by (u_1^*, u_2^*). We characterize the necessary conditions for (u_1^*, u_2^*) to be an optimal solution by finding which constraint is binding.

Given (u_1^*, u_2^*), find an attorney i such that:

$$r(q_i) + E_\lambda^i(c(e_2, d_j)) = R_\lambda.$$

At least one attorney with this property exists. Denote an attorney with this property by i^*.

If the (IR) constraint for i^* is satisfied, then all individual rationality constraints are satisfied. Because the left-hand side of (IR) is the same for all $i \in I$, the right-hand side is maximized for attorney i^*.

We claim that (IR) for i^* binds. Suppose (IR) does not bind for i^*. Then we can reduce u_1^* and u_2^* uniformly to $u_1^* - \epsilon, u_2^* - \epsilon$ such that (IR) is still satisfied for all $i \in I$. Note that this process does not affect (IC) constraints. Therefore, we can reduce the objective function, i.e., we can reduce the cost of the government, a contradiction with optimality of (u_1^*, u_2^*).

If (IC) for the highest d is satisfied, then all incentive constraints are satisfied. Because the left-hand side of the (IC) constraint is the same for all $d \in D$, the right-hand side is maximized at d_N, the largest element in D.

Finally, we claim that (IC) for the highest d binds. However, suppose (IC) for the highest

d does not bind. Consider the following relaxed problem:

$$\min_{u_1,u_2} \sum_{i\in I} p_h h(u_2) + (1-p_h)h(u_1)$$

$$s.t.$$

$$u_1 + p_h(u_2 - u_1) \geq R_\lambda.$$

After simplifying the constraint, we get:

$$\min_{u_1,u_2} \sum_{i\in I} p_h h(u_2) + (1-p_h)h(u_1)$$

$$s.t.$$

$$(1-p_h)u_1 + p_h u_2 \geq R_\lambda.$$

Suppose $(\tilde{u_1}, \tilde{u_2})$ is a solution to this relaxed problem, define $u_1^\dagger = \tilde{u_1} - \frac{\epsilon}{1-p_h}$ and $u_2^\dagger = \tilde{u_2} + \frac{\epsilon}{p_h}$. Because the constraint is satisfied at $(\tilde{u_1}, \tilde{u_2})$, it is also satisfied at $(u_1^\dagger, u_2^\dagger)$. $(\tilde{u_1}, \tilde{u_2})$ is an optimal solution to this relaxed problem; therefore, the following problem must be optimized at $\epsilon = 0$:

$$\min_\epsilon \sum_{i\in I} p_h h(\tilde{u_2} + \tfrac{\epsilon}{p_h}) + (1-p_h)h(\tilde{u_1} - \tfrac{\epsilon}{1-p_h})$$

$$s.t.$$

$$(1-p_h)\tilde{u_1} + p_h \tilde{u_2} \geq R_\lambda.$$

Taking first-order condition with respect to ϵ and evaluating it at $\epsilon = 0$, we get:

$$\sum_{i\in I} p_h(\tfrac{1}{p_h})h'(\tilde{u_2}) - \tfrac{1-p_h}{1-p_h}h'(\tilde{u_1}) = 0$$

$$\Rightarrow h'(\tilde{u_2}) = h'(\tilde{u_1})$$

$$\Rightarrow \tilde{u_1} = \tilde{u_2}.$$

Consider the (IC) in the original problem. At $\tilde{u_1} = \tilde{u_2}$ left-hand side of (IC) is zero. Under Assumption 1, high effort costs more than low effort; hence, the right-hand side of (IC) is strictly positive. Thus, at $(\tilde{u_1}, \tilde{u_2})$, (IC) is violated. Therefore, (IC) constraints bind in the original problem.

Using this binding constraint, we can find an optimal wage contract. There is only one wage contract that satisfies all these necessary conditions. The optimal wage contract is:

$$u_1^* = R_\lambda - p_h\big(\tfrac{c(e_h,d_N)-c(e_l,d_N)}{p_h-p_l}\big),$$
$$u_2^* = R_\lambda + (1-p_h)\big(\tfrac{c(e_h,d_N)-c(e_l,d_N)}{p_h-p_l}\big).$$

∎

Proof of Theorem 3. Given an allocation λ and a set of hired attorneys I, from Lemma 1 the optimal wage contract is:

$$u_1^* = R_\lambda - p_h\big(\tfrac{c(e_h,d_N)-c(e_l,d_N)}{p_h-p_l}\big),$$
$$u_2^* = R_\lambda + (1-p_h)\big(\tfrac{c(e_h,d_N)-c(e_l,d_N)}{p_h-p_l}\big).$$

Hence, the government's problem for finding the optimal allocation is:

$$\min_\lambda \textstyle\sum_{i\in I} p_h h(u_2^*) + (1-p_h)h(u_1^*),$$

or simply:

$$\min_\lambda p_h h(u_2^*) + (1-p_h)h(u_1^*).$$

The objective function depends on λ only through R_λ, and it is strictly increasing in R_λ.

Therefore, the government's problem is:

$$\min_\lambda R_\lambda.$$

Thus, the optimal allocation given the set I solves:

$$\min_{\lambda}\left[\max\left\{r(q_i)+\sum_{j\in J}\phi_\lambda(i,j)(c(e_h,d_j))\right\}_{i\in I}\right],$$

which is equivalent to the linear program (LP*). Given the solution to this program, $\{\phi^*(i,j)\}_{i\in I,j\in J}$, we can use the Birkhoff-von Neumann decomposition algorithm to find the optimal allocation λ. ■

Proof of Lemma 5. (If direction:) Suppose $r(q_i) = r$ for all $i \in I$, and the uniform random allocation is not optimal. Then there exists λ' such that:

$$\max\left\{r+E_{\lambda'}(c(e_h,d))\right\}_{i\in I} < \max\left\{r+E_{\lambda_u}(c(e_h,d))\right\}_{i\in I}.$$

$r+E_{\lambda_u}(c(e_h,d))$ is constant for all $i\in I$ and equal to $r+\frac{\sum_{j\in J}(c(e_h,d_j))}{N}$. Hence,

$$\max\left\{r+E_{\lambda'}(c(e_h,d))\right\}_{i\in I} < r+\frac{\sum_{j\in J}(c(e_h,d_j))}{N}.$$

Then under λ' we have:

$$\begin{aligned}
& r+\textstyle\sum_{j\in J}\phi_{\lambda'}(i,j)c(e_h,d_j) < r+\frac{\sum_{j\in J}(c(e_h,d_j))}{N}\forall i\in I \\
\Rightarrow\quad & \textstyle\sum_{i\in I}\left(r+\sum_{j\in J}\phi_{\lambda'}(i,j)c(e_h,d_j)\right) < \sum_{i\in I}\left(r+\frac{\sum_{j\in J}(c(e_h,d_j))}{N}\right) \\
\Rightarrow\quad & \textstyle\sum_{i\in I}\sum_{j\in J}\phi_{\lambda'}(i,j)c(e_h,d_j) < \sum_{i\in I}\frac{\sum_{j\in J}(c(e_h,d_j))}{N} \\
\Rightarrow\quad & \textstyle\sum_{j\in J}\sum_{i\in I}\phi_{\lambda'}(i,j)c(e_h,d_j) < \sum_{j\in J}\frac{\sum_{i\in I}(c(e_h,d_j))}{N} \\
\Rightarrow\quad & \textstyle\sum_{j\in J}c(e_h,d_j) < \sum_{j\in J}(c(e_h,d_j)),
\end{aligned}$$

a contradiction.

(Only if direction:) If $r(q_i)$ is not constant, there exist $i', i'' \in I$ such that $r(q_{i'}) < r(q_{i''})$. Hence, under the uniform random allocation:

$$r(q_{i'}) + E_{\lambda_u}(c(e_h, d)) < R_{\lambda_u},$$

and:

$$r(q_{i'}) + E_{\lambda_u}(c(e_h, d)) < r(q_{i''}) + E_{\lambda_u}(c(e_h, d)).$$

Fix an arbitrary small $\epsilon > 0$ and construct a new allocation $\bar{\lambda}$ such that:

$$\begin{aligned}
\phi_{\bar{\lambda}}(i', 1) &= \tfrac{1-\epsilon}{N}, \\
\phi_{\bar{\lambda}}(i', N) &= \tfrac{1+\epsilon}{N}, \\
\phi_{\bar{\lambda}}(i, 1) &= \tfrac{1}{N} + \tfrac{\epsilon}{(N-1)N} \quad \forall i \in I, i \neq i', \\
\phi_{\bar{\lambda}}(i, N) &= \tfrac{1}{N} - \tfrac{\epsilon}{(N-1)N} \quad \forall i \in I, i \neq i', \\
\phi_{\bar{\lambda}}(i, j) &= \tfrac{1}{N} \quad \forall i \in I, j \in J, j \neq 1, N.
\end{aligned}$$

Intuitively, the new allocation rule is constructed from the uniform random allocation with a few changes. i' gets the easiest case with lower probability under $\bar{\lambda}$. Everyone else gets the easiest case with higher probability under $\bar{\lambda}$. However, i' gets the hardest case with higher probability under $\bar{\lambda}$. Everyone else gets the hardest case with lower probability under $\bar{\lambda}$.

Note that $\bar{\lambda}$ is indeed an allocation because each row and column of Φ adds up to 1. For an arbitrary small $\epsilon > 0$ we have:

$$r(q_{i'}) + E_{\bar{\lambda}}(c(e_h, d)) < r(q_{i''}) + E_{\bar{\lambda}}(c(e_h, d)).$$

For any $i \neq i'$:

$$r(q_i) + E_{\bar{\lambda}}(c(e_h, d)) < r(q_i) + E_{\lambda_u}(c(e_h, d)),$$

one such i is $i = i''$:

$$r(q_i'') + E_{\bar{\lambda}}(c(e_h, d)) < r(q_i'') + E_{\lambda_u}(c(e_h, d)) \leq R_{\lambda_u}.$$

However, for i':

$$r(q_{i'}) + E_{\bar{\lambda}}(c(e_h, d)) > r(q_{i'}) + E_{\lambda_u}(c(e_h, d)).$$

Therefore:

$$r(q_{i'}) + E_{\lambda_u}(c(e_h, d)) < r(q_{i''}) + E_{\bar{\lambda}}(c(e_h, d)) \leq R_{\lambda_u}.$$

We can conclude that

$$R_{\bar{\lambda}} < R_{\lambda_u}.$$

This is a contradiction with optimality of the uniform random allocation. ■

Proof of Theorem 4. From Lemma 5 we know that the uniform random allocation is the optimal allocation. We need to show that a nondiscriminatory wage contract is optimal even if the government can use discriminatory wage contracts.

Consider a selection rule, an allocation rule Λ, and an optimal discriminatory wage contract $\{u_1^\dagger(i,j), u_2^\dagger(i,j)\}_{i \in I^a, j \in J}$. We need to show that the cost of government under this contract is higher than the cost of the government under the status quo contract. We restrict our attention to optimal discriminatory wage contracts only because if this claim is true for any contract that uses an optimal discriminatory wage contract then it is true for any other contract, too.

Denote the set of hired attorneys under this contract by I, the allocation by λ, and the optimal discriminatory wage contract by $\{u_1^\dagger(i,j), u_2^\dagger(i,j)\}_{i \in I, j \in J}$. After the contract is signed, attorneys and indigent defendants are matched based on λ. Note that each attorney is assigned to one indigent defendant, and denote this realized matching by μ. The cost of government if μ is realized is:

$$\mathcal{C}_\mu = \sum_{i \in I} (p_2 h(u_2^*(i, \mu^{-1}(i))) + (1 - p_2) h(u_1^*(i, \mu^{-1}(i))),$$

where:

$$u_1^\dagger(i, \mu^{-1}(i)) = r(q_i) + c(e_h, d_{\mu^{-1}(i)}) - p_h\left(\tfrac{c(e_h) - c(e_l)}{p_h - p_l}\right),$$
$$u_2^\dagger(i, \mu^{-1}(i)) = r(q_i) + c(e_h, d_{\mu^{-1}(i)}) + (1 - p_h)\left(\tfrac{c(e_h) - c(e_l)}{p_h - p_l}\right).$$

Finding an optimal discriminatory wage contract is simple. For each possible match $\mu(\tilde{j}) = \tilde{i}$, there are two binding constraints, an incentive compatibility constraint for $\tilde{i}$ when he/she is matched to indigent defendant $\tilde{j}$, and an individual rationality constraint for $\tilde{i}$ when he/she is matched to indigent defendant $\tilde{j}$. One can show these constraints bind at optimality. Hence, the optimal wage contract $u_1^\dagger(i, j), u_2^\dagger(i, j)$ is derived by solving each possible match under the allocation λ. Note that $c(e, d_j) = c(e) - g(d_j)$ implies $c(e_h, d_j) - c(e_l, d_j) = c(e_h) - c(e_l)$.

For any set of hired attorneys I, we know that $r(q_i) > r(q_1) = r \forall i \in I$. Therefore:

$$u_1^*(i, \mu^{-1}(i)) \geq r + c(e_h, d_{\mu^{-1}(i)}) - p_h\left(\tfrac{c(e_h) - c(e_l)}{p_h - p_l}\right),$$
$$u_2^*(i, \mu^{-1}(i)) \geq r + c(e_h, d_{\mu^{-1}(i)}) + (1 - p_h)\left(\tfrac{c(e_h) - c(e_l)}{p_h - p_l}\right).$$

Define

$$\xi = r - p_h\left(\tfrac{c(e_h) - c(e_l)}{p_h - p_l}\right),$$
$$\zeta = r + (1 - p_h)\left(\tfrac{c(e_h) - c(e_l)}{p_h - p_l}\right).$$

Hence,

$$u_1^*(i, \mu^{-1}(i)) \geq c(e_h, d_{\mu^{-1}(i)}) + \xi, \tag{A.1}$$

$$u_2^*(i, \mu^{-1}(i)) \geq c(e_h, d_{\mu^{-1}(i)}) + \zeta. \tag{A.2}$$

The cost of the government under the uniform random allocation with nondiscriminatory wage contract is

$$\mathcal{C}_u = \sum_{i \in I} (p_2 h(u_2^\dagger) + (1 - p_2) h(u_1^\dagger)),$$

where:

$$u_1^\dagger = r + E_d(c(e_h, d)) - p_h\big(\tfrac{c(e_h) - c(e_l)}{p_h - p_l}\big) = E_d(c(e_h, d)) + \xi,$$
$$u_2^\dagger = r + E_d(c(e_h, d)) + (1 - p_h)\big(\tfrac{c(e_h) - c(e_l)}{p_h - p_l}\big) = E_d(c(e_h, d)) + \zeta.$$

Suppose there exists $i' \in I$ such that $u_2^*(i', \mu^{-1}(i')) \neq \frac{\sum_{i \in I} u_2^*(i, \mu^{-1}(i))}{N}$, i.e., at least two attorneys are getting different wages under the high signal. Under Assumption 1, $h(.)$ is a strictly convex function. By Jensen's inequality, we have:

$$h\left(\frac{\sum_{i \in I} u_2^*(i, \mu^{-1}(i))}{N}\right) < \frac{\sum_{i \in I} h(u_2^*(i, \mu^{-1}(i)))}{N}. \tag{A.3}$$

From equation (A.2) we have:

$$\frac{\sum_{i \in I} u_2^*(i, \mu^{-1}(i))}{N} \geq \frac{\sum_{i \in I} (c(e_h, d_{\mu^{-1}(i)}) + \zeta)}{N} \tag{A.4}$$

$$= \zeta + \frac{\sum_{i \in I} c(e_h, d_{\mu^{-1}(i)})}{N} = \zeta + \frac{\sum_{j \in J} c(e_h, d_j)}{N} = \zeta + E_d(c(e_h, d)) = u_2^\dagger. \tag{A.5}$$

Under Assumption 1, $h(.)$ is a strictly increasing function; therefore:

$$h(u_2^\dagger) \leq h\left(\frac{\sum_{i \in I} u_2^*(i, \mu^{-1}(i))}{N}\right). \tag{A.6}$$

Observe that that if there does not exist $i' \in I$ such that $u_2^*(i', \mu^{-1}(i')) \neq \frac{\sum_{i\in I} u_2^*(i,\mu^{-1}(i))}{N}$, then for at least one attorney $i \in I$ we must have that $r(q_i) > r$. In this case, the Jensen's inequality is a weak inequality; however, inequity (A.5) is a strict inequality. Therefore, inequality (A.6) holds with strict inequality.

By comparing inequality (A.3) and inequality (A.6), we get:

$$\sum_{i\in I} h(u_2^\dagger) < \sum_{i\in I} h(u_2^*(i, \mu^{-1}(i)).$$

Similarly:

$$\sum_{i\in I} h(u_1^\dagger) < \sum_{i\in I} h(u_1^*(i, \mu^{-1}(i)).$$

Therefore:

$$\mathcal{C}_u < \mathcal{C}_\mu.$$

We can conclude that the status quo system, i.e., the merit-based selection rule, the uniform random allocation rule, and nondiscriminatory wage contracts, is the optimal contract even among discriminatory wage contracts. ■

Proof of Lemma 6

Consider two classes $\theta' = (\theta'_1, ..., \theta'_n)$ and $\theta'' = (\theta''_1, ..., \theta''_n)$. Order all elements of these two vectors in descending order, denoted by $\bar{\theta}_1, \bar{\theta}_2, ..., \bar{\theta}_{2n}$, i.e., $\bar{\theta}_1$ is greater than or equal to all elements of both classes, $\bar{\theta}_2$ is the second greatest element of all elements of both classes, and so on. $\bar{\theta}_1$ is either in class θ' or in class θ''. Consider the permutation of these two classes such that $\bar{\theta}_1$ is the first element of one these two classes. Similarly $\bar{\theta}_2$ is either class θ' or in class θ''. Consider the permutation of these two classes such that $\bar{\theta}_1$ is the first element of one of these two classes and $\bar{\theta}_2$ is the second element of one of these two classes. We can do the same for $\bar{\theta}_1, \bar{\theta}_2, ..., \bar{\theta}_n$. Therefore, we have a permutation of two classes, where $\bar{\theta}_1$ is the first element of one class. Moreover, the first element of the other class is one of the following: $\bar{\theta}_{n+1}, \bar{\theta}_{n+2}, ..., \bar{\theta}_{2n}$. Note that $\bar{\theta}_1 \geq \bar{\theta}_{n+1}, \bar{\theta}_{n+2}, ..., \bar{\theta}_{2n}$. Similarly, $\bar{\theta}_2$ is the second element of one of the two classes, and the second element of the other class is one of $\bar{\theta}_{n+1}, \bar{\theta}_{n+2}, ..., \bar{\theta}_{2n}$. The same is true for any element i between 1 and n. $\bar{\theta}_i$ is the ith element of one these classes, and the ith element of the other class is one of $\bar{\theta}_{n+1}, \bar{\theta}_{n+2}, ..., \bar{\theta}_{2n}$. Furthermore, $\bar{\theta}_i \geq \bar{\theta}_{n+1}, \bar{\theta}_{n+2}, ..., \bar{\theta}_{2n}$. Hence, using coordinate-wise maximum and minimum on these permutations results in having $\bar{\theta}_1, \bar{\theta}_2, ..., \bar{\theta}_n$ in one class and $\bar{\theta}_{n+1}, \bar{\theta}_{n+2}, ..., \bar{\theta}_{2n}$ in the other class. Therefore, by using these permutations for these two classes, one-step sorting is equivalent to sorting. ■

Proof of Theorem 5

Consider a function $f(e, \theta)$; maximizing with respect to e we have:

$$f_e(e^*, \theta) = 0.$$

By the Implicit Function Theorem, we have:

$$e^* = g(\theta) \Rightarrow \frac{\partial e^*}{\partial \theta_i} = -\frac{\frac{\partial f_e}{\partial \theta_i}}{\frac{\partial f_e}{\partial e}} = -\frac{f_{e\theta_i}}{f_{ee}}. \tag{A.7}$$

Therefore,

$$\frac{\partial^2 e^*}{\partial \theta_i \partial \theta_j} = -\frac{f_{ee}^2 f_{e\theta_i\theta_j} + f_{eee} f_{e\theta_i} f_{e\theta_j} - f_{ee\theta_i} f_{e\theta_j} f_{ee} - f_{ee\theta_j} f_{e\theta_i} f_{ee}}{f_{ee}^3}. \tag{A.8}$$

Condition 8

$$f_{ee}^2 f_{e\theta_i\theta_j} + f_{eee} f_{e\theta_i} f_{e\theta_j} - f_{ee\theta_i} f_{e\theta_j} f_{ee} - f_{ee\theta_j} f_{e\theta_i} f_{ee} > 0, \forall e \in [0,1], i, j, \theta_i \in \mathcal{R}_+, \theta_j \in \mathcal{R}_+.$$

Condition 9

$$f_{ee}^2 f_{e\theta_i\theta_j} + f_{eee} f_{e\theta_i} f_{e\theta_j} - f_{ee\theta_i} f_{e\theta_j} f_{ee} - f_{ee\theta_j} f_{e\theta_i} f_{ee} < 0, \forall e \in [0,1], i, j, \theta_i \in \mathcal{R}_+, \theta_j \in \mathcal{R}_+.$$

If Condition 8 is satisfied, then (A.8) is strictly positive. Moreover, if Condition 1 is satisfied, then Condition 8 is satisfied. Hence, the argmax is strictly supermodular if Condition 1 is satisfied. If Condition 9 is satisfied, then (A.8) is strictly negative. Moreover, if condition 2 is satisfied, then Condition 9 is satisfied. Hence, the argmax is strictly submodular if Condition 2 is satisfied.

Note that

$$f_{ee}^2 f_{e\theta_i\theta_j} + f_{eee} f_{e\theta_i} f_{e\theta_j} - f_{ee\theta_i} f_{e\theta_j} f_{ee} - f_{ee\theta_j} f_{e\theta_i} f_{ee},$$

is either zero, strictly positive, or strictly negative locally. Therefore, locally the argmax is either strictly supermodular, strictly submodular, or modular. ∎

Proof of Example 1

Define $\bar{\theta} = (\frac{1}{n}) \sum_{i \in \mu(t)} \theta_i$. We have:

$$\begin{aligned}
P_{\theta_i} &= \tfrac{e}{n}. \\
P_e &= \bar{\theta}. \\
f_{\theta_i} &= \tfrac{e}{n} u'(.). \\
f_{\theta_i,\theta_j} &= (\tfrac{e}{n})^2 u''(.). \\
f_{\theta_i,\theta_j,e} &= 2(\tfrac{e}{n}) u''(.) + \bar{\theta}(\tfrac{e}{n})^2 u'''(.). \\
f_e &= \bar{\theta} u'(.) - c_e. \\
f_{e,e} &= (\bar{\theta})^2 u''(.) - c_{e,e}. \\
f_{e,e,e} &= (\bar{\theta})^3 u'''(.) - c_{e,e,e}. \\
f_{\theta_i,e,e} &= \tfrac{2}{n}(\bar{\theta}) u''(.) + \tfrac{e}{n}(\bar{\theta})^2 u'''(.).
\end{aligned}$$

Note that $f_{\theta_i,\theta_j,e}$ and $f_{\theta_i,e,e}$ are strictly negative, and $f_{e,e,e}$ is negative. Therefore, Condition 2 is satisfied.

$$\begin{aligned} P_i^* &= P_e e_i + p_i, \\ P_{i,j}^* &= p_{e,e}e_ie_j + p_{e,j}e_i + p_e e_{i,j} + p_{e,i}e_j + P_{i,j} \\ &= \tfrac{1}{n}(e_i + e_j) + \bar{\theta}e_{i,j}. \end{aligned}$$

Want to show $P_{i,j}^* < 0$:

$$\begin{aligned} & \tfrac{1}{n}(e_i + e_j) + \bar{\theta}e_{i,j} < 0 \Leftrightarrow 2e_i < -n\bar{\theta}e_{i,j} \\ \Leftrightarrow \quad & -2\tfrac{f_{e,i}}{f_{e,e}} < -n\bar{\theta}\tfrac{-1}{f_{e,e}^3}(\Phi) \\ \Leftrightarrow \quad & 2f_{e,i} < -n\bar{\theta}\tfrac{1}{f_{e,e}^2}(\Phi) \\ \Leftrightarrow \quad & \tfrac{2}{n\bar{\theta}} < \tfrac{-(\Phi)}{f_{e,e}^2 f_{e,i}}, \end{aligned}$$

where

$$\begin{aligned} -\Phi &= -f_{e,e}^2 f_{e,\theta_i,\theta_j} - f_{e,e,e}f_{e,\theta_i}f_{e,\theta_j} + f_{e,e,\theta_i}f_{e,\theta_j}f_{e,e} + f_{e,e,\theta_j}f_{e,\theta_i}f_{e,e} \\ &= -f_{e,e}^2 f_{e,\theta_i,\theta_j} - f_{e,e,e}f_{e,\theta_i}f_{e,\theta_j} + 2f_{e,e,\theta_i}f_{e,\theta_j}f_{e,e}. \end{aligned}$$

Note that

$$-f_{e,e}^2 f_{e,\theta_i,\theta_j} \geq 0, -f_{e,e,e}f_{e,\theta_i}f_{e,\theta_j} \geq 0.$$

Therefore,

$$-\Phi \geq 2f_{e,e,\theta_i}f_{e,\theta_j}f_{e,e}.$$

Because $f_{e,e}^2 f_{e,i} \geq 0$, it is enough to show:

$$\frac{2}{n\bar{\theta}} < \frac{2f_{e,e,\theta_i} f_{e,\theta_j} f_{e,e}}{f_{e,e}^2 f_{e,i}} \Leftrightarrow \frac{1}{n\bar{\theta}} < \frac{f_{e,e,\theta_i}}{f_{e,e}}.$$

Note that $f_{e,e} = (\bar{\theta})^2 u''(.) - c_{e,e}$ and $c_{e,e} > 0$; hence, $f_{e,e} \leq (\bar{\theta})^2 u''(.)$. Therefore, $\frac{f_{e,e,\theta_i}}{f_{e,e}} \geq \frac{f_{e,e,\theta_i}}{(\bar{\theta})^2 u''(.)}$. It is enough to show:

$$\begin{aligned}
& \tfrac{1}{n\bar{\theta}} < \tfrac{f_{e,e,\theta_i}}{(\bar{\theta})^2 u''(.)} \\
\Leftrightarrow \quad & \tfrac{1}{n} < \tfrac{f_{e,e,\theta_i}}{(\bar{\theta}) u''(.)} \\
& = \tfrac{\frac{2}{n}(\bar{\theta}) u''(.) + \frac{e}{n}(\bar{\theta})^2 u'''(.)}{(\bar{\theta}) u''(.)} \\
& = \tfrac{2}{n} + \tfrac{\frac{e}{n}(\bar{\theta})^2 u'''(.)}{(\bar{\theta}) u''(.)} \\
& = \tfrac{2}{n} + \tfrac{\frac{e}{n}(\bar{\theta})^2}{(\bar{\theta})} \tfrac{u'''(.)}{u''(.)}.
\end{aligned}$$

Because $\frac{u'''(.)}{u''(.)} \geq 0$, we have:

$$\frac{2}{n} + \frac{\frac{e}{n}(\bar{\theta})^2}{(\bar{\theta})} \frac{u'''(.)}{u''(.)} > \frac{2}{n}.$$

which is what we wanted to show. ■

Proof of Theorem 6

Consider a general performance function $p(e, \theta)$:

$$\frac{\partial(p\big|_{e=e^*})}{\partial\theta_i} = \frac{\partial p}{\partial e}\frac{\partial(e\big|_{e=e^*})}{\partial\theta_i} + \frac{\partial p}{\partial\theta_i}\Big|_{e=e^*} \Rightarrow \frac{\partial(p\big|_{e=e^*})}{\partial\theta_i\partial\theta_j} =$$

$$\left(\left(\frac{\partial^2 p}{\partial e\partial e}\frac{\partial e\big|_{e=e^*}}{\partial\theta_j} + \frac{\partial^2 p}{\partial e\partial\theta_j}\right)\frac{\partial(e\big|_{e=e^*})}{\partial\theta_i} + \frac{\partial p}{\partial e}\frac{\partial^2(e\big|_{e=e^*})}{\partial\theta_i\partial\theta_j} + \frac{\partial p}{\partial e\partial\theta_i}\frac{\partial(e\big|_{e=e^*})}{\partial\theta_j} + \frac{\partial^2 p}{\partial\theta_i\partial\theta_j}\right)\Bigg|_{e=e^*}. \quad \text{(A.9)}$$

Using (A.9) when performance is modular, i.e., $p_{e\theta_i}(e, \theta) = 0, p_{\theta_i\theta_j}(e, \theta) = 0 \quad \forall e \in [0, 1]$,

$\forall i, j, \theta_i \in \mathcal{R}_+, \theta_j \in \mathcal{R}_+$, we have:

$$\frac{\partial(p\big|_{e=e^*})}{\partial\theta_i\partial\theta_j} = \left(\left(\frac{\partial^2 p}{\partial e \partial e} \frac{\partial e\big|_{e=e^*}}{\partial\theta_j}\right) \frac{\partial(e\big|_{e=e^*})}{\partial\theta_i} + \frac{\partial p}{\partial e} \frac{\partial^2(e\big|_{e=e^*})}{\partial\theta_i\partial\theta_j} \right)\Big|_{e=e^*}. \tag{A.10}$$

1. By Theorem 5, under Condition 1, we have $\frac{\partial(e\big|_{e=e^*})}{\partial\theta_i\partial\theta_j} > 0$. Performance is convex, hence, we have $\frac{\partial^2 p}{\partial e \partial e} \geq 0$. Moreover, $\frac{\partial e\big|_{e=e^*}}{\partial\theta_i} > 0$ by Lemma 7. Therefore, $\frac{\partial(p\big|_{e=e^*})}{\partial\theta_i\partial\theta_j} > 0$.

2. By Theorem 5, under Condition 2, we have $\frac{\partial(e\big|_{e=e^*})}{\partial\theta_i\partial\theta_j} < 0$. Performance is concave, hence, we have $\frac{\partial^2 p}{\partial e \partial e} \leq 0$. Moreover, $\frac{\partial e\big|_{e=e^*}}{\partial\theta_i} > 0$ by Lemma 7. Therefore, $\frac{\partial(p\big|_{e=e^*})}{\partial\theta_i\partial\theta_j} < 0$.

■

Proof of Theorem 7

Consider a general performance function $p(e, \theta)$, and recall (A.9):

$$\frac{\partial(p\big|_{e=e^*})}{\partial\theta_i\partial\theta_j} = \left(\left(\frac{\partial^2 p}{\partial e \partial e} \frac{\partial e\big|_{e=e^*}}{\partial\theta_j} + \frac{\partial^2 p}{\partial e \partial\theta_j}\right) \frac{\partial(e\big|_{e=e^*})}{\partial\theta_i} + \frac{\partial p}{\partial e} \frac{\partial(e\big|_{e=e^*})}{\partial\theta_i\partial\theta_j} + \frac{\partial p}{\partial e\partial\theta_i} \frac{\partial(e\big|_{e=e^*})}{\partial\theta_j} + \frac{\partial^2 p}{\partial\theta_i\partial\theta_j} \right)\Big|_{e=e^*}.$$

1. $\frac{\partial(p\big|_{e=e^*})}{\partial\theta_i\partial\theta_j} > 0$ because:

   i) By convexity of performance, we have $\frac{\partial^2 p}{\partial e \partial e} \geq 0$.

   ii) By Lemma 7, we have $\frac{\partial e\big|_{e=e^*}}{\partial\theta_i} > 0 \forall i \in I$.

   iii) Performance is supermodular; therefore, we have $\frac{\partial^2 p}{\partial e \partial\theta_i} \geq 0 \forall i \in I$.

   iv) By Assumption 1, performance is increasing in effort $\frac{\partial p}{\partial e} > 0$.

   v) By Theorem 5, under Condition 1, the optimal effort is strictly supermodular $\frac{\partial(e\big|_{e=e^*})}{\partial\theta_i\partial\theta_j} > 0$.

   vi) Performance is supermodular; therefore, we have $\frac{\partial^2 p}{\partial\theta_i\partial\theta_j} \geq 0$.

Therefore, $\frac{\partial(p\big|_{e=e^*})}{\partial\theta_i\partial\theta_j} > 0$, i.e., sorting increases the total performance of students.

2. $\frac{\partial(p\big|_{e=e^*})}{\partial\theta_i\partial\theta_j} < 0$ because:

   i) By concavity of performance, we have $\frac{\partial^2 p}{\partial e\partial e} \leq 0$.

   ii) By Lemma 7, we have $\frac{\partial e\big|_{e=e^*}}{\partial\theta_i} > 0 \forall i \in I$.

   iii) Performance is submodular; therefore, we have $\frac{\partial^2 p}{\partial e\partial\theta_i} \leq 0 \forall i \in I$.

   iv) By Assumption 1, performance is increasing in effort $\frac{\partial p}{\partial e} > 0$.

   v) By Theorem 5, under Condition 2, the optimal effort is strictly submodular $\frac{\partial(e\big|_{e=e^*})}{\partial\theta_i\partial\theta_j} < 0$.

   vi) Performance is submodular; therefore, we have $\frac{\partial^2 p}{\partial\theta_i\partial\theta_j} \leq 0$.

   Therefore, $\frac{\partial(p\big|_{e=e^*})}{\partial\theta_i\partial\theta_j} < 0$, i.e., sorting decreases the total performance of students.

■

General Method

Using (A.7), (A.8), and (A.9), we have:

$$\frac{\partial(p\big|_{e=e^*})}{\partial\theta_i\partial\theta_j} = \left(\left(\left(\frac{\partial^2 p}{\partial e\partial e}\right) \left(-\frac{\frac{\partial^2 f}{\partial e\partial\theta_j}}{\frac{\partial^2 f}{\partial e\partial e}}\right) + \frac{\partial^2 p}{\partial e\partial\theta_j} \right) \left(-\frac{\frac{\partial^2 f}{\partial e\partial\theta_i}}{\frac{\partial^2 f}{\partial e\partial e}}\right) \right.$$

$$+ \quad \frac{\partial p}{\partial e}\left(-\frac{(\frac{\partial^2 f}{\partial e\partial e})^2(\frac{\partial^3 f}{\partial e\partial\theta_i\partial\theta_j})+(\frac{\partial^3 f}{\partial e\partial e\partial e})(\frac{\partial^2 f}{\partial e\partial\theta_i})(\frac{\partial^2 f}{\partial e\partial e\theta_j})-(\frac{\partial^3 f}{\partial e\partial e\partial\theta_i})(\frac{\partial^2 f}{\partial e\partial\theta_j})(\frac{\partial^2 f}{\partial e\partial e})-(\frac{\partial^3 f}{\partial e\partial e\partial\theta_j})(\frac{\partial^2 f}{\partial e\partial\theta_i})(\frac{\partial^2 f}{\partial e\partial e})}{(\frac{\partial^2 f}{\partial e\partial e})^3}\right)$$

$$\left. + \quad \frac{\partial p}{\partial e\partial\theta_i}\left(-\frac{\frac{\partial^2 f}{\partial e\partial\theta_j}}{\frac{\partial^2 f}{\partial e\partial e}}\right) + \frac{\partial^2 p}{\partial\theta_i\partial\theta_j} \right)\Bigg|_{e=e^*}.$$

Note that $\frac{\partial(p\big|_{e=e^*})}{\partial\theta_i\partial\theta_j}$ depends only on the primitives in this equation. Define:

$$\Psi(e,\theta) = \left(\left(\frac{\partial^2 p}{\partial e\partial e}\right)\left(-\frac{\frac{\partial^2 f}{\partial e\partial\theta_j}}{\frac{\partial^2 f}{\partial e\partial e}}\right) + \frac{\partial^2 p}{\partial e\partial\theta_j}\right)\left(-\frac{\frac{\partial^2 f}{\partial e\partial\theta_i}}{\frac{\partial^2 f}{\partial e\partial e}}\right)$$
$$+ \quad \frac{\partial p}{\partial e}\Bigg(-\frac{(\frac{\partial^2 f}{\partial e\partial e})^2(\frac{\partial^3 f}{\partial e\partial\theta_i\partial\theta_j})+(\frac{\partial^3 f}{\partial e\partial e\partial e})(\frac{\partial^2 f}{\partial e\partial\theta_i})(\frac{\partial^2 f}{\partial e\partial e\theta_j})-(\frac{\partial^3 f}{\partial e\partial e\partial\theta_i})(\frac{\partial^2 f}{\partial e\partial\theta_j})(\frac{\partial^2 f}{\partial e\partial e})-(\frac{\partial^3 f}{\partial e\partial e\partial\theta_j})(\frac{\partial^2 f}{\partial e\partial\theta_i})(\frac{\partial^2 f}{\partial e\partial e})}{(\frac{\partial^2 f}{\partial e\partial e})^3}\Bigg)$$

$$+ \quad \frac{\partial p}{\partial e\partial\theta_i}\left(-\frac{\frac{\partial^2 f}{\partial e\partial\theta_j}}{\frac{\partial^2 f}{\partial e\partial e}}\right) + \frac{\partial^2 p}{\partial\theta_i\partial\theta_j}. \tag{A.11}$$

If $\Psi(e,\theta) \geq (\leq)0$, then $\frac{\partial(p\big|_{e=e^*})}{\partial\theta_i\partial\theta_j} \geq (\leq)0$, i.e., sorting increases (decreases) the total performance of students. ■

Proof of Theorem 8

First, we extend Theorem 5:

Lemma 13

1. *If Condition 3 holds, sorting strictly increases the total effort of teachers.*

2. *If Condition 4 holds, sorting strictly decreases the total effort of teachers.*

Proof: Consider a function $f(e,q,\theta)$, maximizing with respect to e:

$$f_e(e,q,\theta) = 0 \Rightarrow e^* = g(q,\theta)$$
$$\Rightarrow \qquad \frac{\partial e^*}{\partial\theta_i} = -\frac{\frac{\partial f_e}{\partial\theta_i}}{\frac{\partial f_e}{\partial e}} = -\frac{f_{e\theta_i}}{f_{ee}}.$$

Therefore,

$$\frac{\partial^2 e^*}{\partial\theta_i\partial q} = -\frac{f_{ee}^2 f_{e\theta_i q} + f_{eee} f_{e\theta_i} f_{eq} - f_{ee\theta_i} f_{eq} f_{ee} - f_{eeq} f_{e\theta_i} f_{ee}}{f_{ee}^3}. \tag{A.12}$$

If Condition 3 is satisfied, then (A.12) is strictly positive. Moreover, Condition 3 implies Condition 1. Therefore, if Condition 3 is satisfied, then (A.8) is strictly positive, by the same argument as in proof of Theorem 5. Hence, the argmax is strictly supermodular if Condition 3 is satisfied and sorting strictly increases the total effort of teachers. If Condition 4 is satisfied, then (A.12) is strictly negative. Moreover, Condition 4 implies Condition 2. Therefore, if Condition 4 is satisfied, then (A.8) is strictly negative, by the same argument as in proof of Theorem 5. Hence, the argmax is strictly submodular if Condition 2 is satisfied and sorting strictly decreases the total effort of teachers. ∎

Consider a general performance function $p(e, q, \theta)$, and recall (A.9):

$$\frac{\partial(p\big|_{e=e^*})}{\partial\theta_i\partial q} = \left(\left(\frac{\partial^2 p}{\partial e\partial e}\frac{\partial e\big|_{e=e^*}}{\partial q} + \frac{\partial^2 p}{\partial e\partial q}\right)\frac{\partial(e\big|_{e=e^*})}{\partial\theta_i} + \frac{\partial p}{\partial e}\frac{\partial(e\big|_{e=e^*})}{\partial\theta_i\partial q} + \frac{\partial p}{\partial e\partial\theta_i}\frac{\partial(e\big|_{e=e^*})}{\partial q} + \frac{\partial^2 p}{\partial\theta_i\partial q}\right)\Bigg|_{e=e^*}.$$

1. To prove the first part of the theorem, we need to show that $p(e, q, \theta)$ is pairwise supermodular in students' types and pairwise supermodular in each student's type and the teacher's type at the optimal effort. In Theorem 7, we established that under these conditions, $p(e, q, \theta)$ is pairwise supermodular in students' types.

   We need to show that $\frac{\partial(p\big|_{e=e^*})}{\partial\theta_i\partial q} > 0$:

   i) By convexity of performance, we have $\frac{\partial^2 p}{\partial e\partial e} \geq 0$.

   ii) By Lemma 7, we have $\frac{\partial e\big|_{e=e^*}}{\partial\theta_i} > 0 \forall i \in I$.

   iii) Because performance is supermodular, we have $\frac{\partial^2 p}{\partial e\partial\theta_i} \geq 0 \forall i \in I$ and $\frac{\partial^2 p}{\partial e\partial q} \geq 0 \forall t$.

   iv) By Assumption 1, performance is increasing in effort $\frac{\partial p}{\partial e} > 0$.

v) By Lemma 13, under Condition 3, the optimal effort is strictly supermodular $\frac{\partial(e\big|_{e=e^*})}{\partial\theta_i\partial q} > 0$.

vi) Because performance is supermodular, we have $\frac{\partial^2 p}{\partial\theta_i\partial q} \geq 0$.

Therefore, $\frac{\partial(p\big|_{e=e^*})}{\partial\theta_i\partial q} > 0$. We can conclude that sorting increases the total performance of students.

2. Similarly, to prove the second part of the theorem, we need to show that $p(e,q,\theta)$ is pairwise submodular in students' types and pairwise submodular in each student's type and the teacher's type at the optimal effort. In Theorem 7, we established that under these conditions, $p(e,q,\theta)$ is pairwise submodular in students' types. We need to show that $\frac{\partial(p\big|_{e=e^*})}{\partial\theta_i\partial q} < 0$:

i) By concavity of performance, we have $\frac{\partial^2 p}{\partial e\partial e} \leq 0$.

ii) By Lemma 7, we have $\frac{\partial e\big|_{e=e^*}}{\partial\theta_i} > 0 \forall i \in I$.

iii) Because performance is submodular, we have $\frac{\partial^2 p}{\partial e\partial\theta_i} \leq 0 \forall i \in I$.

iv) By Assumption 1, performance is increasing in effort $\frac{\partial p}{\partial e} > 0$.

v) By Lemma 13, under Condition 4, the optimal effort is strictly submodular $\frac{\partial(e\big|_{e=e^*})}{\partial\theta_i\partial\theta_j} < 0$.

vi) Because performance is submodular, we have $\frac{\partial^2 p}{\partial\theta_i\partial\theta_j} \leq 0$.

Therefore, $\frac{\partial(p\big|_{e=e^*})}{\partial\theta_i\partial\theta_j} < 0$. We can conclude that sorting decreases the total performance of students.

■

Proof of Theorem 9

Consider a general performance function $p(e,q,\theta)$, and recall (A.9):

$$\frac{\partial(p\big|_{e=e^*})}{\partial\theta_i\partial q} = \left(\left(\frac{\partial^2 p}{\partial e\partial e}\frac{\partial e\big|_{e=e^*}}{\partial q} + \frac{\partial^2 p}{\partial e\partial q}\frac{\partial(e\big|_{e=e^*})}{\partial\theta_i} + \frac{\partial p}{\partial e}\frac{\partial(e\big|_{e=e^*})}{\partial\theta_i\partial q} + \frac{\partial p}{\partial e\partial\theta_i}\frac{\partial(e\big|_{e=e^*})}{\partial q} + \frac{\partial^2 p}{\partial\theta_i\partial q} \right) \right|_{e=e^*}.$$

By proof of Theorem 8, we know that $\frac{\partial(p\big|_{e=e^*})}{\partial\theta_i\partial q} < 0$.

Consider two classes after student sorting with NAM. By sorting these two classes, we get two new classes with student sorting with PAM. By Theorem 8, sorting decreases the total performance of students, i.e., the total performance of students under NAM is higher than under PAM. Furthermore, by Theorem 8, sorting increases inequality in students' performance, i.e., inequality in students' performance under NAM is higher than under PAM. To put it differently, when two classes are sorted, i.e., student sorting with PAM, changing the matching of teachers and class from PAM to NAM has the opposite effect of sorting. Because when two classes are sorted with NAM, sorting results in two classes that are sorted with PAM. ■

Proof of Lemma 10

We need to show that there is no blocking pair. Suppose i', j' and $\{w'(i', j', e)\}_{e\in[0,1]}$ is a blocking pair. First observe that if this wage schedule doesn't induce the efficient effort then the same worker and firm can block with a different wage schedule that induces the efficient effort. Under this wage schedule the worker will choose an effort e' and will receive w'.

Note that under the efficient effort firm j is indifferent between matching with worker $i = j$ and worker $i = j + 1$:

$$\pi(i, i, e^*(i, i)) = \pi(i + 1, i, e^*(i + 1, i)).$$

Moreover, worker $i+1$ is indifferent between matching with firm $j=i+1$ and firm $j=i$:

$$u(i+1,i+1,e^*(i+1,i+1)) = u(i+1,i,e^*(i+1,i)).$$

Furthermore, because total surplus is strictly supermodular, we have:

$$s^*(i,i)+s^*(i+1,i+1) > s^*(i,i+1)+s^*(i+1,i) \Leftrightarrow$$
$$\pi(i,i,e^*(i,i))+u(i,i,e^*(i,i))+\pi(i+1,i+1,e^*(i+1,i+1))$$
$$+u(i+1,i+1,e^*(i+1,i+1)) > s^*(i,i+1)+s^*(i+1,i) \Leftrightarrow$$
$$\pi(i+1,i,e^*(i+1,i)+u(i,i,e^*(i,i))+\pi(i+1,i+1,e^*(i+1,i+1))+$$
$$u(i+1,i,e^*(i+1,i)) > s^*(i,i+1)+s^*(i+1,i) \Leftrightarrow$$
$$u(i,i,e^*(i,i))+\pi(i+1,i+1,e^*(i+1,i+1)) > s^*(i,i+1)$$

Similarly we can show:

$$u(i+1,i+1,e^*(i+1,i+1))+\pi(i+2,i+2,e^*(i+2,i+2)) > s^*(i+1,i+2)$$

Adding these two inequalities and using the fact that $u(i+1,i+1,e^*(i+1,i+1))+\pi(i+1,i+1,e^*(i+1,i+1)) = s^*(i+1,i+1)$ we have:

$$s^*(i+1,i+1)+u(i,i,e^*(i,i))+\pi(i+2,i+2,e^*(i+2,i+2))$$
$$> s^*(i+1,i+2)+s^*(i,i+1) \Leftrightarrow$$
$$u(i,i,e^*(i,i))+\pi(i+2,i+2,e^*(i+2,i+2))$$
$$> s^*(i+1,i+2)+s^*(i,i+1)-s^*(i+1,i+1).$$

Moreover, by supermodularity of s^* (lemma 9) we have:

$$s^*(i,i+2) < s^*(i+1,i+2)+s^*(i,i+1)-s^*(i+1,i+1).$$

Hence:

$$u(i,i,e^*(i,i)) + \pi(i+2,i+2,e^*(i+2,i+2)) > s^*(i,i+2).$$

We can do the same process for any $i' > i$. Hence no worker i and firm j were $i > j$ can block this CE.

Now consider $i < j$. First observe that:

$$u(i+1,i+1,e^*(i+1,i+1)) + \pi(i,i,e^*(i,i)) =$$
$$u(i+1,i,e^*(i+1,i)) + \pi(i+1,i,e^*(i+1,i) = s^*(i+1,i)$$

Hence $i+1, j=i$ can't block. Moreover, by supermodularity of total surplus:

$$s^*(i+2,i+1) + s^*(i+1,i) > s^*(i+2,i) + s^*(i+1,i+1) \Leftrightarrow$$
$$u(i+2,i+2,e^*(i+2,i+2)) + \pi(i+1,i+1,e^*(i+1,i+1)) +$$
$$u(i+1,i+1,e^*(i+1,i+1)) + \pi(i,i,e^*(i,i)) > s^*(i+2,i) + s^*(i+1,i+1) \Leftrightarrow$$
$$u(i+2,i+2,e^*(i+2,i+2)) + s^*(i+1,i+1) + \pi(i,i,e^*(i,i))$$
$$> s^*(i+2,i) + s^*(i+1,i+1) \Leftrightarrow$$
$$u(i+2,i+2,e^*(i+2,i+2)) + \pi(i,i,e^*(i,i)) > s^*(i+2,i)$$

Hence $i+2, j=i$ can't block.

Note that each worker gets at least zero utility and each firm makes positive profit, firms are best replying, and workers are choosing the utility maximizing effort level. ■

Proof of Lemma 11

Suppose not, then there exists at least one pair of worker-firm such that the worker is not choosing the efficient effort. Therefore, there is another contract that induces the efficient effort and increases the surplus of the match. Hence, there is a wage schedule that induces the efficient effort, gives the worker a strictly higher utility, and strictly increases the firm's profit. Therefore, a blocking pair exists, a contradiction. ∎

Proof of Theorem 10

Suppose not, then there exists at least two pairs of worker-firm in a contracting equilibrium such that $i < i'$, $j > j'$, $\hat{\mu}(i) = j$, and $\hat{\mu}(i') = j'$. By Lemma 9, the total surplus is strictly supermodular in the types of firms and workers, given the efficient effort for every firm-worker pair. Since the total surplus is strictly supermodular, if we match i with j' and i' with j and let them choose the efficient effort, the total surplus increases:

$$s(i, j, e^*(i, j)) + s(i', j', e^*(i', j')) < s(i, j', e^*(i, j')) + s(i', j, e^*(i', j))$$

Hence, either (i, j') or (i', j) is a blocking pair. ∎

Proof of Lemma 12

The following algorithm with n steps proves the result. Do the following step for $i = 1, ..., n$: In step i: Select the following two pairs of matched worker-firm pairs: The pair that has firm i as the firm and the pair that has worker i as the worker. If these are different pairs, use meet and joint on these two pair, otherwise go to the next step.

In each step i, worker i will match with firm $j = i$ as the result of joint operation. Hence, after n steps, the matching is positive assortative matching. ∎

Proof of Theorem 11

1. Kamali-Shahdadi (2016) proves that, under Condition 5, the argmax (the efficient effort) is strictly supermodular $e_{ij}^* > 0$. By taking cross partial derivatives of production function at the efficient effort, we have:

$$v_{ij}^* = v_{ij} + v_{ie}e_j^* + v_{je}e_i^* + v_{ee}e_i^*e_j^* + v_e e_{ij}^*$$
$$\Rightarrow \qquad v_{ij}^* = s_{ij}^* + c_{ee}e_i^*e_j^* + v_e e_{ij}^*.$$

Recall that, $s_{ij}^* < 0 >$. By Assumption 1, $c_{ee} > 0, v_e > 0$. Given Assumption 4 or 5 $e_i^*e_j^* > 0$. Therefore, the total output at the efficient effort is supermodular. Hence, PAM has the highest total surplus, the highest total output, the highest total labor, and the lowest total leisure.

2. Because the argmax (the efficient effort) is strictly supermodular, by eliminating all frictions in the labor market, the total effort increases. Given Assumption 4, the efficient effort is increasing in firm's technology. Recall that, starting from any matching, there is a finite sequence of meet and joint operation such that the final matching is PAM. Therefore, it is enough to prove the result for implementing meet and joint operations once. Consider two pairs of worker-firm pairs, $(i, j), (i', j')$, where $i < i', j > j'$. Implementing meet and joint operation is equivalent to assigning better worker i' to the better firm j. The better workers, matched with the better firm, works more than the other worker matched with the better firm. Moreover, this increase in labor is more that the decrease in labor in the lower firm after implementing meet and joint operations. Stated differently, the labor at the higher firm was higher at the initial matching, compared with the labor at the lower firm. Moreover, the increase in labor at the higher firm is greater than the decrease in the lower firm. Because the cost function is convex, the total cost of effort increases as the result of implementing meet and joint operations.

■

Proof of Theorem 12

1. Kamali-Shahdadi (2016) proves that, under Condition 6, the argmax (the efficient effort) is strictly submodular. By taking cross partial derivatives of production function at the efficient effort, we have:

$$
\begin{aligned}
& v^*_{ij} = v_{ij} + v_{ie}e^*_j + v_{je}e^*_i + v_{ee}e^*_i e^*_j + v_e e^*_{ij} \\
\Rightarrow \quad & v^*_{ij} = s_{ij} + s_{ie}(\tfrac{-s_{ej}}{s_{ee}}) + s_{je}(\tfrac{-s_{ei}}{s_{ee}}) + v_{ee}(\tfrac{-s_{ei}}{s_{ee}})(\tfrac{-s_{ej}}{s_{ee}}) \\
& -v_e(\tfrac{s^2_{ee}s_{eij}+s_{eee}s_{ei}s_{ej}-s_{eei}s_{ej}s_{ee}-s_{eej}s_{ei}s_{ee}}{s^3_{ee}}) \\
\Rightarrow \quad & v^*_{ij} = (\tfrac{-1}{s^3_{ee}})\big(s_{ij}(-s^3_{ee}) + 2s_{ie}(s_{ej})(s^2_{ee}) + v_{ee}(s_{ei})(s_{ej})(-s_{ee}) \\
& +v_e s^2_{ee}s_{eij} + v_e s_{eee}s_{ei}s_{ej} - v_e s_{eei}s_{ej}s_{ee} - v_e s_{eej}s_{ei}s_{ee}\big) \\
\Rightarrow \quad & v^*_{ij} = (\tfrac{-1}{s^3_{ee}})\big((s_{ei}s_{ej})(v_e s_{eee} + 2s^2_{ee} - v_{ee}s_{ee}) + \\
& (s^2_{ee})(v_e s_{eij} - s_{ij}s_{ee}) - (v_e s_{ee})(s_{eei}s_{ej} + s_{eej}s_{ei})\big)
\end{aligned}
$$

   Under Condition 6, the total output at the efficient effort is supermodular. Hence, PAM has the highest total surplus, highest total output, lowest total labor, and highest total leisure.

2. Because the argmax (the efficient effort) is submodular, the total effort decreases. Under Assumption 5, the efficient effort is decreasing in firm's technology.

   Recall that, starting from any matching, there is a finite sequence of meet and joint operation such that the final matching is PAM. Therefore, it is enough to prove the result for implementing meet and joint operations once. Consider two pairs of worker-firm pairs, $(i, j), (i', j')$, where $i < i', j > j'$. Implementing meet and joint operation is equivalent to assigning better worker i' to the better firm j. The better workers, matched with the better firm, works less than the other worker matched with the better firm. Moreover, this decrease in labor is more that the increase in labor in the lower firm after implementing meet and joint operations. Stated differently, the labor at the higher firm was lower at the initial matching, compared with the labor at the lower firm. Moreover, the decrease in labor at the higher firm is greater than the increase in the lower firm. Because the cost function is convex, the total cost of effort decreases as the result of implementing meet and joint operations.

■

Proof of Theorem 13

1. Kamali-Shahdadi (2016) proves that, under Condition 6, the argmax (the efficient effort) is strictly submodular. Recall that:

$$v_{ij}^{*} = \left(\frac{-1}{s_{ee}^{3}}\right)\Big((s_{ei}s_{ej})(v_e s_{eee} + 2s_{ee}^{2} - v_{ee}s_{ee}) + (s_{ee}^{2})(v_e s_{eij} - s_{ij}s_{ee}) - (v_e s_{ee})(s_{eei}s_{ej} + s_{eej}s_{ei})\Big)$$

   Under Condition 7, the total output at the efficient effort is submodular. Hence, PAM has the highest total surplus, lowest total output, lowest total labor, and highest total leisure.

2. Eliminating frictions in the labor market results in PAM. By first part of the theorem, PAM has the highest total surplus, lowest total output.

■

Proof of Theorem 14

In a contracting equilibrium, if $i < i'$ and $j < j'$ where (i, j) and (i', j') are feasible matches, then (i, j') and (i', j) is not part of a stable matching; i.e., it is not an equilibrium outcome, because either (i, j) or (i', j') is a blocking pair.

We want to show that for any given labor market with friction, if the total surplus in a contracting equilibrium is strictly supermodular in worker's type and firm's type, then the profit of the firm is strictly supermodular in worker's type and firm's type. Define $\hat{r}(i,j) = \hat{w}(i) - c(e^*(i,j))$ as the rent of worker i when he/she is matched with firm j, in a given contracting equilibrium $< \{\hat{w}(i,j,e^*(i,j))\}_{e\in[0,1]}, \hat{\mu} >$. Consider $i < i'$ and $j < j'$, where (i,j) and (i',j') are part of the induced matching in the contracting equilibrium. By revealed preferences, firm j' prefers its current match compared with the worker assigned to firm j:

$$\pi(i',j',\hat{r}(i',j')) \geq \pi(i,j',\hat{r}(i,\hat{\mu}(i))),$$

$$\pi(i,j,\hat{r}(i,j)) \geq \pi(i',j,\hat{r}(i',\hat{\mu}(i')))$$

$$\Rightarrow \quad \pi(i',j',\hat{r}(i',j')) + \pi(i,j,\hat{r}(i,j)) \geq \pi(i,j',\hat{r}(i,j)) + \pi(i',j,\hat{r}(i',j'))$$

Therefore, the profit at the efficient effort level is strictly supermodular. Similar argument proves the second part of this theorem.

■

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